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ON THE λ-DAEHEE POLYNOMIALS WITH q-PARAMETER

JIN-WOO PARK

ABSTRACT. In this paper, we consider the generalization of Daehee polynomials with q-parameter and investigate some properties of those polynomials.

1. INTRODUCTION

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will respectively denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completions of algebraic closure of $\mathbb{Q}_p$. The p-adic norm is defined $|p|_p = \frac{1}{p}$.

When one talks of q-extension, q is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. Throughout this paper, we use the notation :

$$[x]_q = \frac{1-q^x}{1-q}.$$

Note that $\lim_{q\to 1}[x]_q = x$ for each $x \in \mathbb{Z}_p$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the *p-adic invariant integral on* $\mathbb{Z}_p$ is defined by Kim as follows :

$$I(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_0(x) = \lim_{n\to\infty} \frac{1}{p^n} \sum_{x=0}^{p^n-1} f(x)\,, \text{ (see [4, 5, 6]).} \tag{1.1}$$

Let f_1 be the translation of f with $f_1(x) = f(x+1)$. Then, by (1.1), we get

$$I(f_1) = I(f) + f'(0)\,, \text{ where } f'(0) = \left.\frac{df(x)}{dx}\right|_{x=0}. \tag{1.2}$$

As it is well-known fact, the *Stirling number of the first kind* is defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)\, x^l, \tag{1.3}$$

and the *Stirling number of the second kind* is given by the generating function to be

$$\left(e^t - 1\right)^m = m! \sum_{l=m}^{\infty} S_2(l,m) \frac{t^l}{l!}, \tag{1.4}$$

(see [1, 10]).

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Key words and phrases. Bernoulli polynomials, Daehee polynomials with q-parameter, p-adic invariant integral.

Unsigned Stirling numbers of the first kind is given by

$$x^{\underline{n}} = x(x+1)\cdots(x+n-1) = \sum_{l=0}^{n} |S_1(n,l)|x^l. \tag{1.5}$$

Note that if we replace x to $-x$ in (1.3), then

$$\begin{aligned}(-x)_n =& (-1)^n x^{\underline{n}} = \sum_{l=0}^{n} S_1(n,l)(-1)^l x^l \\ =& (-1)^n \sum_{l=0}^{n} |S_1(n,l)|x^l.\end{aligned} \tag{1.6}$$

Hence $S_1(n,l) = |S_1(n,l)|(-1)^{n-l}$.

For $r \in \mathbb{N}$, the *Bernoulli polynomials of order* r are defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!}, \text{ (see [7, 8, 11])}. \tag{1.7}$$

When $x = 0$, $B_n^{(r)} = B_n^{(r)}(0)$ are called the *Bernoulli numbers of order* r, and in the special case, $r = 1$, $B_n^{(1)}(x) = B_n(x)$ are called the *ordinary Bernoulli polynomials.*

For $n \in \mathbb{N}$, let T_p be the p-adic locally constant space defined by

$$T_p = \bigcup_{n\geq 1} C_{p^n} = \lim_{n\to\infty} C_{p^n},$$

where $C_{p^n} = \left\{\omega | \omega^{p^n} = 1\right\}$ is the cyclic group of order p^n.

We assume that q is an indeterminate in $\mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$. Then we define the q-analogue of falling factorial sequence as follows :

$$(x)_{n,q} = x(x-q)(x-2q)\cdots(x-(n-1)q), \ (n \geq 1), \ (x)_{0,q} = 1.$$

Note that

$$\lim_{q\to 1}(x)_{n,q} = (x)_n = \sum_{l=0}^{n} S_1(n,l)x^l.$$

Recently, D. S. Kim and T. Kim introduced the *Daehee polynomials* as follows :

$$D_n(x) = \int_{\mathbb{Z}_p} (x+y)_n d\mu_0(y), (n \geq 0), \text{ (see [2, 5, 9])}. \tag{1.8}$$

When $x = 0$, $D_n = D_n(0)$ are called the *n's Daehee numbers.* From (1.8), we can derive the generating function to be

$$\left(\frac{\log(1+t)}{t}\right)(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}, \text{ (see [2])}. \tag{1.9}$$

In addition, D. S. Kim et. al. consider the *Daehee polynomials with q-parameter* which is defined by the generating function to be

$$\sum_{n=0}^{\infty} D_{n,q}\frac{t^n}{n!} = (1+qt)^{\frac{x}{q}} \frac{\log(1+qt)}{q\left((1+qt)^{\frac{1}{q}}-1\right)}, \text{ (see [3])}. \tag{1.10}$$

When $x = 0$, $D_{n,q} = D_{n,q}(0)$ are called the *Daehee numbers with q-parameter.*

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In the viewpoint of generalization of the Daehee polynomials with q-parameter, we consider the λ-*Daehee polynomials with* q-*parameter* are defined to be

$$\sum_{n=0}^{\infty} D_{n,q}(\lambda|x)\frac{t^n}{n!} = (1+qt)^{\frac{x}{q}} \frac{\lambda \log(1+qt)}{q\left((1+qt)^{\frac{\lambda}{q}}-1\right)}. \tag{1.11}$$

When $x = 0$, $D_{n,q}(\lambda) = D_{n,q}(\lambda|0)$ are called the λ-*Daehee numbers with* q-*parameter*. In particular, the case $\lambda = 1$ is the Daehee polynomials with q-parameter.

In this paper, we give a p-adic integral representation of the λ- Daehee polynomials with q-parameter, which are called the Witt-type formula for the λ-Daehee polynomials with q-parameter. We can derive some interesting properties related to the λ-Daehee polynomials with q-parameter.

2. Witt-type formula for the n-th λ-Daehee polynomials with q-parameter

In this section, we assume that $t, q \in \mathbb{C}_p$ with $|t|_p < |q|_p p^{-\frac{1}{p-1}}$ and $\lambda \in \mathbb{Z}_p$. First, we consider the following integral representation associated with falling factorial sequences :

$$\int_{\mathbb{Z}_p} (x+\lambda y)_{n,q} d\mu_0(y), \text{ where } n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}. \tag{2.1}$$

By (2.1),

$$\begin{aligned}\sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (x+\lambda y)_{n,q} d\mu_0(y)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} q^n \int_{\mathbb{Z}_p} \left(\frac{x+\lambda y}{q}\right)_n d\mu_0(y)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} q^n \int_{\mathbb{Z}_p} \binom{\frac{x+\lambda y}{q}}{n} d\mu_0(y) t^n \\ &= \int_{\mathbb{Z}_p} (1+qt)^{\frac{x+\lambda y}{q}} d\mu_0(y)\end{aligned} \tag{2.2}$$

where $t \in \mathbb{C}_p$ with $|t|_p < |q|_p p^{-\frac{1}{p-1}}$. For $t \in \mathbb{C}_p$ with $|t|_p < |q|_p p^{-\frac{1}{p-1}}$, we get

$$\begin{aligned}\int_{\mathbb{Z}_p} (1+qt)^{\frac{x+\lambda y}{q}} d\mu_0(y) &= (1+qt)^{\frac{x}{q}} \frac{\lambda \log(1+qt)}{q\left((1+qt)^{\frac{\lambda}{q}}-1\right)} \\ &= \sum_{n=0}^{\infty} D_{n,q}(\lambda|x)\frac{t^n}{n!}.\end{aligned} \tag{2.3}$$

By (2.2) and (2.3), we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$D_{n,q}(\lambda|x) = \int_{\mathbb{Z}_p} (x+\lambda y)_{n,q} d\mu_0(y).$$

In (2.3), by replacing t by $\frac{1}{q}\left(e^t-1\right)$, we have

$$\sum_{n=0}^{\infty}\frac{D_{n,q}(\lambda|x)}{q^n}\frac{\left(e^t-1\right)^n}{n!}=e^{\frac{x}{q}t}\frac{\frac{\lambda}{q}t}{e^{\frac{\lambda}{q}t}-1}=\sum_{n=0}^{\infty}B_n\left(\frac{x}{\lambda}\right)\frac{\lambda^n}{q^n}\frac{t^n}{n!},\tag{2.4}$$

and

$$\sum_{n=0}^{\infty}\frac{D_{n,q}(\lambda|x)}{q^n}\frac{1}{n!}\left(e^t-1\right)^n=\sum_{n=0}^{\infty}\frac{D_{n,q}(\lambda|x)}{q^n}\sum_{m=n}^{\infty}S_2(m,n)\frac{t^m}{m!}=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{m}\frac{D_{m,q}(\lambda|x)}{q^m}S_2(n,m)\right)\frac{t^n}{n!}.\tag{2.5}$$

By (2.4) and (2.5), we obtain the following corollary.

Corollary 2.2. *For $n\geq 0$, we have*

$$B_n\left(\frac{x}{\lambda}\right)=\sum_{m=0}^{n}D_{m,q}(\lambda|x)q^{n-m}\lambda^{-n}S_2(n,m).$$

By the Theorem 2.1,

$$\begin{aligned}D_{n,q}(\lambda|x)&=\int_{\mathbb{Z}_p}(x+\lambda y)_{n,q}d\mu_0(y)\\&=q^n\int_{\mathbb{Z}_p}\left(\frac{x+\lambda y}{q}\right)_n d\mu_0(y)\\&=q^n\sum_{l=0}^{n}\frac{1}{q^l}S_1(n,l)\int_{\mathbb{Z}_p}(x+\lambda y)^l d\mu_0(y).\end{aligned}\tag{2.6}$$

By (1.2), we can derive easily that

$$\begin{aligned}\int_{\mathbb{Z}_p}e^{(x+\lambda y)t}d\mu_0(y)&=\frac{\lambda t}{e^{\lambda t}-1}e^{xt}=\sum_{n=0}^{\infty}B_n\left(\frac{x}{\lambda}\right)\frac{(\lambda t)^n}{n!}\\&=\sum_{l=0}^{\infty}\int_{\mathbb{Z}_p}(x+\lambda y)^l d\mu_0(y)\frac{t^l}{l!},\end{aligned}\tag{2.7}$$

and so

$$B_n\left(\frac{x}{\lambda}\right)=\int_{\mathbb{Z}_p}\left(\frac{x}{\lambda}+y\right)^n d\mu_0(y),\ (n\geq 0).\tag{2.8}$$

By (1.6), (2.7) and (2.8), we obtain the following corollary.

Corollary 2.3. *For $n\geq 0$, we have*

$$\begin{aligned}D_{n,q}(\lambda|x)&=\sum_{l=0}^{n}q^{n-l}S_1(n,l)\lambda^l B_l\left(\frac{x}{\lambda}\right)\\&=\sum_{l=0}^{n}|S_1(l,n)|(-q)^{n-l}\lambda^l B_l\left(\frac{x}{\lambda}\right).\end{aligned}$$

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From now on, we consider *λ-Daehee polynomials of order $k(\in\mathbb{N})$ with q-parameter*. λ-Daehee polynomials of order k with q-parameter are defined by the multivariant p-adic invariant integral on $\mathbb{Z}_p$:

$$D_{n,q}^{(k)}(\lambda|x)=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(\lambda(x_1+\cdots+x_k)+x)_{n,q}\,d\mu_0(x_1)\cdots d\mu_0(x_k) \tag{2.9}$$

where n is an nonnegative integer and $k\in\mathbb{N}$. In the special case, $x=0$, $D_{nq}^{(k)}(\lambda)=D_{n,q}^{(k)}(\lambda|0)$ are called the *λ-Daehee numbers of order k with q-parameter.*

From (2.9), we can derive the generating function of $D_{n,q}^{(k)}(x)$ as follows:

$$\begin{aligned}
&\sum_{n=0}^{\infty}D_{n,q}^{(k)}(\lambda|x)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}q^n\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{\frac{\lambda(x_1+\cdots+x_k)+x}{q}}{n}d\mu_0(x_1)\cdots d\mu_0(x_k)t^n\\
&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+qt)^{\frac{\lambda(x_1+\cdots+x_k)+x}{q}}\,d\mu_0(x_1)\cdots d\mu_0(x_k)\\
&=(1+qt)^{\frac{x}{q}}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+qt)^{\frac{\lambda(x_1+\cdots+x_k)}{q}}\,d\mu_0(x_1)\cdots d\mu_0(x_k)\\
&=(1+qt)^{\frac{x}{q}}\left(\frac{\lambda\log(1+qt)}{q\left((1+qt)^{\frac{\lambda}{q}}-1\right)}\right)^k.
\end{aligned} \tag{2.10}$$

Note that, by (2.9),

$$\begin{aligned}
&D_{n,q}^{(k)}(\lambda|x)\\
=&q^n\sum_{m=0}^{n}\frac{S_1(n,m)}{q^m}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(\lambda(x_1+\cdots+x_k)+x)^m d\mu_0(x_1)\cdots d\mu_0(x_k).
\end{aligned} \tag{2.11}$$

Since

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}e^{(x_1+\cdots+x_k+x)t}d\mu_0(x_1)\cdots d\mu_0(x_k)\\
=&\left(\frac{t}{e^t-1}\right)^k e^{xt}=\sum_{n=0}^{\infty}B_n^{(k)}(x)\frac{t^n}{n!},
\end{aligned}$$

we can derive easily

$$B_n^{(k)}(x)=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_k+x)^n d\mu_0(x_1)\cdots d\mu_0(x_k). \tag{2.12}$$

Thus, by (2.11) and (2.12), we have

$$\begin{aligned}
D_{n,q}^{(k)}(\lambda|x)=&q^n\sum_{m=0}^{n}\frac{S_1(n,m)}{q^m}\lambda^m B_m^{(k)}\left(\frac{x}{\lambda}\right)\\
=&\sum_{m=0}^{n}q^{n-m}S_1(n,m)B_m^{(k)}\left(\frac{x}{\lambda}\right)\\
=&\sum_{m=0}^{n}|S_1(n,m)|(-q)^{n-m}B_m^{(k)}\left(\frac{x}{\lambda}\right).
\end{aligned} \tag{2.13}$$

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In (2.10), by replacing t by $\frac{1}{q}(e^t-1)$, we get

$$\begin{aligned}\sum_{n=0}^{\infty} D_{n,q}^{(k)}(\lambda|x)\frac{(e^t-1)^n}{q^n n!} &= e^{\frac{x}{q}t}\left(\frac{\frac{\lambda t}{q}}{e^{\frac{\lambda}{q}t}-1}\right)^k \\ &= \sum_{n=0}^{\infty} \lambda^n \frac{B_n^{(k)}\left(\frac{x}{\lambda}\right)}{q^n}\frac{t^n}{n!},\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{D_{n,q}^{(k)}(\lambda|x)}{q^n}\frac{1}{n!}(e^t-1)^n &= \sum_{n=0}^{\infty} \frac{D_{n,q}^{(k)}(\lambda|x)}{q^n}\sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!} \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\frac{D_{n,q}^{(k)}(\lambda|x)}{q^n}S_2(m,n)\right)\frac{t^m}{m!}.\end{aligned} \tag{2.15}$$

By (2.13), (2.14) and (2.15), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$ and $k \in \mathbb{N}$, we have*

$$\begin{aligned}D_{n,q}^{(k)}(\lambda|x) &= \sum_{m=0}^{n} q^{n-m}S_1(n,m)B_m^{(k)}\left(\frac{x}{\lambda}\right) \\ &= \sum_{m=0}^{n} |S_1(n,m)|(-q)^{n-m}B_m^{(k)}\left(\frac{x}{\lambda}\right),\end{aligned}$$

and

$$B_n^{(k)}\left(\frac{x}{\lambda}\right) = \lambda^{-n}\sum_{m=0}^{n} D_{m,q}^{(k)}(\lambda|x)q^{n-m}S_2(n,m).$$

Now, we consider the λ-*Daehee polynomials of the second kind with q-parameter* as follows :

$$\widehat{D}_{n,\xi,q}(\lambda|x) = \int_{\mathbb{Z}_p}(-\lambda y+x)_{n,q}d\mu_0(y),\ (n\geq 0). \tag{2.16}$$

In the special case, $x=0$, $\widehat{D}_{n,q}(\lambda) = \widehat{D}_{n,q}(\lambda|0)$ are called the λ-*Daehee numbers of the second kind with q-parameter.*

By (2.16), we have

$$\widehat{D}_{n,q}(\lambda|x) = q^n\int_{\mathbb{Z}_p}\left(\frac{-\lambda y+x}{q}\right)_n d\mu_0(y), \tag{2.17}$$

and so we can derive the generating function of $\widehat{D}_{n,q}(x)$ by (1.1) as follows :

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{D}_{n,q}(\lambda|x)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} q^n\int_{\mathbb{Z}_p}\left(\frac{-\lambda y+x}{q}\right)_n d\mu_0(y)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} q^n\int_{\mathbb{Z}_p}\binom{\frac{-\lambda y+x}{q}}{n}d\mu_0(y)t^n \\ &= \int_{\mathbb{Z}_p}(1+qt)^{\frac{-\lambda y+x}{q}}d\mu_0(y) \\ &= (1+qt)^{\frac{x+\lambda}{q}}\frac{\lambda\log(1+qt)}{q\left((1+qt)^{\frac{\lambda}{q}}-1\right)}.\end{aligned} \tag{2.18}$$

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From (1.3), (1.6) and (2.17), we get

$$\begin{aligned}\widehat{D}_{n,q}(\lambda|x) =& q^n \int_{\mathbb{Z}_p} \left(\frac{-\lambda y + x}{q}\right)_n d\mu_0(y)\\ =& q^n \int_{\mathbb{Z}_p} \sum_{l=0}^{n} \frac{S_1(n,l)}{q^l}(-\lambda y + x)^l d\mu_0(y)\\ =& \sum_{l=0}^{n} S_1(n,l)(-\lambda)^l \int_{\mathbb{Z}_p} \left(y - \frac{x}{\lambda}\right)^l d\mu_0(y) q^{n-l}\\ =& \sum_{l=0}^{n} S_1(n,l)(-\lambda)^l B_l\left(-\frac{x}{\lambda}\right) q^{n-l}\\ =& (-1)^n \sum_{l=0}^{n} |S_1(n,l)|\lambda^l B_l\left(-\frac{x}{\lambda}\right) q^{n-l}.\end{aligned} \tag{2.19}$$

By replacing qt to $e^t - 1$ in the equation (2.18), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \widehat{D}_{n,q}(\lambda|x)\frac{1}{n!}\left(e^t - 1\right)^n =& \frac{\frac{\lambda}{q}t}{q\left(e^{\frac{\lambda}{q}t} - 1\right)} e^{\frac{(x+\lambda)t}{q}}\\ =& \sum_{n=0}^{\infty} B_n\left(1 + \frac{x}{\lambda}\right)\lambda^n q^{-n}\frac{t^n}{n!},\end{aligned} \tag{2.20}$$

and, by (1.4),

$$\sum_{n=0}^{\infty} \widehat{D}_{n,q}(\lambda|x)\frac{1}{n!}\left(e^t - 1\right)^n = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} \widehat{D}_{m,q}(\lambda|x)S_2(n,m)\right)\frac{t^m}{m!}. \tag{2.21}$$

Note that , by (1.10), it is easy to show that $B_n(-x) = (-1)^n B_n(x+1)$. Thus, from (2.19), (2.20) and (2.21), we have the following theorem.

Theorem 2.5. *For $n \geq 0$, we have*

$$\begin{aligned}\widehat{D}_{n,q}(\lambda|x) =& \sum_{l=0}^{n} S_1(n,l)(-\lambda)^l B_l\left(-\frac{x}{\lambda}\right) q^{n-l}\\ =& (-1)^n \sum_{l=0}^{n} |S_1(n,l)|\lambda^l q^{n-l} B_l\left(-\frac{x}{\lambda}\right).\end{aligned}$$

and

$$\lambda^n B_n\left(1 + \frac{x}{\lambda}\right) = q^n \sum_{m=0}^{n} \widehat{D}_{m,q}(\lambda|x)S_2(n,m).$$

By Theorem 2.5, we obtain the following corollary.

Corollary 2.6. *For $n \geq 0$,*

$$\widehat{D}_{n,q}(\lambda|x) = q^n \sum_{l=0}^{n}\sum_{m=0}^{l} \widehat{D}_{m,q}(\lambda|x)S_1(n,l)S_2(l,m). \tag{2.22}$$

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Now, we observe that

$$\begin{aligned} q^{-n}(-1)^n\frac{D_{n,q}(\lambda|x)}{n!} =&(-1)^n\int_{\mathbb{Z}_p}\binom{\frac{x+\lambda y}{q}}{n}d\mu_0(y)\\ =&\int_{\mathbb{Z}_p}\binom{-\frac{x+\lambda y}{q}+n-1}{n}d\mu_0(y)\\ =&\sum_{m=1}^{n}\binom{n-1}{n-m}\int_{\mathbb{Z}_p}\binom{\frac{-x-\lambda y}{q}}{m}d\mu_0(y)\\ =&\sum_{m=1}^{n}\binom{n-1}{n-m}\frac{q^{-n}\widehat{D}_{m,q}(\lambda|-x)}{m!}, \end{aligned} \tag{2.23}$$

and, by the similar method to (2.23), we have

$$q^{-n}(-1)^n\frac{\widehat{D}_{n,q}(\lambda|x)}{n!}=\sum_{m=1}^{n}\binom{n-1}{n-m}\frac{D_{n,q}(\lambda|-x)}{m!}q^{-n}. \tag{2.24}$$

Hence, by (2.23) and (2.24), we obtain the following theorem.

Theorem 2.7. *For $n\geq 1$, we have*

$$q^{-n}(-1)^n\frac{D_{n,q}(\lambda|x)}{n!}=\sum_{m=1}^{n}\binom{n-1}{n-m}\frac{\widehat{D}_{m,q}(\lambda|-x)}{m!}q^{-n}$$

and

$$q^{-n}(-1)^n\frac{\widehat{D}_{n,q}(\lambda|x)}{n!}=\sum_{m=1}^{n}\binom{n-1}{n-m}\frac{D_{n,q}(\lambda|-x)}{m!}q^{-n}.$$

Now, we consider *higher-order λ-Daehee polynomials of second kind with q-parameter*. Higher-order λ-Daehee polynomials of second kind with q-parameter are defined by the multivariant p-adic invariant integral on $\mathbb{Z}_p$:

$$\widehat{D}_{n,\xi,q}^{(k)}(\lambda|x)=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-\lambda(x_1+\cdots+x_k)+x)_{n,q}d\mu_0(x_1)\cdots d\mu_0(x_k) \tag{2.25}$$

where n is an nonnegative integer and $k\in\mathbb{N}$. In the special case, $x=0$, $\widehat{D}_{n,q}^{(k)}(\lambda)=\widehat{D}_{n,q}^{(k)}(\lambda|0)$ are called the *higher-order λ-Daehee numbers of second kind with q-parameter*.

From (2.25), we can derive the generating function of $\widehat{D}_{n,q}^{(k)}(\lambda|x)$ as follows:

$$\begin{aligned} &\sum_{n=0}^{\infty}\widehat{D}_{n,q}^{(k)}(\lambda|x)\frac{t^n}{n!}\\ =&\sum_{n=0}^{\infty}q^n\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{\frac{-\lambda(x_1+\cdots+x_k)+x}{q}}{n}d\mu_0(x_1)\cdots d\mu_0(x_k)t^n\\ =&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+qt)^{\frac{-\lambda(x_1+\cdots+x_k)+x}{q}}d\mu_0(x_1)\cdots d\mu_0(x_k)\\ =&(1+qt)^{\frac{x+\lambda k}{q}}\left(\frac{\lambda\log(1+qt)}{q\left((1+qt)^{\frac{\lambda}{q}}-1\right)}\right)^k. \end{aligned} \tag{2.26}$$

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By (2.25),

$$\begin{aligned}
&\widehat{D}_{n,q}^{(k)}(\lambda|x)\\
=&q^n\sum_{m=0}^{n}\frac{S_1(n,m)}{q^m}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-\lambda(x_1+\cdots+x_k)+x)^m d\mu_0(x_1)\cdots d\mu_0(x_k)\\
=&q^n\sum_{m=0}^{n}\frac{S_1(n,m)}{q^m}(-\lambda)^m\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(x_1+\cdots+x_k-\frac{x}{\lambda}\right)^m d\mu_0(x_1)\cdots d\mu_0(x_k)\\
=&q^n\sum_{m=0}^{n}\frac{S_1(n,m)}{q^m}(-\lambda)^m B_m^{(k)}\left(-\frac{x}{\lambda}\right)\\
=&(-1)^n\sum_{m=0}^{n}q^{n-m}\lambda^m|S_1(n,m)|B_m^{(k)}\left(-\frac{x}{\lambda}\right).
\end{aligned}\tag{2.27}$$

From (1.10), we know that $B_n^{(k)}(-x)=(-1)^nB_n^{(k)}(k+x)$. Hence, by (2.27), we obtain the following theorem.

Theorem 2.8. *For $n\geq 0$, we have*

$$\begin{aligned}
\widehat{D}_{n,q}^{(k)}(\lambda|x)=&\sum_{m=0}^{n}S_1(n,m)q^{n-m}(-\lambda)^mB_m^{(k)}\left(-\frac{x}{\lambda}\right)\\
=&(-1)^n\sum_{m=0}^{n}(-\lambda)^mq^{n-m}|S_1(n,m)|B_m^{(k)}\left(k+\frac{x}{\lambda}\right).
\end{aligned}$$

In (2.26), by replacing t by $\frac{1}{q}(e^t-1)$, we get

$$\begin{aligned}
\sum_{n=0}^{\infty}\widehat{D}_{n,q}^{(k)}(\lambda|x)\frac{(e^t-1)^n}{q^nn!}=&e^{\frac{t}{q}(x+\lambda k)}\left(\frac{\frac{\lambda t}{q}}{e^{\frac{\lambda t}{q}}-1}\right)^k\\
=&\sum_{n=0}^{\infty}\frac{\lambda^nB_n^{(k)}\left(\frac{x}{\lambda}+k\right)}{q^n}\frac{t^n}{n!},
\end{aligned}\tag{2.28}$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{\widehat{D}_{n,q}^{(k)}(\lambda|x)}{q^n}\frac{1}{n!}\left(e^t-1\right)^n=&\sum_{n=0}^{\infty}\frac{\widehat{D}_{n,q}^{(k)}(\lambda|x)}{q^n}\sum_{l=n}^{\infty}S_2(l,n)\frac{t^l}{l!}\\
=&\sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\frac{\widehat{D}_{n,q}^{(k)}(\lambda|x)}{q^n}S_2(m,n)\right)\frac{t^m}{m!}.
\end{aligned}\tag{2.29}$$

By (2.28) and (2.29), we obtain the following theorem.

Theorem 2.9. *For $n\geq 0$ and $k\in\mathbb{N}$, we have*

$$B_n^{(k)}\left(\frac{x}{\lambda}+k\right)=\lambda^{-n}\sum_{m=0}^{n}\widehat{D}_{m,q}^{(k)}(\lambda|x)q^{n-m}S_2(n,m).$$

By Theorem 2.8 and Theorem 2.9, we obtain the following corollary.

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Corollary 2.10. *For $n \geq 0$, we have*

$$\widehat{D}_{n,q}^{(k)}(\lambda|x) = \sum_{m=0}^{n}\sum_{l=0}^{m} \widehat{D}_{l,q}^{(k)}(\lambda|x) q^{n-l} S_1(n,m) S_2(m,l).$$

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Stability of ternary quadratic derivation on ternary Banach algebras: revisited

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Abstract. In [6], Shagholi et al. defined ternary quadratic derivations on ternary Banach algebras and proved the Hyers-Ulam stability of ternary quadratic derivations on ternary Banach algebras. But the definition is not well-defined and so the proofs of the main results are wrong.

In this paper, we correct the definition of ternary quadratic derivation and the proofs of the main results.

1. Introduction

The study of stability problems for functional equations is related to a question of Ulam [7] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [3]. Subsequently, the result of Hyers was generalized by Aoki [1] for additive mappings and by Th. M. Rassias [4] for linear mappings by considering an unbounded Cauchy difference.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic mapping. The Hyers-Ulam stability problem for the quadratic functional equation (1.1) was proved by Skof for mappings $f: A \to B$, where A is a normed space and B is a Banach space (see [5]).

In [6], Shagholi et al. defined a ternary quadratic derivation D from a ternary Banach algebra A into a ternary Banach algebra B such that

$$D[x,y,z]=[D(x),y^2,z^2]+[x^2,D(y),z^2]+[x^2,y^2,D(z)]$$

for all $x,y,z \in A$. But x^2,y^2,z^2 are not defined and the brackets of the right side are not defined, since A is not an algebra and $D(x) \in B$ and $y^2,z^2 \in A$. So we correct them as follows.

Definition 1.1. Let A be an algebra and ternary Banach algebra with norm $\|\cdot\|$. A mapping $D: A \to A$ is called a ternary quadratic derivation if

(1) D is a quadratic mapping,

(2) $D[x,y,z]=[D(x),y^2,z^2]+[x^2,D(y),z^2]+[x^2,y^2,D(z)]$ for all $x,y,z \in A$.

In this paper, the proofs of the main results given in [6] are corrected.

2. Stability of ternary quadratic derivations

Let A be an algebra and ternary Banach algebra with norm $\|\cdot\|$.

Theorem 2.1. *Let $f: A \to A$ be a mapping for which there exists a function $\phi: A\times A\times A \to [0,\infty)$ such that*

$$\tilde{\phi}(x,y,z) := \sum_{j=0}^{\infty} \frac{1}{4^j}\phi(2^j x, 2^j y, 2^j z) < \infty \tag{2.1}$$

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \le \phi(x,y,0), \tag{2.2}$$

$$\|f([x,y,z])-[f(x),y^2,z^2]-[x^2,f(y),z^2]-[x^2,y^2,f(z)])\| \le \phi(x,y,z) \tag{2.3}$$

for all $x,y,z \in A$. Then there exists a unique ternary quadratic derivation $D: A \to A$ such that

$$\|f(x)-D(x)\| \le \frac{1}{4}\tilde{\phi}(x,x,0), \tag{2.4}$$

for all $x \in A$.

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Proof. Putting $x = y = 0$ in (2.2), we get $f(0) = 0$. If we replace y in (2.2) by x and multiply both sides of (2.2) by $\frac{1}{4}$, we get

$$\|\frac{f(2x)}{4} - f(x)\| \le \frac{\phi(x,x,0)}{4} \tag{2.5}$$

for all $x \in A$. Now we use the Rassias' method on inequality (2.5) (see [2]). One can use induction on n to show that

$$\|\frac{f(2^n x)}{2^{2n}} - f(x)\| \le \frac{1}{4}\sum_{j=0}^{n-1} \frac{\phi(2^j x, 2^j x, 0)}{4^j} \tag{2.6}$$

for all $x \in A$ and all nonnegative integers n. Hence

$$\|\frac{f(2^{n+m}x)}{2^{2(n+m)}} - \frac{f(2^m x)}{2^{2m}}\| \le \frac{1}{4}\sum_{j=m}^{n+m-1} \frac{\phi(2^j x, 2^j x, 0)}{4^j}$$

for all nonnegative integers n and m with $n \ge m$ and all $x \in A$. It follows from (2.1) that the sequence $\{\frac{f(2^n x)}{2^{2n}}\}$ is Cauchy. Due to the completeness of A, this sequence is convergent. So one can define the mapping $D : A \to A$ by

$$D(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^{2n}} \tag{2.7}$$

for all $x \in A$. Replacing x, y by $2^n x, 2^n y$, respectively, in (2.2) and multiplying both sides of (2.2) by $\frac{1}{2^{2n}}$, we get

$$\begin{aligned}
&\|D(x+y) + D(x-y) - 2D(x) - 2D(y)\| \\
&= \lim_{n\to\infty} \frac{1}{2^{2n}}\|f(2^n(x+y)) + f(2^n(x-y)) - 2f(2^n x) - 2f(2^n y)\| \\
&\le \lim_{n\to\infty} \frac{\phi(2^n x, 2^n y, 0)}{2^{2n}} = 0
\end{aligned}$$

for all $x, y \in A$ and all nonnegative integers n. So

$$D(x+y) + D(x-y) = 2D(x) + 2D(y)$$

for all $x, y \in A$. Moreover, it follows from (2.6) and (2.7) that

$$\|f(x) - D(x)\| \le \frac{1}{4}\tilde{\phi}(x,x,0)$$

for all $x \in A$. It follows from (2.3) we get

$$\begin{aligned}
&\|D([x,y,z]) - [D(x), y^2, z^2] - [x^2, D(y), z^2] - [x^2, y^2, D(z)]\| \\
&\le \lim_{n\to\infty} \frac{1}{4^{3n}}\|f([2^n x, 2^n y, 2^n z]) - [f(2^n x), (2^n y)^2, (2^n z)^2] - [(2^n x)^2, f(2^n y), (2^n z)^2] - [(2^n x)^2, (2^n y)^2, f(2^n z)]\| \\
&\le \lim_{n\to\infty} \frac{\phi(2^n x, 2^n y, 2^n z)}{4^{3n}} \le \lim_{n\to\infty} \frac{\phi(2^n x, 2^n y, 2^n z)}{4^n} = 0
\end{aligned}$$

for all $x, y, z \in A$. So

$$D[x,y,z] = [D(x), y^2, z^2] + [x^2, D(y), z^2] + [x^2, y^2, D(z)]$$

for all $x, y, z \in A$.

Now, let $D' : A \to A$ be another ternary quadratic derivation satisfying (2.4). Then we have

$$\begin{aligned}
\|D(x) - D'(x)\| &= \frac{1}{2^{2n}}\|D(2^n x) - D'(2^n x)\| \\
&\le \frac{1}{2^{2n}}(\|D(2^n x) - f(2^n x)\|_B + \|f(2^n x) - D'(2^n x)\|) \\
&\le \frac{2}{2^{2n}}\phi(2^n x, 2^n x, 0)
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in A$. So we can conclude that $D(x) = D'(x)$ for all $x \in A$. This proves the uniqueness of D. Thus, the mapping $D : A \to A$ is a unique ternary quadratic derivation satisfying (2.4). □

Theorem 2.2. *Let $f : A \to A$ be a mapping for which there exists a function $\phi : A \times A \times A \to [0,\infty)$ satisfying* (2.2), (2.3) *and*

$$\sum_{j=0}^{\infty} 4^{3j}\phi(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}) < \infty \tag{2.8}$$

for all $x, y, z \in A$. Then there exists a unique ternary quadratic derivation $D : A \to A$ such that

$$\|f(x) - D(x)\| \leq \tilde{\phi}(\frac{x}{2}, \frac{x}{2}, 0), \tag{2.9}$$

for all $x \in A$. Here,

$$\tilde{\phi}(x, y, z) := \sum_{j=0}^{\infty} 4^j \phi(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j})9$$

for all $x, y, z \in A$.

Proof. It follows from (2.5) that

$$\|f(x) - 4f(\frac{x}{2})\| \leq \phi(\frac{x}{2}, \frac{x}{2}, 0)$$

for all $x \in A$.

By the same reasoning as in the proof od Theorem 2.1, one can define a quadartic unique mapping $D : A \to A$ by

$$D(x) := \lim_{n\to\infty} 2^{2n} f(\frac{x}{2^n}) \tag{2.10}$$

for all $x \in A$. It follows from (2.8) and (2.10) that

$$\begin{aligned}
&\|D([x, y, z]) - [D(x), y^2, z^2] - [x^2, D(y), z^2] - [x^2, y^2, D(z)]\| \\
&\leq \lim_{n\to\infty} 4^{3n} \|f([\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}]) - [f(\frac{x}{2^n}), (\frac{y}{2^n})^2, (\frac{z}{2^n})^2] - [(\frac{x}{2^n})^2, f(\frac{y}{2^n}), (\frac{z}{2^n})^2] - [(\frac{x}{2^n})^2, (\frac{y}{2^n})^2, f(\frac{z}{2^n})]\| \\
&\leq \lim_{n\to\infty} 4^{3n} \phi(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}) = 0
\end{aligned}$$

for all $x, y, z \in A$. So

$$D[x, y, z] = [D(x), y^2, z^2] + [x^2, D(y), z^2] + [x^2, y^2, D(z)]$$

for all $x, y, z \in A$. Thus the mapping $D : A \to A$ is a unique ternary quadratic derivation satisfying (2.9). □

From Theorems 2.1 and 2.2, we obtain the following corollary concerning the Hyers-Ulam stability of the functional equation (1.1).

Corollary 2.3. *Let p and θ be nonnegative real numbers with $p \neq 2$, and let $f : A \to A$ be a mapping such that*

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \theta(\|x\|^p + \|y\|^p),$$

$$\|f([x, y, z]) - [f(x), y^2, z^2] - [x^2, f(y), z^2] - [x^2, y^2, f(z)])\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p),$$

for all $x, y, z \in A$. Then there exists a unique ternary quadratic derivation $D : A \to A$ such that

$$\|f(x) - D(x)\| \leq \frac{2\theta}{4 - 2^p} \|x\|^p$$

holds for all $x \in X$, where $p < 2$, and the inequality

$$\|f(x) - D(x)\| \leq \frac{2\theta}{2^p - 4} \|x\|^p$$

holds for all $x \in X$, where $p > 6$.

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SOME PROPERTIES OF MODULAR S-METRIC SPACES AND ITS FIXED POINT RESULTS

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ABSTRACT. In this paper, we introduce modular S-metric spaces and deal with their some properties. We also prove some fixed point theorems on complete modular S-metric spaces.

1. INTRODUCTION

Fixed point theory in metric spaces begins with the Banach Contraction Principle which is published in 1922 [6]. Since it is simple and useful, it has become a very popular tool to solve existence problems in mathematical analysis. There are some authors introduced the generalization of metric spaces such as Gahler [16], which is called 2-metric space, and Dhage [14], which is called D-metric space. In 2013, Mustafa and Sims [24] found that the fundamental topology properties of the metric spaces are incorrect. They [25] introduced a generalization of metric spaces which is called G-metric spaces.

The concept of S-metric spaces was firstly introduced by Sedghi et al. [28] in 2012. Sedghi and Dung [29] proved a general fixed point theorem in S-metric spaces which is a generalization [[28], Theorem 3.1]. Gupta [17] introduced the concepts of cyclic contraction on S-metric space and proved some fixed point theorems on S-metric spaces. Chouhan [12] proved a common unique fixed point theorem for expansive mappings in S-metric space. Hieu et al. [18] gave a fixed point theorem for a class of maps depending on another map on S-metric spaces.

The notion of modular space was firstly introduced by Nakano [26] and developed by Koshi, Shimogaki, Yamamuro (see [22, 30]) and others. Recently, many researchers have been interested in fixed point of modular space. In 2008, Chistyakov [7] introduced the notion of modular metric space generated by F-modular and developed the theory of this space. He also defined the notion of a modular on an arbitrary set and the modular metric spaces in 2010 [8]. Abdou [1] studied and proved some new fixed points theorems for pointwise and asymptotic pointwise contraction mappings in modular metric spaces. Azadifer et. al. [3] introduced the notion of modular G-metric spaces and proved some fixed point theorems of contractive in this space. Many authors studied on modular metric spaces [4],[5],[10],[11],[19],[20],[21].

In this paper we introduce the concept of modular S-metric spaces and their properties. Then we give fixed point theorems for self mappings on complete modular S-metric spaces.

2. PRELIMINARIES

Definition 2.1. [27]. A modular on a real linear space X is a functional $\rho : X \longrightarrow [0,\infty]$ satisfying the followings:

(A1) $\rho(0) = 0$;
(A2) If $x \in X$ and $\rho(\alpha x) = 0$ for all numbers $\alpha > 0$, then $x = 0$;
(A3) $\rho(-x) = \rho(x)$ for all $x \in X$;
(A4) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $x, y \in X$.

Let X be a non-empty set and $\lambda \in (0,\infty)$. We remark that the function $\omega : (0,\infty)\times X\times X \longrightarrow [0,\infty]$ is denoted by $\omega_\lambda(x, y) = \omega(\lambda, x, y)$ for all $\lambda > 0$ and $x, y \in X$.

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Definition 2.2. [8]. Let X be a non-empty set, a function $\omega : (0,\infty) \times X \times X \longrightarrow [0,\infty]$ is said to be a metric modular on X if satisfying, for all $x, y, z \in X$ the following conditions hold:

(i) $\omega_\lambda(x,y) = 0$ for all $\lambda > 0 \;\Leftrightarrow\; x = y$;
(ii) $\omega_\lambda(x,y) = \omega_\lambda(y,x)$ for all $\lambda > 0$;
(iii) $\omega_{\lambda+\mu}(x,y) \leq \omega_\lambda(x,z) + \omega_\mu(z,y)$ for all $\lambda, \mu > 0$.

Definition 2.3. [28] Let X be a non-empty set. An S-metric on X is a function $S : X^3 \to [0,\infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,

(i) $S(x,y,z) \geq 0$;
(ii) $S(x,y,z) = 0$ if and only if $x = y = z$;
(iii) $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$.

The pair (X,S) is called an S-metric space.

3. Modular S-metric spaces

We define a new concept combining with S-metric and modular metric space.

Definition 3.1. Let X be a non-empty set. An modular S-metric on X is a function

$$s_\lambda : (0,\infty) \times X \times X \times X \to [0,\infty]$$

that satisfies the following conditions for all $x, y, z \in X$ and $\lambda > 0$:

$(S1)$ $s_\lambda(x,y,z) \geq 0$;
$(S2)$ $s_\lambda(x,y,z) = 0$ if and only if $x = y = z$;
$(S3)$ $s_{\lambda+\mu+\nu}(x,y,z) \leq s_\lambda(x,x,a) + s_\mu(y,y,a) + s_\nu(z,z,a)$ for all $\lambda, \mu, \nu > 0$ and $a \in X$.

Example 3.2. **(1)** $s_\lambda(x,y,z) = 0$ if $x = y = z$ and $s_\lambda(x,y,z) = \infty$ if $x \neq y \neq z$.
(2) If S is an modular S-metric on X, we can get:

(a) $s_\lambda(x,y,z) = 0$ if $\lambda > S(x,y,z)$ and $s_\lambda(x,y,z) = \infty$ if $\lambda \leq S(x,y,z)$.
(b) $s_\lambda(x,y,z) = 0$ if $\lambda \geq S(x,y,z)$ and $s_\lambda(x,y,z) = \infty$ if $\lambda < S(x,y,z)$.
(c) $s_\lambda(x,y,z) = \dfrac{S(x,y,z)}{\varphi(\lambda)}$; where $\varphi : (0,\infty) \to (0,\infty)$ is non-decreasing function.

Lemma 3.3. *If the function $0 < \lambda \to s_\lambda(x,y,z)$ is continuous on $(0,\infty)$ where $x, y, z \in X$, then we have $s_\lambda(x,x,y) = s_\lambda(y,y,x)$.*

Proof. There exists $\varepsilon > 0$ such that

$$s_\lambda(x,x,y) \leq s_\varepsilon(x,x,x) + s_\varepsilon(x,x,x) + s_{\lambda-2\varepsilon}(y,y,x).$$

If we take limit as $\varepsilon \to 0$, we get $s_\lambda(x,x,y) \leq s_\lambda(y,y,x)$. Similarly $s_\lambda(y,y,x) \leq s_\lambda(x,x,y)$. So we get

$$s_\lambda(x,x,y) \leq s_\lambda(y,y,x) \leq s_\lambda(x,x,y)$$

and

$$s_\lambda(x,x,y) = s_\lambda(y,y,x).$$

□

Remark 3.4. The function $s_\lambda(x,y,z)$ for $\lambda > 0$ is non-increasing on $(0,\infty)$ where $x, y, z \in X$, if it is continuous on $(0,\infty)$. In fact if $0 < \nu < \mu < \lambda$, $(S3)$ implies

$$s_\lambda(x,x,y) \leq s_{\lambda-\mu}(x,x,x) + s_{\mu-\nu}(x,x,x) + s_\nu(y,y,x)$$

and we have

$$s_\lambda(x,x,y) \leq s_\nu(y,y,x)$$

from $(S2)$.

From Lemma 3.3, we conclude that $s_\lambda(x,x,y) \leq s_\nu(x,x,y)$. From that inequality the function $s_\lambda(x,y,z)$ is non-increasing on $(0,\infty)$. It follows that at each point $\lambda > 0$ the right limit

$$s_{\lambda+0}(x,y,z) = \lim_{\mu \to \lambda+0} s_\mu(x,y,z)$$

and the left limit

$$s_{\lambda-0}(x,y,z)=\lim_{\varepsilon\to 0}s_{\lambda-\varepsilon}(x,y,z)$$

exist in $[0,\infty]$ and the following two inequalities hold:

$$s_{\lambda+0}(x,y,z)\le s_\lambda(x,y,z)\le s_{\lambda-0}(x,y,z).$$

Definition 3.5. Let s_λ be a modular S-metric on X. The binary relation $\overset{s}{\sim}$ on X defined for $x,y\in X$ by

$$x\overset{s}{\sim}y\Leftrightarrow \lim_{\lambda\to\infty}s_\lambda(x,x,y)=0 \tag{3.1}$$

is an equivalence relation. Indeed $x\overset{s}{\sim}x$ is clear by virtue of $(S2)$. From Lemma 3.3, we have

$$x\overset{s}{\sim}y\Leftrightarrow \lim_{\lambda\to\infty}s_\lambda(x,x,y)=0=\lim_{\lambda\to\infty}s_\lambda(y,y,x)\Leftrightarrow y\overset{s}{\sim}x.$$

If $x\overset{s}{\sim}y$ and $y\overset{s}{\sim}z$, we get $\lim_{\lambda\to\infty}s_\lambda(x,x,y)=0$ and $\lim_{\lambda\to\infty}s_\lambda(y,y,z)=0$. By $(S3)$ and Lemma 3.3, we conclude that

$$\begin{aligned}\lim_{\lambda\to\infty}s_{3\lambda}(x,x,z)\le&\lim_{\lambda\to\infty}s_\lambda(x,x,y)+\lim_{\lambda\to\infty}s_\lambda(x,x,y)+\lim_{\lambda\to\infty}s_\lambda(y,y,z)\\=&0+0+0.\end{aligned}$$

It is clear that

$$\lim_{\lambda\to\infty}s_{3\lambda}(x,x,z)=0\Leftrightarrow x\overset{s}{\sim}z$$

by $(S1)$. The equivalence class of the element $x\in X$ in the quotient set $X/\overset{s}{\sim}$ is defined by

$$X_s\equiv X_s(x)=\{y\in X: y\overset{s}{\sim}x\}.$$

For $x_0\in X$, the set X_s^* is defined as follows:

$$X_s^*\equiv X_s^*(x_0)=\{x\in X:\exists\lambda=\lambda(x)>0\ \text{ such that }\ s_\lambda(x,x,x_0)<\infty\}.$$

From Remark 3.4, the function

$$\widetilde{S}:(X/\overset{s}{\sim})\times(X/\overset{s}{\sim})\times(X/\overset{s}{\sim})\to[0,\infty]$$

given by

$$\widetilde{S}(X_s(x),X_s(x),X_s(y))=s_\lambda(x,x,y)$$

is well-defined and satisfies the axioms of S-metric.

Theorem 3.6. *If s_λ is a modular S-metric on X, then the modular set X_s is an modular S-metric space with S-metric given by*

$$S^\circ(x,x,y)=\inf\{\lambda>0:s_\lambda(x,x,y)\le\lambda\},$$

for all $x,y\in X_s$.

Proof. Since $x\overset{s}{\sim}y$, there exists $\lambda_0>0$ such that

$$s_\lambda(x,x,y)\le 1$$

for all $\lambda\ge\lambda_0$ by (3.1). Taking $\lambda_1=\max\{1,\lambda_0\}$, we get

$$s_{\lambda_1}(x,x,y)\le 1\le\lambda_1$$

which together with the definition of $S^\circ(x,x,y)$ gives

$$S^\circ(x,x,y)\le\lambda_1<\infty.$$

Given $x\in X_s$, $(S2)$ implies that $s_\lambda(x,x,x)=0$ for all $\lambda>0$, and so, $S^\circ(x,x,x)=0$. Let s_λ satisfy $(S2)$, $x,y\in X_s$ and $S^\circ(x,x,y)=0$. Then $s_\mu(x,x,y)$ does not exceed μ for all $\mu>0$. Hence for any $\lambda>0$ and $0<\mu<\lambda$, from Remark 3.4 we have $s_\lambda(x,x,y)\le s_\mu(x,x,y)\le\mu\to 0$ as $\mu\to+0$. It follows that $s_\lambda(x,x,y)=0$ for all $\lambda>0$. Thus axiom $(S2)$ implies $x=y$.

It is clear from $(S1)$ that $S^\circ(x,x,y) \geq 0$. Now we show the triangle inequality:

$$S^\circ(x,x,y) \leq 2S^\circ(x,x,z) + S^\circ(y,y,z)$$

for some $z \in X_s$. In fact by the definition of S° for any $\lambda > S^\circ(x,x,z)$ and $\mu > S^\circ(y,y,z)$, we find $s_\lambda(x,x,z) \leq \lambda$ and $s_\mu(y,y,z) \leq \mu$. As a result, we get

$$s_{2\lambda+\mu}(x,x,y) \leq 2s_\lambda(x,x,z) + s_\mu(y,y,z) \leq 2\lambda + \mu$$

by the axiom $(S3)$. It follows from the definition of S° that $S^\circ(x,x,y) \leq 2\lambda + \mu$ and it remains to pass limit as $\lambda \to S^\circ(x,x,z)$ and $\mu \to S^\circ(y,y,z)$. □

Theorem 3.7. *Let s_λ be a modular S-metric on a set X and*

$$S^1(x,x,y) = \inf\{\lambda + s_\lambda(x,x,y) : \lambda > 0\}$$

be defined for all $x, y \in X_s$. Then S^1 is an S-metric on X_s such that $S^\circ \leq S^1 \leq 2S^\circ$.

Proof. Since, for $x, y \in X_s$, the value $s_\lambda(x,x,y)$ is finite due to (3.1) for $\lambda > 0$ large enough, then the set $\{\lambda + s_\lambda(x,x,y) : \lambda > 0\} \subset \mathbb{R}^+$ is non-empty and bounded from below, therefore $S^1(x,x,y) \in \mathbb{R}^+$.

Since $s_\lambda(x,x,x) = 0$, then from the definition of S^1, $S^1(x,x,x) = \inf\{\lambda + \underbrace{s_\lambda(x,x,x)}_{0} : \lambda > 0\} = 0$.

Let s_λ satisfy $(S2)$, $x, y \in X_s$ and $S^1(x,x,y) = 0$. The equality $x = y$ will follow from $(S2)$ if we show that $s_\lambda(x,x,y) = 0$ for all $\lambda > 0$. On the contrary, suppose that $s_{\lambda_0}(x,x,y) > 0$ for some $\lambda_0 > 0$. Then for $\lambda \geq \lambda_0$ we find $\lambda + s_\lambda(x,x,y) \geq \lambda_0$, and if $0 < \lambda < \lambda_0$, then

$$0 < s_{\lambda_0}(x,x,y) \leq s_\lambda(x,x,y) \leq \lambda + s_\lambda(x,x,y)$$

from Remark 3.4. Thus, $\lambda + s_\lambda(x,x,y) \geq \lambda_1 = \min\{\lambda_0, s_{\lambda_0}(x,x,y)\}$ for all $\lambda > 0$. By the definition of S^1, $S^1(x,x,y) \geq \lambda_1 > 0$, which contradicts the assumption.

Now let us show that triangle inequality: $S^1(x,x,y) \leq 2S^1(x,x,z) + S^1(y,y,z)$. For any $\varepsilon > 0$ we find $\lambda = \lambda(\varepsilon) > 0$ and $\mu = \mu(\varepsilon) > 0$ such that

$$\lambda + s_\lambda(x,x,z) \leq S^1(x,x,z) + \varepsilon \quad \text{and} \quad \mu + s_\mu(y,y,z) \leq S^1(y,y,z) + \varepsilon$$

from the definition of S^1. Applying axiom $(S3)$,

$$\begin{aligned} S^1(x,x,y) \leq& (2\lambda + \mu) + s_{2\lambda+\mu}(x,x,y) \leq 2\lambda + \mu + 2s_\lambda(x,x,z) + s_\mu(y,y,z) \\ \leq& 2S^1(x,x,z) + 2\varepsilon + S^1(y,y,z) \end{aligned}$$

and it remains to take into account the arbitrariness of $\varepsilon > 0$.

Let us prove that metrics S° and S^1 are equivalent on X_s. In order to obtain the left-hand side inequality, suppose that $\lambda > 0$ is arbitrary. If $s_\lambda(x,x,y) \leq \lambda$, then the definition of S° implies $S^\circ(x,x,y) \leq \lambda$. If $s_\lambda(x,x,y) > \lambda$, then $S^\circ(x,x,y) \leq s_\lambda(x,x,y)$. Setting $\mu = s_\lambda(x,x,y)$ we find $\mu > \lambda$. Thus it follows from Remark 3.4 that

$$s_\mu(x,x,y) \leq s_\lambda(x,x,y) = \mu.$$

Hence

$$S^\circ(x,x,y) \leq \mu = s_\lambda(x,x,y).$$

Therefore for any $\lambda > 0$ we have

$$S^\circ(x,x,y) \leq \max\{\lambda, s_\lambda(x,x,y)\} \leq \lambda + s_\lambda(x,x,y).$$

Taking the infimum over all $\lambda > 0$, we get the inequality

$$S^\circ(x,x,y) \leq S^1(x,x,y)$$

To obtain the right-hand side inequality, we note that given $\lambda > 0$ such that $S^\circ(x,x,y) < \lambda$ by the definition of S°. We get $s_\lambda(x,x,y) \leq \lambda$. So $S^1(x,x,y) \leq \lambda + s_\lambda(x,x,y) \leq 2\lambda$. Passing to the limit as $\lambda \to S^\circ(x,x,y)$, we get

$$S^1(x,x,y) \leq 2S^\circ(x,x,y).$$

□

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Theorem 3.8. *Let s_λ be a modular S-metric on a set X, $x, y \in X_s$ and $\lambda > 0$. We have*

(a) *If $S^\circ(x,x,y) < \lambda$, then $s_\lambda(x,x,y) \le S^\circ(x,x,y) < \lambda$.*
(b) *If $s_\lambda(x,x,y) = \lambda$, then $S^\circ(x,x,y) = \lambda$.*
(c) *If $\lambda = S^\circ(x,x,y) > 0$, then $s_{\lambda+0}(x,x,y) \le \lambda \le s_{\lambda-0}(x,x,y)$.*
(d) *If the function $\mu \to s_\mu(x,x,y)$ is continuous from the right on $(0,\infty)$, then along with $(a)-(c)$ we have:*

$$S^\circ(x,x,y) \le \lambda \Leftrightarrow s_\lambda(x,x,y) \le \lambda.$$

(e) *If the function $\mu \to s_\mu(x,x,y)$ is continuous from the left on $(0,\infty)$, then along with $(a)-(c)$ we have:*

$$S^\circ(x,x,y) < \lambda \Leftrightarrow s_\lambda(x,x,y) < \lambda.$$

(f) *If the function $\mu \to s_\mu(x,x,y)$ is continuous on $(0,\infty)$, then along with $(a)-(e)$ we have:*

$$S^\circ(x,x,y) = \lambda \Leftrightarrow s_\lambda(x,x,y) = \lambda.$$

Proof. (a) For any $\mu > 0$ such that $S^\circ(x,x,y) < \mu < \lambda$ by the definition of S° and Remark 3.4, we have $s_\mu(x,x,y) \le \mu$ and $s_\lambda(x,x,y) \le s_\mu(x,x,y)$. Hence $s_\lambda(x,x,y) \le \mu$ and it remains to pass to the limit as $\mu \to S^\circ(x,x,y)$.

(b) By the definition, $S^\circ(x,x,y) \le \lambda$ and item (a) implies $S^\circ(x,x,y) = \lambda$.

(c) For any $\mu > \lambda = S^\circ(x,x,y)$, the definition of S° implies $s_\mu(x,x,y) \le \mu$ and so

$$s_{\lambda+0}(x,x,y) = \lim_{\mu\to\lambda+0} s_\mu(x,x,y) \le \lim_{\mu\to\lambda+0} \mu = \lambda.$$

For any $0 < \mu < \lambda$ we find $s_\mu(x,x,y) > \mu$ and so

$$s_{\lambda-0}(x,x,y) = \lim_{\mu\to\lambda-0} s_\mu(x,x,y) \ge \lim_{\mu\to\lambda-0} \mu = \lambda.$$

(d) The sufficient condition follows from the definition of S°. Let us prove the reverse implication. If $S^\circ(x,x,y) < \lambda$, then by virtue of item (a), $s_\lambda(x,x,y) < \lambda$ and if $S^\circ(x,x,y) = \lambda$, then

$$s_\lambda(x,x,y) = s_{\lambda+0}(x,x,y) \le \lambda$$

which is a consequence of the continuity from the right of the function $\mu \to s_\mu(x,x,y)$ and item (c).

(e) By item (a), it suffices to prove the sufficient condition. The definition of S° gives $S^\circ(x,x,y) \le \lambda$ but if $S^\circ(x,x,y) = \lambda$, then by item (c) we would have

$$s_\lambda(x,x,y) = s_{\lambda-0}(x,x,y) \ge \lambda$$

which contradicts the assumption.

(f) Sufficient condition follows from (b). For the reverse asertion the two inequalities

$$s_\lambda(x,x,y) \le \lambda \le s_\lambda(x,x,y)$$

follows from (c). □

Definition 3.9. Let s_λ be a modular S-metric on a set X.

(1) A sequence $(x_n) \subset X_s^*$ converges to $x \in X_s^*$ if $s_\lambda(x_n,x_n,x) \to 0$ as $n \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ we have $s_\lambda(x_n,x_n,x) < \varepsilon$. We write $x_n \to x$.
(2) A sequence $(x_n) \subset X_s^*$ is a s-Cauchy if $s_\lambda(x_n,x_n,x_m) \to 0$ as $m,n \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ we have $s_\lambda(x_n,x_n,x_m) < \varepsilon$.
(3) The modular S-metric space X_s^* is s-complete if every s-Cauchy is a s-convergent in X_s^*.

Lemma 3.10. *Let s_λ be a modular S-metric on a set X. If $x_n \to x$ and $y_n \to y$, then*

$$s_\lambda(x_n,x_n,y_n) \to s_\lambda(x,x,y).$$

Proof. Since $x_n \to x$ and $y_n \to y$, then for each $\varepsilon > 0$ there exist $n_1, n_2 \in \mathbb{N}$ such that

$$\begin{aligned}\forall n \geq & n_1,\ s_\lambda(x_n, x_n, x) < \varepsilon \\ \forall n \geq & n_2,\ s_\lambda(y_n, y_n, y) < \varepsilon.\end{aligned}$$

Without loss of generality we can assume

$$\begin{aligned}\forall n \geq & n_1, s_\delta(x_n, x_n, x) < \varepsilon(\delta) = \frac{\varepsilon}{4} \\ \forall n \geq & n_2, s_\delta(y_n, y_n, y) < \varepsilon(\delta) = \frac{\varepsilon}{4}.\end{aligned}$$

If we set $n_0 = \max\{n_1, n_2\}$, therefore for every $n \geq n_0$ we get

$$\begin{aligned}s_\lambda(x_n, x_n, y_n) \leq & 2s_\delta(x_n, x_n, x) + s_{\lambda-2\delta}(y_n, y_n, x) \\ \leq & 2s_\delta(x_n, x_n, x) + 2s_\delta(y_n, y_n, y) + s_{\lambda-4\delta}(x, x, y)\end{aligned}$$

for $\lambda > \delta > 0$ by triangle inequality. If we take $\delta \to 0$, we have

$$\begin{aligned}s_\lambda(x_n, x_n, y_n) \leq & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + s_\lambda(x, x, y) \\ s_\lambda(x_n, x_n, y_n) \leq & \varepsilon + s_\lambda(x, x, y)\end{aligned}$$

$$s_\lambda(x_n, x_n, y_n) - s_\lambda(x, x, y) \leq \varepsilon.$$

On the other hand we get

$$\begin{aligned}s_\lambda(x, x, y) \leq & 2s_\delta(x, x, x_n) + s_{\lambda-2\delta}(y, y, x_n) \\ \leq & 2s_\delta(x, x, x_n) + 2s_\delta(y, y, y_n) + s_{\lambda-4\delta}(x_n, x_n, y_n).\end{aligned}$$

From Lemma 3.3 and taking the limit as $\delta \to 0$ we have:

$$\begin{aligned}s_\lambda(x, x, y) \leq & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + s_\lambda(x_n, x_n, y_n) \\ \leq & \varepsilon + s_\lambda(x_n, x_n, y_n)\end{aligned}$$

$$s_\lambda(x, x, y) - s_\lambda(x_n, x_n, y_n) \leq \varepsilon.$$

So we get from that inequalities $|s_\lambda(x_n, x_n, y_n) - s_\lambda(x, x, y)| < \varepsilon$, that is, $s_\lambda(x_n, x_n, y_n) \to s_\lambda(x, x, y)$. □

4. Fixed Point Theorems

In this section we introduce some fixed point theorems on modular S-metric space.

Definition 4.1. Let s_λ be a modular S-metric on a set X. A map $T : X_s^* \to X_s^*$ is said to be a s-contraction if there exists a constant $0 \leq k < 1$ such that

$$s_\lambda(Tx, Tx, Ty) \leq k s_\lambda(x, x, y)$$

for all $x, y \in X$.

Corollary 4.2. *Let X_s^*, Y_s^* modular S-metric spaces and $f : X_s^* \to Y_s^*$ be a map. Then f is continuous at $x \in X_s^*$ if and only if $f(x_n) \to f(x)$ where $x_n \to x$.*

Theorem 4.3. *Let X_s^* be a s-complete and $T : X_s^* \to X_s^*$ be s-contraction. Then T has a unique fixed point $u \in X_s^*$.*

Proof. First, we show uniqueness. Suppose that there exist $x, y \in X_s^*$ with $x = Tx$ and $y = Ty$. Then

$$s_\lambda(x, x, y) = s_\lambda(Tx, Tx, Ty) \leq k s_\lambda(x, x, y).$$

Therefore $s_\lambda(x, x, y) = 0$.

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To show the existence, we select $x \in X_s^*$ and show that $(T^n x)$ is a Cauchy sequence. For $n = 0, 1, 2, \cdots$, we get by induction

$$\begin{aligned} s_\lambda(T^n x, T^n x, T^{n+1}x) \leq & k s_\lambda(T^{n-1}x, T^{n-1}x, T^n x) \\ & \vdots \\ \leq & k^n s_\lambda(x, x, Tx). \end{aligned}$$

Taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} s_\lambda(T^n x, T^n x, T^{n+1}x) = 0.$$

Thus there exists $\varepsilon > 0$ such that

$$s_\lambda(T^n x, T^n x, T^{n+1}x) \leq \varepsilon.$$

Without loss of generality, we can assume that there exists $\frac{\varepsilon}{m-n}$ for $\frac{\lambda}{m-n}$ such that

$$\begin{aligned} s_\lambda(T^n x, T^n x, T^m x) \leq & 2\sum_{i=n}^{m-2} s_{\frac{\lambda}{m-n}}(T^i x, T^i x, T^{i+1}x) + s_{\frac{\lambda}{m-n}}(T^{m-1}x, T^{m-1}x, T^m x) \\ \leq & 2\sum_{i=n}^{m-2} k^i s_{\frac{\lambda}{m-n}}(x, x, Tx) + k^{m-1} s_{\frac{\lambda}{m-n}}(x, x, Tx) \\ \leq & 2(\frac{\varepsilon}{m-n} + \cdots + \frac{\varepsilon}{m-n}) \\ \leq & 2\varepsilon. \end{aligned}$$

That is for $m > n$,

$$s_\lambda(T^n x, T^n x, T^m x) \leq 2\varepsilon.$$

This shows that $(T^n x)$ is a Cauchy sequence and since X_s^* is s-complete, there exists $u \in X_s^*$ with $\lim\limits_{n\to\infty} T^n x = u$.

From the continuity of T, we get

$$u = \lim_{n\to\infty} T^{n+1}x = \lim_{n\to\infty} T(T^n x) = Tu.$$

Therefore u is a fixed point of T. □

Let $\mathcal{M}$ be the family of all continuous functions of five variables $M : \mathbb{R}_+^5 \to \mathbb{R}_+$. For some $k \in [0, 1)$, we consider the following conditions:

(C1) For all $x, y, z \in \mathbb{R}_+$, if $y \leq M(x, x, 0, z, y)$ with $z \leq 2x + y$, then $y \leq kx$.
(C2) If $y \leq M(y, 0, y, y, 0)$ for all $y \in \mathbb{R}_+$, then $y = 0$.

Theorem 4.4. *Let T be a self-map on s-complete X_s^* and*

$$s_\lambda(Tx, Tx, Ty) \leq M(s_\lambda(x, x, y), s_\lambda(Tx, Tx, x), s_\lambda(Tx, Tx, y), s_{3\lambda}(Ty, Ty, x), s_\lambda(Ty, Ty, y)) \tag{4.1}$$

for all $x, y, z \in X_s^*$ and some $M \in \mathcal{M}$. Then we have

1. *If M satisfies the condition $(C1)$, then T has a fixed point.*
2. *If M satisfies the condition $(C2)$ and T has a fixed point x, then the fixed point is unique.*

Proof. **(1)** For each $x_0 \in X_s^*$ and $n \in \mathbb{N}$, we take $x_{n+1} = Tx_n$. It follows from (4.1) and Lemma 3.3 that

$$\begin{aligned} s_\lambda(x_{n+1}, x_{n+1}, x_{n+2}) = & s_\lambda(Tx_n, Tx_n, Tx_{n+1}) \\ \leq & M(s_\lambda(x_n, x_n, x_{n+1}), s_\lambda(x_{n+1}, x_{n+1}, x_n), s_\lambda(x_{n+1}, x_{n+1}, x_{n+1}), \\ & s_{3\lambda}(x_{n+2}, x_{n+2}, x_n), s_\lambda(x_{n+2}, x_{n+2}, x_{n+1})) \\ = & M(s_\lambda(x_n, x_n, x_{n+1}), s_\lambda(x_n, x_n, x_{n+1}), 0, s_{3\lambda}(x_n, x_n, x_{n+2}), s_\lambda(x_{n+1}, x_{n+1}, x_{n+2})). \end{aligned}$$

By triangle inequality and Lemma 3.3, we have

$$s_{3\lambda}(x_n, x_n, x_{n+2}) \leq 2s_\lambda(x_n, x_n, x_{n+1}) + s_\lambda(x_{n+1}, x_{n+1}, x_{n+2}) \tag{4.2}$$

From (4.2), we see that $z \leq 2x + y$. Since M satisfies the condition $(C1)$, there exists $k \in [0, 1)$ such that

$$s_\lambda(x_{n+1}, x_{n+1}, x_{n+2}) \leq ks_\lambda(x_n, x_n, x_{n+1}) \leq \cdots \leq k^{n+1}s_\lambda(x_0, x_0, x_1). \tag{4.3}$$

Taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} s_\lambda(x_n, x_n, x_{n+1}) = 0.$$

Hence there exists $\varepsilon > 0$ for $\lambda > 0$ such that

$$s_\lambda(x_n, x_n, x_{n+1}) \leq \varepsilon.$$

Without loss of generality, we can assume that there exists $\frac{\varepsilon}{m-n}$ for $\frac{\lambda}{m-n} > 0$ such that

$$s_{\frac{\lambda}{m-n}}(x_n, x_n, x_{n+1}) \leq \frac{\varepsilon}{m-n}.$$

Thus for all $n < m$ by using $(S3)$, Remark 3.4 and (4.3) we have

$$\begin{aligned}
s_\lambda(x_n, x_n, x_m) \leq& 2s_{\frac{\lambda}{3}}(x_n, x_n, x_{n+1}) + s_{\frac{\lambda}{3}}(x_m, x_m, x_{n+1})\\
\leq& 2s_{\frac{\lambda}{3}}(x_n, x_n, x_{n+1}) + s_{\frac{\lambda}{3}}(x_{n+1,}x_{n+1}, x_m)\\
&\vdots\\
\leq& 2\left(\frac{\varepsilon}{m-n} + \frac{\varepsilon}{m-n} + \cdots + \frac{\varepsilon}{m-n}\right)\\
\leq& 2\varepsilon.
\end{aligned}$$

This proves that (x_n) is s-Cauchy in the s-complete X_s^*. Then (x_n) converges an $x \in X_s^*$.

Now we prove that x is a fixed point of T. By using (4.1), we get

$$\begin{aligned}
s_\lambda(x_{n+1}, x_{n+1}, Tx) =& s_\lambda(Tx_n, Tx_n, Tx)\\
\leq& M(s_\lambda(x_n, x_n, x), s_\lambda(Tx_n, Tx_n, x_n), s_\lambda(Tx_n, Tx_n, x), s_{3\lambda}(Tx, Tx, x_n), s_\lambda(Tx, Tx, x)).
\end{aligned}$$

Since $M \in \mathcal{M}$, then using Lemma 3.10 and taking the limit as $n \to \infty$, we obtain

$$s_\lambda(x, x, Tx) \leq M(0, 0, 0, s_{3\lambda}(Tx, Tx, x), s_\lambda(Tx, Tx, x)).$$

From Remark 3.4, we can rewrite

$$s_{3\lambda}(Tx, Tx, x) \leq s_\lambda(Tx, Tx, x).$$

Then the inequality can be written as follows:

$$s_\lambda(x, x, Tx) \leq M(0, 0, 0, s_\lambda(Tx, Tx, x), s_\lambda(Tx, Tx, x)).$$

Since M satisfies the condition $(C1)$, then $s_\lambda(x, x, Tx) \leq k.0 = 0$. This proves that $x = Tx$.

(2) Let x, y be fixed points of T. We prove that $x = y$. It follows from (4.1) that

$$\begin{aligned}
s_\lambda(x, x, y) =& s_\lambda(Tx, Tx, Ty)\\
\leq& M(s_\lambda(x, x, y), s_\lambda(Tx, Tx, x), s_\lambda(Tx, Tx, y), s_{3\lambda}(Ty, Ty, x), s_\lambda(Ty, Ty, y))\\
\leq& M(s_\lambda(x, x, y), 0, s_\lambda(x, x, y), s_{3\lambda}(y, y, x), 0).
\end{aligned}$$

From Remark 3.4 and Lemma 3.3, we get

$$s_\lambda(x, x, y) \leq M(s_\lambda(x, x, y), 0, s_\lambda(x, x, y), s_\lambda(x, x, y), 0).$$

Since M satisfies the condition $(C2)$,

$$s_\lambda(x, x, y) = 0 \Longleftrightarrow x = y.$$

□

Remark 4.5. Theorem 4.3 is a corollary of Theorem 4.4 when we take $M(x,y,z,s,t) = k.x$ for $k \in [0,1)$ and $x,y,z,s,t \in \mathbb{R}_+$.

Now we will give a new corollary of Theorem 4.4.

Corollary 4.6. *Let T be a self map on s-complete X_s^* and*

$$s_\lambda(Tx,Tx,Ty) \leq a(s_\lambda(Tx,Tx,x) + s_\lambda(Ty,Ty,y))$$

for some $a \in [0,\frac{1}{2})$ and all $x,y \in X_s^*$. Then T has a unique fixed point in X_s^*.

Proof. We must show that $M(x,y,z,s,t) = a(y+t)$ satisfies conditions $(C1)$ and $(C2)$. First, we have

$$M(x,x,0,z,y) = a(x+y).$$

So, if $y \leq M(x,x,0,z,y)$ with $z \leq 2x+y$, then

$$\begin{aligned} y \leq & M(x,x,0,z,y) = a(x+y) \\ y \leq & ax + ay \\ y \leq & \frac{a}{1-a}x \end{aligned}$$

with $\dfrac{a}{1-a} \in [0,1)$. Therefore, M satisfies condition $(C1)$.

If $y \leq M(y,0,y,y,0) = 0$, then $y = 0$. Therefore, M satisfies the condition $(C2)$.

Since

$$\begin{aligned} s_\lambda(Tx,Tx,Ty) \leq & M(s_\lambda(x,x,y), s_\lambda(Tx,Tx,x), s_\lambda(Tx,Tx,y), s_\lambda(Ty,Ty,x), s_\lambda(Ty,Ty,y)) \\ = & a(s_\lambda(Tx,Tx,x) + s_\lambda(Ty,Ty,y)), \end{aligned}$$

T has a unique fixed point in X_s^* by Theorem 4.4. $\square$

Open problems How can obtain some similar results for the papers (see [2, 15]) in fuzzy metric spaces with the help of this technique?

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The strong converse inequality for de la Vallée Poussin means on the sphere *

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Abstract

This paper discusses the approximation by de la Vallée Poussin means $V_n f$ on the unit sphere. Especially, the lower bound of approximation is studied. As a main result, the strong converse inequality for the means is established. Namely, it is proved that there are constants C_1 and C_2 such that $C_1\omega\left(f, \frac{1}{\sqrt{n}}\right)_p \leq \|V_n f - f\|_p \leq C_2\omega\left(f, \frac{1}{\sqrt{n}}\right)_p$ for any p-th Lebesgue integrable or continuous function f defined on the sphere, where $\omega(f,t)_p$ is the modulus of smoothness of f.

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Keywords: sphere; de la Vallée Poussin means; approximation; modulus of smoothness; lower bound

1 Introduction

Motivated by geoscience, meteorology, and oceanography, sphere-oriented mathematics has gained increasing attention in recent decades. As main tools, spherical positive polynomial operators play prominent roles in the approximation and the interpolation on the sphere by means of orthonormal spherical harmonics. Several authors such as Ditzian [5], Dai and Ditzian [4], Bernes and Li [3], Wang and Li [16], Nikol'skiĭ and Lizorkin [10, 8] introduced and studied some spherical versions of some known one-dimensional polynomial operators, for example, spherical Jackson operators [8], spherical de la Vallée Poussin operators [3, 16], spherical delay mean operators [13] and best approximation operators [5, 4, 16] etc..

The main aim of the present paper is to study the approximation by the de la Vallée Poussin means on the unit sphere.

For to formulate our results, we first give some notations. Let $\mathbb{R}^d$, $d \geq 3$, be the Euclidean space of the points $x := (x_1, x_2, \ldots, x_d)$ endowed with the scalar product $x \cdot x' = \sum_{j=1}^d x_j x'_j (x, x' \in \mathbb{R}^d)$ and let $\sigma := \sigma^{d-1}$ be the unit sphere in $\mathbb{R}^d$ consisting of the points x satisfying $x^2 = x \cdot x = 1$.

We shall denote the points of σ by μ, and the elementary surface piece on σ by $d\sigma$. If it is necessary, we shall write $d\sigma :\equiv d\sigma(\mu)$ referring to the variable of integration. The surface area of σ^{d-1} is denoted by $|\sigma^{d-1}|$, and it is easy to deduce that $|\sigma^{d-1}| = \int_\sigma d\sigma = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$.

By $C(\sigma)$ and $L^p(\sigma)$, $1 \leq p < +\infty$, we denote the space of continuous, real valued functions and the space of (the equivalence classes of) p-integrable functions defined on σ endowed with the respective norms $\|f\|_\infty := \max_{\mu\in\sigma} |f(\mu)|$, $\|f\|_p := \left(\int_\sigma |f(\mu)|^p d\sigma(\mu)\right)^{1/p}$ $(1 \leq p < \infty)$. In the following, $L^p(\sigma)$ will always be one of the spaces $L^p(\sigma)$ for $1 \leq p < \infty$, or $C(\sigma)$ for $p = \infty$.

Now we state some properties of spherical harmonics (see [16], [7], [9]). For integer $k \geq 0$, the restriction of a homogeneous harmonic polynomial of degree k on the unit sphere is called a spherical harmonic of degree k. The class of all spherical harmonics of degree k will be denoted by $\mathcal{H}_k$, and the class of all spherical harmonics of degree $k \leq n$ will be denoted by Π_n^d. Of course,

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$\Pi_n^d = \bigoplus_{k=0}^n \mathcal{H}_k$, and it comprises the restriction to σ of all algebraic polynomials in d variables of total degree not exceeding n. The dimension of $\mathcal{H}_k$ is given by

$$d_k^d := \dim\, \mathcal{H}_k^d := \begin{cases} \frac{2k+d-2}{k+d-2}\binom{k+d-2}{k}, & k \geq 1; \\ 1, & k = 0, \end{cases}$$

and that of Π_n^d is $\sum_{k=0}^n d_k^d$.

The spherical harmonics have an intrinsic characterization. To describe this, we first introduce the Laplace-Beltrami operators (see [9]) to sufficiently smooth functions f defined on σ, which is the restriction of Laplace operator $\Delta := \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ on the sphere σ, and can be expressed as $Df(\mu) := \Delta f\left(\frac{\mu}{|\mu|}\right)\Big|_{\mu\in\sigma}$. Clearly, the operator D is an elliptic, (unbounded) selfadjoint operator on $L^2(\sigma)$, is invariant under arbitrary coordinate changes, and its spectrum comprises distinct eigenvalues $\lambda_k := -k(k+d-2),\ k = 0, 1, \ldots,$ each having finite multiplicity. The space $\mathcal{H}_k$ can be characterized intrinsically as the eigenspace corresponding to λ_k, i.e.

$$\mathcal{H}_k = \{\Psi \in C^\infty(\sigma) : D\Psi = -k(k+d-2)\Psi\}.$$

Since the λ_k's are distinct, and the operator is selfadjoint, the spaces $\mathcal{H}_k$ are mutually orthogonal; also, $L^2(\sigma) =$ closure $\{\bigoplus_k \mathcal{H}_k\}$. Hence, if we choose an orthogonal basis $\{Y_{k,l} : l = 1, \ldots, d_k^d\}$ for each $\mathcal{H}_k$, then the set $\{Y_{k,l} : k = 0, 1, \ldots, l = 1, \ldots, d_k^d\}$ is an orthogonal basis for $L^2(\sigma)$.

The orthogonal projection $Y_k : L^1(\sigma) \to \mathcal{H}_k$ is given by

$$Y_k(f;\mu) := \frac{\Gamma(\lambda)(k+\lambda)}{2\pi^{\lambda+1}} \int_\sigma P_k^\lambda(\mu\cdot\nu) f(\nu) d\sigma(\nu),$$

where $2\lambda = d-2$, and P_k^λ are the ultraspherical (or Gegenbauer) polynomials defined by the generating equation $(1-2r\cos\theta + r^2)^{-\lambda} = \sum_{k=0}^\infty r^k P_k^\lambda(\cos\theta)(0 \leq \theta \leq \pi)$. The further details for the ultraspherical polynomials can be found in [15].

For an arbitrary number θ, $0 < \theta < \pi$, we define the spherical translation operator of the function $f \in L^p(\sigma)$ with a step θ by the aid of the following equation (see [12], [2]):

$$S_\theta(f) := S_\theta(f;\mu) := \frac{1}{|\sigma^{d-2}|\sin^{d-2}\theta} \int_{\mu\cdot\nu=\cos\theta} f(\nu) d\sigma(\nu), \tag{1.1}$$

where $|\sigma^{d-2}|$ means the $(d-2)$-dimensional surface area of the unit sphere of $\mathbb{R}^{d-1}$. Here we integrate over the family of points $\nu \in \sigma$ whose spherical distance from the given point $\mu \in \sigma$ (i.e. the length of minor arc between μ and ν on the great circle passing through them) is equal to θ. Thus $S_\theta(f;\mu)$ can be interpreted as the mean value of the function f on the surface of $(d-2)$-dimensional sphere with radius $\sin\theta$.

The properties of spherical translation operator (1.1) are well known; see e.g., [2]. In particular, it can be expressed as the following series

$$S_\theta(f;\mu) = \sum_{k=0}^\infty \frac{P_k^\lambda(\cos\theta)}{P_k^\lambda(1)} Y_k(f;\mu) := \sum_{k=0}^\infty Q_k^\lambda(\cos\theta) Y_k(f;\mu)$$

where $Q_k^\lambda(\cos\theta) := \frac{P_k^\lambda(\cos\theta)}{P_k^\lambda(1)}$, and for any $f \in L^p(\sigma)$, $\|S_\theta(f)\|_p \leq \|f\|_p$, $\lim_{\theta\to 0}\|S_\theta f - f\|_p = 0$. We usually apply the translation operator to define spherical modulus of smoothness of a function $f \in L^p(\sigma)$, i.e. (see [16]) $\omega(f,t)_p := \sup_{0<\theta\leq t}\|f - S_\theta f\|_p$. Clearly, the modulus is meaningful to describe the approximation degree and the smoothness of functions on σ, which has been widely used in the study of approximation on sphere.

We also need a K-functional on sphere σ defined by (see [5], [16])

$$K(f;t)_p := \inf\left\{\|f-g\|_p + t^2\|Dg\|_p :\ g, Dg \in L^p(\sigma)\right\}. \quad 0 < t < t_0. \tag{1.2}$$

For the modulus of smoothness and K-functional, the following equivalent relationship has been proved (see [5])

$$\omega(f,t)_p \approx K(f,t)_p. \tag{1.3}$$

Here and in the following, $a \approx b$ means that there are positive constants C_1 and C_2 such that $C_1 \leq a \leq C_2 b$. We denote by $C_i (i = 1, 2, \ldots)$ the positive constants independent of f and n, and by $C(a)$ the positive constants depending only on a. Their value will be different at different occurrences, even within the same formula.

Define the kernel of de la Vallée Poussin as

$$v_n(t) = \frac{1}{I_{n,d}} \Big(\cos \frac{t}{2} \Big)^{2n}, \quad n \in \mathbb{N}, \tag{1.4}$$

where the constant $I_{n,d}$ satisfies $\int_\sigma v_n(\breve{\mu\nu}) d\sigma(\nu) = |\sigma^{d-2}|$, and $\breve{\mu\nu}$ is the spherical distance between the points μ and ν, i.e. the length of minor arc of great circle crossing μ and ν. Then the convolution resulted by the kernel is

$$V_n(f; \mu) = (f * v_n)(\mu) = \frac{1}{|\sigma^{d-2}|} \int_\sigma f(\nu) v_n(\breve{\mu\nu}) d\sigma(\nu), \quad f \in L^1(\sigma), \tag{1.5}$$

which is called de la Vallée Poussin means on the sphere.

The means were introduced by de la Vallée Poussin in 1908 for one dimensional Fourier series and were generalized to ultraspherical and Jacobi series by Kogbeliantz and Bavinck in 1925 and 1972, respectively (see also [16]). In 1993, Berens and Li [3] established the relation between the means and the best spherical polynomial approximation on the sphere, and discussed their approximation behavior by various of smoothness. Especially, they proved (see also [16]) the relation:

$$\max_{k \geq n} \|V_k f - f\|_p \approx \omega\Big(f, \frac{1}{\sqrt{n}}\Big)_p, \quad f \in L^p(\sigma). \tag{1.6}$$

Motivated by [1] and [6], we will improve the above result. Indeed, we will prove

$$\|V_n f - f\|_p \approx \omega\Big(f, \frac{1}{\sqrt{n}}\Big)_p$$

for any $f \in L^p(\sigma)$, $1 \leq p \leq +\infty$.

2 The kernel of de la Vallée Poussin

In the definition of de la Vallée-Poussin kernel v_n given by (1.4), the constants $I_{n,d}$ is requested to satisfy $\int_\sigma v_n(\breve{\mu\nu}) d\sigma(\nu) = |\sigma^{d-2}|$, which implies that $\int_0^\pi v_n(\theta) \sin^{2\lambda} \theta d\theta = 1 (2\lambda = d - 2)$.

By computation, we have $I_{n,d} = 2^{2\lambda} \frac{\Gamma(\lambda+1/2)\Gamma(n+\lambda+1/2)}{\Gamma(n+2\lambda+1)}$, where $\Gamma(\lambda)$ is Gamma function. So,

$$v_n(t) = \frac{\Gamma(n + 2\lambda + 1)}{2^{2\lambda} \Gamma(\lambda + 1/2) \Gamma(n + \lambda + 1/2)} \left(\cos \frac{t}{2} \right)^{2n}.$$

Since $v_n(t)$ are even trigonometric polynomials with degree n, $V_n(f, \mu)$ are spherical polynomials with degree n. So we also call (1.5) spherical de la Vallée Poussin polynomial operators.

We can translate de la Vallée Poussin means given by (1.5) into the multiplier form:

$$V_n(f; \mu) = \sum_{k=0}^\infty \omega_{n,k}^{(\lambda)} Y_k(f; \mu) \tag{2.1}$$

where

$$\omega_{n,k}^{(\lambda)} := \begin{cases} \frac{n!(n+2\lambda)!}{(n-k)!(n+k+2\lambda)!}, & 0 \leq k \leq n; \\ 0, & k > n. \end{cases}$$

Since the means can be rewritten as

$$\begin{aligned} V_n(f; \mu) &= \int_0^\pi v_n(\theta) S_\theta(f; \mu) \sin^{2\lambda} \theta d\theta = \int_0^\pi v_n(\theta) \left(\sum_{k=0}^\infty \frac{P_k^\lambda(\cos\theta)}{P_k^\lambda(1)} Y_k(f; \mu) \right) \sin^{2\lambda} \theta d\theta \\ &= \sum_{k=0}^\infty \left(\int_0^\pi v_n(\theta) \frac{P_k^\lambda(\cos\theta)}{P_k^\lambda(1)} \sin^{2\lambda} \theta d\theta \right) Y_k(f; \mu), \end{aligned}$$

it is sufficient to prove $\int_0^\pi v_n(\theta)\frac{P_k^\lambda(\cos\theta)}{P_k^\lambda(1)}\sin^{2\lambda}\theta\, d\theta=\omega_{n,k}^{(\lambda)}(k\geq 0)$. Indeed, when $k=1$, one has

$$\begin{aligned}
&\int_0^\pi v_n(\theta)\frac{P_1^\lambda(\cos\theta)}{P_1^\lambda(1)}\sin^{2\lambda}\theta d\theta=\int_0^\pi v_n(\theta)\left(1-2\sin^2\frac{\theta}{2}\right)\sin^{2\lambda}\theta d\theta\\
=&\ \frac{2^{2\lambda}}{I_{n,d}}\left(\int_0^\pi\left(\cos\frac{\theta}{2}\right)^{2(n+\lambda)}\sin^{2\lambda}\frac{\theta}{2}d\theta-2\int_0^\pi\left(\cos\frac{\theta}{2}\right)^{2(n+\lambda)}\sin^{2(\lambda+1)}\frac{\theta}{2}d\theta\right)\\
=&\ \frac{2^{2\lambda+1}}{I_{n,d}}\left(\frac{1}{2}B\left(\lambda+\frac{1}{2},n+\lambda+\frac{1}{2}\right)-B\left(\lambda+1+\frac{1}{2},n+\lambda+\frac{1}{2}\right)\right)\\
=&\ \frac{n}{n+2\lambda+1}=\frac{n!(n+2\lambda)!}{(n-1)!(n+1+2\lambda)!}=\omega_{n,1}^{(\lambda)},
\end{aligned}$$

where $B(a,b)$ is Beta function.

Now, we suppose for $k\leq n$ that $\int_0^\pi v_n(\theta)\frac{P_k^\lambda(\cos\theta)}{P_k^\lambda(1)}\sin^{2\lambda}\theta d\theta=\omega_{n,k}^{(\lambda)}$. Then for $k+1$ we first recall the relation (see page 81 of [15])

$$(k+1)P_{k+1}^\lambda(x)-2(\lambda+k)xP_k^\lambda(x)+(2\lambda+k-1)P_{k-1}^\lambda(x)=0\quad (k\geq 1),$$

i.e.,

$$P_{k+1}^\lambda(\cos\theta)=\frac{1}{k+1}\left(2(\lambda+k)\cos\theta P_k^\lambda(\cos\theta)-(2\lambda+k-1)P_{k-1}^\lambda(\cos\theta)\right).$$

Then,

$$\begin{aligned}
&\int_0^\pi v_n(\theta)\frac{P_{k+1}^\lambda(\cos\theta)}{P_{k+1}^\lambda(1)}\sin^{2\lambda}\theta d\theta=\frac{1}{P_{k+1}^\lambda(1)(k+1)}\left(2(\lambda+k)\int_0^\pi v_n(\theta)\cos\theta P_k^\lambda(\cos\theta)\sin^{2\lambda}\theta d\theta\right.\\
&\left.-(2\lambda+k-1)\int_0^\pi v_n(\theta)P_{k-1}^\lambda(\cos\theta)\sin^{2\lambda}\theta d\theta\right):=\frac{1}{P_{k+1}^\lambda(1)(k+1)}\left(2(\lambda+k)J_2-J_1\right).
\end{aligned}$$

By the assumption, we obtain

$$J_1=(2\lambda+k-1)P_{k-1}^\lambda(1)\int_0^\pi v_n(\theta)\frac{P_{k-1}^\lambda(\cos\theta)}{P_{k-1}^\lambda(1)}\sin^{2\lambda}\theta d\theta=\frac{(2\lambda+k-1)!n!(n+2\lambda)!}{\Gamma(2\lambda)(k-1)!(n-k+1)!(n+k-1+2\lambda)!}.$$

For J_2 we have

$$\begin{aligned}
J_2\ =&\ \frac{1}{I_{n,d}}\int_0^\pi\left(\cos\frac{\theta}{2}\right)^{2n}\left(2\cos^2\frac{\theta}{2}-1\right)P_k^\lambda(\cos\theta)\sin^{2\lambda}\theta d\theta\\
=&\ \frac{2I_{n+1,d}}{I_{n,d}}P_k^\lambda(1)\int_0^\pi v_{n+1}(\theta)\frac{P_k^\lambda(\cos\theta)}{P_k^\lambda(1)}\sin^{2\lambda}\theta d\theta-P_k^\lambda(1)\int_0^\pi v_n(\theta)\frac{P_k^\lambda(\cos\theta)}{P_k^\lambda(1)}\sin^{2\lambda}\theta d\theta\\
:=&\ J_{21}-J_{22},
\end{aligned}$$

which implies from the assumption that $J_{22}=\frac{\Gamma(k+2\lambda)}{k!\Gamma(2\lambda)}\frac{n!(n+2\lambda)!}{(n-k)!(n+k+2\lambda)!}$, and

$$J_{21}=\frac{2\Gamma(n+1+\lambda+1/2)\Gamma(n+2\lambda+1)}{\Gamma(n+\lambda+1/2)\Gamma(n+1+2\lambda+1)}\frac{\Gamma(k+2\lambda)}{k!\Gamma(2\lambda)}\frac{(n+1)!(n+1+2\lambda)!}{(n+1-k)!(n+1+k+2\lambda)!}.$$

Therefore,

$$J_2=\frac{\Gamma(k+2\lambda)}{k!\Gamma(2\lambda)}\frac{n!(n+2\lambda)!}{(n+1-k)!(n+1+k+2\lambda)!}\left(n(n+1)+k(2\lambda+k)\right).$$

So,

$$\int_0^\pi v_n(\theta)\frac{P_{k+1}^\lambda(\cos\theta)}{P_{k+1}^\lambda(1)}\sin^{2\lambda}\theta d\theta=\frac{n!(n+2\lambda)!}{(n-k-1)!(n+k+1+2\lambda)!}=\omega_{n,k+1}^{(\lambda)}.$$

On the other hand, it is clear that for $k > n$, $\omega_{n,k}^{(\lambda)} = 0$. Hence, de la Vallée Poussin means $V_n(f;\mu)$ have the form of multiplier expression given in (2.1).

Now we give some properties for the de la Vallée Poussin kernel v_n.

Lemma 2.1. Let $v_n(t)$ be the kernel of de la Vallée Poussin defined by (1.4), $2\lambda = d-2$ and $d \geq 3$. Then there hold

$$\int_0^\pi \theta^{-\lambda} v_n(\theta)\sin^{2\lambda}\theta d\theta \leq C(d)n^{\frac{\lambda}{2}}, \tag{2.2}$$

and

$$\int_0^\pi \theta^{-\frac{2}{m}} v_n(\theta)\sin^{2\lambda}\theta d\theta \leq C(d)n^{\frac{1}{m}}, \quad m = 1,2,\dots. \tag{2.3}$$

Proof. We only prove (2.2). The proof of (2.3) is similar. First, a direct computation implies

$$I_{n,d} = C(d)\frac{(2n+d-3)!!}{(2n+2d-4)!!} \approx n^{-\frac{d-1}{2}}.$$

Then, $\int_0^\pi \theta^{-\lambda} v_n(\theta)\sin^{2\lambda}\theta d\theta = \frac{1}{I_{n,d}}\int_0^\pi \theta^{-\lambda}\Big(\cos\frac{\theta}{2}\Big)^{2n}\sin^{2\lambda} d\theta = \frac{J_{n,d}^{(-\lambda)}}{I_{n,d}}$, where

$$J_{n,d}^{(-\lambda)} = \int_0^{\frac{\pi}{2}} \theta^{-\frac{d-2}{2}}\sin^{d-2}\theta\cos^{2n}\frac{\theta}{2}d\theta.$$

So, we have

$$\begin{aligned} J_{n,d}^{(-\lambda)} &\leq 2^{\frac{d}{2}}\int_0^{\frac{\pi}{2}} \sin^{\frac{d-2}{2}} t\cos^{2n+d-2} tdt = 2^{\frac{d}{2}-1}B\Big(\frac{\frac{d-2}{2}+1}{2}, \frac{2n+d-2+1}{2}\Big) \\ &= 2^{\frac{d}{2}-1}\Gamma(\frac{d}{4})\frac{\Gamma(n+\frac{d-1}{2})}{\Gamma(n+\frac{3d-2}{4})} = C(d)\frac{\Gamma(n+\frac{d-1}{2})}{\Gamma(n+\frac{3d-2}{4})} \approx n^{-\frac{d}{4}}. \end{aligned}$$

Therefore

$$\int_0^\pi \theta^{-\lambda} v_n(\theta)\sin^{2\lambda}\theta d\theta = \frac{J_{n,d}^{(-2)}}{I_{n,d}} \leq C(d)\frac{n^{-\frac{d}{4}}}{n^{-\frac{d-1}{2}}} = C(d)n^{\frac{d-2}{4}}.$$

The proof of Lemma 2.1 is completed. □

Lemma 2.2. For the kernel of de la Vallée Poussin $v_n(t)$ defined by (1.4), we have

$$\int_0^\pi \theta^4 v_n(\theta)\sin^{2\lambda}\theta d\theta \leq C(d)n^{-2}.$$

Proof. Since

$$\begin{aligned} J_{n,d}^{(4)} &= \int_0^\pi \theta^4\cos^{2n}\frac{\theta}{2}\sin^{2\lambda}\theta d\theta = 2^{d-1}\pi^4\int_0^{\frac{\pi}{2}} \sin^{d+2}\theta\cos^{2n+d-2}\theta d\theta \\ &= \begin{cases} 2^{d-2}\pi^5\frac{(2n+d-3)!!(d+1)!!}{(2n+2d)!!}, & \text{if } d \text{ is even;} \\ 2^{d-1}\pi^4\frac{(2n+d-3)!!(d+1)!!}{(2n+2d)!!}, & \text{if } d \text{ is odd} \end{cases} \\ &= C(d)\frac{(2n+d-3)!!}{(2n+2d)!!}, \end{aligned}$$

we have

$$\int_0^\pi \theta^4 v_n(\theta)\sin^{2\lambda}\theta d\theta = \frac{J_{n,d}^{(4)}}{I_{n,d}} = C(d)\frac{(2n+2d-4)!!}{(2n+2d)!!} \leq C(d)n^{-2}.$$

This finishes the proof of Lemma 2.2. □

3 Lower bound of approximation for de la Vallée Poussin means

In this section we prove the main result of this paper, which can be stated as follows.

Theorem 3.1. Let $V_n(f;\mu)$ be de la Vallée Poussin means on the sphere given by (1.5). Then for $f \in L^p(\sigma), 1 \le p \le +\infty$, there exists a constant C which is independent of f and n, such that

$$\omega\left(f, \frac{1}{\sqrt{n}}\right)_p \le C\|V_n f - f\|_p.$$

In order to prove the result, we first prove the following lemma.

Lemma 3.1. For any $g, Dg, D^2g \in L^p(\sigma)$, $1 \le p \le \infty$, there exist the constants A, B and C_2 which are are independent of n and g, such that $\|V_n g - g - \alpha(n)Dg\|_p \le C_2 n^{-2}\|D^2 g\|_p$, where $0 < \frac{A}{n} \le \alpha(n) \le \frac{B}{n}$.

Proof. Since (see (3.6) of [11]) $S_\theta(g;\mu) - g(\mu) = \int_0^\theta \sin^{-2\lambda} t\, dt \int_0^t \sin^{2\lambda} u\, S_u(Dg;\mu)du$, we have

$$S_u(Dg;\mu) - Dg(\mu) = \int_0^u \sin^{-2\lambda}\gamma d\gamma \int_0^\gamma \sin^{2\lambda}\nu\, S_\nu(D^2g;\mu)d\nu.$$

Observing that

$$\begin{aligned}
V_n(g;\mu) - g(\mu) &= \int_0^\pi v_n(\theta)\sin^{2\lambda}\theta d\theta \int_0^\theta \sin^{-2\lambda} tdt \int_0^t \sin^{2\lambda} u\, S_u(Dg;\mu)du \\
&= Dg(\mu)\int_0^\pi v_n(\theta)\sin^{2\lambda}\theta d\theta \int_0^\theta \sin^{-2\lambda} tdt \int_0^t \sin^{2\lambda} udu \\
&\quad + \int_0^\pi v_n(\theta)\sin^{2\lambda}\theta d\theta \int_0^\theta \sin^{-2\lambda} tdt \int_0^t \sin^{2\lambda} u\Big(S_u(Dg;\mu) - Dg(\mu)\Big)du \\
&:= Dg(\mu)\alpha(n) + \Psi(g;\mu),
\end{aligned}$$

where $\alpha(n) = C(d)n^{-1}$ satisfies $0 < An^{-1} \le C(d)n^{-1} \le Bn^{-1}$, we obtain that from the Hölder-Minkowski's inequality and the contractility of translation operator

$$\begin{aligned}
\|\Psi g\|_p &\le \|D^2g\|_p \int_0^\pi v_n(\theta)\sin^{2\lambda}\theta d\theta \int_0^\theta \sin^{-2\lambda} tdt \int_0^t \sin^{2\lambda} udu \int_0^u \sin^{-2\lambda}\gamma d\gamma \int_0^\gamma \sin^{2\lambda}\nu d\nu \\
&\le C_3\|D^2g\|_p \int_0^\pi v_n(\theta)\theta^4 \sin^{2\lambda}\theta d\theta.
\end{aligned}$$

Thus, from Lemma 2.2 it follows that $\|\Psi g\|_p \le C_4 n^{-2}\|D^2g\|_p$. The Lemma 3.1 has been proved. □

Now we turn to the proof of Theorem 3.1. We first introduce an operator V_n^m given by

$$V_n^m(f;\mu) = \sum_{k=0}^n \Big(\int_0^\pi v_n(\theta)Q_k^\lambda(\cos\theta)\sin^{2\lambda}\theta d\theta\Big)^m Y_k(f;\mu).$$

Then, form the orthogonality of projection operator Y_k, it follows that

$$\begin{aligned}
V_n^{m+l}f &= \sum_{k=0}^n \Big(\int_0^\pi v_n(\theta)Q_k^\lambda(\cos\theta)\sin^{2\lambda}\theta d\theta\Big)^m Y_k\Big(\sum_{s=0}^n \Big(\int_0^\pi v_n(\theta)Q_s^\lambda(\cos\theta)\sin^{2\lambda}\theta d\theta\Big)^l Y_s f\Big) \\
&= V_n^m(V_n^l f).
\end{aligned}$$

Thus, we take $g = V_n^m f$ and obtain that

$$\|f - g\|_p = \|f - V_n^m f\|_p \le \sum_{k=1}^m \|V_n^{k-1}f - V_n^k f\|_p \le m\|f - V_n f\|_p,$$

where $V_n^0 f = f$.

Next, we prove the estimate: $\|DV_n^m f\|_p \leq \frac{A}{2C_2} C_1 n \|f\|_p$, where A and C_2 are the same as that in Lemma 3.1. In fact, we have

$$\|DV_n^m f\|_p \leq \left\| \sum_{k=0}^{n} k(k+d-2)\Big(\int_0^\pi v_n(\theta) \Big| Q_k^\lambda(\cos\theta) \Big| \sin^{2\lambda}\theta d\theta \Big)^m Y_k(f) \right\|_p .$$

Since (see [1])

$$|Q_k^\lambda(\cos\theta)| \equiv \Big| \frac{P_k^\lambda(\cos\theta)}{P_k^\lambda(1)} \Big| \leq C_5 \min\Big((k\theta)^{-\lambda}, 1 \Big),$$

we use (2.2) and obtain for $k\theta \geq 1$ and $\theta \leq \frac{\pi}{2}$, that

$$\begin{aligned} \|DV_n^m f\|_p &\leq C_6 \left\| \sum_{k=0}^{n} k(k+d-2) k^{-\frac{d-2}{2}m} \Big(\int_0^\pi v_n(\theta) \theta^{-\lambda} \sin^{2\lambda}\theta d\theta \Big)^m Y_k(f) \right\|_p \\ &\leq C_7 n^{\frac{d-2}{4}m} \|f\|_p \sum_{k=0}^{\infty} k^{2-\frac{d-2}{2}m}. \end{aligned}$$

For $2 - \frac{d-2}{2}m < -1$, i.e. $m > \frac{6}{d-2}$, it is clear that the series $\sum_{k=0}^{\infty} k^{2-\frac{d-2}{2}m}$ is convergence. Thus

$$\|DV_n^m f\|_p \leq C_8 n^{\frac{d-2}{4}m} \|f\|_p.$$

For $k\theta \leq 1$, then (2.3) implies that

$$\|DV_n^m f\|_p \leq \left\| \sum_{k=0}^{n} \Big(\int_0^\pi v_n(\theta) \theta^{-\frac{2}{m}} (\theta^2 k(k+d-2))^{\frac{1}{m}} \Big| Q_k^\lambda(\cos\theta) \Big| \sin^{2\lambda}\theta d\theta \Big)^m Y_k(f) \right\|_p$$

$$\leq C_9 \left\| \sum_{k=0}^{n} \Big(\int_0^\pi v_n(\theta) \theta^{-\frac{2}{m}} \sin^{2\lambda}\theta d\theta \Big)^m Y_k(f) \right\|_p \leq C_{10} n \left\| \sum_{k=0}^{\infty} Y_k(f) \right\|_p = \frac{A}{2C_2} C_1 n \|f\|_p,$$

where A and C_2 are the same as that in Lemma 3.1. Therefore, when $m > \frac{6}{d-2}$, we have

$$\|DV_n^m f\|_p \leq \frac{A}{2C_2} C_1 n \|f\|_p.$$

In the next, without loss generality, we assume $m_1 > \frac{6}{d-2}$, and $m > \frac{6}{d-2} + m_1$. According to Lemma 3.1 we see that

$$\alpha(n)\|DV_n^m f\|_p \leq \|V_n^m f - f\|_p + C_2 n^{-2} \|D^2 V_n^m f\|_p \leq m\|V_n f - f\|_p + \frac{AC_1}{2} n^{-1} \|DV_n^{m-m_1} f\|_p$$

$$\leq m\|V_n f - f\|_p + \frac{AC_1}{2} n^{-1} \|DV_n^m f\|_p + \frac{AC_1}{2} n^{-1} \|DV_n^{m-m_1}(V_n^{m_1} f - f)\|_p$$

$$\leq m\|V_n f - f\|_p + \frac{AC_1}{2n} \|DV_n^m f\|_p + \frac{AC_1 C_{11}}{2} \|V_n^{m_1} f - f\|_p$$

$$= C_{12} \|V_n f - f\|_p + \frac{AC_1}{2n} \|DV_n^m f\|_p.$$

Taking $\alpha(n) = \frac{AC_1}{n}$, one has

$$\frac{1}{n} \|DV_n^m f\|_p \leq \frac{2C_{12}}{AC_1} \|V_n f - f\|_p.$$

So from the definition of K-functional it follows

$$\begin{aligned} K\left(f, \frac{1}{\sqrt{n}}\right) &\leq \|f - V_n^m f\|_p + \Big(\frac{1}{\sqrt{n}}\Big)^2 \|DV_n^m f\|_p \\ &\leq m\|f - V_n f\|_p + \frac{2C_{12}}{AC_1} \|f - V_n f\|_p \leq C_{14} \|f - V_n f\|_p, \end{aligned}$$

which together with (1.3) implies

$$\omega\left(f, \frac{1}{\sqrt{n}}\right)_p \leq C\|f - V_n f\|_p.$$

This finishes the proof of Theorem 3.1. □

From (1.6) and Theorem 3.1, the following Corollary 3.1 follows directly.

Corollary 3.1. For any $f \in L^p(\sigma), 1 \leq p \leq \infty$, there holds

$$\|V_n f - f\|_p \approx \omega\left(f, \frac{1}{\sqrt{n}}\right)_p.$$

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On the fixed point method for stability of a mixed type AQ-functional equation

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Abstract

In this article, we take into account the stability for the following functional equation of additive-quadratic type

$$f(x-y)-f(-x+y)-4f(x)+f(2x)-f(-y)+f(y)=0$$

with the fixed point method.

Keywords: Stability; Fixed point method; Additive-quadratic mapping.
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1 Introduction

Ulam [9] proposed the following question concerning the stability of homomorphisms:

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon>0$, does there exist a $\delta>0$ such that if a function $h: G_1\to G_2$ satisfies the inequality $d(h(xy),h(x)h(y))<\delta$ for all $x,y\in G_1$, then there exists a homomorphism $H: G_1\to G_2$ with $d(h(x),H(x))<\varepsilon$ for all $x\in G_1$?

Hyers [5] answers the problem of Ulam under the assumption that the groups are Banach spaces. A generalized version of the theorem of Hyers for approximately additive mappings was given by Aoki [1], and for approximately linear mappings was presented by Rassias [7] by considering an unbounded Cauchy difference. Thereafter, many interesting results of the stability of several functional equation have been extensively investigated.

On the contrary, Cădariu and Radu [2] observed that the existence of the solution for a functional equation and the estimation of the difference with the given mapping can be obtained from the fixed point alternative. This method is called a fixed point method. In particular, they [3, 4] applied this method to prove the stability theorems of the additive functional equation and the quadratic functional equation by using the fixed point method.

Now we consider the stability of the following mixed type additive-quadratic functional equation (briefly, AQ–functional equation)

$$f(x-y)-f(-x+y)-4f(x)+f(2x)-f(-y)+f(y)=0. \tag{1.1}$$

by using the fixed point method. In this case, every solution of the functional equation (1.1) is said to be an additive-quadratic mapping.

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2 Stability of Eq. (1.1) and its applications

Throughout this article, let V be a real or complex linear space and Y a Banach space. For a given mapping $f : V \to Y$, we use the following abbreviation

$$Df(x,y) := f(x-y) - f(-x+y) - 4f(x) + f(2x) - f(-y) + f(y)$$

for all $x, y \in V$. We first prove the following lemma.

Lemma 2.1 *Let $f : V \to Y$ be a mapping with $f(0) = 0$ such that $Df(x,y) = 0$ for all $x, y \in V\backslash\{0\}$. Then f is an additive-quadratic mapping.*

Proof. Since $f(0) = 0$, we get $Df(x,0) = Df(x,x) = 0$ for all $x \in V\backslash\{0\}$, and $Df(0,y) = 0$ for all $y \in V$. This completes the proof. □

For explicitly later use, we state the following theorem:

Theorem 2.2 (The alternative of fixed point) ([6] or [8]) *Suppose that a complete generalized metric space (X,d), which means that the metric d may assume infinite values, and a strictly contractive mapping $J : X \to X$ with the Lipschitz constant $0 < L < 1$ are given. Then, for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = +\infty, \ \forall n \in \mathbb{N} \cup \{0\},$$

or there exists a nonnegative integer k such that:

(1) *$d(J^n x, J^{n+1}x) < +\infty$ for all $n \geq k$;*

(2) *the sequence $\{J^n x\}$ is convergent to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in $Y := \{y \in X, d(J^k x, y) < +\infty\}$;*

(4) *$d(y, y^*) \leq (1/(1-L))d(y, Jy)$ for all $y \in Y$.*

Now, by the use of fixed point method, we obtain the main results as follow.

Theorem 2.3 *Let $\varphi : (V\backslash\{0\})^2 \to [0,\infty)$ be a function with $\varphi(x,y) = \varphi(-x,-y)$ for all $x, y \in V\backslash\{0\}$. Suppose that a mapping $f : V \to Y$ satisfies*

$$\|Df(x,y)\| \leq \varphi(x,y) \tag{2.1}$$

for all $x, y \in V\backslash\{0\}$ with $f(0) = 0$. If there exists a constant $0 < L < 1$ such that a function φ has the property

$$\varphi(2x, 2y) \leq 2L\varphi(x,y) \tag{2.2}$$

for all $x, y \in V\backslash\{0\}$, then there exists a unique additive-quadratic mapping $F : V \to Y$ such that

$$\|f(x) - F(x)\| \leq \frac{\varphi(x,x)}{2(1-L)} \tag{2.3}$$

for all $x \in V\backslash\{0\}$. In particular, F is represented by

$$F(x) = \lim_{n\to\infty} \left(\frac{f(2^n x) + f(-2^n x)}{2 \cdot 4^n} + \frac{f(2^n x) - f(-2^n x)}{2^{n+1}} \right) \tag{2.4}$$

for all $x \in V$.

Proof. Consider the set

$$S := \{g : g : V \to Y, \ g(0) = 0\}$$

and introduce a generalized metric on S by

$$d(g,h) = \inf\{K \in \mathbb{R}^+ : \|g(x) - h(x)\| \leq K\varphi(x,x) \ \text{ for all } \ x \in V\backslash\{0\}\}.$$

It is easy to see that (S,d) is a generalized complete metric space.

Now we define a mapping $J : S \to S$ by

$$Jg(x) := \frac{g(2x) - g(-2x)}{4} + \frac{g(2x) + g(-2x)}{8}$$

for all $x \in V$. Note that

$$J^n g(x) = \frac{g(2^n x) - g(-2^n x)}{2^{n+1}} + \frac{g(2^n x) + g(-2^n x)}{2 \cdot 4^n}$$

for all $n \in \mathbb{N}$ and all $x \in V$. Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g, h) \le K$. From the definition of d, we have

$$\begin{aligned}\|Jg(x) - Jh(x)\| =& \frac{3}{8}\|g(2x) - h(2x)\| + \frac{1}{8}\|g(-2x) - h(-2x)\| \\ &\le \frac{1}{2}K\varphi(2x, 2x) \\ &\le KL\varphi(x, x)\end{aligned}$$

for all $x \in V\backslash\{0\}$, which implies that

$$d(Jg, Jh) \le Ld(g, h)$$

for any $g, h \in S$, that is, J is a strictly contractive self-mapping of S with the Lipschitz constant L. Moreover, by (2.1), we see that

$$\|f(x) - Jf(x)\| = \frac{1}{8}\| - 3Df(x, x) + Df(-x, -x)\| \le \frac{\varphi(x, x)}{2}$$

for all $x \in V\backslash\{0\}$. It means that $d(f, Jf) \le \frac{1}{2} < \infty$ by the definition of d. Therefore, according to Theorem 2.2, the sequence $\{J^n f\}$ converges to the unique fixed point $F : V \to Y$ of J in the set $T = \{g \in S : d(f, g) < \infty\}$, which is represented by (2.4).

Note that

$$d(f, F) \le \frac{1}{1 - L} d(f, Jf) \le \frac{1}{2(1 - L)},$$

which implies (2.3).

By the definition of F, together with (2.1) and (2.4), we find that

$$\begin{aligned}\|DF(x, y)\| = \lim_{n\to\infty} \Bigg\| & \frac{Df(2^n x, 2^n y) - Df(-2^n x, -, 2^n y)}{2^{n+1}} \\ &+ \frac{Df(2^n x, 2^n y) + Df(-2^n x, -2^n y)}{2 \cdot 4^n}\Bigg\| \\ \le& \lim_{n\to\infty} \frac{2^n + 1}{2 \cdot 4^n} \left(\varphi(2^n x, 2^n y) + \varphi(-2^n x, -2^n y)\right) \\ =& 0\end{aligned}$$

for all $x, y \in V\backslash\{0\}$. By Lemma 2.1, we have proved that $DF(x, y) = 0$ for all $x, y \in V$. This completes the proof. $\Box$

We continue our investigation with the following theorem.

Theorem 2.4 *Let $\varphi : (V\backslash\{0\})^2 \to [0, \infty)$ with $\varphi(x, y) = \varphi(-x, -y)$ for all $x, y \in V\backslash\{0\}$. Suppose that $f : V \to Y$ satisfies the inequality $\|Df(x, y)\| \le \varphi(x, y)$ for all $x, y \in V\backslash\{0\}$ with $f(0) = 0$. If there exists $0 < L < 1$ such that the mapping φ has the property*

$$L\varphi(2x, 2y) \ge 4\varphi(x, y) \tag{2.5}$$

for all $x, y \in V\backslash\{0\}$, then there exists a unique additive-quadratic mapping $F : V \to Y$ such that

$$\|f(x) - F(x)\| \le \frac{L\varphi(x, x)}{4(1 - L)} \tag{2.6}$$

for all $x \in V\backslash\{0\}$. In particular, F is given by

$$F(x) = \lim_{n\to\infty} \left(2^{n-1}\left(f\left(\frac{x}{2^n}\right) - f\left(-\frac{x}{2^n}\right)\right) + \frac{4^n}{2}\left(f\left(\frac{x}{2^n}\right) + f\left(-\frac{x}{2^n}\right)\right)\right) \tag{2.7}$$

for all $x \in V$.

Proof. Let (S,d) be the set as in the proof of Theorem 2.3, and we consider the mapping $J : S \to S$ defined by

$$Jg(x) := g\left(\frac{x}{2}\right) - g\left(-\frac{x}{2}\right) + 2\left(g\left(\frac{x}{2}\right) + g\left(-\frac{x}{2}\right)\right)$$

for all $g \in S$ and all $x \in V$. Observe that

$$J^n g(x) = 2^{n-1}\left(g\left(\frac{x}{2^n}\right) - g\left(-\frac{x}{2^n}\right)\right) + \frac{4^n}{2}\left(g\left(\frac{x}{2^n}\right) + g\left(-\frac{x}{2^n}\right)\right)$$

and $J^0 g(x) = g(x)$ for all $x \in V$. Let $g, h \in S$ and let $K \in [0,\infty]$ be an arbitrary constant with $d(g,h) \le K$. The definition of d yields

$$\begin{aligned}\|Jg(x) - Jh(x)\| &= 3\left\|g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right)\right\| + \left\|g\left(-\frac{x}{2}\right) - h\left(-\frac{x}{2}\right)\right\| \\ &\le 4K\varphi\left(\frac{x}{2},\frac{x}{2}\right) \\ &\le LK\varphi(x,x)\end{aligned}$$

for all $x \in V\backslash\{0\}$. So we get

$$d(Jg, Jh) \le Ld(g,h)$$

for any $g, h \in S$, that is, J is a strictly contractive self-mapping of S with the Lipschitz constant L. Also we see that

$$\|f(x) - Jf(x)\| = \left\|Df\left(\frac{x}{2},\frac{x}{2}\right)\right\| \le \varphi\left(\frac{x}{2},\frac{x}{2}\right) \le \frac{L}{4}\varphi(x,x)$$

for all $x \in V\backslash\{0\}$, which implies that $d(f, Jf) \le \frac{L}{4} < \infty$.

Therefore, according to Theorem 2.2, the sequence $\{J^n f\}$ converges to the unique fixed point F of J in the set $T := \{g \in S : d(f,g) < \infty\}$, which is given by (2.7).

Since

$$d(f,F) \le \frac{1}{1-L}d(f,Jf) \le \frac{L}{4(1-L)}$$

the inequality (2.6) holds.

From the definition of F with (2.1) and (2.5), we have

$$\begin{aligned}\|DF(x,y)\| &= \lim_{n\to\infty}\left\|2^{n-1}\left(Df\left(\frac{x}{2^n},\frac{y}{2^n}\right) - Df\left(-\frac{x}{2^n},-\frac{y}{2^n}\right)\right)\right. \\ &\quad \left. + \frac{4^n}{2}\left(Df\left(\frac{x}{2^n},\frac{y}{2^n}\right) + Df\left(-\frac{x}{2^n},-\frac{y}{2^n}\right)\right)\right\| \\ &\le \lim_{n\to\infty}\frac{2^n+4^n}{2}\left(\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right) + \varphi\left(-\frac{x}{2^n},-\frac{y}{2^n}\right)\right) \\ &= 0\end{aligned}$$

for all $x, y \in V\backslash\{0\}$. So, by Lemma 2.1, F is an additive-quadratic mapping, which completes the proof. □

From now on, given a mapping $f : V \to Y$, we set

$$\begin{aligned}Af(x,y) &:= f(x+y) - f(x) - f(y), \\ Qf(x,y) &:= f(x+y) + f(x-y) - 2f(x) - 2f(y)\end{aligned}$$

for all $x, y \in V$. Using Theorem 2.3 and Theorem 2.4, we will prove the stability of the additive functional equation $Af \equiv 0$, and the quadratic functional equation $Qf \equiv 0$ in the following results.

Corollary 2.5 *Let $f_i : V \to Y$, $i = 1,2$, be mappings for which there exist functions $\phi_i : (V\backslash\{0\})^2 \to [0,\infty)$, $i = 1,2$, such that*

$$\|Af_i(x,y)\| \le \phi_i(x,y) \tag{2.8}$$

for all $x, y \in V\backslash\{0\}$. *If* $f_i(0) = 0$, $\phi_i(x, y) = \phi_i(-x, -y)$, $i = 1, 2$, *for all* $x, y \in V\backslash\{0\}$, *and there exists* $0 < L < 1$ *such that*

$$\frac{1}{L}\phi_1(x, y) \le \phi_1(2x, 2y) \le 2L\phi_1(x, y), \tag{2.9}$$

$$\phi_2(2x, 2y) \le L\phi_2(2x, 2y) \tag{2.10}$$

for all $x, y \in V\backslash\{0\}$, *then there exist unique additive mappings* $F_i : V \to Y$, $i = 1, 2$, *such that*

$$\|f_1(x) - F_1(x)\| \le \frac{\phi_1(x, x) + 3\phi_1(x, -x)}{2(1 - L)}, \tag{2.11}$$

$$\|f_2(x) - F_2(x)\| \le \frac{L(\phi_2(x, x) + 3\phi_2(x, -x))}{4(L - 1)} \tag{2.12}$$

for all $x \in V\backslash\{0\}$. *In particular, the mappings* F_i, $i = 1, 2$, *are represented by*

$$F_1(x) = \lim_{n\to\infty} \frac{f_1(2^n x)}{2^n}, \tag{2.13}$$

$$F_2(x) = \lim_{n\to\infty} 2^n f_2\left(\frac{x}{2^n}\right) \tag{2.14}$$

for all $x \in V$.

Proof. We first note that

$$Df_i(x, y) = Af_i(x, -y) - Af_i(-x, y) + Af_i(x, x) + Af_i(x, -x)$$

for all $x, y \in V$ and $i = 1, 2$. Put

$$\varphi_i(x, y) := \phi_i(x, -y) + \phi_i(-x, y) + \phi_i(x, x) + \phi_i(x, -x)$$

for all $x, y \in V\backslash\{0\}$ and $i = 1, 2$, then φ_1 satisfies (2.2) and φ_2 fulfills (2.5). Therefore $\|Df_i(x, y)\| \le \varphi_i(x, y)$ for all $x, y \in V\backslash\{0\}$ and $i = 1, 2$. According to Theorem 2.3, there exists a unique mapping $F_1 : V \to Y$ satisfying (2.11), which is represented by (2.4).

Observe that, by virtue of (2.8) and (2.9),

$$\begin{aligned}
\lim_{n\to\infty} \left\|\frac{f_1(2^n x) + f_1(-2^n x)}{2^{n+1}}\right\| &= \lim_{n\to\infty} \left\|\frac{f_1(2^n x) + f_1(-2^n x) - f_1(0)}{2^{n+1}}\right\| \\
&= \lim_{n\to\infty} \frac{1}{2^{n+1}} \|Af_1(2^n x, -2^n x)\| \\
&\le \lim_{n\to\infty} \frac{1}{2^{n+1}} \phi_1(2^n x, -2^n x) \\
&\le \lim_{n\to\infty} \frac{L^n}{2} \phi_1(x, -x) = 0
\end{aligned}$$

and

$$\lim_{n\to\infty} \left\|\frac{f_1(2^n x) + f_1(-2^n x)}{2 \cdot 4^n}\right\| \le \lim_{n\to\infty} \frac{2^n L^n}{2 \cdot 4^n} \phi_1(x, -x) = 0$$

for all $x \in V\backslash\{0\}$. This inequality and (2.4) guarantees (2.13).

Moreover, we have

$$\left\|\frac{Af_1(2^n x, 2^n y)}{2^n}\right\| \le \frac{\phi_1(2^n x, 2^n y)}{2^n} \le L^n \phi_1(x, y)$$

for all $x, y \in V\backslash\{0\}$. Sending the limit as $n \to \infty$ in the above inequality, and using $F_1(0) = 0$, we get $AF_1(x, y) = 0$ for all $x, y \in V$.

On the other hand, according to Theorem 2.4, we see that there exists a unique mapping $F_2 : V \to Y$ satisfying (2.12), which is given by (2.7).

Notice that, by (2.8) and (2.11),

$$\begin{aligned}\lim_{n\to\infty} 2^{2n-1}\left\|f_2\left(\frac{x}{2^n}\right)+f_2\left(\frac{-x}{2^n}\right)\right\| &= \lim_{n\to\infty} 2^{2n-1}\left\|Af_2\left(\frac{x}{2^n},-\frac{x}{2^n}\right)\right\| \\ &\le \lim_{n\to\infty} 2^{2n-1}\phi_2\left(\frac{x}{2^n},-\frac{x}{2^n}\right) \\ &\le \lim_{n\to\infty}\frac{L^n}{2}\phi_2(x,-x)=0.\end{aligned}$$

as well as

$$\lim_{n\to\infty} 2^{n-1}\left\|f_2\left(\frac{x}{2^n}\right)+f_2\left(\frac{-x}{2^n}\right)\right\| \le \lim_{n\to\infty}\frac{L^n}{2^{n+1}}\phi_2(x,-x)=0$$

for all $x\in V\backslash\{0\}$. From these and (2.7), we obtain (2.14).

Moreover, we have

$$\left\|2^nAf_2\left(\frac{x}{2^n},\frac{y}{2^n}\right)\right\| \le 2^n\phi_2\left(\frac{x}{2^n},\frac{y}{2^n}\right)\le \frac{L^n}{2^n}\phi_2(x,y)$$

for all $x,y\in V\backslash\{0\}$. Taking the limit as $n\to\infty$ in the above inequality, and using $F_2(0)=0$, we fee that $AF_2(x,y)=0$ for all $x,y\in V$. The proof is ended. □

Corollary 2.6 *Let $f_i : V\to Y$, $i=1,2$, be mappings for which there exist functions $\phi_i : (V\backslash\{0\})^2\to[0,\infty)$, $i=1,2$, such that*

$$\|Qf_i(x,y)\|\le\phi_i(x,y)$$

for all $x,y\in V\backslash\{0\}$. If $f_i(0)=0$, $\phi_i(x,y)=\phi_i(-x,-y)$, $i=1,2$, for all $x,y\in V\backslash\{0\}$, and there exists $0<L<1$ such that the mapping ϕ_1 satisfies (2.9) and ϕ_2 satisfies (2.10) for all $x,y\in V\backslash\{0\}$, then there exist unique quadratic mappings $F_i : V\to Y$, $i=1,2$, such that

$$\|f_1(x)-F_1(x)\|\le\frac{3\phi_1(x,x)+5\phi_1(x,-x)}{4(1-L)}, \tag{2.15}$$

$$\|f_2(x)-F_2(x)\|\le\frac{L(3\phi_2(x,x)+5\phi_2(x,-x))}{8(1-L)} \tag{2.16}$$

for all $x\in V\backslash\{0\}$. In particular, the mappings $F_i, i=1,2$, are given by

$$F_1(x)=\lim_{n\to\infty}\frac{f_1(2^nx)}{4^n}, \tag{2.17}$$

$$F_2(x)=\lim_{n\to\infty}4^nf_2\left(\frac{x}{2^n}\right) \tag{2.18}$$

for all $x\in V$.

Proof. Note that

$$Df_i(x,y)=Qf_i(x,y)-Qf_i(y,-x)+f_i(x,-x)+\frac{1}{2}Qf_i(y,-y)-\frac{1}{2}Qf_i(y,y)$$

for all $x,y\in V$ and $i=1,2$. Put $\varphi_i(x,y):=\phi_i(x,y)+\phi_i(y,-x)+\phi_i(x,-x)+\frac{1}{2}\phi_i(y,y)+\frac{1}{2}\phi_i(y,-y)$ for all $x,y\in V\backslash\{0\}$ and $i=1,2$, then φ_1 (resp. φ_2) satisfies (2.2) (resp. (2.5)). Moreover,

$$\|Df_i(x,y)\|\le\varphi_i(x,y)$$

for all $x,y\in V\backslash\{0\}$ and $i=1,2$. It follows from Theorem 2.3 that there exists a unique mapping $F_1 : V\to Y$ satisfying (2.15), which is represented by (2.4).

Observe that

$$\begin{aligned}\lim_{n\to\infty}\left\|\frac{f_1(2^nx)-f_1(-2^nx)}{2^{n+1}}\right\| &= \lim_{n\to\infty}\frac{1}{2^{n+1}}\left\|Qf_1(2^{n-1}x,-2^{n-1}x)-Qf_1(-2^{n-1}x,2^{n-1}x)\right\| \\ &\le \lim_{n\to\infty}\frac{1}{2^{n+1}}\left(\phi_1(2^{n-1}x,-2^{n-1}x)+\phi_1(-2^{n-1}x,2^{n-1}x)\right) \\ &\le \lim_{n\to\infty}\frac{L^n}{2}\left(\phi_1\left(\frac{x}{2},-\frac{x}{2}\right)+\phi_1\left(-\frac{x}{2},\frac{x}{2}\right)\right) \\ &=0\end{aligned}$$

and

$$\lim_{n\to\infty}\left\|\frac{f_1(2^nx)-f_1(-2^nx)}{2\cdot 4^n}\right\| \le \lim_{n\to\infty}\frac{L^n}{2^{n+1}}\left(\phi_1\left(\frac{x}{2},-\frac{x}{2}\right)+\phi_1\left(-\frac{x}{2},\frac{x}{2}\right)\right)=0$$

for all $x \in V\backslash\{0\}$. Due to this fact and (2.4), we get (2.17).

Moreover, we have

$$\left\|\frac{Qf_1(2^nx,2^ny)}{4^n}\right\| \le \frac{\phi_1(2^nx,2^ny)}{4^n} \le \frac{L^n}{2^n}\phi_1(x,y)$$

for all $x, y \in V\backslash\{0\}$. As $n \to \infty$ in the above inequality, we see that $QF_1(x,y) = 0$ for all $x, y \in V\backslash\{0\}$. By using $F_1(0) = 0$, then we have

$$QF_1(x,0)=0,\ QF_1(0,y)=-\,QF_1\left(\frac{y}{2},-\frac{y}{2}\right)+QF_1\left(-\frac{y}{2},\frac{y}{2}\right)=0$$

for all $x, y \in V\backslash\{0\}$. Therefore, $QF_1(x,y) = 0$ for all $x, y \in V$.

On the other hand, Theorem 2.4 guarantees that there exists a unique mapping $F_2 : V \to Y$ satisfying (2.16), which is represented by (2.7).

Observe that

$$\begin{aligned}4^n\left\|f_2\left(\frac{x}{2^n}\right)-f_2\left(-\frac{x}{2^n}\right)\right\| =&4^n\left\|Qf_2\left(\frac{x}{2^{n+1}},-\frac{x}{2^{n+1}}\right)-Qf_2\left(-\frac{x}{2^{n+1}},\frac{x}{2^{n+1}}\right)\right\|\\ \le&4^n\left(\phi_2\left(\frac{x}{2^{n+1}},-\frac{x}{2^{n+1}}\right)+\phi_2\left(-\frac{x}{2^{n+1}},\frac{x}{2^{n+1}}\right)\right)\\ \le&L^n\left(\phi_2\left(\frac{x}{2},-\frac{x}{2}\right)+\phi_2\left(-\frac{x}{2},\frac{x}{2}\right)\right)\end{aligned}$$

for all $x \in V\backslash\{0\}$. It leads us to get

$$\lim_{n\to\infty}4^n\left(f_2\left(\frac{x}{2^n}\right)-f_2\left(-\frac{x}{2^n}\right)\right)=0,\ \lim_{n\to\infty}2^n\left(f_2\left(\frac{x}{2^n}\right)-f_2\left(-\frac{x}{2^n}\right)\right)=0$$

for all $x \in V\backslash\{0\}$. Based on these facts and (2.7), we obtain (2.18).

Moreover, we have

$$\left\|4^nQf_2\left(\frac{x}{2^n},\frac{y}{2^n}\right)\right\| \le 4^n\phi_2\left(\frac{x}{2^n},\frac{y}{2^n}\right) \le L^n\phi_2(x,y)$$

for all $x, y \in V\backslash\{0\}$. Going the limit as $n \to \infty$ in the previous inequality, and using $F_2(0) = 0$, we get $QF_2(x,y) = 0$ for all $x, y \in V$, which complete the proof.

Now, we obtain the stability in the framework of normed spaces using Theorem 2.3 and Theorem 2.4.

Corollary 2.7 *Let X be a normed space and Y a Banach space. Suppose that the mapping $f : X \to Y$ satisfies the inequality*

$$\|Df(x,y)\| \le \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X\backslash\{0\}$ with $f(0) = 0$, where $\theta \ge 0$ and $p \in (-\infty, 1) \cup (2,\infty)$. Then there exists a unique quadratic-additive mapping $F : X \to Y$ such that

$$\|f(x)-F(x)\| \le= \begin{cases}\frac{2\theta}{2^p-4}\|x\|^p & \text{if } p>2,\\ \frac{2\theta}{2-2^p}\|x\|^p & \text{if } p<1,\end{cases}$$

for all $x \in X\backslash\{0\}$.

Proof. This follows from Theorem 2.3 and Theorem 2.4 by putting

$$\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X\backslash\{0\}$ with $L = 2^{p-1} < 1$ if $p < 1$ and $L = 2^{2-p} < 1$ if $p > 2$.

Corollary 2.8 *Let X be a normd space and Y a Banach space. Suppose that the mapping $f : X \to Y$ satisfies the inequality*

$$\|Df(x,y)\| \le \theta\|x\|^p\|y\|^q$$

for all $x, y \in X\backslash\{0\}$ with $f(0) = 0$, where $\theta \geq 0$ and $p + q \in (-\infty, 1) \cup (2, \infty)$. Then there exists a unique quadratic-additive mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \leq = \begin{cases} \frac{\theta\|x\|^{p+q}}{2^{p+q}-4} & \text{if } p+q > 2, \\ \frac{\theta\|x\|^{p+q}}{2(2-2^{p+q})} & \text{if } p+q < 1 \end{cases}$$

for all $x \in X\backslash\{0\}$.

Proof. By considering

$$\varphi(x, y) := \theta\|x\|^p\|y\|^q$$

for all $x, y \in X\backslash\{0\}$ with $L = 2^{p+q-1} < 1$ if $p + q < 1$ and $L = 2^{2-p-q} < 1$ if $p + q > 2$, then by Theorem 2.3 and Theorem 2.4, we arrive at the conclusion of the corollary.

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DIFFERENCES OF COMPOSITION OPERATORS FROM LIPSCHITZ SPACE TO WEIGHTED BANACH SPACES IN POLYDISK

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ABSTRACT. Let φ and ψ be holomorphic self-maps of the unit polydisk $\mathbb{D}_n$ in the n-dimensional complex space C^n, denote by C_φ and C_ψ the induced composition operators. In this paper, we estimate the essential norm of the differences of composition operators $C_\varphi - C_\psi$ from Lipschitz space to weighted Banach space in the unit polydisk.

1. INTRODUCTION

The algebra of all holomorphic functions on domain Ω will be denoted by $H(\Omega)$, where Ω is a bounded domain in C^n, where $n \geq 1$ is a fixed integer. Let $\mathbb{D}_n = \{z = (z_1, ..., z_n) \in C^n, \ |z_i| < 1, \ 1 \leq i \leq n\}$ be the open unit polydisk of the complex n-dimensional Euclidean space C^n and $H(\mathbb{D}_n)$ be the space of all holomorphic functions on $\mathbb{D}_n$. For $z = (z_1, ..., z_n)$ and $w = (w_1, ..., w_n)$ in C^n, the inner product of z and w is $\langle z, w\rangle = z_1\overline{w}_1 + ... + z_n\overline{w}_n$, where $\langle ., .\rangle$ denotes the inner product. Moreover, $|||z||| = \max_j\{|z_j|\}$ stands for the supremum norm on $\mathbb{D}_n$.

For $z, w \in \mathbb{D}$, the *pseudo-hyperbolic* distance between z and w is defined by

$$\rho(z, w) = |(z - w)/(1 - \overline{w}z)|.$$

It is well known that if $f \in H(\mathbb{D})$, then $\rho(f(z), f(w)) \leq \rho(z, w)$. The Bergman metric on the unit polydisk is given by

$$H_z(u, v) = \sum_{j=1}^{n} \frac{u_j\overline{v_j}}{(1 - |z_j|^2)^2}.$$

The *Kobayashi* distance $k_{\mathbb{D}_n}$ on $\mathbb{D}_n$ is defined by

$$k_{\mathbb{D}_n}(z, w) = \frac{1}{2} \log \frac{1 + |||\phi_z(w)|||}{1 - |||\phi_z(w)|||}, \tag{1.1}$$

where $\phi_z : \mathbb{D}_n \to \mathbb{D}_n$ is the automorphism of $\mathbb{D}_n$ given by

$$\phi_z(w) = \Big(\frac{w_1 - z_1}{1 - \overline{z_1}w_1}, ..., \frac{w_n - z_n}{1 - \overline{z_n}w_n}\Big).$$

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Let v be a strictly positive bounded continuous function (weight) on the open unit polydisk $\mathbb{D}_n$ in $C^n, n \geq 1$. We first introduce the weighted Banach spaces of analytic functions of the following form:

$$H_v^\infty := \Big\{ f \in H(\mathbb{D}_n);\ \|f\|_v = \sup_{z \in \mathbb{D}_n} v(z)|f(z)| < \infty \Big\}$$

endowed with the sup-norm $\|.\|_v$. Spaces of this type appear in the study of growth conditions of analytic functions and have been studied in various articles, see, e.g. [2, 8, 10].

For $0 \leq \alpha < 1$, an $f \in H(\mathbb{D}_n)$ belongs to the Lipschitz space $Lip_\alpha(\mathbb{D}_n)$, if

$$\|f\|_\alpha = |f(0)| + \sup_{z \in \mathbb{D}_n} \sum_{l=1}^{n} \Big| \frac{\partial f}{\partial z_l}(z) \Big| (1 - |z_l|^2)^{1-\alpha} < \infty. \tag{1.2}$$

It is easy to show that $Lip_\alpha(\mathbb{D}_n)$ is a Banach space endowed with the norm $\|.\|_\alpha$ (see, e.g.[13, 14]).

Let $\varphi = (\varphi_1(z), ..., \varphi_n(z))$ and $\psi(z) = (\psi_1(z), ..., \psi_n(z))$ be holomorphic self-maps of $\mathbb{D}_n$. The composition operator C_φ induced by φ is defined by

$$(C_\varphi)f(z) = f(\varphi(z))$$

for $z \in \mathbb{D}_n$ and $f \in H(\mathbb{D}_n)$ (see, e.g.[3]). The essential norm of a continuous linear operator T is the distance from T to the set of all compact operators, that is, $\|T\|_e = \inf\{\|T - K\| :\ K \text{ is compact }\}$. Notice that $\|T\|_e = 0$ if and only if T is compact, so estimates on $\|T\|_e$ lead to conditions for T to be compact (see, e.g.[6, 14, **?**]). In the past few years, many authors have been interested in studying the mapping properties of the differences of two composition operators, that is, an operator of the form

$$T = C_\varphi - C_\psi.$$

The primary motivation for this has been the desire to understand the topological structure of the whole set of composition operators. Most papers in this area have focused on the classical reflexive spaces, but some classical nonreflexive spaces in the unit disc in the complex plane have also recently been discussed. We refer the readers to the recent papers [1, 4, 5, 6, 7, 9, 12] to learn more about the propertied about the differences.

Building on the above foundations we estimate the essential norm for the differences of composition operators induced by φ and ψ acting from Lipschitz space to weighted Banach space in the unit polydisk $\mathbb{D}_n$, where φ and ψ are two holomorphic self-maps of the unit polydisk in n-dimensional complex space C^n. The paper is organized as following: Some lemmas are given in section 2. Section 3 is devoted to the main results.

Throughout the remainder of this paper, C will denote a positive constant, the exact value of which will vary from one appearance to the next.

2. Some Lemmas

Lemma 1. *Assume that* $f \in Lip_\alpha(\mathbb{D}_n)$, *then*

$$|f(z) - f(w)| \leq n\|f\|_\alpha k_{\mathbb{D}_n}(z, w)$$

for any $z, w \in \mathbb{D}_n$.

Proof. Empolying the definitions in (1.1) and (1.2) we have that

$$\begin{aligned}
|f(z)-f(0)| &= \Big|\int_0^1 \frac{\Re f(tz)}{t}dt\Big| = \Big|\sum_{j=1}^n \int_0^1 z_j \frac{\partial f}{\partial \zeta_j}(tz)dt\Big| \\
&\leq \sum_{j=1}^n \int_0^1 \frac{|z_j|}{(1-|tz_j|^2)^{1-\alpha}}\Big|\frac{\partial f}{\partial \zeta_j}(tz)\Big|(1-|tz_j|^2)^{1-\alpha}dt \\
&\leq \|f\|_\alpha \sum_{j=1}^n \int_0^{|z_j|} \frac{1}{(1-t^2)^{1-\alpha}}dt \\
&\leq \|f\|_\alpha \sum_{j=1}^n \int_0^{|z_j|} \frac{1}{1-t^2}dt \\
&= \frac{1}{2}\|f\|_\alpha \sum_{j=1}^n \log\frac{1+|z_j|}{1-|z_j|} \\
&\leq n\|f\|_\alpha \frac{1}{2}\log\frac{1+|||z|||}{1-|||z|||}. \qquad (2.3)
\end{aligned}$$

The last inequality in (2.3) follows from the fact the map $t \to \log((1+t)/(1-t))$ is strictly increasing on $[0,1)$. Setting $z = \phi_w(z)$ and using (1.2), it's evident that

$$|f\circ\phi_w(z) - f\circ\phi_w(w)| \leq n\|f\circ\phi_w\|_\alpha \frac{1}{2}\log\frac{1+|||\phi_w(z)|||}{1-|||\phi_w(z)|||}.$$

Replacing $f\circ\phi_w$ by $f\circ\phi_w\circ\phi_w^{-1}$,

$$|f(z)-f(w)| \leq n\|f\|_\alpha \frac{1}{2}\log\frac{1+|||\phi_w(z)|||}{1-|||\phi_w(z)|||} \leq n\|f\|_\alpha k_{\mathbb{D}_n}(z,w).$$

This completes the proof. □

Lemma 2. *For $f \in Lip_\alpha(\mathbb{D}_n)$ and a fixed $0<\delta<1$, define $G = \{z \in \mathbb{D}_n : |||z||| \leq \delta\}$. Then*

$$\lim_{r\to 1} \sup_{\|f\|_\alpha \leq 1} \sup_{z\in G} |f(z)-f(rz)| = 0.$$

Proof. Using the definition in (1.2) we obtain that

$$\begin{aligned}
&\sup_{z\in G}|f(z)-f(rz)| \\
&= \sup_{z\in G}\Big|\sum_{j=1}^n \Big(f(rz_1,rz_2,...,rz_{j-1},z_j,...,z_n) - f(rz_1,rz_2,...,z_{j+1},...,z_n)\Big)\Big| \\
&\leq \sup_{z\in G}\sum_{j=1}^n \Big|\int_r^1 z_j\frac{\partial f}{\partial z_j}(rz_1,...,rz_{j-1},tz_j,z_{j+1},...,z_n)dt\Big| \\
&\leq (1-r)n\|f\|_\alpha \sup_{z\in G}\frac{1}{(1-|||z|||^2)^{1-\alpha}} \\
&\leq \frac{(1-r)n\|f\|_\alpha}{(1-\delta^2)^{1-\alpha}} \to 0,\ r\to 1
\end{aligned}$$

This ends the proof. □

3. Main result

In this section we estimate the essential norm of $C_\varphi - C_\psi : Lip_\alpha(\mathbb{D}_n) \to H_v^\infty(\mathbb{D}_n)$. We denote $F_\delta = \{z \in \mathbb{D}_n,\ \max\{|||\varphi(z)|||,\ |||\psi(z)|||\} \le 1-\delta\}$ and $E_\delta = \mathbb{D}_n - F_\delta$ for $0 < \delta < 1$. We consider the following two conditions

$$M_1 := \max_{1\le l\le n} \lim_{\delta\to 0} \sup_{z\in E_\delta} v(z)|(1-|\psi_l(z)|^2)^\alpha |\varphi_{\varphi_l(z)}(\psi_l(z))|$$

$$M_2 := \max_{1\le l\le n} \lim_{\delta\to 0} \sup_{z\in E_\delta} v(z)|(1-|\varphi_l(z)|^2)^\alpha |\varphi_{\psi_l(z)}(\varphi_l(z))|.$$

Theorem 1. *For any $0<\delta<1$, denote $F_\delta = \{z \in \mathbb{D}_n,\ \max\{|||\varphi(z)|||,\ |||\psi(z)|||\} \le 1-\delta\}$. Suppose $C_\varphi - C_\psi : Lip_\alpha(\mathbb{D}_n) \to H_v^\infty(\mathbb{D}_n)$ is bounded. Then*

$$\max\{M_1,\ M_2\} \le \|C_\varphi - C_\psi\|_e \le 2n \lim_{\delta\to 0} \sup_{z\in E_\delta} v(z) k_{\mathbb{D}_n}(\varphi(z), \psi(z)). \tag{3.4}$$

Proof. The upper estimate. For a fixed $0<r<1$, we have that both $C_{r\varphi}$ and $C_{r\psi}$ are compact operators. For any $0<\delta<1$,

$$\begin{aligned}
&\|C_\varphi - C_\psi\|_{e,Lip_\alpha\to H_v^\infty} \le \|C_\varphi - C_\psi - C_{r\varphi} + C_{r\psi}\|_{Lip_\alpha\to H_v^\infty}\\
&= \sup_{\|f\|_\alpha\le 1} \|(C_\varphi - C_\psi - C_{r\varphi} + C_{r\psi})f\|_{H_v^\infty}\\
&= \sup_{\|f\|_\alpha\le 1} \sup_{z\in\mathbb{D}_n} v(z)|f(\varphi(z)) - f(r\varphi(z)) + f(r\psi(z)) - f(\psi(z))|\\
&\le \sup_{\|f\|_\alpha\le 1} \sup_{z\in F_\delta} v(z)|f(\varphi(z)) - f(r\varphi(z)) + f(r\psi(z)) - f(\psi(z))|\\
&+ \sup_{\|f\|_\alpha\le 1} \sup_{z\in E_\delta} v(z)|f(\varphi(z)) - f(r\varphi(z)) + f(r\psi(z)) - f(\psi(z))|.
\end{aligned} \tag{3.5}$$

Since the weight $v(z)$ is a strictly positive bounded continuous function on the open unit polydisc $\mathbb{D}_n$ and using lemma 2 and we can choose r sufficiently close to 1 such that the first term in (3.5) is less than any given $\varepsilon > 0$, and we denote the second term in (3.5) by I. Empolying lemma 1, it follows that

$$\begin{aligned}
I &\le \sup_{\|f\|_\alpha\le 1} \sup_{z\in E_\delta} v(z)(|f(\varphi(z)) - f(\psi(z))| + |f(r\varphi(z)) - f(r\psi(z))|)\\
&\le \sup_{\|f\|_\alpha\le 1} \sup_{z\in E_\delta} v(z)n\|f\|_\alpha (k_{\mathbb{D}_n}(\varphi(z),\psi(z)) + k_{\mathbb{D}_n}(r\varphi(z), r\psi(z)))\\
&\le 2n \sup_{z\in E_\delta} v(z) k_{\mathbb{D}_n}(\varphi(z),\psi(z)),
\end{aligned} \tag{3.6}$$

the last inequality is obtained from $k_{\mathbb{D}_n}(r\varphi(z), r\psi(z)) \le k_{\mathbb{D}_n}(\varphi(z),\psi(z))$. Firstly letting $r\to 1$ and then $\delta \to 0$, the upper estimate yeilds.

The lower estimate. For $l = 1, 2, .., n$, set

$$E_\delta^l = \{z \in \mathbb{D}_n : \max(|\varphi_l(z)|, |\psi_1(z)|) > 1-\delta\}.$$

It is easy to see that $E_\delta = \bigcup_{l=1}^n E_\delta^l$. For a fixed l $(1 \le l \le n)$, define

$$a_l = \lim_{\delta\to 0} \sup_{z\in E_\delta^l} v(z)(1-|\varphi_l(z)|^2)^\alpha |\varphi_{\psi_l(z)}(\varphi_l(z))|.$$

If we put $\delta_m = 1/m$, then $\delta_m \to 0$ as $m\to\infty$. For the case $\|\varphi_l\|_\infty = 1$ or $\|\psi_l\|_\infty = 1$, then for large enough m with $E_{\delta_m}^l \ne \emptyset$, there exists $z^m \in E_{\delta_m}^l$ such that

$$\lim_{m\to\infty} v(z^m)(1-|\varphi_l(z^m)|^2)^\alpha |\varphi_{\psi_l(z^m)}(\varphi_l(z^m))| = a_l. \tag{3.7}$$

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Since $z^m \in E^l_{\delta_m}$ implies that $|\varphi_l(z^m)| > 1-\delta_m$ or $|\psi_l(z^m)| > 1-\delta_m$, without loss of generality we assume that $|\varphi_l(z^m)| \to 1$. Set

$$f_m(z) = \frac{1-|\varphi_l(z^m)|^2}{(1-\overline{\varphi_l(z^m)}z_l)^{1-\alpha}} \cdot \frac{\langle \varphi_{\psi_l(z^m)}(z), \varphi_{\psi_l(z^m)}(\varphi_l(z^m))\rangle}{|\varphi_{\psi_l(z^m)}(\varphi_l(z^m))|}.$$

We can easily obtain that $(f_m)_{m\in\mathbb{N}}$ converges to zero uniformly on compact subsets of $\mathbb{D}_n$ as $m\to\infty$ and $\sup_{k\in\mathbb{N}} \|f_m\|_\alpha \le C$. Thus for any compact operator $K : Lip_\alpha \to H_v^\infty$, we get $\|Kf_m\|_{H_v^\infty} \to 0,\ m\to\infty$. Moreover, it is obvious that

$$f_m(\varphi(z^m)) = (1-|\varphi_l(z^m)|^2)^\alpha |\varphi_{\psi_l(z^m)}(\varphi_l(z^m))|,\ f_m(\psi(z^m)) = 0. \tag{3.8}$$

Thus using the above results, (3.7) and (3.8), it is clear that

$$\begin{aligned}
&\|C_\varphi - C_\psi - K\|_{Lip_\alpha\to H_v^\infty} \ge C \limsup_{m\to\infty} \|(C_\varphi - C_\psi - K)f_m\|_{H_v^\infty} \\
&\ge C \limsup_{m\to\infty} (\|(C_\varphi - C_\psi)f_m\|_{H_v^\infty} - \|Kf_m\|_{H_v^\infty}) \\
&= C \limsup_{m\to\infty} \|(C_\varphi - C_\psi)f_m\|_{H_v^\infty} \\
&= C \limsup_{m\to\infty} \sup_{z\in\mathbb{D}_n} v(z)|f_m(\varphi(z)) - f_m(\psi(z))| \\
&\ge C \limsup_{m\to\infty} v(z^m)|f_m(\varphi(z^m)) - f_m(\psi(z^m))| \\
&= C \limsup_{m\to\infty} v(z^m)|(1-|\varphi_l(z^m)|^2)^\alpha |\varphi_{\psi_l(z^m)}(\varphi_l(z^m))| \\
&= Ca_l = C \lim_{\delta\to 0} \sup_{z\in E^l_\delta} v(z)(1-|\varphi_l(z)|^2)^\alpha |\varphi_{\psi_l(z)}(\varphi_l(z))|
\end{aligned}$$

From the above inequality we obtain that

$$\|C_\varphi - C_\psi\|_{e,Lip_\alpha\to H_v^\infty} \ge C \lim_{\delta\to 0} \sup_{z\in E^l_\delta} v(z)(1-|\varphi_l(z)|^2)^\alpha |\varphi_{\psi_l(z)}(\varphi_l(z))|. \tag{3.9}$$

If both $\|\varphi_l\|_\infty < 1$ and $\|\psi_l\|_\infty < 1$, in this condition, when δ is small enough, E^l_δ is empty, and without loss of generality we may assume that

$$\lim_{\delta\to 0} \sup_{z\in E^l_\delta} v(z)(1-|\varphi_l(z)|^2)^\alpha |\varphi_{\psi_l(z)}(\varphi_l(z))| = 0. \tag{3.10}$$

Since the above inequality (3.9) and (3.10) holds for every $1 \le l \le n$, thus we obtain that

$$\|C_\varphi - C_\psi\|_{e,Lip_\alpha\to H_v^\infty} \ge C \max_{1\le l\le n} \lim_{\delta\to 0} \sup_{z\in E^l_\delta} v(z)|(1-|\varphi_l(z)|^2)^\alpha |\varphi_{\psi_l(z)}(\varphi_l(z))|. \tag{3.11}$$

Now for each $l = 1, 2, ..., n$, we define

$$b_l = \lim_{\delta\to 0} \sup_{z\in E_\delta} v(z)(1-|\varphi_l(z)|^2)^\alpha |\varphi_{\psi_l(z)}(\varphi_l(z))|.$$

Then for any $\varepsilon > 0$, there exists a δ_0 with $0 < \delta_0 < 1$ such that

$$v(z)(1-|\varphi_l(z)|^2)^\alpha |\varphi_{\psi_l(z)}(\varphi_l(z))| > b_l - \varepsilon \tag{3.12}$$

whenever $z \in E_{\delta_0}$ and $l = 1, 2, ..., n$. From the above definition we know that $z \in E^l_{\delta_0}$ implies that $z \in E_{\delta_0}$, then by (3.11) and (3.12) we obtain that

$$\begin{aligned}
&\|C_\varphi - C_\psi\|_{e,Lip_\alpha\to H_v^\infty} \ge C \max_{1\le l\le n} (b_l - \varepsilon) \\
&= C \max_{1\le l\le n} \lim_{\delta\to 0} \sup_{z\in E_\delta} v(z)|(1-|\varphi_l(z)|^2)^\alpha |\varphi_{\psi_l(z)}(\varphi_l(z))| - C\varepsilon.
\end{aligned}$$

Now letting $\varepsilon \to 0$ in the above inequality we obtain that

$$\|C_\varphi - C_\psi\|_{e,Lip_\alpha \to H_v^\infty} \geq C \max_{1\leq l\leq n} \lim_{\delta\to 0} \sup_{z\in E_\delta} v(z)(1-|\varphi_l(z)|^2)^\alpha |\varphi_{\psi_l(z)}(\varphi_l(z))|. \quad (3.13)$$

Using the similar proof of (3.13) we can get

$$\|C_\varphi - C_\psi\|_{e,Lip_\alpha \to H_v^\infty} \geq C \max_{1\leq l\leq n} \lim_{\delta\to 0} \sup_{z\in E_\delta} v(z)|(1-|\psi_l(z)|^2)^\alpha |\psi_{\varphi_l(z)}(\psi_l(z))|. \quad (3.14)$$

Combining (3.13) and (3.14), we get the lower estimate for the essential norm of the differences. □

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THE PATH COMPONENT OF THE SET OF GENERALIZED COMPOSITION OPERATORS ON THE BLOCH TYPE SPACES

LIU YANG

ABSTRACT. In this note, we give a characterization of the path component of the set of generalized composition operator on Bloch type spaces.

Keywords: Path component, composition operator, Bloch type spaces

1. INTRODUCTION

Let $\mathbb{D}$ be the unit disk of the complex plane $\mathbb{C}$, and $H(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$. $f \in H(\mathbb{D})$ belongs to the Bloch type space $\mathcal{B}^\alpha$, if

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f'(z)| < \infty,$$

where $0 < \alpha < \infty$. It is known that $\mathcal{B}^\alpha$ is a Banach space under the $\|\cdot\|_{\mathcal{B}^\alpha}$ norm. If $\alpha = 1$, $\mathcal{B}^\alpha$ is just the well-known Bloch space. More details about properties on Bloch type space are given in [4], [32] and [16].

We denote $S(\mathbb{D})$ be the set of analytic self-maps of $\mathbb{D}$. Every analytic self-map $\varphi \in S(\mathbb{D})$ induces a linear composition operator C_φ from $H(\mathbb{D})$ to itself. A general and concerning problem in the investigation of composition operator is to characterize operator theoretic properties of C_φ in terms of function theoretic properties of φ. To learn more conclusions about the composition operator, see [6].

For $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$, Li and Stevic [10] defined the generalized composition operator C_φ^g as follows:

$$C_\varphi^g(f)(z) = \int_0^z f'(\varphi(w))g(w)dw,\ f \in H(\mathbb{D}).$$

The boundedness and compactness of the generalized composition operator from Zygmund spaces to Bloch-type spaces were considered in [10]. Lindstrom and Sanatpour [15] gave the characterization of the generalized composition operator between Zygmund spaces. We can also refer to [11–14], [21–30] for the study of the operator C_φ^g and its generalizations. The composition operators between Bloch type spaces have been studied by several authors, for example [1–3, 5, 17, 19].

Recently, lots of researchers are interested in the difference of two composition operators, that is, an operator of the form $T = C_\varphi - C_\psi$, where $\varphi, \psi \in S(\mathbb{D})$. For example, Shapiro

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and Sundberg [20] studied the difference of composition operators on Hardy spaces. In [18], MacCluer, Ohno and Zhao considered it on H^∞. In [7] and [8], Hosokawa and Ohno investigated it on Bloch spaces. The purpose of studying the difference of composition operators is to investigate the topological structure of the set of composition operators acting on a given function space. Li [9] gave the sufficient and necessary conditions for the boundedness and compactness of the differences of generalized composition operator on the Bloch space. Yang, Luo, and Zhu [31] generalized Li's results between Bloch type spaces, which help us to study the topological structure of the set of generalized composition operators on the Bloch type spaces. In fact, we give a sufficient condition for the path component of the set of generalized composition operator on Bloch type spaces.

2. NOTATIONS AND AUXILIARY RESULTS

For $w, z \in \mathbb{D}$, the pseudo-hyperbolic distance between z and w is defined by

$$\rho(w, z) = |\frac{w - z}{1 - \bar{w}z}|.$$

Let

$$u_s(z, w) = (1 - s)z + sw, \phi_s(\varphi(z), \psi(z)) = (1 - s)\varphi(z) + s\psi(z),$$

where $s \in [0, 1], w \in \mathbb{D}$, $\varphi, \psi \in S(\mathbb{D})$ and simply denote $\phi_s(\varphi(z), \psi(z))$ by $\phi_s(z)$.

Let

$$\Gamma(\varphi) = \{\{z_n\} \in \mathbb{D} : |\varphi(z_n)| \to 1\},$$
$$\Gamma(\psi) = \{\{z_n\} \in \mathbb{D} : |\psi(z_n)| \to 1\}.$$

Obviously, $\Gamma(\phi_s) \subset \Gamma(\varphi) \cap \Gamma(\psi)$.

Define

$$D_\alpha^{\varphi,g}(z) = \frac{g(z)}{(1 - |\varphi(z)|^2)^\alpha}, \ D_{\alpha,\beta}^{\varphi,g}(z) = \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^\alpha} g(z),$$

$$D_\alpha^{\psi,h}(z) = \frac{h(z)}{(1 - |\psi(z)|^2)^\alpha}, \ D_{\alpha,\beta}^{\psi,h}(z) = \frac{(1 - |z|^2)^\beta}{(1 - |\psi(z)|^2)^\alpha} h(z),$$

and

$$D_\alpha^{\phi_s}(z) = \frac{1 - |z|^\alpha}{(1 - |\phi(z)|^2)^\alpha}[(1 - s)g(z) + sh(z)],$$

$$C_{\phi_s} f(z) = \int_0^z f'((1 - s)\varphi(w) + s\psi(w))[(1 - s)g(w) + sh(w)]dw, f \in \mathcal{B}^\alpha.$$

Let

$$I_1(z) = D_{\alpha,\beta}^{\varphi,g}\rho(\varphi(z), \psi(z)),$$
$$I_2(z) = D_{\alpha,\beta}^{\psi,h}\rho(\varphi(z), \psi(z)),$$

and

$$I_3(z) = D_{\alpha,\beta}^{\varphi,g}(z) - D_{\alpha,\beta}^{\psi,h}(z).$$

Lemma 2.1. *([7, Lemma 4.1]) Let $z, w \in \mathbb{D}$ and $\rho(z,w) = \lambda < 1$. Then the map $s \mapsto \rho(u_s, w)$ is continuous and decreasing on $[0,1]$.*

Lemma 2.2. *([31, Theorem 1.]) The following statements are equivalent:*
(1) $C_\varphi^g - C_\psi^h : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ *is bounded.*
(2) $\sup_{z\in\mathbb{D}} |I_1(z)| < \infty$ *and* $\sup_{z\in\mathbb{D}} |I_3(z)| < \infty$.
(3) $\sup_{z\in\mathbb{D}} |I_2(z)| < \infty$ *and* $\sup_{z\in\mathbb{D}} |I_3(z)| < \infty$.

Lemma 2.3. *([31, Theorem 4.]) Let $0 < \alpha, \beta < \infty$ and $\varphi, \psi \in S(\mathbb{D})$, $g, h \in H(\mathbb{D})$, if $C_\varphi^g - C_\psi^h : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded, and C_φ^g, C_ψ^h: $\mathcal{B}^\alpha \to \mathcal{B}^\beta$ are not compact, then $C_\varphi^g - C_\psi^h : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is compact if and only if the following two conditions hold.*
(1) $D(g,\varphi) = D(h,\psi) \neq \emptyset$, $D(g,\varphi) \subset \Gamma(\psi)$.
(2) *For arbitrary* $\{z_n\} \subset \Gamma(\varphi) \cap \Gamma(\psi)$,

$$\lim_{n\to\infty} |I_1(z_n)| = \lim_{n\to\infty} |I_2(z_n)| = \lim_{n\to\infty} |I_3(z_n)| = 0.$$

Lemma 2.4. *If $t < 0$ or $t > 1$, then $1 - x^t \leq t(1-x)$.*

Proof. Let $f(x) = 1 - x^t - t(1-x)$, then

$$f'(x) = -tx^(t-1) + t, f''(x) = -t(t-1)x(t-2).$$

Obviously, $f'(1) = 0$, $f''(1) \neq 0$, $f''(x) > 0$ for $t < 0$, $f''(x) < 0$ for $t > 1$. □

Lemma 2.5. *Let φ, ψ be analytic self maps of the unit disk $\mathbb{D}$, then*
(1) *For any $z \in \mathbb{D}$, when $\alpha < 1$, we have*

$$\left|D_\alpha^{\varphi,g}(z) - D_\alpha^{\phi_s}(z)\right| \leq \left|D_\alpha^{\varphi,g}(z) - D_\alpha^{\psi,h}(z)\right| + (2-\alpha)\left|D_\alpha^{\varphi,g}(z)\right|\rho^2(\varphi(z),\psi(z)).$$

(2) *For any $z \in \mathbb{D}$, when $\alpha \geq 1$, we have*

$$\left|D_\alpha^{\varphi,g}(z) - D_\alpha^{\phi_s}(z)\right| \leq \left|D_\alpha^{\varphi,g}(z) - D_\alpha^{\psi,h}(z)\right| + \alpha\left(\left|D_\alpha^{\varphi,g}(z)\right| + \left|D_\alpha^{\psi,h}(z)\right|\right)\rho^2(\varphi(z),\psi(z)).$$

Proof. (1) The lemma is trivially for $s = 0$ or 1. In the following, we assume $0 < s < 1$. For arbitrary $z \in \mathbb{D}$, denote $\zeta = \frac{1-|\varphi(z)|^2}{1-|\phi_s(z)|^2}$ and $\xi = \frac{1-|\psi(z)|^2}{1-|\phi_s(z)|^2}$. By the definition of $D_\alpha^{\varphi,g}(z)$, $D_\alpha^{\psi,h}(z)$ and $D_\alpha^{\phi_s}(z)$, it is easy to see

$$\begin{aligned} D_\alpha^{\phi_s}(z) &= \frac{1-|z|^\alpha}{(1-|\phi(z)|^2)^\alpha}[(1-s)g(z) + sh(z)] \\ &= (1-s)\frac{(1-|\varphi(z)|^2)^\alpha}{(1-|\phi_s(z)|^2)^\alpha}D_\alpha^{\varphi,g}(z) + s\frac{(1-|\psi(z)|^2)^\alpha}{(1-|\phi_s(z)|^2)^\alpha}D_\alpha^{\psi,h}(z) \\ &= D_\alpha^{\varphi,g} - (1-s)\zeta^\alpha D_\alpha^{\varphi,g}(z) - s\xi^\alpha D_\alpha^{\psi,h}(z) \end{aligned}$$

and

$$\begin{aligned} \left|D_\alpha^{\varphi,g}(z) - D_\alpha^{\phi_s}(z)\right| &= \left|D_\alpha^{\varphi,g}(z) - (1-s)\frac{(1-|\varphi(z)|^2)^\alpha}{(1-|\phi_s(z)|^2)^\alpha}D_\alpha^{\varphi,g}(z) - s\frac{(1-|\psi(z)|^2)^\alpha}{(1-|\phi_s(z)|^2)^\alpha}D_\alpha^{\psi,h}(z)\right| \\ &= \left|D_\alpha^{\varphi,g}(z) - (1-s)\zeta^\alpha D_\alpha^{\varphi,g}(z) - s\xi^\alpha D_\alpha^{\psi,h}(z)\right| \\ &= \left|D_\alpha^{\varphi,g}(z)(1-(1-s)\zeta^\alpha) - D_\alpha^{\psi,h}(1-(1-s)\zeta^\alpha) + D_\alpha^{\psi,h}(1-(1-s)\zeta^\alpha) - s\xi^\alpha D_\alpha^{\psi,h}(z)\right| \\ &\leq \left|D_\alpha^{\varphi,g}(z) - D_\alpha^{\psi,h}\right|\left|(1-(1-s)\zeta^\alpha)\right| + \left|D_\alpha^{\psi,h}\right|\left|(1-(1-s)\zeta^\alpha) - s\xi^\alpha\right| \\ &\leq \left|D_\alpha^{\varphi,g}(z) - D_\alpha^{\psi,h}\right|\left|s\zeta^\alpha\right| + \left|D_\alpha^{\varphi,g}\right|\left|(1-(1-s)\zeta^\alpha) - s\xi^\alpha\right|. \end{aligned} \tag{2.1}$$

$$\left|D_\alpha^{\varphi,g} - D_\alpha^{\phi_s}(z)\right| \le \left|D_\alpha^{\varphi,g} - D_\alpha^{\psi,h}\right|\left|s\zeta^\alpha\right| + \left|D_\alpha^{\varphi,g}\right|\left|(1-(1-s)\zeta^\alpha) - s\xi^\alpha\right|. \tag{2.2}$$

By simply calculating and the proving process of Proposition 4.2 in [7], we get

$$0 \le \frac{s(1-s)|\varphi(z)-\psi(z)|^2}{1-|\phi_s(t)|^2} = 1-(1-s)\zeta - s\xi \le \rho^2(\varphi(z),\psi(z)). \tag{2.3}$$

Firstly, we consider the case $0<\alpha<1$.

Since $s\zeta = s\frac{1-|\varphi(z)|^2}{1-|\phi_s(z)|^2} \le 1$, then

$$s\zeta^\alpha \le s^{1-\alpha} \le 1. \tag{2.4}$$

Now, we estimate $(1-(1-s)\zeta^\alpha) - s\xi^\alpha$.

Choosing

$$f(\zeta) = 1-(1-s)\zeta^\alpha - s\xi^\alpha - (1-(1-s)\zeta - s\xi), \tag{2.5}$$

then

$$\begin{aligned} f(\zeta) &= (1-s)\zeta(1-\zeta^{\alpha-1}) + s\xi(1-\xi^{\alpha-1}) \\ &\le (\alpha-1)((1-s)\zeta(1-\zeta) + s\xi(1-\xi)) \\ &= (\alpha-1)((1-s)\zeta^2 - s\xi^2) - (\alpha-1)(1-(1-s)\zeta - s\xi). \end{aligned} \tag{2.6}$$

The last inequality above is obtained by Lemma 2.4. Uniting (2.5) and (2.6), we obtain

$$\begin{aligned} &1-(1-s)\zeta^\alpha - s\xi^\alpha - (1-(1-s)\zeta - s\xi) \\ \le&(1-(1-s\zeta)\alpha - s\xi) - (\alpha-1)(1-(1-s)\zeta^2 - s\xi^2) - (\alpha-1)(1-(1-s\zeta) - s\xi) \\ =&(2-\alpha)(1-(1-s)\zeta - s\xi) + (\alpha-1)(1-(1-s)\zeta^2 - s\xi^2). \end{aligned} \tag{2.7}$$

and

$$1-(1-s)\zeta^2 - s\xi^2 = \frac{s|\psi(z)|^2(1-|\psi(z)|^2) + (1-s)|\varphi(z)|^2(1-|\varphi(z)|^2)) + s(1-s)|\varphi(z)-\psi(z)|^2}{(1-|\phi_s(z)|^2)^2} > 0 \tag{2.8}$$

Hence,

$$1-(1-s)\zeta^\alpha - s\xi^\alpha \le (2-\alpha)(1-(1-s)\zeta - s\xi) \le (2-\alpha)\rho^2(\varphi(z),\psi(z)). \tag{2.9}$$

Combining (2.1), (2.4) and (2.9), we get

$$\left|D_\alpha^{\varphi,g}(z) - D_\alpha^{\phi_s}(z)\right| \le \left|D_\alpha^{\varphi,g}(z) - D_\alpha^{\psi,h}(z)\right| + (2-\alpha)\left|D_\alpha^{\varphi,g}(z)\right|\rho^2(\varphi(z),\psi(z)).$$

This complete the proof of (1).

Next, we are going to prove (2).

If $\alpha = 1$, then by (2.3), we have

$$1-(1-s)\zeta^\alpha - s\xi^\alpha = 1-(1-s)\zeta - s\xi \le \rho^2(\varphi(z),\psi(z)) = \alpha\rho^2(\varphi(z),\psi(z). \tag{2.10}$$

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If $\alpha > 1$, then

$$\begin{aligned} 1-(1-s)\zeta^\alpha - s\xi^\alpha &= 1-s-(1-s)\zeta^\alpha + s - s\xi^\alpha \\ &= (1-s)(1-\zeta^\alpha) + s(1-\xi^\alpha) \\ &\le \alpha(1-s)(1-\zeta) + s(1-\xi)) \\ &= \alpha(1-(1-s)\zeta - s\xi) \\ &\le \alpha\rho^2(\varphi(z),\psi(z)). \end{aligned} \tag{2.11}$$

The first inequality in (2.11) above is obtained by Lemma 2.4.

If $\xi \le 1$, using (2.1) and (2.11), we obtain

$$\left|D_\alpha^{\varphi,g}(z) - D_\alpha^{\phi_s}(z)\right| \le \left|D_\alpha^{\varphi,g}(z) - D_\alpha^{\psi,h}(z)\right| + \alpha\left|D_\alpha^{\varphi,g}(z)\right|\rho^2(\varphi(z),\psi(z)). \tag{2.12}$$

If $\xi \ge 1$, for any $s \in (0,1)$, we have $|\psi(z)| \le |\phi_s(z)| \le (1-s)|\varphi(z)| + s|\psi(z)|$ and $|\psi(z)| \le |\varphi(z)|$. Then $|\phi_s(z)| \le |\varphi(z)|$ and $\frac{1-|\varphi(z)|^2}{1-|\phi_s(z)|^2} = \zeta \le 1$. Combing (2.1), (2.10) with (2.11), it is obvious that

$$\left|D_\alpha^{\varphi,g}(z) - D_\alpha^{\phi_s}(z)\right| \le \left|D_\alpha^{\varphi,g}(z) - D_\alpha^{\psi,h}(z)\right| + \alpha\left|D_\alpha^{\psi,h}(z)\right|\big)\rho^2(\varphi(z),\psi(z)). \tag{2.13}$$

Due to (2.11), (2.12) and (2.13) above, we infer that

$$\left|D_\alpha^{\varphi,g}(z) - D_\alpha^{\phi_s}(z)\right| \le \left|D_\alpha^{\varphi,g}(z) - D_\alpha^{\psi,h}(z)\right| + \alpha\big(\left|D_\alpha^{\varphi,g}(z)\right| + \left|D_\alpha^{\psi,h}(z)\right|\big)\rho^2(\varphi(z),\psi(z)).$$

□

3. MAIN RESULTS

Proposition 3.1. *Let φ,ψ be analytic self maps of the unit disk $\mathbb{D}$, $g,h \in H(\mathbb{D})$. Suppose that C_φ^g and C_ψ^h are bounded but not compact on $\mathcal{B}^\alpha$. For any $s \in [0,1]$, when $C_\varphi^g - C_\psi^h$ is compact on $\mathcal{B}^\alpha$, then we have*

(1) *$D_{\phi_s}^\alpha \subset \Gamma(\varphi)\cap\Gamma(\psi)$, where $D_{\phi_s}^\alpha = \{\{z_n\} \subset \mathbb{D} : |\varphi(z_n)| \to 1, |D_{\phi_s}^\alpha(z_n)| \not\to 1\}$.*

(2) *For any $\{z\}_n \subset \Gamma(\varphi)\cap\Gamma(\psi)$, we have*

$$\lim_{n\to\infty}(D_\alpha^{\varphi,g}(z_n) - D_\alpha^{\phi_s}(z_n)) = \lim_{n\to\infty}(D_\alpha^{\varphi,g}(z_n)\rho(\varphi(z_n),\phi_s(z_n)) = 0.$$

Proof. (1) It is trivial.

(2) For any $\{z_n\} \subset \Gamma(\varphi)\cap\Gamma(\psi)$, it follows from Lemma 2.3 that

$$\begin{aligned} \lim_{n\to\infty}\left|D_\alpha^{\varphi,g}(z_n) - D_\alpha^{\phi_s}(z_n)\right| &= \lim_{n\to\infty}\left|(D_\alpha^{\varphi,g}(z_n)\right|\rho(\varphi(z_n),\phi_s(z_n)) \\ &= \lim_{n\to\infty}\left|D_\alpha^{\psi,h}(z_n)\right|\rho(\varphi(z_n),\phi_s(z_n)) \\ &= 0. \end{aligned}$$

Applying Lemma 2.5,

$$\lim_{n\to\infty}|D_\alpha^{\varphi,g}(z_n) - D_\alpha^{\phi_s}(z_n)| = 0,$$

then by Lemma 2.1,

$$|D_\alpha^{\varphi,g}(z_n)|\rho(\varphi(z_n),\phi_s(z_n)) \le |D_\alpha^{\varphi,g}|\rho(\varphi(z_n),\psi(z_n)) \to 0.$$

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Equivalently,

$$\lim_{n\to\infty}(D_\alpha^{\varphi,g}(z_n)-D_\alpha^{\phi_s}(z_n))=\lim_{n\to\infty}(D_\alpha^{\varphi,g}(z_n)\rho(\varphi(z_n),\phi_s(z_n))=0.$$

□

Theorem 3.2. *Let φ,ψ be analytic self maps of the unit disk $\mathbb{D}$, $g,h\in H(\mathbb{D})$. Suppose that C_φ^g and C_ψ^h are bounded but not compact on $\mathcal{B}^\alpha$. If $C_\varphi^g-C_\psi^h$ is compact on $\mathcal{B}^\alpha$, then the following two conclusions are equivalent:*

(1) *For any $\{z_n\}\subset\Gamma(\psi)\backslash\Gamma(\varphi)$, $D_\alpha^{\varphi,g}(z_n)\to 0$ as $n\to\infty$ and for any $\{z_n\}\subset\Gamma(\varphi)\backslash\Gamma(\psi)$, $D_\alpha^{\phi,h}(z_n)\to 0$ as $n\to\infty$.*

(2) *The map $s\mapsto C_{\phi_s}:[0,1]\to C_{\phi_s}(\mathcal{B}^\alpha)$ is continous.*

Proof. (1) $\Longrightarrow$ (2) We only need to prove the continuity at $s=0$.

Let

$$t(s)=\sup_{z\in\mathbb{D}}|D_\alpha^{\varphi,g}(z)-D_\alpha^{\phi_s}(z)|+\sup_{z\in\mathbb{D}}|D_\alpha^{\varphi,g}(z)|\rho(\varphi(z)),\psi(z)).$$

Then, it is easy to see that $\|C_\varphi^g-C_{\phi_s}\|_{\mathcal{B}^\alpha}\le t(s)$. By lemma 2.3 and the conditions of (1), we have

$$\begin{aligned}\lim_{n\to\infty}|D_\alpha^{\varphi,g}(z_n)-D_\alpha^{\psi,h}(z_n)|&=\lim_{n\to\infty}|D_\alpha^{\varphi,g}(z_n)|\rho(\varphi(z_n),\psi(z_n))\\&=\lim_{n\to\infty}|D_\alpha^{\psi,h}(z_n)|\rho(\varphi(z_n),\psi(z_n))\\&=0\end{aligned}$$

Hence, for any $\varepsilon>0$, there exists $r_1\in(0,1)$ such that for every $z\in\Gamma_{r_1}(\varphi)=\{z\in\mathbb{D}:|\varphi(z)|>r_1\}$,

$$|D_\alpha^{\varphi,g}(z)-D_\alpha^{\psi,h}(z)|<\frac{\varepsilon}{2},$$

and

$$|D_\alpha^{\varphi,g}(z)|\rho(\varphi(z),\psi(z)<\frac{\varepsilon}{2}.$$

Applying Lemma 2.5, we obtain that

$$|D_\alpha^{\varphi,g}(z)-D_\alpha^{\psi,h}(z)|<\frac{\varepsilon}{2}+\alpha\varepsilon=(\frac{1}{2}+\alpha)\varepsilon. \tag{3.1}$$

If $z\in\mathbb{D}\backslash\Gamma_{r_1}(\varphi)$, $D_\alpha^{\varphi,g}-D_\alpha^{\phi_s}$ is uniformly convergence to 0 when s approaches to 0, then there exists an s_1 very close to 0 such that for any $s<s_1$,

$$\sup_{z\in\mathbb{D}\backslash\Gamma_{r_1}}|D_\alpha^{\varphi,g}(z)-D_\alpha^{\phi_s}(z)|<\varepsilon. \tag{3.2}$$

For any $s<s_1$, uniting (3.1) and (3.2), we get

$$\sup_{z\in\mathbb{D}}|D_\alpha^{\varphi,g}(z)-D_\alpha^{\phi_s}(z)|<\varepsilon. \tag{3.3}$$

Hence,

$$\sup_{z\in\mathbb{D}}|D_\alpha^{\varphi,g}(z)-D_\alpha^{\phi_s}(z)|\to 0\ as\ s\to 0. \tag{3.4}$$

Next, we are going to prove that

$$\sup_{z\in\mathbb{D}} |D_\alpha^{\varphi,g}(z)|\rho(\varphi(z)), \psi(z)) \to 0 \ as \ s \to 0.$$

For any $\{z_n\} \subset \Gamma(\varphi)$, applying Proposition 3.1 and Lemma 2.3, we have

$$\lim_{n\to\infty} (D_\alpha^{\varphi,g}\rho(\varphi(z_n), \phi_s(z_n)) = 0.$$

This implies that there exists an $r_2 \in (0,1)$, such that for any $z \in \Gamma_{r_2}(\varphi) = \{z \in \mathbb{D} : |\varphi(z)| > r_2\}$,

$$|D_\alpha^{\varphi,g}(z)|\rho(\varphi(z), \psi(z)) \leq |D_\alpha^{\varphi,g}(z)|\rho(\varphi(z), \psi(z)) < \varepsilon.$$

And because $\rho(\varphi(z), \psi(z))$ uniformly converges to 0 on $\mathbb{D}\backslash\Gamma_{r_2}(\varphi)$, we can find a sufficiently small positive number s_2, such that for any $s < s_2$,

$$\sup_{z\in\mathbb{D}\backslash\Gamma_{r_2}(\varphi)} |D_\alpha^{\varphi,g}(z)|\rho(\varphi(z), \phi_s(z)) < \varepsilon.$$

Then,

$$\sup_{z\in\mathbb{D}} |D_\alpha^{\varphi,g}(z)|\rho(\varphi(z), \phi_s(z)) \to 0 \ as \ s \to 0. \tag{3.5}$$

Combing (3.4) with (3.5), we obtain that $t(s)$ converges to 0 as s approaches to 0, which finishes the proof of continuity.

(2) $\Longrightarrow$ (1) Assume there is a sequence $\{z_n\} \subset \Gamma(\psi)\backslash\Gamma(\varphi)$, such that $D_\alpha^{\varphi,g}(z_n) \to \delta \neq 0$ as $n \to \infty$. Let $\lambda \in \mathbb{D}$ and $\lambda \neq 0$, define the test function f_λ and g_λ respectively as follows:

$$f_\lambda(z) = \frac{1}{2^{\alpha+1}} \frac{1-|\lambda|^2}{\alpha\bar{\lambda}(1-\bar{\lambda}z)^\alpha},$$

$$g_\lambda(z) = \frac{1-|\lambda|^2}{(\alpha+1)2^{\alpha+1}} \left(\frac{\lambda - z}{\bar{\lambda}(1-\bar{\lambda}z)^{\alpha+1}} + \frac{1}{\alpha\bar{\lambda}^2(1-\bar{\lambda}z)^{\alpha+1}}\right).$$

Then $\|f_\lambda\|_{\mathcal{B}^\alpha} \leq 1$, $\|g_\lambda\|_{\mathcal{B}^\alpha} \leq 1$,

$$\begin{aligned} \|C_\varphi^g - C_{\phi_s}\| &\geq \|(C_\varphi^g - C_{\phi_s})g_{\varphi(z_n)}\|_{\mathcal{B}^\alpha} \\ &\geq \frac{1}{2^{\alpha+1}} \Big| D_\alpha^{\phi_s}(z_n) \frac{(1-|\varphi(z_n)|^2)(1-|\varphi_s(z_n)|^2)^\alpha}{(1-\overline{\varphi(z_n)}\phi_s(z_n))^{\alpha+1}} \rho(\varphi(z_n), \phi_s(z_n)) \Big|. \end{aligned} \tag{3.6}$$

Because $z_n \in \Gamma(\psi)\backslash\Gamma(\varphi)$, then $\phi_s(z_n) \not\to 1$ and $\lim_{n\to\infty} \rho(\varphi(z_n), \phi_s(z_n)) \neq 0$. And $s \mapsto C_{\phi_s}$ is continous at 0, then by (3.6), we have

$$\Big| D_\alpha^{\phi_s}(z_n) \frac{(1-|\varphi(z_n)|^2)(1-|\varphi_s(z_n)|^2)^\alpha}{(1-\overline{\varphi(z_n)}\phi_s(z_n))^{\alpha+1}} \rho(\varphi(z_n), \phi_s(z_n) \Big| \to 0, n \to \infty, s \to 0.$$

By the compactness of $C_\varphi^g - C_\psi^h$, it is bounded. It follows from Lemma 2.1, Lemma 2.2 and lemma 2.5 that $C_\varphi^g - C_{\phi_s}$ is bounded. So

$$\begin{aligned} \|C_\varphi^g - C_{\phi_s}\| &\geq \|(C_\varphi^g - C_{\phi_s})g_{\varphi(z_n)}\|_{\mathcal{B}^\alpha} \\ &\geq \frac{1}{2^{\alpha+1}} \Big(\Big| D_\alpha^{\varphi,g}(z_n) \Big| - \Big| D_\alpha^{\phi_s}(z_n) \Big| \Big| \frac{(1-|\varphi(z_n)|^2)(1-|\varphi_s(z_n)|^2)^\alpha}{(1-\overline{\varphi(z_n)}\phi_s(z_n))^{\alpha+1}} \Big| \Big). \end{aligned} \tag{3.7}$$

Letting $n \to \infty$ and $s \to 0$, we have

$$\|C_\varphi^g - C_{\phi_s}\| \geq \frac{\delta}{2^{\alpha+1}} > 0. \tag{3.8}$$

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For $\varphi(z_n) \equiv 0$, suppose $\lambda \in \mathbb{D}, \lambda \neq 0$ and

$$h_\lambda(z) = \frac{1}{2^{\alpha+1}} \frac{1}{\alpha\bar{\lambda}(1-\bar{\lambda}z)^\alpha}. \tag{3.9}$$

Then $h_\lambda \in \mathcal{B}^\alpha$ and $\|h_\lambda\|_{\mathcal{B}^\alpha} \leq 1$. If $s \neq 0$, then $\phi_s(z_n) \to s \neq 0$. Choosing $\lambda = \phi_s(z_n)$, we have

$$\begin{aligned} \|(C_\varphi^g - C_{\phi_s})h_{\phi_s(z_n)}\|_{\mathcal{B}^\alpha} &\geq \|(C_\varphi^g - C_{\phi_s})h_{\phi_s(z_n)}\|_{\mathcal{B}^\alpha} \\ &\geq \frac{1}{2^{\alpha+1}}\Big(|(1-|z_n|^2)^\alpha \varphi'(z_n)|D_\alpha^{\phi_s}(z_n) - \frac{D_\alpha^{\phi_s}(z_n)}{1-|\phi_s(z_n)|^\alpha}\Big). \end{aligned}$$

For $\Gamma(\psi)\backslash\Gamma(\varphi)$, Proposition 3.1 implies that $D_\alpha^{\phi_s}(z_n) \to 0$. Letting $n \to \infty$ and $s \to 0$, we get

$$\|C_\varphi^g - C_{\phi_s}\| \geq \delta > 0. \tag{3.10}$$

It follows from (3.8) and (3.10) that the map $s \mapsto C_{\phi_s}$ is not continuous at 0, which is a contradiction. So we complete the proof. □

Corollary 3.3. *Let φ, ψ be two analytic self maps of the unit disk $\mathbb{D}$, $g, h \in H(\mathbb{D})$. Suppose C_φ^g and C_ψ^h are bounded but not compact on $\mathcal{B}^\alpha$. If $C_\varphi^g - C_\psi^h$ is compact on $\mathcal{B}^\alpha$, then C_φ^g and C_ψ^h are in the same path component of $\mathcal{B}^\alpha$.*

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THE GENERALIZED HYERS-ULAM STABILITY OF QUADRATIC FUNCTIONAL EQUATIONS ON RESTRICTED DOMAINS

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Abstract. In this paper, we prove the generalized Hyers-Ulam stability for the functional equation

$$f(ax+by)+abf(x-y)=a(a+b)f(x)+b(a+b)f(y)$$

for some real numbers a, b with $2a+b=1$ on a restricted domain using the fixed point theorem.

Key words. Generalized Hyers-Ulam stability, Quadratic functional equation, Banach space, Restricted domains, Fixed point theorem

1. Introduction

In 1940, S. M. Ulam [15] proposed the following stability problem :

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists a unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In 1941, Hyers [7] answered this problem under the assumption that the groups are Banach spaces. Aoki [1] and Rassias [11] generalized the result of Hyers. Rassias [11] solved the generalized Hyers-Ulam stability of the functional inequality

$$\|f(x+y)-f(x)-f(y)\| \leq \epsilon(\|x\|^p+\|y\|^p)$$

for some $\epsilon \geq 0$ and p with $0 < p < 1$ and all $x, y \in X$, where $f : X \longrightarrow Y$ is a function between Banach spaces. A generalization of the Rassias theorem was obtained by Găvruta [6] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassis approach.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is called *a quadratic functional equation* and a solution of a quadratic functional equation is called *quadratic*. The generalized Hyers-Ulam stability problem for a quadratic functional equation was proved by Skof [13] for mappings $f : X \longrightarrow Y$, where X is a normed space and Y is a Banach space. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [3] proved the generalized Hyers-Ulam stability for a quadratic functional equation.

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Skof [14] was the first author to slove the Hyers-Ulam problem for additive mappings on a restricted domain and in 1998, Jung [8] investigated the Hyers-Ulam stability for additive and quadratic mappimgs on resticted domains. In 2002, Rassias [12] proved that if $f : X \longrightarrow Y$ satisfies the following inequality

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \le \delta, \tag{1.2}$$

then there exists a unique quadratic mapping which is approximately. Recently, Najati and Jung [9] showed that the functional equation

$$f(ax+by) + abf(x-y) = af(x) + bf(y) \tag{1.3}$$

is equivalent to (1.1) if a, b are non-zero real numbers with $a+b=1$ and proved that the Hyers-Ulam stability for the functional equation (1.3) on a resticted domain if f is even. Elhoucien and Youssef [5] showed the results in [9] by removing the Najati-Jung's assumption that f is even.

In this paper, we consider the functional equation

$$f(ax+by) + abf(x-y) = a(a+b)f(x) + b(a+b)f(y) \tag{1.4}$$

for fixed non-zero real numbers a, b with $2a+b=1$, $a \neq 1$ and we prove the generalized Hyers-Ulam stability of it on a restricted domain. Throughout this paper, we assume that X is a normed space and Y is a Banach space.

2. Solutions of (1.4)

Najati and Jung [9] showed that if an even mapping $f : X \longrightarrow Y$ satisfies (1.3), then f is quadratic and that if a, b are rational numbers, then f satisfies (1.3) if and only if f is quadratic. Elhoucien and Youssef [5] showed that if a mapping $f : X \longrightarrow Y$ satisfies (1.3), then f is additive-quadratic. In this section, we will show that if a mapping $f : X \longrightarrow Y$ satisfies (1.4), then f is quadratic.

Theorem 2.1. *Let $f : X \longrightarrow Y$ be a mapping satisfying (1.4). Then f is a quadratic mapping.*

Proof. Letting $x = y = 0$ in (1.4), since $2a+b=1$, we have $(a^2+ab+b^2-1)f(0) = 3a(a-1)f(0) = 0$. Since $a \neq 0, 1$, $f(0) = 0$. Letting $y = 0$ in (1.4), we have

$$f(ax) = a^2 f(x) \tag{2.1}$$

for all $x \in X$. Letting $x = 0$ in (1.4), we have

$$f(by) = b(a+b)f(y) - abf(-y) \tag{2.2}$$

for all $y \in X$. Let $f_o(x) = \frac{f(x)-f(-x)}{2}$. Then f_o satisfies (1.4), (2.1) and (2.2) and hence by (2.2), we have

$$f_o(bx) = bf_o(x) \tag{2.3}$$

for all $x \in X$. By (1.4), we have

$$f_o(ax+by) + f_o(ax-by) = 2a(a+b)f_o(x) - ab[f_o(x+y) + f_o(x-y)] \tag{2.4}$$

for all $x, y \in X$. Letting $y = ay$ in (2.4), by (2.1), we have

$$a[f_o(x+by) + f_o(x-by)] = 2(a+b)f_o(x) - b[f_o(x+ay) + f_o(x-ay)] \tag{2.5}$$

for all $x, y \in X$ and letting $x = bx$ in (2.5), by (2.3), we have

$$f_o(bx+ay) + f_o(bx-ay) = 2(a+b)f_o(x) - a[f_o(x+y) + f_o(x-y)] \tag{2.6}$$

for all $x, y \in X$. Interchanging x and y in (1.4), we have

$$f_o(bx+ay) + f_o(bx-ay) = 2b(a+b)f_o(x) + ab[f_o(x+y) + f_o(x-y)] \tag{2.7}$$

for all $x, y \in X$. By (2.6) and (2.7), since $a(a+b) \neq 0$, we have

$$f_o(x+y) + f_o(x-y) - 2f_o(x) = 0$$

for all $x, y \in X$ and hence f_o is additive. By (2.1), we have $a^2 f_o(x) = af_o(x)$ and since $a \neq 0, 1$, $f_o(x) = 0$ for all $x \in X$.

Let $f_e(x) = \frac{f(x)+f(-x)}{2}$. Then $f_e : X \longrightarrow Y$ is an even mapping satisfying (1.4) and so f_e satisfies (2.1) and (2.2). Replacing x and y by $2x$ and $x+y$ in (1.4), we have

$$f_e(x+by) + abf_e(x-y) - a(a+b)f_e(2x) - b(a+b)f_e(x+y) = 0 \tag{2.8}$$

for all $x, y \in X$. Since $a(a+b) \neq 0$ and f_e is even, by (2.8), we have

$$f_e(2x) = 4f_e(x), \;\; f_e(bx) = b^2 f(x) \tag{2.9}$$

for all $x \in X$. Letting $x = bx$ in (2.8), by (2.9), we have

$$bf_e(x+y) + af_e(bx-y) - 4ab(a+b)f_e(x) - (a+b)f_e(bx+y) = 0 \tag{2.10}$$

for all $x, y \in X$. Interchanging x and y in (2.10), we have

$$bf_e(x+y) + af_e(x-by) - 4ab(a+b)f_e(y) - (a+b)f_e(x+by) = 0 \tag{2.11}$$

for all $x, y \in X$. Letting $y = -y$ in (2.8), we have

$$f_e(x-by) + abf_e(x+y) - 4a(a+b)f_e(x) - b(a+b)f_e(x-y) = 0 \tag{2.12}$$

for all $x, y \in X$. Since $b(1 - 2a^2 - 2ab - b^2) = 2ab(a+b)$, by (2.8), (2.11), and (2.12), we have

$$f_e(x+y) + f_e(x-y) = 2f_e(x) + 2f_e(y)$$

for all $x, y \in X$ and so f_e is quadraric. Since $f = f_o + f_e = f_e$, f is quadratic. $\square$

Corollary 2.2. *Let $f : X \longrightarrow Y$ be a mapping. If a, b are rational numbers, then f is quadratic if and only if f satisfies (1.4).*

3. Stability of (1.4)

In this section, we investigate the generalized Hyers-Ulam stability of (1.4) on a restricted domain. Jung [8] proved the Hyers-Ulam stability for additive and quadratic mappings on a resticted domain and Najati and Jung [9] proved the Hyers-Ulam syability of (1.3) on a resticted domain if f is an even mapping. Rahimi, Najati and Bae [10] investigated the generalized Hyers-Ulam syability of (1.1) with the bounded function $\delta + \epsilon(\|x\|^{2p} + \|y\|^{2p}) + \theta\|x\|^p\|y\|^p$ on a resticted domain.

Theorem 3.1. *Let $\phi : X^2 \longrightarrow [0, \infty)$ be a mapping and M a non-negative real number. Let $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$. Suppose that f satisfies the following inequality*

$$\|f(ax+by) + abf(x-y) - a(a+b)f(x) - b(a+b)f(y)\| \leq \delta + \phi(x,y) \tag{3.1}$$

for all $x, y \in X$ with $\|x\| + \|y\| \geq M$ and for some non-negative real number δ. Then we have

$$\|f(2x) - 4f(x)\| \leq \Phi(x,y) \tag{3.2}$$

for all $x, y \in X$ *with* $\|y\| \geq M$, *where*

$$\begin{aligned}
&\Phi(x,y) = \{\phi(2x-2by, x+(1-b)y) + \phi(2x-2by, x-(1+b)y) \\
&+ \phi(2x+2by, x+(1+b)y) + \phi(2x+2by, x-(1-b)y) \\
&+ |b|[\phi(2x+2y, x+2y) + \phi(2x+2y, x) + \phi(2x-2y, x) + \phi(2x-2y, x-2y)] \\
&+ \phi(2x, x+2y) + \phi(2x, x-2y) + 4(|b|+2)\delta\} \times |2a(a+b)|^{-1}.
\end{aligned}$$

Proof. Let $x, y \in X$ with $\|x\| + \|y\| \geq M$. Then $\|2x\| + \|x+y\| \geq \|x\| + \|y\| \geq M$. Hence by (3.1), we have

$$\begin{aligned}
&\|f(x+by) + abf(x-y) - a(a+b)f(2x) - b(a+b)f(x+y)\| \\
&\leq \delta + \phi(2x, x+y)
\end{aligned} \tag{3.3}$$

and letting $y = -y$ in (3.3), we have

$$\begin{aligned}
&\|f(x-by) + abf(x+y) - a(a+b)f(2x) - b(a+b)f(x-y)\| \\
&\leq \delta + \phi(2x, x-y).
\end{aligned} \tag{3.4}$$

By (3.3) and (3.4), we have

$$\begin{aligned}
&\|f(x+by) - f(x-by) + bf(x-y) - bf(x+y)\| \\
&\leq 2\delta + \phi(2x, x+y) + \phi(2x, x-y).
\end{aligned} \tag{3.5}$$

Let $x, y \in X$ with $\|y\| \geq M$. Since $\|x-by\| + \|y\| \geq M$ and $\|x+by\| + \|y\| \geq M$, by (3.5), we have

$$\begin{aligned}
&\|f(x) - f(x-2by) + bf(x-(1+b)y) - bf(x+(1-b)y)\| \\
&\leq 2\delta + \phi(2x-2by, x+(1-b)y) + \phi(2x-2by, x-(1+b)y)
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
&\|f(x+2by) - f(x) + bf(x-(1-b)y) - bf(x+(1+b)y)\| \\
&\leq 2\delta + \phi(2x+2by, x+(1+b)y) + \phi(2x+2by, x-(1-b)y).
\end{aligned} \tag{3.7}$$

Since $\|x+y\| + \|y\| \geq M$ and $\|x-y\| + \|-y\| \geq M$, by (3.5), we have

$$\begin{aligned}
&\|f(x+(1+b)y) - f(x+(1-b)y) + bf(x) - bf(x+2y)\| \\
&\leq 2\delta + \phi(2x+2y, x+2y) + \phi(2x+2y, x)
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
&\|f(x-(1+b)y) - f(x-(1-b)y) + bf(x) - bf(x-2y)\| \\
&\leq 2\delta + \phi(2x-2y, x) + \phi(2x-2y, x-2y).
\end{aligned} \tag{3.9}$$

Since $\|x\| + \|2y\| \geq M$, by (3.3) and (3.4), we have

$$\begin{aligned}
&\|f(x+2by) + abf(x-2y) - a(a+b)f(2x) - b(a+b)f(x+2y)\| \\
&\leq \delta + \phi(2x, x+2y).
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
&\|f(x-2by) + abf(x+2y) - a(a+b)f(2x) - b(a+b)f(x-2y)\| \\
&\leq \delta + \phi(2x, x-2y).
\end{aligned} \tag{3.11}$$

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Note that

$$\begin{aligned}
&2a(a+b)[f(2x)-4f(x)] \\
&= -[f(x)-f(x-2by)+bf(x-(1+b)y)-bf(x+(1-b)y)] \\
&+ [f(x+2by)-f(x)+bf(x-(1-b)y)-bf(x+(1+b)y)] \\
&+ b[f(x+(1+b)y)-f(x+(1-b)y)+bf(x)-bf(x+2y)] \\
&+ b[f(x-(1+b)y)-f(x-(1-b)y)+bf(x)-bf(x-2y)] \\
&- [f(x+2by)+abf(x-2y)-a(a+b)f(2x)-b(a+b)f(x+2y)] \\
&- [f(x-2by)+abf(x+2y)-a(a+b)f(2x)-b(a+b)f(x-2y)]
\end{aligned} \tag{3.12}$$

for all $x, y \in X$ with $\|y\| \geq M$. By (3.6), (3.7), (3.8), (3.9), (3.10), (3.11), we have (3.2). □

We apply the fixed point method to investigate the generalized Hyers-Ulam stability for the functional equation (1.4).

Definition 3.2. Let X be a set. A function $d : X \times X \longrightarrow [0, \infty]$ is called *a generalized metric on* X if d satisfies
(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$, and
(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Now, we consider the following fixed point theorem :

Theorem 3.3. [4] *Let (X, d) be a complete generalized metric space and $J : X \longrightarrow X$ a strictly contractive mapping with a Lipschitz constant L with $0 < L < 1$. Then for each element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty \tag{3.13}$$

for all nonnegative integers n or there is a nonnegative integer k such that
(1) $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq k$,
(2) a sequence $\{J^n x\}$ converges to a fixed point y^* of J,
(3) y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^k x, y) < \infty\}$, and
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

Now, we will prove the stability of (1.4) on a restrcted domain.

Theorem 3.4. *Let $\phi : X^2 \longrightarrow [0, \infty)$ be a function such that*

$$\phi(2x, 2y) \leq L\phi(x, y) \tag{3.14}$$

for all $x, y \in X$ for some positive real number L with $L < 1$. Let $f : X \longrightarrow Y$ be a mapping with (3.1). Then there exists a unique quadratic mapping $Q : X \longrightarrow Y$ such that f satisfies (1.4) and

$$\|Q(x) - f(x)\| \leq \frac{1}{4(1-L)} \Phi(x, y) \tag{3.15}$$

for all $x \in X$ and $y \in X$ with $\|y\| \geq M$.

Proof. By Theorem 3.1, the following inequality

$$\|f(x) - 2^{-2} f(2x)\| \leq 2^{-2} \Phi(x, y) \tag{3.16}$$

holds for all $x, y \in X$ with $\|y\| \geq M$.

Let $\Omega = \{g : X \longrightarrow Y \mid g(0) = 0\}$. Define a generalized metric d on Ω by $d(g, h) = inf\{C \in [0, \infty) \mid \|g(x) - h(x)\| \le C\Phi(x, y), \forall x, y \in X \ \ with \ \ \|y\| \ge M\}$. We claim that (Ω, d) is a complete metric space. Let $\{g_n\}$ be a Cauchy sequence in (Ω, d) and $\epsilon > 0$. Then there is a positive integer k such that $d(g_n, g_m) \le \epsilon$ for all $n, m \ge k$. Pick $y_0 \in X$ with $\|y_0\| \ge M$ and let $x \in X$. Since $\|g_n(x) - g_m(x)\| \le \epsilon\Phi(x, y_0)$ for all $n, m \ge k$, $\{g_n(x)\}$ is a Cauchy sequence in Y and hence we can define a mapping $g : X \longrightarrow Y$ by $g(x) = \lim_{n\longrightarrow\infty} g_n(x)$. Clearly, $g \in \Omega$ and $\lim_{n\longrightarrow\infty} g_n = g$. Thus (Ω, d) is a complete metric space.

Define a map $J : \Omega \longrightarrow \Omega$ by $Jh(x) = \frac{1}{4}h(2x)$ for all $x \in X$. Let $g, h \in \Omega$. Suppose that C is a positive real number such that

$$\|g(x) - h(x)\| \le C\Phi(x, y)$$

for all $x, y \in X$ with $\|y\| \ge M$. By (3.14), we have

$$\|Jg(x) - Jh(x)\| = \frac{1}{4}\|g(2x) - h(2x)\| \le \frac{1}{4}C\Phi(2x, 2y) \le \frac{1}{4}CL\Phi(x, y)$$

for all $x, y \in X$ with $\|y\| \ge M$ and hence we have

$$d(Jg, Jh) \le \frac{L}{4}d(g, h)$$

for all $g, h \in \Omega$. Since $0 < L < 4$, J is a strictly contractive mapping and by (3.16), we have

$$d(Jf, f) \le \frac{1}{4}.$$

By Theorem 3.3, $\{J^n f\}$ converges to the unique fixed element Q of J in $Y = \{h \in \Omega \mid d(f, h) < \infty\}$ and (3.15) holds. Further, we have

$$Q(x) = \lim_{n\longrightarrow\infty} J^n f(x) = \lim_{n\longrightarrow\infty} 2^{-2n} f(2^n x)$$

for all $x \in X$ and we have (3.15). Moreover, $Q(0) = 0$, because $f(0) = 0$.

Now, we claim that Q satisafies (1.4). First, suppose that $x \ne 0$ or $y \ne 0$. Replacing x and y by $2^n x$ and $2^n y$ in (3.1), respectively and deviding both sides of (3.1) by 2^{2n}, we have

$$\begin{aligned} &\|2^{-2n} f(2^n(ax + by)) + 2^{-2n} abf(2^n(x - y)) \\ &- a(a + b)2^{-2n} f(2^n x) - b(a + b)2^{-2n} f(2^n y)\| \le \frac{1}{4^n}[L^n\phi(x, y) + \delta] \end{aligned} \tag{3.17}$$

for all $x, y \in X$ and sufficiently large positive integer n. Letting $n \longrightarrow \infty$ in (3.17), Q satisfies (1.4). Clealy, if $x = 0$ and $y = 0$, then Q satisfies (1.4). By Theorem 2.1, Q is quadratic.

Assume that $Q_1 : X \longrightarrow Y$ is another quadratic mapping satisfying (1.4) and (3.15). Then we have

$$\|Q_1(x) - f(x)\| \le \frac{1}{4(1 - L)}\Phi(x, y)$$

for all $x \in X$ and $y \in X$ with $\|y\| \ge M$ and so

$$d(Q_1, f) \le \frac{1}{4(1 - L)} < \infty.$$

By (3) of Theorem 3.3, $Q = Q_1$. □

Skof [13](Jung [8], resp.) proved an asymptotic property of aditive (quadratic, resp.) mappings. We consider such property for (1.4).

Corollary 3.5. *A mapping $f : X \longrightarrow Y$ satisfies (1.4) if and only if the asymptotic condition*

$$\|f(ax+by)+abf(x-y)-a(a+b)f(x)-b(a+b)f(y)\| \longrightarrow 0 \ \ as \ \ \|x\|+\|y\| \longrightarrow \infty$$

holds.

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Hesitant fuzzy soft set and its lattice structures

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Abstract: Hesitant fuzzy set and soft set were introduced by Torra and Molodtsov, respectively. The two sets have been used successfully as effective mathematical tools for dealing with vagueness and uncertainties. By combining hesitant fuzzy set and soft set, in this paper, we propose a new model named hesitant fuzzy soft set, which can be regarded as an extension of many models, such as hesitant fuzzy set, soft set, fuzzy soft set, interval-valued fuzzy soft set and multi-fuzzy soft set. Some basic operations of hesitant fuzzy soft set are defined and some desirable properties of those operations are investigated. Furthermore, the lattice structures of hesitant fuzzy soft set are discussed.

Keywords: Hesitant fuzzy set; soft set; fuzzy soft set; hesitant fuzzy soft set; lattice

1 Introduction

Soft set was firstly proposed by Molodtsov [1], it is a new mathematical tool for modeling vagueness and uncertainty. Since its appearance, soft set theory has attracted more and more attention from many researchers and many important results on soft set have been achieved in theory and application. Maji and Biswas et al. [2] defined some basic operations. Ali et al. [3, 4] gave some new operations on soft sets and studied some algebraic structures of soft sets. Yang and Guo [5] introduced some kernels and closures of soft set relations. Many authors applied soft sets to some algebraic structures such as groups, rings, fields and modules [6–8]. The applications of soft set in decision making and other areas could be found in [9–12].

At the same time, in order to extend the application ranges of soft set, fuzzy extension of soft set theory has become a hot research topic. Maji et al. [13] introduced the notions of fuzzy soft set. Jiang et al. [14] and Majumdar and Samanta [15] further generalized fuzzy soft set to intuitionistic fuzzy soft set and generalised fuzzy soft set, respectively. Yang et al. [16] proposed the concept of interval-valued fuzzy soft set by combining the interval-valued fuzzy set and soft set. Some other generalized models of soft set could be seen in [17–19]

Recently, Torra [20] introduced hesitant fuzzy set which is a new extension of fuzzy set. It permits the membership degree of an element to a set to be represented as some possible values between 0 and 1. Presently, work on hesitant fuzzy set is making progress rapidly and lots of results on hesitant fuzzy set have been obtained [21–25]. The main goal of this paper is to combine the hesitant fuzzy set and soft set and obtain a new hybrid model named hesitant fuzzy soft set. It can be viewed as a hesitant fuzzy extension of the soft set or a generalization of the hesitant fuzzy set.

The rest of this paper is structured as follows. The following section briefly reviews some basic notions of soft set, fuzzy soft set and hesitant fuzzy set. Two new operations on hesitant fuzzy element are defined, and some of their properties are investigated. In Section 3, the concept of

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hesitant fuzzy soft set is first proposed by combining hesitant fuzzy set and soft set. Some operations on hesitant fuzzy soft set are given and some of their properties are studied. In Section 4, we discuss the lattice structures of hesitant fuzzy soft set. The conclusion is finally reached in Section 5.

2 Preliminary

Let U be an initial universe of objects and E the set of parameters in relation to objects in U. Parameters are often attributes, characteristics, or properties of objects. Let $P(U)$ denote the power set of U and $A \subseteq E$. Molodtsov [1] first gave the definition of soft set as follows.

Definition 2.1. [1] A pair (F, A) is called a soft set over U, where $A \subseteq E$ and F is a set valued mapping given by $F : A \to P(U)$.

Maji [13] introduced fuzzy soft set which is an fuzzy extension of soft set.

Definition 2.2. [13] Let $\mathcal{P}(U)$ be the set of all fuzzy subsets of U. A pair $(\mathcal{F}, A)$ is called a fuzzy soft set over U, where $\mathcal{F}$ is a set valued mapping given by $\mathcal{F} : A \to \mathcal{P}(U)$.

As a generalization form of fuzzy set, hesitant fuzzy set (HFS) was first introduced by Torra [20] as follows.

Definition 2.3. [20] Let X be a reference set, an HFS on X is in terms of a function that when applied to X returns a subset of $[0, 1]$, which can be represented as $H = \left\{\frac{h_H(x)}{x}|x \in X\right\}$, where $h_H(x)$ is a set of some values in $[0, 1]$, denoting the possible membership degrees of the element $x \in X$ to the set H.

For convenience, Xu and Xia [21,22] called $h_H(x)$ an hesitant fuzzy element (HFE) with respect to x of H. It is worth noting that the number of values of different $HFEs$ may be different, in this paper, let $l(h_H(x))$ denote the number of values of $h_H(x)$. We arrange the values of $h_H(x)$ in increasing order, and let $h_H^{\sigma(j)}(x)$ be the jth largest value of $h_H(x)$.

Definition 2.4. [20] Let $H = \left\{\frac{h_H(x)}{x}|x \in X\right\}$ be an HFS. Then
(1) H is said to be an empty hesitant set, denoted by Φ, if $h_H(x) = 0$ for all $x \in X$;
(2) H is said to be a full hesitant set, denoted by $\mathcal{I}$, if $h_H(x) = 1$ for all $x \in X$;
(3) H is said to be a complete ignorance set, denoted by $\mathcal{W}$, if $h_H(x) = [0, 1]$ for all $x \in X$.

Definition 2.5. [20] Let $\lambda > 0$, h, h_1 and h_2 be three $HFEs$, some operations on them are given as follows:
(1) $h_1 \cup h_2 = \cup_{\gamma_1\in h_1,\gamma_2\in h_2}\{max(\gamma_1, \gamma_2)\}$;
(2) $h_1 \cap h_2 = \cup_{\gamma_1\in h_1,\gamma_2\in h_2}\{min(\gamma_1, \gamma_2)\}$;
(3) $h^c = \cup_{\gamma\in h}\{1 - \gamma\}$.

We further define the strict union and the strict intersection for $HFEs$ h_1 and h_2, which will be useful in the sequel.

Definition 2.6. *Let h_1 and h_2 be two $HFEs$, $h_i^- = min\{\gamma_i|\gamma_i \in h_i\}$ and $h_i^+ = max\{\gamma_i|\gamma_i \in h_i\} (i = 1, 2)$. The strict union and the strict intersection of h_1 and h_2 are defined as follows:*
(1) $h_1 \sqcup h_2 = \cup_{\gamma_i\in h_i,i=1,2}\{\gamma_i|\gamma_i > min(h_1^+, h_2^+)$ *or* $\gamma_1 = \gamma_2\}$;
(2) $h_1 \sqcap h_2 = \cup_{\gamma_i\in h_i,i=1,2}\{\gamma_i|\gamma_i < max(h_1^-, h_2^-)$ *or* $\gamma_1 = \gamma_2\}$;

For example, let $h_1 = \{0.2, 0.3, 0.6, 0.8\}$ and $h_2 = \{0.4, 0.5, 0.8, 0.9\}$, then $h_1 \sqcup h_2 = \{0.8, 0.9\} \neq \{0.4, 0.5, 0.6, 0.8, 0.9\} = h_1 \cup h_2$, $h_1 \sqcap h_2 = \{0.2, 0.3\} \neq \{0.2, 0.3, 0.4, 0.5, 0.6, 0.8\} = h_1 \cap h_2$.

In fact, all the above operations on $HFEs$ can be suitable for $HFSs$. Some relationships can be further established for these operations on $HFEs$.

Theorem 2.7. *For three $HFEs$ h, h_1 and h_2, the following is valid:*
(1) $h_1^c \sqcup h_2^c = (h_1 \sqcap h_2)^c$;
(2) $h_1^c \sqcap h_2^c = (h_1 \sqcup h_2)^c$.

Proof. (1) Since $h_1 \sqcap h_2 = \cup_{\gamma_i \in h_i, i=1,2}\{\gamma_i | \gamma_i < max(h_1^-, h_2^-) \ or \ \gamma_1 = \gamma_2\}$, then

$$(h_1 \sqcap h_2)^c = \cup_{\gamma_i \in h_i, i=1,2}\{1-\gamma_i | \gamma_i < max(h_1^-, h_2^-) \ or \ \gamma_1 = \gamma_2\}$$

Since $h_1^c = \cup_{\gamma_1 \in h_1}\{1-\gamma_1\}$ and $h_2^c = \cup_{\gamma_2 \in h_2}\{1-\gamma_2\}$, then

$$\begin{aligned} h_1^c \sqcup h_2^c &= \{\cup_{\gamma_1 \in h_1}\{1-\gamma_1\}\} \sqcup \{\cup_{\gamma_2 \in h_2}\{1-\gamma_2\}\} \\ &= \cup_{\gamma_i \in h_i, i=1,2}\{1-\gamma_i | 1-\gamma_i < min(1-h_1^-, 1-h_2^-) \ or \ 1-\gamma_1 = 1-\gamma_2\} \\ &= \cup_{\gamma_i \in h_i, i=1,2}\{1-\gamma_i | \gamma_i < 1-min(1-h_1^-, 1-h_2^-) \ or \ \gamma_1 = \gamma_2\} \\ &= \cup_{\gamma_i \in h_i, i=1,2}\{1-\gamma_i | \gamma_i < max(h_1^-, h_2^-) \ or \ \gamma_1 = \gamma_2\} \end{aligned}$$

□

Theorem 2.8. *For three $HFEs$ h_1, h_2 and h_3, the following is valid:*
(1) $(h_1 \cup h_2) \cup h_3 = h_1 \cup (h_2 \cup h_3)$;
(2) $(h_1 \cap h_2) \cap h_3 = h_1 \cap (h_2 \cap h_3)$;
(3) $h_1 \cup (h_2 \cap h_3) = (h_1 \cup h_2) \cap (h_1 \cup h_3)$;
(4) $h_1 \cap (h_2 \cup h_3) = (h_1 \cap h_2) \cup (h_1 \cap h_3)$.

Proof. (2) and (4) are similar to (1) and (3), respectively, so we only prove (1) and (3).
(1) Since $(h_1 \cup h_2) = \cup_{\gamma_i \in h_i, i=1,2}\{max(\gamma_1, \gamma_2)\}$, then

$$\begin{aligned} (h_1 \cup h_2) \cup h_3 &= \{\cup_{\gamma_i \in h_i, i=1,2}\{max(\gamma_1, \gamma_2)\}\} \cup h_3 \\ &= \cup_{\gamma_i \in h_i, i=1,2,3}\{max(max(\gamma_1, \gamma_2), \gamma_3)\} \\ &= \cup_{\gamma_i \in h_i, i=1,2,3}\{max(\gamma_1, \gamma_2, \gamma_3)\} \\ &= \cup_{\gamma_i \in h_i, i=1,2,3}\{max(\gamma_1, max(\gamma_2, \gamma_3))\} \\ &= h_1 \cup (h_2 \cup h_3). \end{aligned}$$

(3) Since $(h_2 \cap h_3) = \cup_{\gamma_i \in h_i, i=2,3}\{min(\gamma_2, \gamma_3)\}$, then

$$\begin{aligned} h_1 \cup (h_2 \cap h_3) &= h_1 \cup \{\cup_{\gamma_i \in h_i, i=2,3}\{min(\gamma_2, \gamma_3)\}\} \\ &= \cup_{\gamma_i \in h_i, i=1,2,3}\{max(\gamma_1, min(\gamma_2, \gamma_3))\} \\ &= \cup_{\gamma_i \in h_i, i=1,2,3}\{min(max(\gamma_1, \gamma_2), max(\gamma_2, \gamma_3))\} \\ &= \{\cup_{\gamma_i \in h_i, i=1,2}\{max(\gamma_1, \gamma_2)\}\} \cap \{\cup_{\gamma_i \in h_i, i=1,3}\{max(\gamma_1, \gamma_3)\}\} \\ &= (h_1 \cup h_2) \cap (h_1 \cup h_3) \end{aligned}$$

□

Theorem 2.9. *For three $HFEs$ h_1, h_2 and h_3, the following is valid:*
(1) $(h_1 \sqcap h_2) \sqcap h_3 = h_1 \sqcap (h_2 \sqcap h_3)$;
(2) $(h_1 \sqcup h_2) \sqcup h_3 = h_1 \sqcup (h_2 \sqcup h_3)$;
(3) $(h_1 \sqcup h_2) \sqcap h_1 = h_1$;
(4) $(h_1 \sqcap h_2) \sqcup h_1 = h_1$.

Proof. (2) and (4) are similar to (1) and (3), respectively, so we only prove (1) and (3).
(1) Since $h_1 \sqcap h_2 = \cup_{\gamma_i \in h_i, i=1,2}\{\gamma_i | \gamma_i < max(h_1^-, h_2^-) \ or \ \gamma_1 = \gamma_2\}$, then

$$\begin{aligned} (h_1 \sqcap h_2) \sqcap h_3 &= \{\cup_{\gamma_i \in h_i, i=1,2}\{\gamma_i | \gamma_i < max(h_1^-, h_2^-) \ or \ \gamma_1 = \gamma_2\}\} \sqcap h_3 \\ &= \cup_{\gamma_i \in h_i, i=1,2,3}\{\gamma_i | \gamma_i < max(max(h_1^-, h_2^-), h_3^-) \ or \ \gamma_1 = \gamma_2 = \gamma_3\} \\ &= \cup_{\gamma_i \in h_i, i=1,2,3}\{max(h_1^-, h_2^-, h_3^-) \ or \ \gamma_1 = \gamma_2 = \gamma_3\} \\ &= \cup_{\gamma_i \in h_i, i=1,2,3}\{\gamma_i | \gamma_i < max(h_1^-, max(h_2^-, h_3^-)) \ or \ \gamma_1 = \gamma_2 = \gamma_3\} \\ &= h_1 \sqcap \{\cup_{\gamma_i \in h_i, i=2,3}\{\gamma_i | \gamma_i < max(h_2^-, h_3^-) \ or \ \gamma_2 = \gamma_3\}\} \\ &= h_1 \sqcap (h_2 \sqcap h_3). \end{aligned}$$

(3) Since $h_1 \sqcup h_2 = \cup_{\gamma_i \in h_i, i=1,2}\{\gamma_i | \gamma_i > min(h_1^+, h_2^+) \ or \ \gamma_1 = \gamma_2\}$,
i) If $h_1^+ \leq h_2^+$, then $min(h_1^+, h_2^+) = h_1^+$. It follows that $h_1 \sqcup h_2 = \cup_{\gamma_i \in h_i, i=1,2}\{\gamma_2 | \gamma_2 > h_1^+ \ or \ \gamma_2 = \gamma_1\}$.
By Definition 2.6, we have $(h_1 \sqcup h_2) \sqcap h_1 = h_1$.
ii) If $h_1^+ > h_2^+$, then $min(h_1^+, h_2^+) = h_2^+$. It follows that $h_1 \sqcup h_2 = \cup_{\gamma_i \in h_i, i=1,2}\{\gamma_1 | \gamma_1 > h_2^+ \ or \ \gamma_1 = \gamma_2\}$.
By Definition 2.6, we have $(h_1 \sqcup h_2) \sqcap h_1 = h_1$.

□

3 Hesitant fuzzy soft set

In this section, we present an extended soft set model which is called hesitant fuzzy soft set by combining the hesitant fuzzy set and soft set. Some operations and their properties on hesitant fuzzy soft set will also be discussed.

Definition 3.1. *Let $HF(U)$ be the class of all HFSs of the universe U, $A \subseteq E$. A pair $(\widetilde{F}, A)$ is called a hesitant fuzzy soft set ($HFSS$), where $\widetilde{F} : A \to HF(U)$ is a mapping.*

In other words, a hesitant fuzzy soft set over U is a parameterized family of hesitant fuzzy set of the universe U. To illustrate this idea, let us consider the following example.

Example 3.2. *Let $U = \{u_1, u_2, u_3\}$ be a set of mobile telephones and $A = \{e_1, e_2, e_3\} \subseteq E$ be a set of parameters. The $e_i (i = 1, 2, 3)$ stands for the parameters "expensive", "beautiful" and "multifunctional", respectively. Let $\widetilde{F} : A \to HF(U)$ be a function given as follows:*

$$\widetilde{F}(e_1) = \left\{ \frac{\{0.2, 0.7, 0.8\}}{u_1}, \frac{\{0.5, 0.8\}}{u_2}, \frac{\{0.4, 0.6, 0.8\}}{u_3} \right\},$$
$$\widetilde{F}(e_2) = \left\{ \frac{\{0.3, 0.5, 0.7\}}{u_1}, \frac{\{0.4, 0.6, 0.9\}}{u_2}, \frac{\{0.5, 0.7\}}{u_3} \right\},$$
$$\widetilde{F}(e_3) = \left\{ \frac{\{0.5, 0.8\}}{u_1}, \frac{\{0.3, 0.5, 0.8\}}{u_2}, \frac{\{0.5, 0.6, 0.9\}}{u_3} \right\}.$$

Then $(\widetilde{F}, A)$ is a hesitant fuzzy soft set.

Remark 3.3. (1) *If A has only an element, i.e. $A = \{e\}$, then hesitant fuzzy soft set becomes hesitant fuzzy set* [20];
(2) If $h_{\widetilde{F}(e)}(u)$ has only one value for all $e \in A$ and $u \in U$, then hesitant fuzzy soft set degenerates to traditional fuzzy soft set [13];
(3) If $h_{\widetilde{F}(e)}(u)$ is a subinterval of $[0, 1]$ for all $e \in A$ and $u \in U$, then hesitant fuzzy soft set reduces to interval-valued fuzzy soft set [17];
(4) For all $e \in A$, if $h_{\widetilde{F}(e)}(u)$ has the same number of values with respect to $u \in U$, then hesitant fuzzy soft set transforms to multi-fuzzy soft set [19].

Definition 3.4. *The complement of an $HFSS$ $(\widetilde{F}, A)$ is denoted by $(\widetilde{F}, A)^c$ and is defined by $(\widetilde{F}, A)^c = (\widetilde{F}^c, A)$, where $\widetilde{F}^c : A \to HF(U)$ is a mapping given by $\widetilde{F}^c(e) = \left\{ \frac{h_{\widetilde{F}^c(e)}(u)}{u} | u \in U \right\}$, where $h_{\widetilde{F}^c(e)}(u) = \bigcup_{\gamma \in h_{\widetilde{F}(e)}(u)} \{1 - \gamma\}$.*

Example 3.5. (continued) The complement of $(\widetilde{F}, A)$ is following as:

$$\widetilde{F}^c(e_1) = \left\{ \frac{\{0.2, 0.3, 0.8\}}{u_1}, \frac{\{0.2, 0.5\}}{u_2}, \frac{\{0.2, 0.4, 0.6\}}{u_3} \right\},$$
$$\widetilde{F}^c(e_2) = \left\{ \frac{\{0.3, 0.5, 0.7\}}{u_1}, \frac{\{0.1, 0.4, 0.6\}}{u_2}, \frac{\{0.3, 0.5\}}{u_3} \right\},$$
$$\widetilde{F}^c(e_3) = \left\{ \frac{\{0.2, 0.5\}}{u_1}, \frac{\{0.2, 0.5, 0.7\}}{u_2}, \frac{\{0.1, 0.4, 0.5\}}{u_3} \right\}.$$

Definition 3.6. *Let $(\widetilde{F}, A)$ be an $HFSS$ over U. Then*
(1) $(\widetilde{F}, A)$ *is said to be an empty hesitant soft set, denoted by* $\widetilde{\Phi}_A$*, if* $h_{F(e)}(u) = 0$ *for all* $u \in U$ *and* $e \in A$*;*
(2) $(\widetilde{F}, A)$ *is said to be a full hesitant soft set, denoted by* $\widetilde{\mathcal{I}}_A$*, if* $h_{F(e)}(u) = 1$ *for all* $u \in U$ *and* $e \in A$*;*
(3) $(\widetilde{F}, A)$ *is said to be a complete hesitant soft set, denoted by* $\widetilde{\mathcal{W}}_A$*, if* $h_{F(e)}(u) = [0, 1]$ *for all* $u \in U$ *and* $e \in A$.

Proposition 3.7. *Let $A \subseteq E$. Then*
(1) $\widetilde{\Phi}^c_A = \widetilde{\mathcal{I}}_A$;
(2) $\widetilde{\mathcal{I}}^c_A = \widetilde{\Phi}_A$;
(3) $\widetilde{\mathcal{W}}^c_A = \widetilde{\mathcal{W}}_A$.

Definition 3.8. *Let $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ be two $HFSSs$ over U and $A, B \subseteq E$. We define a mapping $\widetilde{H} : A \cup B \to HF(U)$ such that for all $e \in A \cup B \neq \emptyset$,*

$$\widetilde{H}(e) = \begin{cases} \widetilde{F}(e), & \text{if } e \in A - B, \\ \widetilde{G}(e), & \text{if } e \in B - A, \\ \widetilde{H}(e), & \text{if } e \in A \cap B. \end{cases}$$

(1) *If* $\widetilde{H}(e) = \widetilde{F}(e) \cup \widetilde{G}(e)$*, then* $(\widetilde{H}, A \cup B)$ *is called the extended union of* $(\widetilde{F}, A)$ *and* $(\widetilde{G}, B)$*, denoted by* $(\widetilde{F}, A)\widetilde{\cup}(\widetilde{G}, B)$.
(2) *If* $\widetilde{H}(e) = \widetilde{F}(e) \cap \widetilde{G}(e)$*, then* $(\widetilde{H}, A \cup B)$ *is called the extended intersection of* $(\widetilde{F}, A)$ *and* $(\widetilde{G}, B)$*, denoted by* $(\widetilde{F}, A)\widetilde{\cap}(\widetilde{G}, B)$.
(3) *If* $\widetilde{H}(e) = \widetilde{F}(e) \sqcup \widetilde{G}(e)$*, then* $(\widetilde{H}, A \cup B)$ *is called the extended-strict union of* $(\widetilde{F}, A)$ *and* $(\widetilde{G}, B)$*, denoted by* $(\widetilde{F}, A)\widetilde{\sqcup}(\widetilde{G}, B)$.
(4) *If* $\widetilde{H}(e) = \widetilde{F}(e) \sqcap \widetilde{G}(e)$*, then* $(\widetilde{H}, A \cup B)$ *is called the extended-strict intersection of* $(\widetilde{F}, A)$ *and* $(\widetilde{G}, B)$*, denoted by* $(\widetilde{F}, A)\widetilde{\sqcap}(\widetilde{G}, B)$.
If $A \cup B = \emptyset$*, then* $(\widetilde{F}, A)\widetilde{\cup}(\widetilde{G}, B) = \widetilde{\Phi}_\emptyset$, $(\widetilde{F}, A)\widetilde{\cap}(\widetilde{G}, B) = \widetilde{\Phi}_\emptyset$, $(\widetilde{F}, A)\widetilde{\sqcup}(\widetilde{G}, B) = \widetilde{\Phi}_\emptyset$ *and* $(\widetilde{F}, A)\widetilde{\sqcap}(\widetilde{G}, B) = \widetilde{\Phi}_\emptyset$.

Definition 3.9. *Let $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ be two $HFSSs$ over U and $A, B \subseteq E$. We define a mapping $\widetilde{H} : A \cap B \to HF(U)$ such that for all $e \in A \cap B \neq \emptyset$,*
(1) *If* $\widetilde{H}(e) = \widetilde{F}(e) \cup \widetilde{G}(e)$*, then* $(\widetilde{H}, A \cap B)$ *is called the strict union of* $(\widetilde{F}, A)$ *and* $(\widetilde{G}, B)$*, denoted by* $(\widetilde{F}, A)\widetilde{\Cup}(\widetilde{G}, B)$.
(2) *If* $\widetilde{H}(e) = \widetilde{F}(e) \cap \widetilde{G}(e)$*, then* $(\widetilde{H}, A \cap B)$ *is called the strict intersection of* $(\widetilde{F}, A)$ *and* $(\widetilde{G}, B)$*, denoted by* $(\widetilde{F}, A)\widetilde{\Cap}(\widetilde{G}, B)$.
(3) *If* $\widetilde{H}(e) = \widetilde{F}(e) \sqcup \widetilde{G}(e)$*, then* $(\widetilde{H}, A \cap B)$ *is called the strict-strict union of* $(\widetilde{F}, A)$ *and* $(\widetilde{G}, B)$*, denoted by* $(\widetilde{F}, A)\widetilde{\uplus}(\widetilde{G}, B)$.
(4) *If* $\widetilde{H}(e) = \widetilde{F}(e) \sqcap \widetilde{G}(e)$*, then* $(\widetilde{H}, A \cap B)$ *is called the strict-strict intersection of* $(\widetilde{F}, A)$ *and* $(\widetilde{G}, B)$*, denoted by* $(\widetilde{F}, A)\widetilde{\pitchfork}(\widetilde{G}, B)$.
If $A \cap B = \emptyset$*, then* $(\widetilde{F}, A) \Cup (\widetilde{G}, B) = \widetilde{\Phi}_\emptyset$, $(\widetilde{F}, A) \Cap (\widetilde{G}, B) = \widetilde{\Phi}_\emptyset$, $(\widetilde{F}, A)\widetilde{\uplus}(\widetilde{G}, B) = \widetilde{\Phi}_\emptyset$ *and* $(\widetilde{F}, A)\widetilde{\pitchfork}(\widetilde{G}, B) = \widetilde{\Phi}_\emptyset$.

Proposition 3.10. *Let $A \subseteq E$, $(\widetilde{F}, A)$ be an $HFSS$ over (U, E), $\theta_1 \in \{\widetilde{\Cap}, \widetilde{\pitchfork}\}$, $\theta_2 \in \{\widetilde{\cap}, \widetilde{\sqcap}\}$, $\theta_3 \in \{\widetilde{\Cup}, \widetilde{\uplus}\}$ and $\theta_4 \in \{\widetilde{\cup}, \widetilde{\sqcup}\}$. Then*
(1) $(\widetilde{F}, A)\ \theta_1\ \widetilde{\mathcal{I}}_E = (\widetilde{F}, A)\ \theta_2\ \widetilde{\mathcal{I}}_A = (\widetilde{F}, A)$;
(2) $(\widetilde{F}, A)\ \theta_3\ \widetilde{\mathcal{I}}_E = (\widetilde{F}, A)\ \theta_4\ \widetilde{\mathcal{I}}_A = \widetilde{\mathcal{I}}_A$;
(3) $(\widetilde{F}, A)\ \theta_1\ \widetilde{\Phi}_E = (\widetilde{F}, A)\ \theta_2\ \widetilde{\Phi}_E = \widetilde{\Phi}_A$;
(4) $(\widetilde{F}, A)\ \theta_3\ \widetilde{\Phi}_E = (\widetilde{F}, A)\ \theta_4\ \widetilde{\Phi}_A = (\widetilde{F}, A)$;
(5) $(\widetilde{F}, A)\ \theta_1\ \widetilde{\Phi}_\phi = (\widetilde{F}, A)\ \theta_3\ \widetilde{\Phi}_\phi = \widetilde{\Phi}_\phi$;
(6) $(\widetilde{F}, A)\ \theta_2\ \widetilde{\Phi}_\phi = (\widetilde{F}, A)\ \theta_4\ \widetilde{\Phi}_\phi = (\widetilde{F}, A)$.

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Theorem 3.11. *Let* $\alpha \in \{\widetilde{\uplus}, \widetilde{\sqcup}, \widetilde{\dot\cap}, \widetilde{\sqcap}, \widetilde{\cap}, \widetilde{\Cap}, \widetilde{\cup}, \widetilde{\Cup}\}$, $A, B, C \subseteq E$, $(\widetilde{F}, A)$, $(\widetilde{G}, B)$ *and* $(\widetilde{H}, C)$ *be* $HFSSs$ *over* (U, E). *Then the following holds:*
(1) $(\widetilde{F}, A)\ \alpha\ (\widetilde{F}, A) = (\widetilde{F}, A)$;
(2) $(\widetilde{F}, A)\ \alpha\ (\widetilde{G}, B) = (\widetilde{G}, B)\ \alpha\ (\widetilde{F}, A)$;
(3) $(\widetilde{F}, A)\ \alpha\ ((\widetilde{G}, B)\ \alpha\ (\widetilde{H}, C)) = ((\widetilde{F}, A)\ \alpha\ (\widetilde{G}, B))\ \alpha\ (\widetilde{H}, C)$.

Proof. (1) and (2) are trivial. We only prove (3). For example, let $\alpha = \widetilde{\uplus}$, the others can be proved analogously.

Suppose that $(\widetilde{F}, A)\widetilde{\uplus}((\widetilde{G}, B)\widetilde{\uplus}(\widetilde{H}, C)) = (\widetilde{J}, M)$ and $((\widetilde{F}, A)\widetilde{\uplus}(\widetilde{G}, B))\widetilde{\uplus}(\widetilde{H}, C) = (\widetilde{K}, N)$, thus $M = N = A \cap B \cap C$. If $M = \phi$, then $(\widetilde{F}, A)\ \widetilde{\uplus}((\widetilde{G}, B)\ \widetilde{\uplus}(\widetilde{H}, C)) = \Phi_\phi = ((\widetilde{F}, A)\ \widetilde{\uplus}(\widetilde{G}, B))\ \widetilde{\uplus}\ (\widetilde{H}, C)$. If $M \neq \phi$, then by (2) in Theorem 2.9, we have $h_{F(e)}(u) \sqcup (h_{G(e)}(u) \sqcup h_{H(e)}(u)) = (h_{F(e)}(u) \sqcup h_{G(e)}(u)) \sqcup h_{H(e)}(u)$ for all $e \in M$ and $u \in U$. It follows that $\widetilde{F}(e) \sqcup (\widetilde{G}(e) \sqcup \widetilde{H}(e)) = (\widetilde{F}(e) \sqcup \widetilde{G}(e)) \sqcup \widetilde{H}(e)$ for all $e \in M$. By the definition of the operation $\widetilde{\uplus}$, we have $(\widetilde{F}, A)\widetilde{\uplus}((\widetilde{G}, B)\widetilde{\uplus}(\widetilde{H}, C)) = ((\widetilde{F}, A)\widetilde{\uplus}(\widetilde{G}, B))\widetilde{\uplus}(\widetilde{H}, C)$. □

Remark 3.12. *Theorem 3.11 shows that the operations* $\widetilde{\cap}$, $\widetilde{\Cap}$, $\widetilde{\cup}$, $\widetilde{\Cup}$, $\widetilde{\uplus}$, $\widetilde{\sqcup}$, $\widetilde{\dot\cap}$ *and* $\widetilde{\sqcap}$ *are idempotent, commutative and associative, respectively.*

Theorem 3.13. *Let* $A, B \subseteq E$, $(\widetilde{F}, A)$ *and* $(\widetilde{G}, B)$ *be* $HFSSs$ *over* (U, E). *Then the following holds:*
(1) $((\widetilde{F}, A)\widetilde{\sqcap}(\widetilde{G}, B))^c = (\widetilde{F}, A)^c\widetilde{\sqcup}(\widetilde{G}, B)^c$;
(2) $((\widetilde{F}, A)\widetilde{\sqcup}(\widetilde{G}, B))^c = (\widetilde{F}, A)^c\widetilde{\sqcap}(\widetilde{G}, B)^c$;
(3) $((\widetilde{F}, A)\widetilde{\dot\cap}(\widetilde{G}, B))^c = (\widetilde{F}, A)^c\widetilde{\uplus}(\widetilde{G}, B)^c$;
(4) $((\widetilde{F}, A)\widetilde{\uplus}(\widetilde{G}, B))^c = (\widetilde{F}, A)^c\widetilde{\dot\cap}(\widetilde{G}, B)^c$;
(5) $((\widetilde{F}, A)\widetilde{\Cap}(\widetilde{G}, B))^c = (\widetilde{F}, A)^c\widetilde{\Cup}(\widetilde{G}, B)^c$;
(6) $((\widetilde{F}, A)\widetilde{\Cup}(\widetilde{G}, B))^c = (\widetilde{F}, A)^c\widetilde{\Cap}(\widetilde{G}, B)^c$;
(7) $((\widetilde{F}, A)\widetilde{\cap}(\widetilde{G}, B))^c = (\widetilde{F}, A)^c\widetilde{\cup}(\widetilde{G}, B)^c$;
(8) $((\widetilde{F}, A)\widetilde{\cup}(\widetilde{G}, B))^c = (\widetilde{F}, A)^c\widetilde{\cap}(\widetilde{G}, B)^c$,

Proof. We only prove (1). By using a similar technique, (2)-(8) can be proved, too.

Suppose that $(\widetilde{F}, A)\widetilde{\sqcap}(\widetilde{G}, B) = (\widetilde{H}, C)$. Then $C = A \cup B$,
(i) if $C = \phi$, then $A = \phi$ and $B = \phi$. Hence $((\widetilde{F}, A)\widetilde{\sqcap}(\widetilde{G}, B))^c = \widetilde{\Phi}_\phi$=$(\widetilde{F}, A)^c\widetilde{\sqcap}(\widetilde{G}, B)^c$.
(ii) if $C \neq \phi$, then for each $e \in C$ and $u \in U$, we have

$$h_{\widetilde{H}(e)}(u) = \begin{cases} h_{\widetilde{F}(e)}(u), & \text{if } e \in A - B, \\ h_{\widetilde{G}(e)}(u), & \text{if } e \in B - A, \\ h_{\widetilde{F}(e)}(u) \sqcap h_{\widetilde{G}(e)}(u), & \text{if } e \in A \cap B. \end{cases}$$

Then

$$h_{\widetilde{H}^c(e)}(u) = \begin{cases} h_{\widetilde{F}^c(e)}(u), & \text{if } e \in A - B, \\ h_{\widetilde{G}^c(e)}(u), & \text{if } e \in B - A, \\ (h_{\widetilde{F}(e)}(u) \sqcap h_{\widetilde{G}(e)}(u))^c, & \text{if } e \in A \cap B. \end{cases}$$

Again suppose that $(\widetilde{F}, A)^c\widetilde{\sqcup}(\widetilde{G}, B)^c = (\widetilde{J}, D)$. Then $D = A \cup B$ and for each $e \in D$ and $u \in U$, we have

$$h_{\widetilde{J}(e)}(u) = \begin{cases} h_{\widetilde{F}^c(e)}(u), & \text{if } e \in A - B, \\ h_{\widetilde{G}^c(e)}(u), & \text{if } e \in B - A, \\ h_{\widetilde{F}^c(e)}(u) \sqcup h_{\widetilde{G}^c(e)}(u), & \text{if } e \in A \cap B. \end{cases}$$

By Theorem 2.7, we have $h_{\widetilde{F}^c(e)}(u) \sqcup h_{\widetilde{G}^c(e)}(u) = (h_{\widetilde{F}(e)}(u) \sqcap h_{\widetilde{G}(e)}(u))^c$, i.e., $h_{\widetilde{J}(e)}(u) = h_{\widetilde{H}^c(e)}(u)$ for all $e \in A$ and $u \in U$.

Therefore, $(\widetilde{H}, C)$ and $(\widetilde{J}, D)$ are the same $HFSS$s. It follows that $((\widetilde{F}, A)\widetilde{\sqcap}(\widetilde{G}, B))^c = (\widetilde{F}, A)^c\widetilde{\sqcup}(\widetilde{G}, B)^c$. □

4 Lattice structures of hesitant fuzzy soft set

In this section, we first recall briefly the necessary definitions and notations. For convenience, we give the following axioms on an algebra $Q = (X, \vee, \wedge)$:
(1) $x \vee x = x, x \wedge x = x$;
(2) $x \vee y = y \vee x, x \wedge y = y \wedge x$;
(3) $(x \vee y) \vee z = x \vee (y \vee z), (x \wedge y) \wedge z = x \wedge (y \wedge z)$;
(4) $(x \vee y) \wedge x = x$, $(x \wedge y) \vee x = x$;
(5) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$,
where $x, y, z \in X$.

The algebra Q is called a quasilattice, if it satisfies the axioms (1),(2) and (3). If a quasilattice further satisfies the axiom (4), then it is called a lattice. If a quasilattice (or lattice) further satisfies the axiom (5), then it is called a distributive quasilattice (or lattice).

For convenience, let $\widetilde{\mathfrak{S}}(U, E)$ denote the set of all $HFSS$s over U, i.e., $\widetilde{\mathfrak{S}}(U, E) = \{(\widetilde{F}, A) | A \subseteq E, \widetilde{F} : A \to HF(U)\}$. Then based on Theorem 3.11, we have the following property.

Proposition 4.1. *Let $\alpha \in \{\widetilde{\cap}, \widetilde{\Cap}, \widetilde{\cap\!\!\!\!+}, \widetilde{\sqcap}\}$ and $\beta \in \{\widetilde{\cup}, \widetilde{\Cup}, \widetilde{\uplus}, \widetilde{\sqcup}\}$, then $(\widetilde{\mathfrak{S}}(U, E), \alpha, \beta)$ is a quasilattice.*

For the operations $\widetilde{\Cap}$ and $\widetilde{\cup}$, the distributive laws hold.

Theorem 4.2. *Let $(\widetilde{F}, A)$, $(\widetilde{G}, B)$, $(\widetilde{H}, C) \in \widetilde{\mathfrak{S}}(U, E)$. Then*
(1) $((\widetilde{F}, A)\widetilde{\Cap}(\widetilde{G}, B))\widetilde{\cup}(\widetilde{H}, C) = ((\widetilde{F}, A)\widetilde{\Cap}(\widetilde{G}, B))\widetilde{\cup}((\widetilde{F}, A)\widetilde{\Cap}(\widetilde{H}, C))$;
(2) $((\widetilde{F}, A)\widetilde{\cup}(\widetilde{G}, B))\widetilde{\Cap}(\widetilde{H}, C) = ((\widetilde{F}, A)\widetilde{\cup}(\widetilde{G}, B))\widetilde{\Cap}((\widetilde{F}, A)\widetilde{\cup}(\widetilde{H}, C))$.

Proof. we only prove (1). (2) can be proved by using a similar technique. Suppose that $(\widetilde{F}, A)\widetilde{\Cap}((\widetilde{G}, B)\widetilde{\cup}(\widetilde{H}, C)) = (\widetilde{J}, M)$ and $((\widetilde{F}, A)\widetilde{\Cap}(\widetilde{G}, B))\widetilde{\cup}((\widetilde{F}, A)\widetilde{\Cap}(\widetilde{H}, C)) = (\widetilde{K}, N)$. Then $M = A \cap (B \cup C) = (A \cap B) \cup (A \cap C) = N$. For each $e \in M$, it follows that $e \in A$ and $e \in B \cup C$.
(i) if $e \in A, e \notin B, e \in C$, then $\widetilde{J}(e) = \widetilde{F}(e) \cap \widetilde{H}(e) = \widetilde{K}(e)$.
(ii) if $e \in A, e \in B, e \notin C$, then $\widetilde{J}(e) = \widetilde{F}(e) \cap \widetilde{G}(e) = \widetilde{K}(e)$.
(iii) if $e \in A, e \in B, e \in C$, then by (4) in Theorem 2.8, we have $h_{\widetilde{F}(e)}(u) \cap (h_{\widetilde{G}(e)}(u) \cup h_{\widetilde{H}(e)}(u)) = (h_{\widetilde{F}(e)}(u) \cap h_{\widetilde{G}(e)}(u) \cup (\widetilde{F}(e) \cap h_{\widetilde{H}(e)}(u))$ for all $u \in U$. It follows that $\widetilde{J}(e) = \widetilde{F}(e) \cap (\widetilde{G}(e) \cup \widetilde{H}(e)) = (\widetilde{F}(e) \cap \widetilde{G}(e) \cup (\widetilde{F}(e) \cap (\widetilde{H}, C)) = \widetilde{K}(e)$.

Thus, $(\widetilde{J}, M)$ and $(\widetilde{K}, N)$ are the same $HFSS$, i.e., $(\widetilde{F}, A)\widetilde{\Cap}((\widetilde{G}, B)\widetilde{\cup}(\widetilde{H}, C)) = ((\widetilde{F}, A)\widetilde{\Cap}(\widetilde{G}, B))\widetilde{\cup}((\widetilde{F}, A)\widetilde{\Cap}(\widetilde{H}, C))$. □

Corollary 4.3. $(\widetilde{\mathfrak{S}}(U, E), \widetilde{\Cap}, \widetilde{\cup})$ *is a distributive quasilattice.*

The operations $\widetilde{\cap}$ and $\widetilde{\Cup}$ have the similar properties with the operations $\widetilde{\Cap}$ and $\widetilde{\cup}$.

Theorem 4.4. *Let $(\widetilde{F}, A)$, $(\widetilde{G}, B)$, $(\widetilde{H}, C) \in \widetilde{\mathfrak{S}}(U, E)$. Then*
(1) $((\widetilde{F}, A)\widetilde{\cap}(\widetilde{G}, B))\widetilde{\Cup}(\widetilde{H}, C) = ((\widetilde{F}, A)\widetilde{\cap}(\widetilde{G}, B))\widetilde{\Cup}((\widetilde{F}, A)\widetilde{\cap}(\widetilde{H}, C))$;
(2) $((\widetilde{F}, A)\widetilde{\Cup}(\widetilde{G}, B))\widetilde{\cap}(\widetilde{H}, C) = ((\widetilde{F}, A)\widetilde{\Cup}(\widetilde{G}, B))\widetilde{\cap}((\widetilde{F}, A)\widetilde{\Cup}(\widetilde{H}, C))$.

Corollary 4.5. $(\widetilde{\mathfrak{S}}(U, E), \widetilde{\Cap}, \widetilde{\cup})$ *is a distributive quasilattice.*

The following theorem shows that the absorption laws with respect to operations $\widetilde{\sqcap}$ and $\widetilde{\uplus}$ hold.

Theorem 4.6. *Let $(\widetilde{F}, A)$, $(\widetilde{G}, B) \in \widetilde{\mathfrak{S}}(U, E)$. Then*
(1) $((\widetilde{F}, A)\widetilde{\sqcap}(\widetilde{G}, B))\widetilde{\uplus}(\widetilde{F}, A) = (\widetilde{F}, A)$;
(2) $((\widetilde{F}, A)\widetilde{\uplus}(\widetilde{G}, B))\widetilde{\sqcap}(\widetilde{F}, A) = (\widetilde{F}, A)$.

Proof. We only prove (1) since (2) can be proved similarly. Suppose that $(\widetilde{F}, A)\widetilde{\sqcap}(\widetilde{G}, B) = (\widetilde{J}, M)$ and $((\widetilde{F}, A) \widetilde{\sqcap} (\widetilde{G}, B)) \widetilde{\cap\!\!\!\!+} (\widetilde{F}, A) = (\widetilde{K}, N)$. Then $M = A \cup B$, $N = (A \cup B) \cap A = A$, and for all $e \in A$ and $u \in U$,
(i) if $e \notin B$, then $h_{\widetilde{J}(e)}(u) = h_{\widetilde{F}(e)}(u)$ and $h_{\widetilde{K}(e)}(u) = h_{\widetilde{J}(e)}(u) \sqcap h_{\widetilde{F}(e)}(u) = h_{\widetilde{F}(e)}(u)$.

(ii) if $e \in B$, then $h_{\widetilde{J}(e)}(u) = h_{\widetilde{F}(e)}(u) \sqcup h_{\widetilde{G}(e)}(u)$ and $h_{\widetilde{K}(e)}(u) = h_{\widetilde{J}(e)}(u) \sqcap h_{\widetilde{F}(e)}(u) = (h_{\widetilde{F}(e)}(u) \sqcup h_{\widetilde{G}(e)}(u)) \sqcap h_{\widetilde{F}(e)}(u)$. By (3) in Theorem 2.9, we have $(h_{\widetilde{F}(e)}(u) \sqcup h_{\widetilde{G}(e)}(u)) \sqcap h_{\widetilde{F}(e)}(u) = h_{\widetilde{F}(e)}(u)$, i.e. $h_{\widetilde{K}(e)}(u) = h_{\widetilde{F}(e)}(u)$.

Thus $(\widetilde{K}, N) = (\widetilde{F}, A)$, i.e. $((\widetilde{F}, A)\widetilde{\sqcap}(\widetilde{G}, B))\widetilde{\uplus}(\widetilde{F}, A) = (\widetilde{F}, A)$. □

Theorem 4.7. $(\widetilde{\mathfrak{S}}(U, E), \widetilde{\sqcap}, \widetilde{\uplus})$ *is a bounded lattice.*

Proof. By Theorem 3.11 and Theorem 4.6, we get that $(\widetilde{\mathfrak{S}}(U, E), \widetilde{\sqcap}, \widetilde{\uplus})$ is a lattice. It is clear that $\widetilde{\mathcal{I}}_E$ and $\widetilde{\Phi}_\phi$ are the maximum element and the minimum element in $(\widetilde{\mathfrak{S}}(U, E)$, respectively. □

Similar to $\widetilde{\sqcap}$ and $\widetilde{\uplus}$, the operations $\widetilde{\sqcup}$ and $\widetilde{⋒}$ have also the following properties.

Theorem 4.8. *Let* $(\widetilde{F}, A)$, $(\widetilde{G}, B) \in \widetilde{\mathfrak{S}}(U, E)$. *Then*
(1) $((\widetilde{F}, A)\widetilde{\sqcup}(\widetilde{G}, B))\widetilde{⋒}(\widetilde{F}, A) = (\widetilde{F}, A)$;
(2) $((\widetilde{F}, A)\widetilde{⋒}(\widetilde{G}, B))\widetilde{\sqcup}(\widetilde{F}, A) = (\widetilde{F}, A)$.

Theorem 4.9. $(\widetilde{\mathfrak{S}}(U, E), \widetilde{\sqcup}, \widetilde{⋒})$ *is a bounded lattice.*

Remark 4.10. *It is worth noting that* $(\widetilde{\mathfrak{S}}(U, E), \widetilde{\sqcup}, \widetilde{\sqcap})$, $(\widetilde{\mathfrak{S}}(U, E), \widetilde{\uplus}, \widetilde{⋒})$ *and* $(\widetilde{\mathfrak{S}}(U, E), \alpha, \beta)$ *are not lattices, as the absorption laws do not hold necessarily, where* $\alpha \in \{\widetilde{\cap}, \widetilde{⋒}\}$ *and* $\beta \in \{\widetilde{\cup}, \widetilde{⋓}\}$. *To illustrate this idea, we give an example below.*

Example 4.11. *Let* $U = \{u_1, u_2, u_3\}$ *be the universe,* $E = \{e_1, e_2, e_3\}$ *the set of parameters,* $A = \{e_1, e_2\}$ *and* $B = \{e_2, e_3\}$. *The HFSSs* $(\widetilde{F}, A)$ *and* $(\widetilde{G}, B)$ *over* U *are given as:*

$$
\begin{aligned}
\widetilde{F}(e_1) &= \left\{ \frac{\{0.2, 0.3, 0.7, 0.8\}}{u_1}, \frac{\{0.5, 0.8\}}{u_2}, \frac{\{0.4, 0.5, 0.6\}}{u_3} \right\}, \\
\widetilde{F}(e_2) &= \left\{ \frac{\{0.3, 0.4, 0.7\}}{u_1}, \frac{\{0.5, 0.7\}}{u_2}, \frac{\{0.1, 0.2, 0.4, 0.7\}}{u_3} \right\}, \\
\widetilde{G}(e_2) &= \left\{ \frac{\{0.5, 0.6\}}{u_1}, \frac{\{0.4, 0.8, 0.9\}}{u_2}, \frac{\{0.3, 0.5, 0.7, 0.8\}}{u_3} \right\}, \\
\widetilde{G}(e_3) &= \left\{ \frac{\{0.1, 0.3, 0.5\}}{u_1}, \frac{\{0.5, 0.6, 0.8\}}{u_2}, \frac{\{0.6, 0.9\}}{u_3} \right\}.
\end{aligned}
$$

(1) *Let* $((\widetilde{F}, A)\widetilde{\sqcup}\ (\widetilde{G}, B))\widetilde{\sqcap}(\widetilde{F}, A) = (\widetilde{J}, M)$, *then* $M = A \cup B = \{e_1, e_2, e_3\} \neq A$. *So* $(\widetilde{J}, M) \neq (\widetilde{F}, A)$, *i.e.* $((\widetilde{F}, A)\widetilde{\sqcup}\ (\widetilde{G}, B))\widetilde{\sqcap}(\widetilde{F}, A) \neq (\widetilde{F}, A)$.

(2) *Let* $((\widetilde{F}, A)\widetilde{\uplus}\ (\widetilde{G}, B))\widetilde{⋒}(\widetilde{F}, A) = (\widetilde{K}, N)$, *then* $N = A \cap B = \{e_2\} \neq A$, *Therefore,* $(\widetilde{K}, N) \neq (\widetilde{F}, A)$, *i.e.* $((\widetilde{F}, A)\widetilde{\uplus}\ (\widetilde{G}, B))\widetilde{⋒}(\widetilde{F}, A) \neq (\widetilde{F}, A)$.

(3) *If* $e_2 \in A \cap B$, *then* $(h_{\widetilde{F}(e_2)}(u_1) \cap h_{\widetilde{G}(e_2)}(u_1)) \cup h_{\widetilde{F}(e_2)}(u_1) = (\{0.3, 0.4, 0.7\} \cap \{0.5, 0.6\}) \cup \{0.3, 0.4, 0.7\} = \{0.3, 0.4, 0.5, 0.6\} \cup \{0.3, 0.4, 0.7\} = \{0.3, 0.4, 0.5, 0.6, 0.7\} \neq \{0.3, 0.4, 0.7\} = h_{\widetilde{F}(e_2)}(u_1)$. *It follows that* $(\widetilde{F}(e_2) \cap \widetilde{G}(e_2)) \cup \widetilde{F}(e_2) \neq \widetilde{F}(e_2)$. *Consequently,* $((\widetilde{F}, A)\ \alpha\ (\widetilde{G}, B))\ \beta\ (\widetilde{F}, A) \neq (\widetilde{F}, A)$, *where* $\alpha \in \{\widetilde{\cap}, \widetilde{⋒}\}$ *and* $\beta \in \{\widetilde{\cup}, \widetilde{⋓}\}$.

5 Conclusion

Considering that soft set and its existing extension models cannot deal with the situations in which the evaluations of parameters have many possible values, in this paper, we have introduced the notion of $HFSS$ as an new extension to the HFS or the soft set model. We have also defined some basic operations on the $HFSS$ and discussed their properties. Finally, The lattice structures of $HFSS$ have been studied in detail based on the proposed operations.

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INCLUSION PROPERTIES FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH BESSEL FUNCTIONS

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Abstract: The purpose of the present paper is to investigate some characterization for generalized Bessel functions of first kind to be in the new subclasses $\mathcal{G}(\lambda, \alpha)$ and $\mathcal{K}(\lambda, \alpha)$ of analytic functions.

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1. Introduction

Let $\mathcal{A}$ be the class of functions f normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. As usual, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are normalized by $f(0) = 0 = f'(0) - 1$ and also univalent in $\mathbb{U}$. Denote by $\mathcal{T}$ [16] the subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \quad n = 2, 3, \ldots. \tag{1.2}$$

Also, for functions $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad (z \in \mathbb{U}). \tag{1.3}$$

The class $\mathcal{S}^*(\alpha)$ of starlike functions of order α $(0 \leq \alpha < 1)$ may be defined as

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \ \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \ z \in \mathbb{U} \right\}.$$

The class $\mathcal{S}^*(\alpha)$ and the class $\mathcal{K}(\alpha)$ of convex functions of order α $(0 \leq \alpha < 1)$

$$\begin{aligned} \mathcal{K}(\alpha) &= \left\{ f \in \mathcal{A} : \ \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \ z \in \mathbb{U} \right\} \\ &= \{ f \in \mathcal{A} : \ zf' \in \mathcal{S}^*(\alpha) \} \end{aligned}$$

were introduced by Robertson in [14]. We also write $\mathcal{S}^*(0) = \mathcal{S}^*$, where $\mathcal{S}^*$ denotes the class of functions $f \in \mathcal{A}$ that $f(\mathbb{U})$ is starlike with respect to the origin. Further, $\mathcal{K}(0) = \mathcal{K}$ is the well-known standard class of convex functions. It is an established fact that $f \in \mathcal{K}(\alpha) \Longleftrightarrow zf' \in \mathcal{S}^*(\alpha)$.

A function $f \in \mathcal{A}$ is said to be in the class $\Re^\tau(A, B)$, $(\tau \in \mathbb{C}\backslash\{0\}, -1 \leq B < A \leq 1)$, if it satisfies the inequality

$$\left| \frac{f'(z) - 1}{(A - B)\tau - B[f'(z) - 1]} \right| < 1 \quad (z \in \mathbb{U}).$$

The class $\Re^\tau(A, B)$ was introduced earlier by Dixit and Pal [7]. If we put

$$\tau = 1, \ A = \alpha \text{ and } \ B = -\alpha \ (0 < \alpha \leq 1),$$

we obtain the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$\left| \frac{f'(z) - 1}{f'(z) + 1} \right| < \alpha \quad (z \in \mathbb{U}; 0 < \alpha \leq 1)$$

which was studied by (among others) Padmanabhan [12] and Caplinger and Causey [5].

We recall here a generalized Bessel function $\omega_{p,b,c}(z) = \omega(z)$ defined in [1] and given by

$$\omega(z) = \omega_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \ c^n}{n! \ \Gamma\left(p + n + \frac{b+1}{2}\right)} \left(\frac{z}{2}\right)^{2n+p} \tag{1.4}$$

which is the particular solution of the second order linear homogeneous differential equation

$$z^2\omega''(z) + bz\omega'(z) + [cz^2 - p^2 + (1 - b)]\omega(z) = 0, \tag{1.5}$$

where $b, p, c \in \mathbb{C}$, which is a natural generalization of Bessel's equation.

The differential equation (1.5) permits the study of Bessel function, modified Bessel function, spherical Bessel function and modified spherical Bessel functions all together.

Solutions of (1.5) are referred to as the generalized Bessel function of order p. The particular solution given by (1.4) is called the generalized Bessel function of the first kind of order p. Although the series defined above is convergent everywhere, the function $\omega_{p,b,c}$ is generally not univalent in $\mathbb{U}$. It is of interest to note that when $b = c = 1$, we reobtain the Bessel function of the first kind $\omega_{p,1,1} = J_p$, and for $c = -1, b = 1$, the function $\omega_{p,1,-1}$ becomes the modified Bessel function $\mathcal{I}_p$. Now, we consider the function $u_{p,b,c}(z)$ defined by the transformation

$$u_{p,b,c}(z) = 2^p\Gamma\left(p + \frac{b+1}{2}\right)\ z^{\frac{-p}{2}}\ \omega_{p,b,c}\ (\sqrt{z}), \quad \sqrt{1} = 1.$$

By using well known Pochhammer symbol (or the shifted factorial) defined, in terms of the familiar Gamma function, by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & (n = 0), \\ a(a+1)(a+2)\cdots(a+n-1) & (n \in \mathbb{N} = \{1, 2, 3, \ldots\}), \end{cases}$$

we can express $u_{p,b,c}(z)$ as

$$u_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c/4)^n}{\left(p + \frac{b+1}{2}\right)_n} \left(\frac{z^n}{n!}\right), \tag{1.6}$$

where $p + \frac{b+1}{2} \neq 0, -1, -2, \cdots$. This function is analytic on $\mathbb{C}$ and satisfies the second-order linear differential equation

$$4z^2u''(z) + 2(2p + b + 1)zu'(z) + cu(z) = 0.$$

Now, we considered the linear operator

$$\mathcal{I}(c, m) : \mathcal{A} \to \mathcal{A}$$

defined by

$$\mathcal{I}(c, m)f(z) = zu_{p,b,c}(z) * f(z) = z + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} a_n\ z^n, \tag{1.7}$$

where $m = p + \frac{b+1}{2} \neq 0, -1, -2, \cdots$. For convenience throughout in the sequel, we use the following notations

$$u_{p,b,c} = u_p, \qquad m = p + \frac{b+1}{2}.$$

and if $c < 0$ and $m > 0$, then we let

$$z(2 - u_p(z)) = z - \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} z^n. \tag{1.8}$$

For $0 \le \lambda < 1$ and $0 \le \alpha < 1$, we let $\mathcal{G}(\lambda, \alpha)$ the subclass of functions $f \in \mathcal{A}$ which satisfy the condition

$$\Re\left(\frac{zf'(z) + \lambda z^2 f''(z)}{f(z)}\right) > \alpha, \ (z \in \mathbb{U}). \tag{1.9}$$

and also let $\mathcal{K}(\lambda, \alpha)$ the subclass of functions $f \in \mathcal{A}$ which satisfy the condition

$$\Re\left(\frac{z[zf'(z) + \lambda z^2 f''(z)]'}{zf'(z)}\right) > \alpha, \ (z \in \mathbb{U}). \tag{1.10}$$

Also denote $\mathcal{G}^*(\lambda, \alpha) = \mathcal{G}(\lambda, \alpha) \cap \mathcal{T}$ and $\mathcal{K}^*(\lambda, \alpha) = \mathcal{K}(\lambda, \alpha) \cap \mathcal{T}$

The study of the generalized Bessel function is a recent interesting topic in geometric function theory (e.g. see the work of [1, 2, 3, 4] and [9]). In this paper, due to Ramesha et al. [13], Padmanabhan [12], and motivated by the works of Srivastava et al. [17], Murugusundaramoorthy and Magesh [11],(see [6, 8, 10, 15]) and by work of Baricz [1, 2, 3, 4], we obtain sufficient conditions for function $z(2 - u_p(z))$ in $\mathcal{G}(\lambda, \alpha)$ and $\mathcal{K}(\lambda, \alpha)$ and connections between $\mathcal{R}^\tau(A, B)$.

Remark 1. *It is of interest to note that for $\lambda = 0$, we have $\mathcal{G}(\lambda, \alpha) \equiv \mathcal{S}^*(\alpha)$ and $\mathcal{K}(\lambda, \alpha) \equiv \mathcal{K}(\alpha)$*

To prove the main results, we need the following Lemmas.

Lemma 1. [18] *A function $f \in \mathcal{A}$ belongs to the class $\mathcal{G}(\lambda, \alpha)$ if*

$$\sum_{n=2}^{\infty}(n + \lambda n(n-1) - \alpha)|a_n| \le 1 - \alpha.$$

Lemma 2. [18] *A function $f \in \mathcal{A}$ belongs to the class $\mathcal{K}(\lambda, \alpha)$ if*

$$\sum_{n=2}^{\infty} n(n + \lambda n(n-1) - \alpha)|a_n| \le 1 - \alpha.$$

Further we can easily prove that the conditions are also necessary if $f \in \mathcal{T}$.

Lemma 3. [18] *A function $f \in \mathcal{T}$ belongs to the class $\mathcal{G}^*(\lambda, \alpha)$ if and only if*

$$\sum_{n=2}^{\infty}(n + \lambda n(n-1) - \alpha)|a_n| \le 1 - \alpha.$$

Lemma 4. [18] *A function $f \in \mathcal{T}$ belongs to the class $\mathcal{K}^*(\lambda, \alpha)$ if and only if*

$$\sum_{n=2}^{\infty} n(n + \lambda n(n-1) - \alpha)|a_n| \le 1 - \alpha.$$

Lemma 5. [4] *If $b, p, c \in \mathbb{C}$* and $m \ne 0, -1, -2, \ldots$ *then the function u_p satisfies the recursive relation*

$$4mu_p'(z) = -cu_{p+1}(z)$$

for all $z \in \mathbb{C}$.

2. Main Results

Theorem 1. *If $c < 0$ and $m > 0$, then $z(2 - u_p(z))$ is in $\mathcal{G}(\lambda, \alpha)$ if*

$$\lambda u_p''(1) + [1 + 2\lambda]u_p'(1) + (1 - \alpha)u_p(1) \leq 2(1 - \alpha). \tag{2.1}$$

Proof. Since

$$z(2 - u_p(z)) = z - \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} z^n$$

and by virtue of Lemma 1, it suffices to show that

$$\mathcal{L}(c, m, \lambda, \alpha) = \sum_{n=2}^{\infty} (n + \lambda n(n-1) - \alpha) \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} \leq 1 - \alpha.$$

Writing $n^2 = (n-1)(n-2) + 3(n-1) + 1$ and $n = (n-1) + 1$, and by simple computation, we get

$$\begin{aligned}
\mathcal{L}(c, m, \lambda, \alpha) &= \sum_{n=2}^{\infty} (n^2\lambda + n(1-\lambda) - \alpha) \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} \\
&\leq \sum_{n=2}^{\infty} \lambda(n-1)(n-2) \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} + (1+2\lambda) \sum_{n=2}^{\infty} (n-1) \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} \\
&\quad + (1-\alpha) \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} \\
&= \lambda \sum_{n=3}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-3)!} + (1+2\lambda) \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-2)!} + (1-\alpha) \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} \\
&= \lambda \sum_{n=1}^{\infty} \frac{(-c/4)^{n+1}}{(m)_{n+1}\ (n-1)!} + (1+2\lambda) \sum_{n=0}^{\infty} \frac{(-c/4)^{n+1}}{(m)_{n+1}\ (n)!} + (1-\alpha) \sum_{n=0}^{\infty} \frac{(-c/4)^{n+1}}{(m)_{n+1}\ (n+1)!} \\
&= \lambda \frac{(-c/4)^2}{m(m+1)} \sum_{n=0}^{\infty} \frac{(-c/4)^n}{(m+2)_n\ n!} + (1+2\lambda) \frac{(-c/4)}{m} \sum_{n=0}^{\infty} \frac{(-c/4)^n}{(m+1)_n\ n!} \\
&\quad + (1-\alpha) \sum_{n=0}^{\infty} \frac{(-c/4)^{n+1}}{(m)_{n+1}\ (n+1)!} \\
&= \lambda \frac{(-c/4)^2}{m(m+1)} u_{p+2}(1) + (1+2\lambda) \frac{(-c/4)}{m} u_{p+1}(1) + (1-\alpha)[u_p(1) - 1] \\
&= \lambda u_p''(1) + (1+2\lambda)u_p'(1) + (1-\alpha)[u_p(1) - 1].
\end{aligned}$$

By a simplification, we see that the last expression is bounded above by $1 - \alpha$ if (2.1) is satisfied. □

By taking $\lambda = 0$, we state the following corollary.

Corollary 1. *If $c < 0$ and $m > 0$, then $z(2 - u_p(z))$ is in $\mathcal{S}^*(\alpha)$ if*

$$u_p'(1) + (1 - \alpha)u_p(1) \leq 2(1 - \alpha). \tag{2.2}$$

Remark 2. *In particular, when $c = -1$ and $b = 1$, the condition (2.1) becomes*

$$2^{p-2}\Gamma(p+1)\left[\lambda\mathcal{I}_{p+2}(1) + [1 + 2\lambda]\mathcal{I}_{p+1}(1) + 2(1 - \alpha)\mathcal{I}_p(1)\right] \leq 1 - \alpha, \tag{2.3}$$

which is necessary and sufficient condition for $z(2 - \zeta_p(z^{1/2})$ to be in $\mathcal{G}^*(\lambda, \alpha)$, where

$$u_p(z^{1/2}) = 2^p\Gamma(p+1)z^{-p/2}\mathcal{I}_p(z^{1/2}).$$

Theorem 2. *If $c < 0$ and $m > 0$, then $z(2 - u_p(z))$ is in $\mathcal{K}(\lambda, \alpha)$ if*

$$\lambda u_p'''(1) + (1 + 5\lambda)u_p''(1) + (3 + 4\lambda - \alpha)u_p'(1) + (1 - \alpha)u_p(1) \leq 2(1 - \alpha). \tag{2.4}$$

Proof. Since

$$z(2 - u_p(z)) = z - \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!}z^n$$

and by virtue of Lemma 2, it suffices to show that

$$\mathcal{L}(c, m, \lambda, \alpha) = \sum_{n=2}^{\infty}(n^3\lambda + n^2(1 - \lambda) - n\alpha)\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} \leq 1 - \alpha.$$

Writing $n^3 = (n-1)(n-2)(n-3) + 6(n-1)(n-2) + 7(n-1) + 1$, $n^2 = (n-1)(n-2) + 3(n-1) + 1$ and $n = (n-1) + 1$, we can rewrite the above terms as

$$\begin{aligned}
\mathcal{L}(c, m, \lambda, \alpha) &\leq \lambda\sum_{n=2}^{\infty}(n-1)(n-2)(n-3)\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} \\
&\quad + (1 + 5\lambda)\sum_{n=2}^{\infty}(n-1)(n-2)\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} + (3 + 4\lambda - \alpha)\sum_{n=2}^{\infty}(n-1)\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} \\
&\quad + (1 - \alpha)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} \\
&= \lambda\sum_{n=4}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-4)!} + (1 + 5\lambda)\sum_{n=3}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-3)!} + (3 + 4\lambda - \alpha)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-2)!} \\
&\quad + (1 - \alpha)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} \\
&= \lambda\sum_{n=2}^{\infty}\frac{(-c/4)^{n+1}}{(m)_{n+1}\ (n-2)!} + (1 + 5\lambda)\sum_{n=1}^{\infty}\frac{(-c/4)^{n+1}}{(m)_{n+1}\ (n-1)!} \\
&\quad + (3 + 4\lambda - \alpha)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(m)_{n+1}\ (n)!} + (1 - \alpha)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(m)_{n+1}\ (n+1)!}
\end{aligned}$$

$$= \lambda\frac{(-c/4)^3}{m(m+1)(m+2)}\sum_{n=2}^{\infty}\frac{(-c/4)^{n-2}}{(m+3)_{n-2}\ (n-2)!} + (1+5\lambda)\frac{(-c/4)^2}{m(m+1)}\sum_{n=1}^{\infty}\left(\frac{(-c/4)^{n-1}}{(m+2)_{n-1}\ (n-1)!}\right)$$
$$+ (3+4\lambda-\alpha)\frac{(-c/4)}{m}\sum_{n=0}^{\infty}\left(\frac{(-c/4)^{n}}{(m+1)_{n}\ (n)!}\right) + (1-\alpha)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(m)_{n+1}\ (n+1)!}$$
$$= \lambda\frac{(-c/4)^3}{m(m+1)(m+2)}u_{p+3}(1) + (1+5\lambda)\frac{(-c/4)^2}{m(m+1)}u_{p+2}(1)$$
$$+ (3+4\lambda-\alpha)\frac{(-c/4)}{m}u_{p+1}(1) + (1-\alpha)[u_p(1)-1]$$
$$= \lambda u_p'''(1) + (1+5\lambda)u_p''(1) + (3+4\lambda-\alpha)u_p'(1) + (1-\alpha)[u_p(1)-1].$$

By a simplification, we see that the last expression is bounded above by $1-\alpha$ if (2.4) is satisfied. □

By taking $\lambda = 0$, we state the following corollary.

Corollary 2. *If $c < 0$ and $m > 0$, then $z(2-u_p(z))$ is in $\in \mathcal{K}(\alpha)$ if*

$$u_p''(1) + (3-\alpha)u_p'(1) + (1-\alpha)u_p(1) \leq 2(1-\alpha). \tag{2.5}$$

Remark 3. *We also note that the function $z(2-u_p(z))$ is in $\mathcal{K}^*(\lambda,\alpha)$ if and only if the condition (2.4) is satisfied.*

3. Inclusion Properties

Making use of the following lemma, we will study the action of the Bessel function on the classes $\mathcal{K}(\lambda,\alpha)$.

Lemma 6. [7] *A function $f \in \mathfrak{R}^{\tau}(A,B)$ is of form (1.1), then*

$$|a_n| \leq (A-B)\frac{|\tau|}{n}, \quad n \in \mathbb{N}\backslash\{1\}. \tag{3.1}$$

The bound given in (3.1) is sharp.

Theorem 3. *Let $c < 0$ and $m > 0$. If $f \in \mathfrak{R}^{\tau}(A,B)$ and the inequality*

$$(A-B)|\tau|\left[\lambda u_p''(1) + (1+2\lambda)u_p'(1) + (1-\alpha)[u_p(1)-1]\right] \leq 1-\alpha \tag{3.2}$$

is satisfied, then $\mathcal{I}(c,m)f \in \mathcal{K}(\lambda,\alpha)$.

Proof. Let f be of the form (1.1) belong to the class $\mathfrak{R}^{\tau}(A,B)$ then by virtue of Lemma 2, it suffices to show that

$$\mathcal{P}(c,m,\lambda,\alpha) = \sum_{n=2}^{\infty} n(n^2\lambda + n(1-\lambda)-\alpha)\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!}|a_n| \leq 1-\alpha. \tag{3.3}$$

Writing $n^2 = (n-1)(n-2) + 3(n-1) + 1$ and $n = (n-1) + 1$, we get

$$\begin{aligned}\mathcal{P}(c,m,\lambda,\alpha) &\leq \sum_{n=2}^{\infty}(n + \lambda n(n-1) - \alpha)\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!}(A-B)|\tau| \\ &= (A-B)|\tau|\sum_{n=2}^{\infty}\lambda(n-1)(n-2)\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} \\ &\quad + (A-B)|\tau|(1+2\lambda)\sum_{n=2}^{\infty}(n-1)\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} \\ &\quad + (A-B)|\tau|(1-\alpha)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!}\end{aligned}$$

$$\begin{aligned}&= (A-B)|\tau|\left[\lambda\sum_{n=3}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-3)!} + (1+2\lambda)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-2)!} + (1-\alpha)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!}\right] \\ &= (A-B)|\tau|\left[\lambda\sum_{n=1}^{\infty}\frac{(-c/4)^{n+1}}{(m)_{n+1}\ (n-1)!} + (1+2\lambda)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(m)_{n+1}\ n!} + (1-\alpha)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(m)_{n+1}\ (n+1)!}\right].\end{aligned}$$

By using the similar method as in the proof of Theorem 1, we have

$$\begin{aligned}&\mathcal{P}(c,m,\lambda,\alpha) = (A-B)|\tau|\left[\lambda\frac{(-c/4)^2}{m(m+1)}u_{p+2}(1) + (1+2\lambda)\frac{(-c/4)}{m}u_{p+1}(1) + (1-\alpha)[u_p(1)-1]\right] \\ &= (A-B)|\tau|\left[\lambda u_p''(1) + (1+2\lambda)u_p'(1) + (1-\alpha)[u_p(1)-1]\right],\end{aligned}$$

the last expression is bounded above by $(1-\alpha)$ if and only if (3.2) is satisfied. Hence the proof is completed. □

Corollary 3. *Let $c < 0$ and $m > 0$. If $f \in \Re^{\tau}(A,B)$, and the inequality*

$$(A-B)|\tau|\left[u_p'(1) + (1-\alpha)\{u_p(1)-1\}\right] \leq 1-\alpha \tag{3.4}$$

is satisfied, then $\mathcal{I}(c,m)f \in \mathcal{K}(\alpha)$.

Theorem 4. *Let $c < 0$ and $m > 0$. Then*

$$\mathcal{L}(m,c,z) = \int_0^z (2 - u_p(t))dt$$

is in $\mathcal{K}^*(\lambda,\alpha)$ if and only if the inequality

$$\lambda u_p''(1) + [1+2\lambda]u_p'(1) + (1-\alpha)[u_p(1)-1] \leq 1-\alpha. \tag{3.5}$$

Proof. Since

$$\mathcal{L}(m,c,z) = z - \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}} \ \frac{z^n}{(n)!},$$

by Lemma 4, we need only to show that

$$\mathcal{L}(c,m,\lambda,\alpha) = \sum_{n=2}^{\infty} n(n^2\lambda + n(1-\lambda) - \alpha)\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n)!} \leq 1-\alpha.$$

Now, we have

$$\mathcal{L}(c,m,\lambda,\alpha) = \sum_{n=2}^{\infty}(n^2\lambda + n(1-\lambda) - \alpha)\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!}.$$

Writing $n^2 = (n-1)(n-2) + 3(n-1) + 1$ and $n = (n-1) + 1$, and proceeding as in Theorem 1, we get

$$\sum_{n=2}^{\infty}(n^2\lambda + n(1-\lambda) - \alpha)\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} = \lambda u_p''(1) + [1+2\lambda]u_p'(1) + (1-\alpha)[u_p(1) - 1],$$

which is bounded above by $1-\alpha$ if and only if (3.5) holds. □

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Barnes-type Narumi of the second kind and poly-Cauchy of the second kind mixed-type polynomials

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Abstract

In this paper, by considering Barnes-type Narumi polynomials of the second kind as well as poly-Cauchy polynomials of the second kind, we define and investigate

the mixed-type polynomials of these polynomials. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

1 Introduction

In this paper, we consider the polynomials $\widehat{N}_n^{(k)}(x|a_1,\dots,a_r)$ called the Barnes-type Narumi of the second kind and poly-Cauchy of the second kind mixed-type polynomials, whose generating function is given by

$$\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)(1+t)^x=\sum_{n=0}^{\infty}\widehat{N}_n^{(k)}(x|a_1,\dots,a_r)\frac{t^n}{n!}\,, \tag{1}$$

where $a_1,\dots,a_r\neq 0$. Here, $\mathrm{Lif}_k(x)$ $(k\in\mathbb{Z})$ is the polyfactorial function ([10]) defined by

$$\mathrm{Lif}_k(x)=\sum_{m=0}^{\infty}\frac{x^m}{m!(m+1)^k}\,.$$

When $x=0$, $\widehat{N}_n^{(k)}(a_1,\dots,a_r)=\widehat{N}_n^{(k)}(0|a_1,\dots,a_r)$ is called the the Barnes-type Narumi of the second kind and poly-Cauchy of the second kind mixed-type number.

Recall that the Barnes-type Narumi polynomials of the second kind, denoted by $\widehat{N}_n(x|a_1,\dots,a_r)$, are given by the generating function as

$$\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)(1+t)^x=\sum_{n=0}^{\infty}\widehat{N}_n(x|a_1,\dots,a_r)\frac{t^n}{n!}\,.$$

If $a_1=\cdots=a_r=1$, then $\widehat{N}_n^{(r)}(x)=\widehat{N}_n(x|\underbrace{1,\dots,1}_{r})$ are the Narumi polynomials of the second kind of order r. Narumi polynomials were mentioned in [14, p.127] and have been investigated in e.g. [9, 12, 15].

The poly-Cauchy polynomials of the second kind, denoted by $\widehat{c}_n^{(k)}(x)$ ([8, 11]), are given by the generating function as

$$\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)(1+t)^x=\sum_{n=0}^{\infty}\widehat{c}_n^{(k)}(x)\frac{t^n}{n!}\,.$$

In this paper, by considering Barnes-type Narumi polynomials of the second kind as well as poly-Cauchy polynomials of the second kind, we define and investigate the mixed-type polynomials of these polynomials. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

2

2 Umbral calculus

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable t:

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \middle| a_k \in \mathbb{C} \right\}. \tag{2}$$

Let $\mathbb{P} = \mathbb{C}[x]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L|p(x)\rangle$ is the action of the linear functional L on the polynomial $p(x)$, and we recall that the vector space operations on $\mathbb{P}^*$ are defined by $\langle L+M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle$, $\langle cL|p(x)\rangle = c\langle L|p(x)\rangle$, where c is a complex constant in $\mathbb{C}$ (see [1-16]). For $f(t) \in \mathcal{F}$, let us define the linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle = a_n, \quad (n \geq 0). \tag{3}$$

In particular,

$$\left\langle t^k|x^n\right\rangle = n!\delta_{n,k} \quad (n, k \geq 0), \tag{4}$$

where $\delta_{n,k}$ is the Kronecker's symbol.

For $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$, we have $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. That is, $L = f_L(t)$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the *umbral algebra* and the *umbral calculus* is the study of umbral algebra. The order $O\big(f(t)\big)$ of a power series $f(t)(\neq 0)$ is the smallest integer k for which the coefficient of t^k does not vanish. If $O\big(f(t)\big) = 1$, then $f(t)$ is called a *delta series*; if $O\big(f(t)\big) = 0$, then $f(t)$ is called an *invertible series*. For $f(t), g(t) \in \mathcal{F}$ with $O\big(f(t)\big) = 1$ and $O\big(g(t)\big) = 0$, there exists a unique sequence $s_n(x)$ ($\deg s_n(x) = n$) such that $\left\langle g(t)f(t)^k|s_n(x)\right\rangle = n!\delta_{n,k}$, for $n, k \geq 0$. Such a sequence $s_n(x)$ is called the *Sheffer sequence* for $\big(g(t), f(t)\big)$ which is denoted by $s_n(x) \sim \big(g(t), f(t)\big)$.

For $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$\langle f(t)g(t)|p(x)\rangle = \langle f(t)|g(t)p(x)\rangle = \langle g(t)|f(t)p(x)\rangle \tag{5}$$

and

$$f(t) = \sum_{k=0}^{\infty} \left\langle f(t)|x^k\right\rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \left\langle t^k|p(x)\right\rangle \frac{x^k}{k!} \tag{6}$$

([14, Theorem 2.2.5]). Thus, by (6), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad \text{and} \quad e^{yt}p(x) = p(x+y). \tag{7}$$

Sheffer sequences are characterized in the generating function ([14, Theorem 2.3.4]).

Lemma 1 *The sequence $s_n(x)$ is Sheffer for $\big(g(t), f(t)\big)$ if and only if*

$$\frac{1}{g\big(\bar{f}(t)\big)} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{s_k(y)}{k!} t^k \quad (y \in \mathbb{C}),$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$.

For $s_n(x) \sim \big(g(t), f(t)\big)$, we have the following equations ([14, Theorem 2.3.7, Theorem 2.3.5, Theorem 2.3.9]):

$$f(t)s_n(x) = ns_{n-1}(x) \quad (n \geq 0), \tag{8}$$

$$s_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \left\langle g\big(\bar{f}(t)\big)^{-1} \bar{f}(t)^j | x^n \right\rangle x^j, \tag{9}$$

$$s_n(x+y) = \sum_{j=0}^{n} \binom{n}{j} s_j(x) p_{n-j}(y), \tag{10}$$

where $p_n(x) = g(t)s_n(x)$.

Assume that $p_n(x) \sim \big(1, f(t)\big)$ and $q_n(x) \sim \big(1, g(t)\big)$. Then the transfer formula ([14, Corollary 3.8.2]) is given by

$$q_n(x) = x \left(\frac{f(t)}{g(t)} \right)^n x^{-1} p_n(x) \quad (n \geq 1).$$

For $s_n(x) \sim \big(g(t), f(t)\big)$ and $r_n(x) \sim \big(h(t), l(t)\big)$, assume that

$$s_n(x) = \sum_{m=0}^{n} C_{n,m} r_m(x) \quad (n \geq 0).$$

Then we have ([14, p.132])

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{h\big(\bar{f}(t)\big)}{g\big(\bar{f}(t)\big)} l\big(\bar{f}(t)\big)^m \middle| x^n \right\rangle. \tag{11}$$

3 Main results

From the definition (1), $\widehat{N}_n^{(k)}(x|a_1, \ldots, a_r)$ is the Sheffer sequence for the pair

$$g(t) = \prod_{j=1}^{r} \left(\frac{te^{a_j t}}{e^{a_j t} - 1} \right) \frac{1}{\mathrm{Lif}_k(-t)} \quad \text{and} \quad f(t) = e^t - 1.$$

So,

$$\widehat{N}_n^{(k)}(x|a_1, \ldots, a_r) \sim \left(\prod_{j=1}^{r} \left(\frac{te^{a_j t}}{e^{a_j t} - 1} \right) \frac{1}{\mathrm{Lif}_k(-t)}, e^t - 1 \right). \tag{12}$$

4

3.1 Explicit expressions

Let $(n)_j = n(n-1)\cdots(n-j+1)$ $(j \geq 1)$ with $(n)_0 = 1$. The (signed) Stirling numbers of the first kind $S_1(n,m)$ are defined by

$$(x)_n = \sum_{m=0}^{n} S_1(n,m)x^m \,.$$

Define the multinomial coefficient by

$$\binom{n}{l_1,\dots,l_r} = \frac{n!}{l_1!\cdots l_r!}$$

where $l_1+\cdots+l_r = n$.

Theorem 1

$$\begin{aligned}
\widehat{N}_n^{(k)}(x|a_1,\dots,a_r) &= \sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{i=0}^{m-l}\sum_{l_1+\cdots+l_r=m-l-i}\frac{(-1)^{m-i}(m-l-i)!}{(m-l-i+r)!(l+1)^k}\\
&\quad\times\binom{m-l-i+r}{l_1+1,\dots,l_r+1}\binom{m}{l}\binom{m-l}{i}S_1(n,m)a_1^{l_1+1}\cdots a_r^{l_r+1}x^i \qquad (13)\\
&= \sum_{j=0}^{n}\sum_{l=j}^{n}\sum_{i=0}^{n-l}\binom{n}{l}\binom{n-l}{i}S_1(l,j)\widehat{c}_i^{(k)}\widehat{N}_{n-l-i}(a_1,\dots,a_r)x^j \qquad (14)\\
&= \sum_{l=0}^{n}\binom{n}{l}\widehat{N}_{n-l}(a_1,\dots,a_r)\widehat{c}_l^{(k)}(x)\,, \qquad (15)\\
&= \sum_{l=0}^{n}\binom{n}{l}\widehat{c}_{n-l}^{(k)}\widehat{N}_l(x|a_1,\dots,a_r)\,. \qquad (16)
\end{aligned}$$

Proof. Since

$$\prod_{j=1}^{r}\left(\frac{te^{a_jt}}{e^{a_jt}-1}\right)\frac{1}{\mathrm{Lif}_k(-t)}\widehat{N}_n^{(k)}(x|a_1,\dots,a_r) \sim (1, e^t-1) \qquad (17)$$

and

$$(x)_n \sim (1, e^t-1)\,, \qquad (18)$$

we have

$$
\begin{aligned}
\widehat{N}_n^{(k)}(x|a_1,\ldots,a_r) &= \prod_{j=1}^{r}\left(\frac{e^{a_jt}-1}{te^{a_jt}}\right)\mathrm{Lif}_k(-t)(x)_n\\
&= \sum_{m=0}^{n} S_1(n,m)\prod_{j=1}^{r}\left(\frac{e^{a_jt}-1}{te^{a_jt}}\right)\mathrm{Lif}_k(-t)x^m\\
&= \sum_{m=0}^{n} S_1(n,m)\prod_{j=1}^{r}\left(\frac{e^{a_jt}-1}{te^{a_jt}}\right)\sum_{l=0}^{m}\frac{(-1)^l t^l}{l!(l+1)^k}x^m\\
&= \sum_{m=0}^{n} S_1(n,m)\prod_{j=1}^{r}\left(\frac{e^{a_jt}-1}{te^{a_jt}}\right)\sum_{l=0}^{m}\frac{(-1)^l (m)_l}{l!(l+1)^k}x^{m-l}\\
&= \sum_{m=0}^{n} S_1(n,m)\sum_{l=0}^{m}\frac{(-1)^l\binom{m}{l}}{(l+1)^k}\prod_{j=1}^{r}\left(\frac{e^{-a_jt}-1}{-t}\right)x^{m-l}\\
&= \sum_{m=0}^{n} S_1(n,m)\sum_{l=0}^{m}\frac{(-1)^l\binom{m}{l}}{(l+1)^k}\\
&\quad\times\sum_{i=0}^{\infty}\sum_{l_1+\cdots+l_r=i}\frac{a_1^{l_1+1}\cdots a_r^{l_r+1}}{(l_1+1)!\cdots(l_r+1)!}(-t)^i x^{m-l}\\
&= \sum_{m=0}^{n} S_1(n,m)\sum_{l=0}^{m}\frac{(-1)^l\binom{m}{l}}{(l+1)^k}\\
&\quad\times\sum_{i=0}^{m-l}\sum_{l_1+\cdots+l_r=i}\frac{a_1^{l_1+1}\cdots a_r^{l_r+1}}{(l_1+1)!\cdots(l_r+1)!}(-1)^i (m-l)_i x^{m-l-i}\\
&= \sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{i=0}^{m-l}\sum_{l_1+\cdots+l_r=i}\frac{(-1)^{l+i} i!}{(i+r)!(l+1)^k}\\
&\quad\times\binom{i+r}{l_1+1,\ldots,l_r+1}\binom{m}{l}\binom{m-l}{i}S_1(n,m)a_1^{l_1+1}\cdots a_r^{l_r+1}x^{m-l-i}\\
&= \sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{i=0}^{m-l}\sum_{l_1+\cdots+l_r=m-l-i}\frac{(-1)^{m-i}(m-l-i)!}{(m-l-i+r)!(l+1)^k}\\
&\quad\times\binom{m-l-i+r}{l_1+1,\ldots,l_r+1}\binom{m}{l}\binom{m-l}{i}S_1(n,m)a_1^{l_1+1}\cdots a_r^{l_r+1}x^{i}.
\end{aligned}
$$

So, we get (13).

By (9) with (12), we get

$$
\begin{aligned}
&\left\langle g\big(\bar{f}(t)\big)^{-1}\bar{f}(t)^j|x^n\right\rangle \\
&=\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^j\Big|x^n\right\rangle \\
&=\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)\Big|j!\sum_{l=j}^{\infty}S_1(l,j)\frac{t^l}{l!}x^n\right\rangle \\
&=j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)\Big|x^{n-l}\right\rangle \\
&=j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\left\langle \sum_{i=0}^{\infty}\widehat{N}_i^{(k)}(a_1,\dots,a_r)\frac{t^i}{i!}\Big|x^{n-l}\right\rangle \\
&=j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\widehat{N}_{n-l}^{(k)}(a_1,\dots,a_r)\,.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\left\langle g\big(\bar{f}(t)\big)^{-1}\bar{f}(t)^j|x^n\right\rangle \\
&=j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\Big|\mathrm{Lif}_k\big(-\ln(1+t)\big)x^{n-l}\right\rangle \\
&=j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\sum_{i=0}^{n-l}\binom{n-l}{i}\widehat{c}_i^{(k)}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\Big|x^{n-l-i}\right\rangle \\
&=j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\sum_{i=0}^{n-l}\binom{n-l}{i}\widehat{c}_i^{(k)}\left\langle \sum_{m=0}^{\infty}\widehat{N}_m(a_1,\dots,a_r)\frac{t^m}{m!}\Big|x^{n-l-i}\right\rangle \\
&=j!\sum_{l=j}^{n}\sum_{i=0}^{n-l}\binom{n}{l}\binom{n-l}{i}S_1(l,j)\widehat{c}_i^{(k)}\widehat{N}_{n-l-i}(a_1,\dots,a_r)\,.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
\widehat{N}_n^{(k)}(x|a_1,\dots,a_r)&=\sum_{j=0}^{n}\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\widehat{N}_{n-l}^{(k)}(a_1,\dots,a_r)x^j \\
&=\sum_{j=0}^{n}\sum_{l=j}^{n}\sum_{i=0}^{n-l}\binom{n}{l}\binom{n-l}{i}S_1(l,j)\widehat{c}_i^{(k)}\widehat{N}_{n-l-i}(a_1,\dots,a_r)x^j\,,
\end{aligned}
$$

which is the identity (14).

Next,

$$
\begin{aligned}
\widehat{N}_n^{(k)}(y|a_1,\ldots,a_r) &= \left\langle \sum_{i=0}^{\infty} \widehat{N}_i^{(k)}(y|a_1,\ldots,a_r)\frac{t^i}{i!}\Big| x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y \Big| x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\Big|\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\Big| \sum_{l=0}^{\infty}\widehat{c}_l^{(k)}(y)\frac{t^l}{l!}x^n \right\rangle \\
&= \sum_{l=0}^{n}\binom{n}{l}\widehat{c}_l^{(k)}(y)\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\Big| x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n}\binom{n}{l}\widehat{c}_l^{(k)}(y)\left\langle \sum_{i=0}^{\infty}\widehat{N}_i(a_1,\ldots,a_r)\frac{t^i}{i!}\Big| x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n}\binom{n}{l}\widehat{c}_l^{(k)}(y)\widehat{N}_{n-l}(a_1,\ldots,a_r)\,.
\end{aligned}
$$

Thus, we obtain (15).
Finally, we obtain that

$$
\begin{aligned}
\widehat{N}_n^{(k)}(y|a_1,\ldots,a_r) &= \left\langle \sum_{i=0}^{\infty} \widehat{N}_i^{(k)}(y|a_1,\ldots,a_r)\frac{t^i}{i!}\Big| x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y \Big| x^n \right\rangle \\
&= \left\langle \mathrm{Lif}_k\big(-\ln(1+t)\big)\Big| \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)(1+t)^y x^n \right\rangle \\
&= \left\langle \mathrm{Lif}_k\big(-\ln(1+t)\big)\Big| \sum_{l=0}^{\infty}\widehat{N}_l(y|a_1,\ldots,a_r)\frac{t^l}{l!}x^n \right\rangle \\
&= \sum_{l=0}^{n}\widehat{N}_l(y|a_1,\ldots,a_r)\binom{n}{l}\left\langle \mathrm{Lif}_k\big(-\ln(1+t)\big)\Big| x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n}\widehat{N}_l(y|a_1,\ldots,a_r)\binom{n}{l}\left\langle \sum_{i=0}^{\infty}\widehat{c}_i^{(k)}\frac{t^i}{i!}\Big| x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n}\binom{n}{l}\widehat{N}_l(y|a_1,\ldots,a_r)\widehat{c}_{n-l}^{(k)}\,.
\end{aligned}
$$

8

Thus, we get the identity (16). ∎

3.2 Sheffer identity

Theorem 2

$$\widehat{N}_n^{(k)}(x+y|a_1,\ldots,a_r)=\sum_{j=0}^{n}\binom{n}{j}\widehat{N}_j^{(k)}(x|a_1,\ldots,a_r)(y)_{n-j}\,. \tag{19}$$

Proof. By (12) with

$$\begin{aligned}p_n(x)&=\prod_{j=1}^{r}\left(\frac{te^{a_jt}}{e^{a_jt}-1}\right)\frac{1}{\mathrm{Lif}_k(-t)}\widehat{N}_n(x|a_1,\ldots,a_r)\\&=(x)_n\sim(1,e^t-1)\,,\end{aligned}$$

using (10), we have (19). ∎

3.3 Difference relations

Theorem 3

$$\widehat{N}_n^{(k)}(x+1|a_1,\ldots,a_r)-\widehat{N}_n^{(k)}(x|a_1,\ldots,a_r)=n\widehat{N}_{n-1}^{(k)}(x|a_1,\ldots,a_r)\,. \tag{20}$$

Proof. By (8) with (12), we get

$$(e^t-1)\widehat{N}_n^{(k)}(x|a_1,\ldots,a_r)=n\widehat{N}_{n-1}^{(k)}(x|a_1,\ldots,a_r)\,.$$

By (7), we have (20). ∎

3.4 Recurrence

Theorem 4

$$
\begin{aligned}
&\widehat{N}_{n+1}^{(k)}(x|a_1,\dots,a_r)\\
&= x\widehat{N}_n^{(k)}(x-1|a_1,\dots,a_r)\\
&\quad+\sum_{m=0}^{n}\sum_{j=1}^{r}\sum_{l=0}^{m}\sum_{i=0}^{l}\sum_{l_1+\cdots+l_r=l-i}\sum_{h=0}^{i}\frac{(-1)^{m+1-h}}{m+1}\frac{(l-i)!}{(l-i+r)!(i-h+1)^k}\\
&\quad\times\binom{m+1}{l}\binom{l-i+r}{l_1+1,\dots,l_r+1}\binom{l}{i}\binom{i}{h}S_1(n,m)B_{m+1-l}a_j^{m+1-l}a_1^{l_1+1}\cdots a_r^{l_r+1}(x-1)^h\\
&\quad-\sum_{j=1}^{r}a_j\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{i=0}^{l}\sum_{l_1+\cdots+l_r=l-i}\frac{(-1)^{m-i}}{(m-l+1)^k}\frac{(l-i)!}{(l-i+r)!}\\
&\quad\times\binom{m}{l}\binom{l-i+r}{l_1+1,\dots,l_r+1}\binom{l}{i}S_1(n,m)a_1^{l_1+1}\cdots a_r^{l_r+1}(x-1)^i\\
&\quad-\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{i=0}^{l}\sum_{l_1+\cdots+l_r=l-i}\frac{(-1)^{m-i+1}}{(m-l+2)^k}\frac{(l-i)!}{(l-i+r)!}\\
&\quad\times\binom{m}{l}\binom{l-i+r}{l_1+1,\dots,l_r+1}\binom{l}{i}S_1(n,m)a_1^{l_1+1}\cdots a_r^{l_r+1}(x-1)^i\,, \qquad (21)
\end{aligned}
$$

where B_n is the nth ordinary Bernoulli number.

Proof. By applying

$$s_{n+1}(x)=\left(x-\frac{g'(t)}{g(t)}\right)\frac{1}{f'(t)}s_n(x) \qquad (22)$$

([14, Corollary 3.7.2]) with (12), we get

$$\widehat{N}_{n+1}^{(k)}(x|a_1,\dots,a_r)=x\widehat{N}_n^{(k)}(x-1|a_1,\dots,a_r)-e^{-t}\frac{g'(t)}{g(t)}\widehat{N}_n^{(k)}(x|a_1,\dots,a_r)\,.$$

Now,

$$
\begin{aligned}
\frac{g'(t)}{g(t)}&=\bigl(\ln g(t)\bigr)'\\
&=\left(r\ln t+\left(\sum_{j=1}^{r}a_j\right)t-\sum_{j=1}^{r}\ln(e^{a_jt}-1)-\ln\mathrm{Lif}_k(-t)\right)'\\
&=\frac{r}{t}+\sum_{j=1}^{r}a_j-\sum_{j=1}^{r}\frac{a_je^{a_jt}}{e^{a_jt}-1}+\frac{\mathrm{Lif}_k'(-t)}{\mathrm{Lif}_k(-t)}\\
&=\frac{\sum_{j=1}^{r}\prod_{i\neq j}(e^{a_it}-1)(e^{a_jt}-1-a_jte^{a_jt})}{t\prod_{j=1}^{r}(e^{a_jt}-1)}+\sum_{j=1}^{r}a_j+\frac{\mathrm{Lif}_k'(-t)}{\mathrm{Lif}_k(-t)}\,,
\end{aligned}
$$

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where

$$\begin{aligned}
&\frac{\sum_{j=1}^{r}\prod_{i\neq j}(e^{a_it}-1)(e^{a_jt}-1-a_jte^{a_jt})}{t\prod_{j=1}^{r}(e^{a_jt}-1)}\\
&=-\frac{\frac{1}{2}\left(\sum_{j=1}^{r}a_1\cdots a_{j-1}a_j^2a_{j+1}\cdots a_r\right)t^{r+1}+\cdots}{(a_1\cdots a_r)t^r+\cdots}\\
&=-\frac{1}{2}\left(\sum_{j=1}^{r}a_j\right)t+\cdots
\end{aligned}$$

is a series with order≥ 1. As seen in the proof of (13)

$$\widehat{N}_n^{(k)}(x|a_1,\ldots,a_r)=\sum_{m=0}^{n}S_1(n,m)\prod_{j=1}^{r}\left(\frac{e^{a_jt}-1}{te^{a_jt}}\right)\mathrm{Lif}_k(-t)x^m\,,$$

so we have

$$\begin{aligned}
&\frac{g'(t)}{g(t)}\widehat{N}_n^{(k)}(x|a_1,\ldots,a_r)\\
&=\sum_{m=0}^{n}S_1(n,m)\frac{g'(t)}{g(t)}\left(\prod_{j=1}^{r}\frac{e^{a_jt}-1}{te^{a_jt}}\right)\mathrm{Lif}_k(-t)x^m\\
&=\sum_{m=0}^{n}S_1(n,m)\mathrm{Lif}_k(-t)\left(\prod_{j=1}^{r}\frac{e^{a_jt}-1}{te^{a_jt}}\right)\frac{\sum_{j=1}^{r}\prod_{i\neq j}(e^{a_it}-1)(e^{a_jt}-1-a_jte^{a_jt})}{t\prod_{j=1}^{r}(e^{a_jt}-1)}x^m\\
&\quad+\sum_{j=1}^{r}a_j\sum_{m=0}^{n}S_1(n,m)\left(\prod_{j=1}^{r}\frac{e^{a_jt}-1}{te^{a_jt}}\right)\mathrm{Lif}_k(-t)x^m\\
&\quad+\sum_{m=0}^{n}S_1(n,m)\left(\prod_{j=1}^{r}\frac{e^{a_jt}-1}{te^{a_jt}}\right)\mathrm{Lif}_k'(-t)x^m\,.
\end{aligned}\tag{23}$$

Since

$$\begin{aligned}
&\frac{\sum_{j=1}^{r}\prod_{i\neq j}(e^{a_it}-1)(e^{a_jt}-1-a_jte^{a_jt})}{t\prod_{j=1}^{r}(e^{a_jt}-1)}x^m\\
&=\frac{\sum_{j=1}^{r}\prod_{i\neq j}(e^{a_it}-1)(e^{a_jt}-1-a_jte^{a_jt})}{\prod_{j=1}^{r}(e^{a_jt}-1)}\frac{x^{m+1}}{m+1}\\
&=\frac{1}{m+1}\sum_{j=1}^{r}\left(1-\frac{a_jte^{a_jt}}{e^{a_jt}-1}\right)x^{m+1}\\
&=\frac{1}{m+1}\sum_{j=1}^{r}\left(1-\sum_{l=0}^{\infty}\frac{(-a_j)^lB_l}{l!}t^l\right)x^{m+1}\\
&=\frac{1}{m+1}\sum_{j=1}^{r}\left(x^{m+1}-\sum_{l=0}^{m+1}\binom{m+1}{l}B_l(-a_j)^lx^{m+1-l}\right)\\
&=-\frac{1}{m+1}\sum_{j=1}^{r}\sum_{l=1}^{m+1}\binom{m+1}{l}B_l(-a_j)^lx^{m+1-l}\\
&=-\frac{1}{m+1}\sum_{j=1}^{r}\sum_{l=0}^{m}\binom{m+1}{l}B_{m+1-l}(-a_j)^{m+1-l}x^l\,,
\end{aligned}$$

the first term in (23) is

$$\begin{aligned}
&-\sum_{m=0}^{n}\sum_{j=1}^{r}\sum_{l=0}^{m}\frac{S_1(n,m)}{m+1}\binom{m+1}{l}B_{m+1-l}(-a_j)^{m+1-l}\mathrm{Lif}_k(-t)\left(\prod_{j=1}^{r}\frac{e^{-a_jt}-1}{-t}\right)x^l\\
&=-\sum_{m=0}^{n}\sum_{j=1}^{r}\sum_{l=0}^{m}\frac{S_1(n,m)}{m+1}\binom{m+1}{l}B_{m+1-l}(-a_j)^{m+1-l}\mathrm{Lif}_k(-t)\\
&\quad\times\sum_{i=0}^{\infty}\sum_{l_1+\cdots+l_r=i}\frac{a_1^{l_1+1}\cdots a_r^{l_r+1}}{(l_1+1)!\cdots(l_r+1)!}(-t)^ix^l\\
&=-\sum_{m=0}^{n}\sum_{j=1}^{r}\sum_{l=0}^{m}\frac{S_1(n,m)}{m+1}\binom{m+1}{l}B_{m+1-l}(-a_j)^{m+1-l}\mathrm{Lif}_k(-t)\\
&\quad\times\sum_{i=0}^{l}\sum_{l_1+\cdots+l_r=i}(-1)^i\frac{a_1^{l_1+1}\cdots a_r^{l_r+1}}{(l_1+1)!\cdots(l_r+1)!}(l)_ix^{l-i}\\
&=-\sum_{m=0}^{n}\sum_{j=1}^{r}\sum_{l=0}^{m}\frac{S_1(n,m)}{m+1}\binom{m+1}{l}B_{m+1-l}(-a_j)^{m+1-l}\\
&\quad\times\sum_{i=0}^{l}\sum_{l_1+\cdots+l_r=i}(-1)^i\frac{a_1^{l_1+1}\cdots a_r^{l_r+1}}{(l_1+1)!\cdots(l_r+1)!}(l)_i\sum_{h=0}^{l-i}\frac{(-1)^h}{h!(h+1)^k}t^hx^{l-i}
\end{aligned}$$

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$$
\begin{aligned}
&= -\sum_{m=0}^{n}\sum_{j=1}^{r}\sum_{l=0}^{m}\frac{S_1(n,m)}{m+1}\binom{m+1}{l}B_{m+1-l}(-a_j)^{m+1-l}\\
&\quad\times\sum_{i=0}^{l}\sum_{l_1+\cdots+l_r=i}(-1)^i\frac{a_1^{l_1+1}\cdots a_r^{l_r+1}}{(l_1+1)!\cdots(l_r+1)!}(l)_i\sum_{h=0}^{l-i}\frac{(-1)^h}{(h+1)^k}\binom{l-i}{h}x^{l-i-h}\\
&= -\sum_{m=0}^{n}\sum_{j=1}^{r}\sum_{l=0}^{m}\sum_{i=0}^{l}\sum_{l_1+\cdots+l_r=i}\sum_{h=0}^{l-i}\frac{(-1)^{m+1-h}}{m+1}\frac{i!}{(i+r)!(l-i-h+1)^k}\\
&\quad\times\binom{m+1}{l}\binom{i+r}{l_1+1,\ldots,l_r+1}\binom{l}{i}\binom{l-i}{h}S_1(n,m)B_{m+1-l}a_j^{m+1-l}a_1^{l_1+1}\cdots a_r^{l_r+1}x^h\\
&= -\sum_{m=0}^{n}\sum_{j=1}^{r}\sum_{l=0}^{m}\sum_{i=0}^{l}\sum_{l_1+\cdots+l_r=l-i}\sum_{h=0}^{i}\frac{(-1)^{m+1-h}}{m+1}\frac{(l-i)!}{(l-i+r)!(i-h+1)^k}\\
&\quad\times\binom{m+1}{l}\binom{l-i+r}{l_1+1,\ldots,l_r+1}\binom{l}{i}\binom{i}{h}S_1(n,m)B_{m+1-l}a_j^{m+1-l}a_1^{l_1+1}\cdots a_r^{l_r+1}x^h\,.
\end{aligned}
$$

The second term in (23) is

$$
\begin{aligned}
&\sum_{j=1}^{r}a_j\sum_{m=0}^{n}S_1(n,m)\left(\prod_{j=1}^{r}\frac{e^{a_jt}-1}{te^{a_jt}}\right)\mathrm{Lif}_k(-t)x^m\\
&=\sum_{j=1}^{r}a_j\sum_{m=0}^{n}S_1(n,m)\left(\prod_{j=1}^{r}\frac{e^{a_jt}-1}{te^{a_jt}}\right)\sum_{l=0}^{m}\frac{(-1)^lt^l}{l!(l+1)^k}x^m\\
&=\sum_{j=1}^{r}a_j\sum_{m=0}^{n}S_1(n,m)\sum_{l=0}^{m}\frac{(-1)^l}{(l+1)^k}\binom{m}{l}\left(\prod_{j=1}^{r}\frac{e^{-a_jt}-1}{-t}\right)x^{m-l}\\
&=\sum_{j=1}^{r}a_j\sum_{m=0}^{n}S_1(n,m)\sum_{l=0}^{m}\frac{(-1)^l}{(l+1)^k}\binom{m}{l}\sum_{i=0}^{m-l}\sum_{l_1+\cdots+l_r=i}\frac{a_1^{l_1+1}\cdots a_r^{l_r+1}}{(l_1+1)!\cdots(l_r+1)!}(-1)^it^ix^{m-l}\\
&=\left(\sum_{j=1}^{r}a_j\right)\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{i=0}^{m-l}\sum_{l_1+\cdots+l_r=i}\frac{(-1)^{i+l}}{(l+1)^k}\frac{i!}{(i+r)!}\\
&\quad\times\binom{m}{l}\binom{i+r}{l_1+1,\ldots,l_r+1}\binom{m-l}{i}S_1(n,m)a_1^{l_1+1}\cdots a_r^{l_r+1}x^{m-l-i}\\
&=\sum_{j=1}^{r}a_j\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{i=0}^{l}\sum_{l_1+\cdots+l_r=l-i}\frac{(-1)^{m-i}}{(m-l+1)^k}\frac{(l-i)!}{(l-i+r)!}\\
&\quad\times\binom{m}{l}\binom{l-i+r}{l_1+1,\ldots,l_r+1}\binom{l}{i}S_1(n,m)a_1^{l_1+1}\cdots a_r^{l_r+1}x^i\,.
\end{aligned}
$$

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The third term in (23) is

$$
\begin{aligned}
&\sum_{m=0}^{n} S_1(n,m)\left(\prod_{j=1}^{r}\frac{e^{a_jt}-1}{te^{a_jt}}\right)\mathrm{Lif}_k'(-t)x^m\\
&=\sum_{m=0}^{n} S_1(n,m)\left(\prod_{j=1}^{r}\frac{e^{a_jt}-1}{te^{a_jt}}\right)\frac{\mathrm{Lif}_{k-1}(-t)-\mathrm{Lif}_k(-t)}{t}x^m\\
&=\sum_{m=0}^{n} \frac{S_1(n,m)}{m+1}\left(\prod_{j=1}^{r}\frac{e^{a_jt}-1}{te^{a_jt}}\right)\big(\mathrm{Lif}_{k-1}(-t)-\mathrm{Lif}_k(-t)\big)x^{m+1}\\
&=\sum_{m=0}^{n} \frac{S_1(n,m)}{m+1}\left(\prod_{j=1}^{r}\frac{e^{a_jt}-1}{te^{a_jt}}\right)\sum_{l=0}^{m}\frac{(-1)^{l+1}t^{l+1}}{l!(l+2)^k}x^{m+1}\\
&=\sum_{m=0}^{n}\sum_{l=0}^{m}\frac{(-1)^{l+1}}{(l+2)^k}\binom{m}{l}S_1(n,m)\left(\prod_{j=1}^{r}\frac{e^{-a_jt}-1}{-t}\right)x^{m-l}\\
&=\sum_{m=0}^{n}\sum_{l=0}^{m}\frac{(-1)^{l+1}}{(l+2)^k}\binom{m}{l}S_1(n,m)\sum_{i=0}^{m-l}\sum_{l_1+\cdots+l_r=i}(-1)^i\frac{a_1^{l_1+1}\cdots a_r^{l_r+1}}{(l_1+1)!\cdots(l_r+1)!}t^ix^{m-l}\\
&=\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{i=0}^{m-l}\sum_{l_1+\cdots+l_r=i}\frac{(-1)^{i+l+1}}{(l+2)^k}\frac{i!}{(i+r)!}\\
&\quad\times\binom{m}{l}\binom{i+r}{l_1+1,\ldots,l_r+1}\binom{m-l}{i}S_1(n,m)a_1^{l_1+1}\cdots a_r^{l_r+1}x^{m-l-i}\\
&=\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{i=0}^{l}\sum_{l_1+\cdots+l_r=l-i}\frac{(-1)^{m-i+1}}{(m-l+2)^k}\frac{(l-i)!}{(l-i+r)!}\\
&\quad\times\binom{m}{l}\binom{l-i+r}{l_1+1,\ldots,l_r+1}\binom{l}{i}S_1(n,m)a_1^{l_1+1}\cdots a_r^{l_r+1}x^i\,.
\end{aligned}
$$

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Thus, we have

$$
\begin{aligned}
&\widehat{N}_{n+1}^{(k)}(x|a_1,\dots,a_r)\\
&= x\widehat{N}_n^{(k)}(x-1|a_1,\dots,a_r)\\
&\quad+\sum_{m=0}^{n}\sum_{j=1}^{r}\sum_{l=0}^{m}\sum_{i=0}^{l}\sum_{l_1+\cdots+l_r=l-i}\sum_{h=0}^{i}\frac{(-1)^{m+1-h}}{m+1}\frac{(l-i)!}{(l-i+r)!(i-h+1)^k}\\
&\quad\times\binom{m+1}{l}\binom{l-i+r}{l_1+1,\dots,l_r+1}\binom{l}{i}\binom{i}{h}S_1(n,m)B_{m+1-l}a_j^{m+1-l}a_1^{l_1+1}\cdots a_r^{l_r+1}(x-1)^h\\
&\quad-\sum_{j=1}^{r}a_j\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{i=0}^{l}\sum_{l_1+\cdots+l_r=l-i}\frac{(-1)^{m-i}}{(m-l+1)^k}\frac{(l-i)!}{(l-i+r)!}\\
&\quad\times\binom{m}{l}\binom{l-i+r}{l_1+1,\dots,l_r+1}\binom{l}{i}S_1(n,m)a_1^{l_1+1}\cdots a_r^{l_r+1}(x-1)^i\\
&\quad-\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{i=0}^{l}\sum_{l_1+\cdots+l_r=l-i}\frac{(-1)^{m-i+1}}{(m-l+2)^k}\frac{(l-i)!}{(l-i+r)!}\\
&\quad\times\binom{m}{l}\binom{l-i+r}{l_1+1,\dots,l_r+1}\binom{l}{i}S_1(n,m)a_1^{l_1+1}\cdots a_r^{l_r+1}(x-1)^i\,,
\end{aligned}
$$

which is the identity (21). ∎

3.5 Differentiation

Theorem 5

$$\frac{d}{dx}\widehat{N}_n^{(k)}(x|a_1,\dots,a_r)=n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}\widehat{N}_l^{(k)}(x|a_1,\dots,a_r)\,. \tag{24}$$

Proof. We shall use

$$\frac{d}{dx}s_n(x)=\sum_{l=0}^{n-1}\binom{n}{l}\left\langle\bar{f}(t)|x^{n-l}\right\rangle s_l(x)$$

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(*Cf.* [14, Theorem 2.3.12]). Since

$$\begin{aligned}\left\langle \bar{f}(t)|x^{n-l}\right\rangle &= \left\langle \ln(1+t)|x^{n-l}\right\rangle \\ &= \left\langle \sum_{m=1}^{\infty}\frac{(-1)^{m-1}t^m}{m}\Big|x^{n-l}\right\rangle \\ &= \sum_{m=1}^{n-l}\frac{(-1)^{m-1}}{m}\left\langle t^m|x^{n-l}\right\rangle \\ &= \sum_{m=1}^{n-l}\frac{(-1)^{m-1}}{m}(n-l)!\delta_{m,n-l} \\ &= (-1)^{n-l-1}(n-l-1)!\,,\end{aligned}$$

with (12), we have

$$\begin{aligned}\frac{d}{dx}\widehat{N}_n^{(k)}(x|a_1,\dots,a_r) &= \sum_{l=0}^{n-1}\binom{n}{l}(-1)^{n-l-1}(n-l-1)!\widehat{N}_l^{(k)}(x|a_1,\dots,a_r) \\ &= n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}\widehat{N}_l^{(k)}(x|a_1,\dots,a_r)\,,\end{aligned}$$

which is the identity (24). ∎

3.6 A more relation

The classical Cauchy numbers c_n are defined by

$$\frac{t}{\ln(1+t)} = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}$$

(see e.g. [3, 10]).

Theorem 6

$$\begin{aligned}&\widehat{N}_n^{(k)}(x|a_1,\dots,a_r) \qquad (25)\\ &= \left(x-\sum_{i=1}^{r}a_i\right)\widehat{N}_{n-1}^{(k)}(x-1|a_1,\dots,a_r) \\ &\quad + \frac{1}{n}\sum_{l=0}^{n}\binom{n}{l}c_l\big(\widehat{N}_{n-l}^{(k-1)}(x-1|a_1,\dots,a_r)-(r+1)\widehat{N}_{n-l}^{(k)}(x-1|a_1,\dots,a_r)\big) \\ &\quad + \frac{1}{n}\sum_{i=1}^{r}\sum_{l=0}^{n}\binom{n}{l}a_i c_l\widehat{N}_{n-l}^{(k)}(x-1|a_1,\dots,a_{i-1},a_{i+1},\dots,a_r)\,. \qquad (26)\end{aligned}$$

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Proof. For $n \geq 1$, we have

$$\begin{aligned}
\widehat{N}_n^{(k)}(y|a_1,\dots,a_r) &= \left\langle \sum_{l=0}^{\infty} \widehat{N}_l^{(k)}(y|a_1,\dots,a_r)\frac{t^l}{l!}\middle| x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)} \right) \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y \middle| x^n \right\rangle \\
&= \left\langle \partial_t \left(\prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)} \right) \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y \right) \middle| x^{n-1} \right\rangle \\
&= \left\langle \left(\partial_t \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)} \right)\right) \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y \middle| x^{n-1} \right\rangle \\
&\quad + \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)} \right) \Big(\partial_t \mathrm{Lif}_k\big(-\ln(1+t)\big)\Big)(1+t)^y \middle| x^{n-1} \right\rangle \\
&\quad + \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)} \right) \mathrm{Lif}_k\big(-\ln(1+t)\big) \left(\partial_t (1+t)^y\right) \middle| x^{n-1} \right\rangle .
\end{aligned}$$

The third term is

$$\begin{aligned}
&y \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)} \right) \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{y-1} \middle| x^{n-1} \right\rangle \\
&= y\widehat{N}_{n-1}^{(k)}(y-1|a_1,\dots,a_r) .
\end{aligned}$$

Since

$$\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big) - \mathrm{Lif}_k\big(-\ln(1+t)\big) = \left(\frac{1}{2^k} - \frac{1}{2^{k-1}} \right) t + \cdots ,$$

the second term is

$$\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\frac{\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)-\mathrm{Lif}_k\big(-\ln(1+t)\big)}{(1+t)\ln(1+t)}(1+t)^y\Big|x^{n-1}\right\rangle$$

$$=\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\right.$$

$$\left.\frac{\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)-\mathrm{Lif}_k\big(-\ln(1+t)\big)}{t}(1+t)^{y-1}\Big|\frac{t}{\ln(1+t)}x^{n-1}\right\rangle$$

$$=\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\right.$$

$$\left.\frac{\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)-\mathrm{Lif}_k\big(-\ln(1+t)\big)}{t}(1+t)^{y-1}\Big|\sum_{l=0}^{\infty}c_l\frac{t^l}{l!}x^{n-1}\right\rangle$$

$$=\sum_{l=0}^{n-1}\binom{n-1}{l}c_l\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\right.$$

$$\left.(1+t)^{y-1}\Big|\frac{\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)-\mathrm{Lif}_k\big(-\ln(1+t)\big)}{t}x^{n-1-l}\right\rangle$$

$$=\sum_{l=0}^{n-1}\frac{1}{n-l}\binom{n-1}{l}c_l\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\right.$$

$$\left.(1+t)^{y-1}\Big|\left(\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)-\mathrm{Lif}_k\big(-\ln(1+t)\big)\right)x^{n-l}\right\rangle$$

$$=\sum_{l=0}^{n-1}\frac{1}{n-l}\binom{n}{l}c_l\left(\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)(1+t)^{y-1}\Big|x^{n-l}\right\rangle\right.$$

$$\left.-\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_{k}\big(-\ln(1+t)\big)(1+t)^{y-1}\Big|x^{n-l}\right\rangle\right)$$

$$=\frac{1}{n}\sum_{l=0}^{n-1}\binom{n}{l}c_l\big(\widehat{N}_{n-l}^{(k-1)}(y-1|a_1,\dots,a_r)-\widehat{N}_{n-l}^{(k)}(y-1|a_1,\dots,a_r)\big).$$

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Since

$$
\begin{aligned}
&\partial_t \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\\
&= \frac{1}{1+t}\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\frac{\sum_{i=1}^{r}\left(\frac{a_i t(1+t)^{a_i}}{(1+t)^{a_i}-1}-\frac{t}{\ln(1+t)}\right)}{t}\\
&\quad -\frac{1}{1+t}\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\sum_{i=1}^{r}a_i\,,
\end{aligned}
$$

the first term is

$$
\begin{aligned}
&\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{y-1}\middle|\frac{\sum_{i=1}^{r}\left(\frac{a_i t(1+t)^{a_i}}{(1+t)^{a_i}-1}-\frac{t}{\ln(1+t)}\right)}{t}x^{n-1}\right\rangle\\
&\quad -\sum_{i=1}^{r}a_i\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{y-1}\middle|x^{n-1}\right\rangle\\
&= \frac{1}{n}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{y-1}\right.\\
&\qquad\qquad \left|\sum_{i=1}^{r}\left(\frac{a_i t(1+t)^{a_i}}{(1+t)^{a_i}-1}-\frac{t}{\ln(1+t)}\right)x^n\right\rangle\\
&\quad -\sum_{i=1}^{r}a_i\widehat{N}_{n-1}^{(k)}(y-1|a_1,\dots,a_r)\\
&= \frac{1}{n}\sum_{i=1}^{r}a_i\left\langle \frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1}\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\right.\\
&\qquad\qquad \left.\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{y-1}\middle|\frac{t}{\ln(1+t)}x^n\right\rangle\\
&\quad -\frac{r}{n}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{y-1}\middle|\frac{t}{\ln(1+t)}x^n\right\rangle\\
&\quad -\sum_{i=1}^{r}a_i\widehat{N}_{n-1}^{(k)}(y-1|a_1,\dots,a_r)
\end{aligned}
$$

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$$
\begin{aligned}
&= \frac{1}{n}\sum_{i=1}^{r} a_i \left\langle \frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1}\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\right. \\
&\qquad\qquad \left. \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{y-1}\Big| \sum_{l=0}^{\infty} c_l \frac{t^l}{l!}x^n \right\rangle \\
&\quad - \frac{r}{n}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{y-1}\Big| \sum_{l=0}^{\infty} c_l \frac{t^l}{l!}x^n \right\rangle \\
&\quad - \sum_{i=1}^{r} a_i \widehat{N}_{n-1}^{(k)}(y-1|a_1,\dots,a_r) \\
&= \frac{1}{n}\sum_{i=1}^{r} a_i \sum_{l=0}^{n}\binom{n}{l} c_l \widehat{N}_{n-l}^{(k)}(y-1|a_1,\dots,a_{i-1},a_{i+1},\dots,a_r) \\
&\quad - \frac{r}{n}\sum_{l=0}^{n}\binom{n}{l} c_l \widehat{N}_{n-l}^{(k)}(y-1|a_1,\dots,a_r) \\
&\quad - \sum_{i=1}^{r} a_i \widehat{N}_{n-1}^{(k)}(y-1|a_1,\dots,a_r)\,.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
&\widehat{N}_{n}^{(k)}(x|a_1,\dots,a_r) \\
&= x\widehat{N}_{n-1}^{(k)}(x-1|a_1,\dots,a_r) \\
&\quad + \frac{1}{n}\sum_{l=0}^{n-1}\binom{n}{l} c_l \big(\widehat{N}_{n-l}^{(k-1)}(x-1|a_1,\dots,a_r) - \widehat{N}_{n-l}^{(k)}(x-1|a_1,\dots,a_r)\big) \\
&\quad + \frac{1}{n}\sum_{i=1}^{r} a_i \sum_{l=0}^{n}\binom{n}{l} c_l \widehat{N}_{n-l}^{(k)}(x-1|a_1,\dots,a_{i-1},a_{i+1},\dots,a_r) \\
&\quad - \frac{r}{n}\sum_{l=0}^{n}\binom{n}{l} c_l \widehat{N}_{n-l}^{(k)}(x-1|a_1,\dots,a_r) \\
&\quad - \sum_{i=1}^{r} a_i \widehat{N}_{n-1}^{(k)}(x-1|a_1,\dots,a_r) \\
&= \left(x - \sum_{i=1}^{r} a_i\right)\widehat{N}_{n-1}^{(k)}(x-1|a_1,\dots,a_r) \\
&\quad + \frac{1}{n}\sum_{l=0}^{n}\binom{n}{l} c_l \big(\widehat{N}_{n-l}^{(k-1)}(x-1|a_1,\dots,a_r) - (r+1)\widehat{N}_{n-l}^{(k)}(x-1|a_1,\dots,a_r)\big) \\
&\quad + \frac{1}{n}\sum_{i=1}^{r}\sum_{l=0}^{n}\binom{n}{l} a_i c_l \widehat{N}_{n-l}^{(k)}(x-1|a_1,\dots,a_{i-1},a_{i+1},\dots,a_r)\,,
\end{aligned}
$$

which is the identity (26). ∎

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3.7 A relation involving the Stirling numbers of the first kind

Theorem 7 *For $n-1 \geq m \geq 1$, we have*

$$\begin{aligned}
& m\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\widehat{N}_l^{(k)}(a_1,\dots,a_r) \\
&= \frac{m}{n}\sum_{i=0}^{n-m}\sum_{l=0}^{i}\binom{n}{i}\binom{i}{l}S_1(n-i,m)c_{i-l} \\
&\quad\times\left(\sum_{j=1}^{r}a_j\widehat{N}_l^{(k)}(-1|a_1,\dots,a_{j-1},a_{j+1},\dots,a_r)-r\widehat{N}_l^{(k)}(-1|a_1,\dots,a_r)\right) \\
&\quad - m\sum_{j=1}^{r}a_j\sum_{l=0}^{n-m-1}\binom{n-1}{l}S_1(n-1-l,m)\widehat{N}_l^{(k)}(-1|a_1,\dots,a_r) \\
&\quad + \sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-1-l,m-1) \\
&\quad\times\left(\widehat{N}_l^{(k-1)}(-1|a_1,\dots,a_r)+(m-1)\widehat{N}_l^{(k)}(-1|a_1,\dots,a_r)\right). \qquad (27)
\end{aligned}$$

Proof. We shall compute

$$\left\langle\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m\Big|x^n\right\rangle$$

in two different ways. On the one hand, it is equal to

$$\begin{aligned}
&\left\langle\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)\Big|\big(\ln(1+t)\big)^m x^n\right\rangle \\
&= \left\langle\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)\Big|m!\sum_{l=0}^{\infty}S_1(l,m)\frac{t^l}{l!}x^n\right\rangle \\
&= m!\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)\left\langle\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)\Big|x^{n-l}\right\rangle \\
&= m!\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)\left\langle\sum_{i=0}^{\infty}\widehat{N}_i^{(k)}(a_1,\dots,a_r)\frac{t^i}{i!}\Big|x^{n-l}\right\rangle \\
&= m!\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)\widehat{N}_{n-l}^{(k)}(a_1,\dots,a_r) \\
&= m!\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\widehat{N}_l^{(k)}(a_1,\dots,a_r).
\end{aligned}$$

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On the other hand, it is equal to

$$\begin{aligned}
&\left\langle \partial_t \left(\prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)} \right) \mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m \right) \Big| x^{n-1} \right\rangle \\
&= \left\langle \left(\partial_t \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)} \right) \right) \mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m \Big| x^{n-1} \right\rangle \\
&\quad + \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)} \right) \big(\partial_t \mathrm{Lif}_k\big(-\ln(1+t)\big)\big) \big(\ln(1+t)\big)^m \Big| x^{n-1} \right\rangle \\
&\quad + \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)} \right) \mathrm{Lif}_k\big(-\ln(1+t)\big) \big(\partial_t \big(\ln(1+t)\big)^m\big) \Big| x^{n-1} \right\rangle . \qquad (28)
\end{aligned}$$

The third term of (28) is equal to

$$\begin{aligned}
&m \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)} \right) \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1} \Big| \big(\ln(1+t)\big)^{m-1} x^{n-1} \right\rangle \\
&= m \Bigg\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)} \right) \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1} \Big| \\
&\qquad (m-1)! \sum_{l=m-1}^{\infty} S_1(l, m-1) \frac{t^l}{l!} x^{n-1} \Bigg\rangle \\
&= m! \sum_{l=m-1}^{n-1} \binom{n-1}{l} S_1(l, m-1) \\
&\quad \times \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)} \right) \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1} \Big| x^{n-1-l} \right\rangle \\
&= m! \sum_{l=m-1}^{n-1} \binom{n-1}{l} S_1(l, m-1) \widehat{N}_{n-1-l}^{(k)}(-1|a_1, \dots, a_r) \\
&= m! \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) \widehat{N}_{l}^{(k)}(-1|a_1, \dots, a_r) .
\end{aligned}$$

The second term of (28) is equal to

$$\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\left(\frac{\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)-\mathrm{Lif}_k\big(-\ln(1+t)\big)}{(1+t)\ln(1+t)}\right)\big(\ln(1+t)\big)^m \middle| x^{n-1}\right\rangle$$
$$=\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)(1+t)^{-1}\middle|\big(\ln(1+t)\big)^{m-1}x^{n-1}\right\rangle$$
$$-\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_{k}\big(-\ln(1+t)\big)(1+t)^{-1}\middle|\big(\ln(1+t)\big)^{m-1}x^{n-1}\right\rangle$$
$$=(m-1)!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{N}_l^{(k-1)}(-1|a_1,\dots,a_r)$$
$$-(m-1)!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{N}_l^{(k)}(-1|a_1,\dots,a_r)\,.$$

The first term of (28) is equal to

$$\left\langle \left(\partial_t\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m\middle| x^{n-1}\right\rangle$$
$$=\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1}\big(\ln(1+t)\big)^m\middle| \frac{\sum_{j=1}^{r}\left(\frac{a_jt(1+t)^{a_j}}{(1+t)^{a_j}-1}-\frac{t}{\ln(1+t)}\right)}{t}x^{n-1}\right\rangle$$
$$-\sum_{j=1}^{r}a_j\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1}\big(\ln(1+t)\big)^m\middle| x^{n-1}\right\rangle$$
$$=\frac{1}{n}\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1}\middle| \sum_{j=1}^{r}\left(\frac{a_jt(1+t)^{a_j}}{(1+t)^{a_j}-1}-\frac{t}{\ln(1+t)}\right)\big(\ln(1+t)\big)^m x^n\right\rangle$$
$$-\sum_{j=1}^{r}a_j\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1}\middle|\big(\ln(1+t)\big)^m x^{n-1}\right\rangle$$

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$$
\begin{aligned}
&= \frac{1}{n}\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1}\right| \\
&\qquad \left.\sum_{j=1}^{r}\left(\frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j}-1}-\frac{t}{\ln(1+t)}\right)m!\sum_{i=m}^{\infty}S_1(i,m)\frac{t^i}{i!}x^n\right\rangle \\
&\quad -\sum_{j=1}^{r}a_j\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1}\middle| m!\sum_{i=m}^{\infty}S_1(i,m)\frac{t^i}{i!}x^{n-1}\right\rangle \\
&= \frac{m!}{n}\sum_{i=m}^{n}\binom{n}{i}S_1(i,m)\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1}\right. \\
&\qquad \left.\sum_{j=1}^{r}\left(\frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j}-1}-\frac{t}{\ln(1+t)}\right)\middle| x^{n-i}\right\rangle \\
&\quad - m!\sum_{j=1}^{r}a_j\sum_{i=m}^{n-1}\binom{n-1}{i}S_1(i,m) \\
&\qquad \times\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1}\middle| x^{n-1-i}\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{m!}{n}\sum_{i=m}^{n}\binom{n}{i}S_1(i,m)\left(\sum_{j=1}^{r}a_j\left\langle \frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\right.\right. \\
&\qquad \left.\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1}\middle|\frac{t}{\ln(1+t)}x^{n-i}\right\rangle \\
&\quad \left.-r\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1}\middle|\frac{t}{\ln(1+t)}x^{n-i}\right\rangle\right) \\
&\quad - m!\sum_{j=1}^{r}a_j\sum_{i=m}^{n-1}\binom{n-1}{i}S_1(i,m)\widehat{N}_{n-1-i}^{(k)}(-1|a_1,\ldots,a_r) \\
&= \frac{m!}{n}\sum_{i=m}^{n}\binom{n}{i}S_1(i,m)\sum_{l=0}^{n-i}\binom{n-i}{l}c_l \\
&\qquad \times\left(\sum_{j=1}^{r}a_j\widehat{N}_{n-i-l}^{(k)}(-1|a_1,\ldots,a_{j-1},a_{j+1},\ldots,a_r)-r\widehat{N}_{n-i-l}^{(k)}(-1|a_1,\ldots,a_r)\right) \\
&\quad - m!\sum_{j=1}^{r}a_j\sum_{i=m}^{n-1}\binom{n-1}{i}S_1(i,m)\widehat{N}_{n-1-i}^{(k)}(-1|a_1,\ldots,a_r)
\end{aligned}
$$

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$$
\begin{aligned}
&= \frac{m!}{n} \sum_{i=m}^{n} \sum_{l=0}^{n-i} \binom{n}{i} \binom{n-i}{l} S_1(i,m) c_{n-i-l} \\
&\quad \times \left(\sum_{j=1}^{r} a_j \widehat{N}_l^{(k)}(-1|a_1,\ldots,a_{j-1},a_{j+1},\ldots,a_r) - r\widehat{N}_l^{(k)}(-1|a_1,\ldots,a_r) \right) \\
&\quad - m! \sum_{j=1}^{r} a_j \sum_{i=0}^{n-m-1} \binom{n-1}{i} S_1(n-1-i,m) \widehat{N}_i^{(k)}(-1|a_1,\ldots,a_r) \\
&= \frac{m!}{n} \sum_{i=0}^{n-m} \sum_{l=0}^{i} \binom{n}{i} \binom{i}{l} S_1(n-i,m) c_{i-l} \\
&\quad \times \left(\sum_{j=1}^{r} a_j \widehat{N}_l^{(k)}(-1|a_1,\ldots,a_{j-1},a_{j+1},\ldots,a_r) - r\widehat{N}_l^{(k)}(-1|a_1,\ldots,a_r) \right) \\
&\quad - m! \sum_{j=1}^{r} a_j \sum_{i=0}^{n-m-1} \binom{n-1}{i} S_1(n-1-i,m) \widehat{N}_i^{(k)}(-1|a_1,\ldots,a_r) .
\end{aligned}
$$

Therefore, we get for $n-1 \geq m \geq 1$

$$
\begin{aligned}
& m! \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l,m) \widehat{N}_l^{(k)}(a_1,\ldots,a_r) \\
&= \frac{m!}{n} \sum_{i=0}^{n-m} \sum_{l=0}^{i} \binom{n}{i} \binom{i}{l} S_1(n-i,m) c_{i-l} \\
&\quad \times \left(\sum_{j=1}^{r} a_j \widehat{N}_l^{(k)}(-1|a_1,\ldots,a_{j-1},a_{j+1},\ldots,a_r) - r\widehat{N}_l^{(k)}(-1|a_1,\ldots,a_r) \right) \\
&\quad - m! \sum_{j=1}^{r} a_j \sum_{i=0}^{n-m-1} \binom{n-1}{i} S_1(n-1-i,m) \widehat{N}_i^{(k)}(-1|a_1,\ldots,a_r) \\
&\quad + (m-1)! \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-1-l,m-1) \widehat{N}_l^{(k-1)}(-1|a_1,\ldots,a_r) \\
&\quad - (m-1)! \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-1-l,m-1) \widehat{N}_l^{(k)}(-1|a_1,\ldots,a_r) \\
&\quad + m! \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-1-l,m-1) \widehat{N}_l^{(k)}(-1|a_1,\ldots,a_r) .
\end{aligned}
$$

Dividing both sides by $(m-1)!$, we obtain, for $n-1 \geq m \geq 1$,

$$
\begin{aligned}
& m \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l,m) \widehat{N}_l^{(k)}(a_1,\dots,a_r) \\
&= \frac{m}{n} \sum_{i=0}^{n-m} \sum_{l=0}^{i} \binom{n}{i} \binom{i}{l} S_1(n-i,m) c_{i-l} \\
&\quad \times \left(\sum_{j=1}^{r} a_j \widehat{N}_l^{(k)}(-1|a_1,\dots,a_{j-1},a_{j+1},\dots,a_r) - r \widehat{N}_l^{(k)}(-1|a_1,\dots,a_r) \right) \\
&\quad - m \sum_{j=1}^{r} a_j \sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1(n-1-l,m) \widehat{N}_l^{(k)}(-1|a_1,\dots,a_r) \\
&\quad + \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-1-l,m-1) \\
&\quad \times \left(\widehat{N}_l^{(k-1)}(-1|a_1,\dots,a_r) + (m-1) \widehat{N}_l^{(k)}(-1|a_1,\dots,a_r) \right).
\end{aligned}
$$

Thus, we get (27). ∎

3.8 A relation with the falling factorials

Theorem 8

$$
\widehat{N}_n^{(k)}(x|a_1,\dots,a_r) = \sum_{m=0}^{n} \binom{n}{m} \widehat{N}_{n-m}^{(k)}(a_1,\dots,a_r)(x)_m . \tag{29}
$$

Proof. For (12) and (18), assume that $\widehat{N}_n^{(k)}(x|a_1,\dots,a_r) = \sum_{m=0}^{n} C_{n,m}(x)_m$. By (11), we have

$$
\begin{aligned}
C_{n,m} &= \frac{1}{m!} \left\langle \frac{1}{\prod_{j=1}^{r} \left(\frac{\ln(1+t) e^{a_j \ln(1+t)}}{e^{a_j \ln(1+t)}-1} \right)} \mathrm{Lif}_k\big(-\ln(1+t)\big) t^m \middle| x^n \right\rangle \\
&= \frac{1}{m!} \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j} \ln(1+t)} \right) \mathrm{Lif}_k\big(-\ln(1+t)\big) \middle| t^m x^n \right\rangle \\
&= \binom{n}{m} \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j} \ln(1+t)} \right) \mathrm{Lif}_k\big(-\ln(1+t)\big) \middle| x^{n-m} \right\rangle \\
&= \binom{n}{m} \widehat{N}_{n-m}^{(k)}(a_1,\dots,a_r).
\end{aligned}
$$

Thus, we get the identity (29). ∎

3.9 A relation with higher-order Frobenius-Euler polynomials

For $\lambda \in \mathbb{C}$ with $\lambda \neq 1$, the Frobenius-Euler polynomials of order r, $H_n^{(r)}(x|\lambda)$ are defined by the generating function

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|\lambda)\frac{t^n}{n!}$$

(see e.g. [6]).

Theorem 9

$$\widehat{N}_n^{(k)}(x|a_1,\dots,a_r) = \sum_{m=0}^{n}\left(\sum_{j=0}^{n-m}\sum_{l=0}^{n-m-j}\binom{s}{j}\binom{n-j}{l}(n)_j \right.$$
$$\left. \times(1-\lambda)^{-j}S_1(n-j-l,m)\widehat{N}_l^{(k)}(a_1,\dots,a_r)\right) H_m^{(s)}(x|\lambda)\,. \tag{30}$$

Proof. For (12) and

$$H_n^{(s)}(x|\lambda) \sim \left(\left(\frac{e^t-\lambda}{1-\lambda}\right)^s, t\right)\,, \tag{31}$$

assume that $\widehat{N}_n^{(k)}(x|a_1,\dots,a_r) = \sum_{m=0}^{n} C_{n,m} H_m^{(s)}(x|\lambda)$. By (11), similarly to the proof of (27), we have

$$C_{n,m} = \frac{1}{m!}\left\langle \frac{\left(\frac{e^{\ln(1+t)}-\lambda}{1-\lambda}\right)^s}{\prod_{j=1}^r\left(\frac{\ln(1+t)e^{a_j\ln(1+t)}}{e^{a_j\ln(1+t)}-1}\right)}\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)\bigl(\ln(1+t)\bigr)^m\Big| x^n\right\rangle$$
$$= \frac{1}{m!(1-\lambda)^s}\left\langle \prod_{j=1}^r\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)\bigl(\ln(1+t)\bigr)^m(1-\lambda+t)^s\Big| x^n\right\rangle$$
$$= \frac{1}{m!(1-\lambda)^s}$$
$$\times\left\langle \prod_{j=1}^r\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)\bigl(\ln(1+t)\bigr)^m\Big| \sum_{i=0}^{\min\{s,n\}}\binom{s}{i}(1-\lambda)^{s-i}t^i x^n\right\rangle$$
$$= \frac{1}{m!(1-\lambda)^s}\sum_{i=0}^{n-m}\binom{s}{i}(1-\lambda)^{s-i}(n)_i$$
$$\times\left\langle \prod_{j=1}^r\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)\bigl(\ln(1+t)\bigr)^m\Big| x^{n-i}\right\rangle$$
$$= \frac{1}{m!(1-\lambda)^s}\sum_{i=0}^{n-m}\binom{s}{i}(1-\lambda)^{s-i}(n)_i\sum_{l=0}^{n-m-i} m!\binom{n-i}{l}S_1(n-i-l,m)\widehat{N}_l^{(k)}(a_1,\dots,a_r)$$
$$= \sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{s}{i}\binom{n-i}{l}(n)_i(1-\lambda)^{-i}S_1(n-i-l,m)\widehat{N}_l^{(k)}(a_1,\dots,a_r)\,.$$

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Thus, we get the identity (30). ∎

3.10 A relation with higher-order Bernoulli polynomials

Bernoulli polynomials $\mathfrak{B}_n^{(r)}(x)$ of order r are defined by

$$\left(\frac{t}{e^t-1}\right)^r e^{xt}=\sum_{n=0}^\infty\frac{\mathfrak{B}_n^{(r)}(x)}{n!}t^n$$

(see e.g. [14, Section 2.2]). In addition, Cauchy numbers of the first kind $\mathfrak{C}_n^{(r)}$ of order r are defined by

$$\left(\frac{t}{\ln(1+t)}\right)^r=\sum_{n=0}^\infty\frac{\mathfrak{C}_n^{(r)}}{n!}t^n$$

(see e.g. [2, (2.1)], [13, (6)]).

Theorem 10

$$\begin{aligned}&\widehat{N}_n^{(k)}(x|a_1,\dots,a_r)\\&=\sum_{m=0}^n\left(\sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{n}{i}\binom{n-i}{l}\mathfrak{C}_i^{(s)}S_1(n-i-l,m)\widehat{N}_l^{(k)}(a_1,\dots,a_r)\right)\mathfrak{B}_m^{(s)}(x)\,.\end{aligned}\tag{32}$$

Proof. For (12) and

$$\mathfrak{B}_n^{(s)}(x)\sim\left(\left(\frac{e^t-1}{t}\right)^s,t\right)\,,\tag{33}$$

assume that $\widehat{N}_n^{(k)}(x|a_1,\dots,a_r)=\sum_{m=0}^n C_{n,m}\mathfrak{B}_m^{(s)}(x)$. By (11), similarly to the proof of

(27), we have

$$
\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \frac{\left(\frac{e^{\ln(1+t)}-1}{\ln(1+t)}\right)^s}{\prod_{j=1}^r \left(\frac{\ln(1+t)e^{a_j\ln(1+t)}}{e^{a_j\ln(1+t)}-1}\right)} \mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m \middle| x^n \right\rangle \\
&= \frac{1}{m!}\left\langle \prod_{j=1}^r \left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right) \mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m \middle| \left(\frac{t}{\ln(1+t)}\right)^s x^n \right\rangle \\
&= \frac{1}{m!}\left\langle \prod_{j=1}^r \left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right) \mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m \middle| \sum_{i=0}^{\infty} \mathfrak{C}_i^{(s)}\frac{t^i}{i!}x^n \right\rangle \\
&= \frac{1}{m!}\sum_{i=0}^{n-m}\mathfrak{C}_i^{(s)}\binom{n}{i}\left\langle \prod_{j=1}^r \left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right) \mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m \middle| x^{n-i} \right\rangle \\
&= \frac{1}{m!}\sum_{i=0}^{n-m}\mathfrak{C}_i^{(s)}\binom{n}{i}\sum_{l=0}^{n-m-i} m!\binom{n-i}{l} S_1(n-i-l,m)\widehat{N}_l^{(k)}(a_1,\dots,a_r) \\
&= \sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{n}{i}\binom{n-i}{l}\mathfrak{C}_i^{(s)} S_1(n-i-l,m)\widehat{N}_l^{(k)}(a_1,\dots,a_r)\,.
\end{aligned}
$$

Thus, we get the identity (32). ∎

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SUPERSTABILITY AND STABILITY OF (r,s,t)-J^*-HOMOMORPHISMS: FIXED POINT AND DIRECT METHODS

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Abstract. In this paper, we introduce the following useful functional equations:

$$f(x+y)+f(x-2y)+f(y-x)=f(x), \tag{0.1}$$

$$f\left(\frac{\sum_{i=1}^{p} x_i}{p-1}\right)+\sum_{i=2}^{p} f\left(\frac{\sum_{\substack{j=1\\ j\neq i}}^{p} x_j - p x_i}{p-1}\right)+f\left(\frac{\sum_{i=2}^{p} x_i - x_1}{p-1}\right)=f(x_1) \tag{0.2}$$

and prove the superstability and the Hyers-Ulam stability of (r,s,t)-J^*-homomorphisms, associated with those, by using the fixed point method and the direct method.

1. Introduction and preliminaries

The stability of functional equations originated from a question of Ulam [54] in 1940. He proposed the following question "when and under what condition does an exact solution of a functional equation near an approximately solution of that exist?" A next year, this question was formulated and answered by Hyers [26] affirmatively, for Cauchy's additive equation on Banach spaces. In 1950, Aoki [1] was the second author to study this problem. In 1978, Rassias [49] obtained a generalization of the result of Hyers by considering the stability problem with unbounded Cauchy differences. For more epochal information and various aspects about the stability of functional equations theory, we refer the reader to monographs (cf. [2, 7, 9, 10, 12, 14, 21, 27, 29, 30, 31, 32, 33, 34, 35, 39, 41, 42, 46, 50, 51, 52, 53]), which also include many interesting results concerning the stability of different functional equations.

We say a functional equation (ξ) is *stable* if any function g satisfying the equation (ξ) *approximately* is near to true solution of (ξ). We say that a functional equation is *superstable* if every approximately solution is an exact solution of that [51].

Throughout this paper, $\mathcal{A}$ and $\mathcal{B}$ denote J^*-algebras and $\{r,s,t\}$ are positive integer constants. The notion of J^*-algebras has been posed by Harris [22] in 1974. By a J^*-algebra we mean a closed subspace $\mathcal{A}$ of a C^*-algebra such that $xx^*x \in \mathcal{A}$ whenever $x \in \mathcal{A}$ [22]. For more study about J^*-algebras, one can refer to (cf. [11, 22, 23, 24, 25]). Moreover, we introduce (r,s,t)-J^*-homomorphisms and (r,s,t)-J^*-derivations, which are an extension of J^*-derivations and J^*-homomorphisms (see [19, 44, 45]).

Definition 1.1. A linear mapping $h:\mathcal{A}\to\mathcal{B}$ is called an (r,s,t)-J^*-*homomorphism* if

$$h(x^r x^{*s} x^t) = h(x)^r h(x)^{*s} h(x)^t$$

for all $x\in\mathcal{A}$, and if $r=s=t=n$, then $h:\mathcal{A}\to\mathcal{B}$ is called an n-J^*-*homomorphism*.

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Definition 1.2. A linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called an (r,s,t)-J^*-*derivation* if

$$\delta(x^r x^{*s} x^t) = \delta(x)^r x^{*s} x^t + x^r \delta(x)^{*s} x^t + x^r x^{*s} \delta(x)^t$$

for all $x \in \mathcal{A}$, and if $r = s = t = n$, then $\delta : \mathcal{A} \to \mathcal{A}$ is called an n-J^*-*derivation.*

With $n = 1$, we have the definitions of J^*-homomorphisms and J^*-derivations.

We will use the following definition and fundamental result of fixed point theory:

Definition 1.3. ([3, 4, 5, 6]) Let $\mathcal{X}$ be a set. A function $d : \mathcal{X} \times \mathcal{X} \to [0, \infty]$ is called a *generalized metric* on $\mathcal{X}$ if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$;
(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in \mathcal{X}$.

Theorem 1.4. ([3, 4, 5, 6]) *Let $(\mathcal{X}, d)$ be a complete generalized metric space and let $\mathcal{J} : \mathcal{X} \to \mathcal{X}$ be a strictly contractive mapping with Lipschitz constant $\mathcal{L} < 1$. Then for each given element $x \in \mathcal{X}$, either*

$$d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) < \infty, \qquad \forall n \ge n_0$*;*
(2) *the sequence $\{\mathcal{J}^n x\}$ converges to a fixed point y^* of $\mathcal{J}$;*
(3) *y^* is the unique fixed point of $\mathcal{J}$ in the set $\mathcal{Y} = \{y \in \mathcal{X} \mid d(\mathcal{J}^{n_0} x, y) < \infty\}$;*
(4) $d(y, y^*) \le \frac{1}{1-\mathcal{L}} d(y, \mathcal{J} y)$ *for all $y \in \mathcal{Y}$.*

This theorem was used by Cădariu and Radu (see [3, 4, 5, 47]) and then others to obtain the applications of fixed point theory in stability problems (cf. [8, 13, 15, 16, 17, 18, 19, 20, 28, 36, 38, 39, 40, 43, 48]).

Now consider the functional equation (0.2), which is a generalized version of the functional equation (0.1). In this paper, in order to investigate the functional equation (0.2), we will suppose that $p \ge 3$.

2. Superstability of (r,s,t)-J^*-homomorphisms

In this section, we prove the superstability of (r,s,t)-J^*-homomorphisms associated with the functional equation (0.2).

For the proof of our results, we first give some useful lemmas.

Lemma 2.1. ([37]) *Let $\mathcal{X}$ and $\mathcal{Y}$ be linear spaces and let $f : \mathcal{X} \to \mathcal{Y}$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $x \in \mathcal{X}$. Then the mapping f is $\mathbb{C}$-linear.*

Lemma 2.2. *Let $n \ge 2$ be a fixed positive integer and $f : \mathcal{A} \to \mathcal{B}$ be a mapping such that*

$$\begin{aligned} &\left\|(n-1) f\left(\frac{x+y+z}{n}\right) + f\left(\frac{x+z-(n+1)y}{n}\right) + f\left(\frac{x+y-(n+1)z}{n}\right)\right\|_{\mathcal{B}} \\ &\qquad \le \left\| f(x) - f\left(\frac{y+z-x}{n}\right)\right\|_{\mathcal{B}} \end{aligned} \tag{2.1}$$

for all $x, y, z \in \mathcal{A}$. Then f is additive.

Proof. From (2.1), it follows that $f(0) = 0$. Putting $x = 0,\ y = x,\ z = -x$ in (2.1), we have $f\left(-\frac{n+2}{n} x\right) + f\left(\frac{n+2}{n} x\right) = 0$ for all $x \in \mathcal{A}$. So $f(-x) = -f(x)$ for all $x \in \mathcal{A}$. Replacing x, y and z by $\frac{x+y}{n+1},\ x$ and y in (2.1), respectively, we get the equality

$$(n-1) f\left(\frac{n+2}{n(n+1)}(x+y)\right) = f\left(\frac{n+2}{n+1} x - \frac{n+2}{n(n+1)} y\right) + f\left(\frac{n+2}{n+1} y - \frac{n+2}{n(n+1)} x\right)$$

for all $x, y \in \mathcal{A}$. By putting $u = \frac{n+2}{n+1}x - \frac{n+2}{n(n+1)}y$ and $v = \frac{n+2}{n+1}y - \frac{n+2}{n(n+1)}x$, we conclude that

$$(n-1)f\left(\frac{1}{n-1}(u+v)\right) = f(u) + f(v)$$

for all $u, v \in \mathcal{A}$. Letting $v = 0$, we see that $(n-1)f\left(\frac{1}{n-1}u\right) = f(u)$ and so $f(u+v) = f(u)+f(v)$ for all $u, v \in \mathcal{A}$. □

Lemma 2.3. *Let $p \geq 3$ be a fixed positive integer and $f : \mathcal{A} \to \mathcal{B}$ be a mapping such that*

$$\left\| f\left(\frac{\sum_{i=1}^p x_i}{p-1}\right) + \sum_{i=2}^p f\left(\frac{\sum_{\substack{j=1\\ j\neq i}}^p x_j - px_i}{p-1}\right)\right\|_{\mathcal{B}} \leq \left\| f(x_1) - f\left(\frac{\sum_{i=2}^p x_i - x_1}{p-1}\right)\right\|_{\mathcal{B}} \tag{2.2}$$

for all $x_1, \cdots, x_p \in \mathcal{A}$. Then f is additive.

Proof. By (2.2), we have $f(0) = 0$. Letting $x_1 = x,\ x_2 = y,\ x_3 = z$ and $x_4 = \cdots = x_p = 0$ in (2.2), we obtain

$$\left\|(p-2)f\left(\frac{x+y+z}{p-1}\right) + f\left(\frac{x+z-py}{p-1}\right) + f\left(\frac{x+y-pz}{p-1}\right)\right\|_{\mathcal{B}} \leq \left\| f(x) - f\left(\frac{y+z-x}{p-1}\right)\right\|_{\mathcal{B}}$$

for all $x, y, z \in \mathcal{A}$, which is (2.1) for the case $n = p-1 \geq 2$. Therefore f is additive, as desired. □

Theorem 2.4. *Let $p \geq 3$ be a fixed positive integer and $\varphi : \mathcal{A}^p \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty} b^{(r+s+t)n}\varphi(b^{-n}x, \cdots, b^{-n}x) = 0$$

for all $x \in \mathcal{A}$, where $b \neq 1$ is a real number. Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying

$$\left\|\mu f\left(\frac{\sum_{i=1}^p x_i}{p-1}\right) + \sum_{i=2}^p f\left(\mu\frac{\sum_{\substack{j=1\\ j\neq i}}^p x_j - px_i}{p-1}\right)\right\|_{\mathcal{B}} \leq \left\| f(\mu x_1) - f\left(\mu\frac{\sum_{i=2}^p x_i - x_1}{p-1}\right)\right\|_{\mathcal{B}} \tag{2.3}$$

$$\|f(x^r x^{*s} x^t) - f(x)^r f(x)^{*s} f(x)^t\|_{\mathcal{B}} \leq \varphi(x, \cdots, x) \tag{2.4}$$

for all $\mu \in \mathbb{T}^1$ and all $x, x_1, \cdots, x_p \in \mathcal{A}$. Then the mapping $f : \mathcal{A} \to \mathcal{B}$ is an (r,s,t)-J^*-homomorphism.

Proof. Let $\mu = 1$ in (2.3). By Lemma 2.3, the mapping $f : \mathcal{A} \to \mathcal{B}$ is additive. From (2.3), for $x_1 = x_3 = \cdots = x_p = x$ and $x_2 = 2x$, we have

$$\frac{p+1}{p-1}\|\mu f(x) - f(\mu x)\|_{\mathcal{B}} = \left\|\mu f\left(\frac{p+1}{p-1}x\right) + f\left(-\mu\frac{p+1}{p-1}\right)\right\|_{\mathcal{B}} \leq \|0\|_{\mathcal{B}} = 0$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. Hence $f(\mu x) = \mu f(x)$ for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. By Lemma 2.1, the mapping $f : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear. From (2.4) and the assumption on φ, it follows that

$$\begin{aligned}
&\|f(x^r x^{*s} x^t) - f(x)^r f(x)^{*s} f(x)^t\|_{\mathcal{B}} \\
&= \lim_{n\to\infty} b^{(r+s+t)n}\left\| f\left(\left(\frac{x}{b^n}\right)^r \left(\frac{x}{b^n}\right)^{*s} \left(\frac{x}{b^n}\right)^t\right) - f\left(\frac{x}{b^n}\right)^r f\left(\frac{x}{b^n}\right)^{*s} f\left(\frac{x}{b^n}\right)^t\right\|_{\mathcal{B}} \\
&\leq \lim_{n\to\infty} b^{(r+s+t)n}\varphi\left(\frac{x}{b^n}, \cdots, \frac{x}{b^n}\right) = 0
\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. Hence $f(x^r x^{*s} x^t) = f(x)^r f(x)^{*s} f(x)^t$ for all $x \in \mathcal{A}$. □

Corollary 2.5. *Let θ be a nonnegative real number and $q_1, \cdots, q_p$ be positive real numbers such that $q_1, \cdots, q_p > r+s+t$ or $q_1, \cdots, q_p < r+s+t$. Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying (2.3) and*

$$\|f(x^r x^{*s} x^t) - f(x)^r f(x)^{*s} f(x)^t\|_{\mathcal{B}} \leq \theta \left(\|x\|_{\mathcal{A}}^{q_1} + \cdots + \|x\|_{\mathcal{A}}^{q_p}\right) \tag{2.5}$$

for all $x \in \mathcal{A}$. Then the mapping $f : \mathcal{A} \to \mathcal{B}$ is an (r, s, t)-J^*-homomorphism.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x_1, \cdots, x_p) := \theta\left(\|x_1\|_{\mathcal{A}}^{q_1} + \cdots + \|x_p\|_{\mathcal{A}}^{q_p}\right)$ with $b > 1$ for the case $q_1, \cdots, q_p > r+s+t$ and with $b < 1$ for the case $q_1, \cdots, q_p < r+s+t$. $\square$

Corollary 2.6. *Let θ be a nonnegative real number and $q_1, \cdots, q_p$ be positive real numbers such that $q_1 + \cdots + q_p \neq r+s+t$. Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying (2.3) and*

$$\|f(x^r x^{*s} x^t) - f(x)^r f(x)^{*s} f(x)^t\|_{\mathcal{B}} \leq \theta \|x\|_{\mathcal{A}}^{q_1 + \cdots + q_p}$$

for all $x \in \mathcal{A}$. Then the mapping $f : \mathcal{A} \to \mathcal{B}$ is an (r, s, t)-J^*-homomorphism.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x_1, \cdots, x_p) := \theta\left(\|x_1\|_{\mathcal{A}}^{q_1} \cdots \|x_p\|_{\mathcal{A}}^{q_p}\right)$ with $b > 1$ for the case $q_1 + \cdots + q_p > r+s+t$ and with $b < 1$ for the case $q_1 + \cdots + q_p < r+s+t$. $\square$

3. Hyers-Ulam stability of (r, s, t)-J^*-homomorphisms: fixed point method

In this section, by using the fixed point method, we prove the Hyers-Ulam stability of (r, s, t)-J^*-homomorphisms associated with the functional equation (0.2).

For a given mapping $f : \mathcal{A} \to \mathcal{B}$, we define

$$\varrho_\mu f(x_1, \cdots, x_p) := f\left(\mu \frac{\sum_{i=1}^p x_i}{p-1}\right) + \sum_{i=2}^p f\left(\mu \frac{\sum_{\substack{j=1 \\ j \neq i}}^p x_j - p x_i}{p-1}\right) + f\left(\mu \frac{\sum_{i=2}^p x_i - x_1}{p-1}\right) - \mu f(x_1)$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \cdots, x_p \in \mathcal{A}$.

Lemma 3.1. *The mapping $f : \mathcal{A} \to \mathcal{B}$ is a $\mathbb{C}$-linear mapping if and only if*

$$\varrho_\mu f(x_1, \cdots, x_p) = 0$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \cdots, x_p \in \mathcal{A}$.

Proof. The proof is easy and thus omitted. $\square$

In the following theorems, we will except the case $p = 3$. This case will be considered individually.

Theorem 3.2. *Let $\varphi : \mathcal{A}^p \to [0, \infty)$ be a function with $\varphi(0, \cdots, 0) = 0$ and $p \neq 3$ such that there exists an $\mathcal{L} < 1$ with*

$$\varphi(x_1, \cdots, x_p) < \frac{\mathcal{L}}{k} \varphi(kx_1, \cdots, kx_p) \tag{3.1}$$

for all $x_1, \cdots, x_p \in \mathcal{A}$, where $k = \frac{2}{p-1}$. Suppose that $f : \mathcal{A} \to \mathcal{B}$ is an odd mapping satisfying (2.4) and

$$\|\varrho_\mu f(x_1, \cdots, x_p)\|_{\mathcal{B}} \leq \varphi(x_1, \cdots, x_p) \tag{3.2}$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \cdots, x_p \in \mathcal{A}$. Then there exists a unique (r, s, t)-J^*-homomorphism $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ such that

$$\|f(x) - \mathcal{H}(x)\|_{\mathcal{B}} \leq \frac{\mathcal{L}}{2(1-\mathcal{L})} \varphi(0, x, \cdots, x) \tag{3.3}$$

for all $x \in \mathcal{A}$.

Proof. We first consider the set $\mathcal{S} := \{g : \mathcal{A} \to \mathcal{B}\}$ and introduce the generalized metric d as follows:

$$d(g,h) = \inf_{x \in \mathcal{A}} \left\{ \mathcal{C} \in \mathbb{R}^+ : \quad \|g(x) - h(x)\|_{\mathcal{B}} \le \mathcal{C}\varphi(0, x, \cdots, x) \right\}.$$

It is easy to show that $(\mathcal{S}, d)$ is complete (see the proof of [35, Lemma 2.1]). Now we define the linear mapping $\mathcal{J} : \mathcal{S} \to \mathcal{S}$ such that

$$\mathcal{J}(g(x)) := kg\left(\frac{x}{k}\right)$$

for all $x \in \mathcal{A}$. From (3.2), we can get $f(0) = 0$. By letting $\mu = 1,\ x_1 = 0$ and $x_2 = \cdots = x_p = x$ in (3.2) and the fact that $f(-x) = -f(x)$, (f is an odd mapping) and then by (3.1), we have

$$\begin{aligned} \left\| 2f(x) + (p-1)f\left(\frac{-2}{p-1}x\right) \right\|_{\mathcal{B}} &\le \varphi(0, x, \cdots, x), \\ \left\| kf\left(\frac{x}{k}\right) - f(x) \right\|_{\mathcal{B}} &\le \frac{k}{2}\varphi\left(0, \frac{x}{k}, \cdots, \frac{x}{k}\right) \le \frac{\mathcal{L}}{2}\varphi(0, x, \cdots, x) \end{aligned}$$

for all $x \in \mathcal{A}$. This means that

$$d(\mathcal{J}(f), f) \le \frac{\mathcal{L}}{2} \tag{3.4}$$

Assume that $g, h \in \mathcal{S}$ are given with $d(g,h) = \varepsilon$. Then we have

$$\begin{aligned} \|\mathcal{J}(g(x)) - \mathcal{J}(h(x))\|_{\mathcal{B}} &= k\left\| g\left(\frac{x}{k}\right) - h\left(\frac{x}{k}\right) \right\|_{\mathcal{B}} \le k\varepsilon\varphi\left(0, \frac{x}{k}, \cdots, \frac{x}{k}\right) \\ &< \mathcal{L}\varepsilon\varphi(0, x, \cdots, x) \end{aligned}$$

for all $x \in \mathcal{A}$. This implies that $d(\mathcal{J}(g), \mathcal{J}(h)) < \mathcal{L}\varepsilon = \mathcal{L}d(g,h)$, which means $\mathcal{J}$ is a strictly contractive mapping.

By Theorem 1.4, we have the following:

(1) $\mathcal{J}$ has a fixed point, i.e., there exists a mapping $\mathcal{H} : \mathcal{A} \to \mathcal{B}$, such that $\mathcal{J}(\mathcal{H}) = \mathcal{H}$. So

$$\mathcal{H}(x) = k\mathcal{H}\left(\frac{x}{k}\right) \tag{3.5}$$

for all $x \in \mathcal{A}$. The mapping $\mathcal{H}$ is also the unique fixed point of $\mathcal{J}$ in the set

$$\mathcal{M} = \{g \in \mathcal{S} : d(f,g) < \infty\}.$$

This signifies that $\mathcal{H}$ is a unique mapping satisfying (3.5), moreover there exists a $\mathcal{C} \in (0, \infty)$ such that

$$\|f(x) - \mathcal{H}(x)\|_{\mathcal{B}} \le \mathcal{C}\varphi(0, x, \cdots, x)$$

for all $x \in \mathcal{A}$;

(2) The sequence $\{\mathcal{J}^n(g)\}$ converges to $\mathcal{H}$, for each given $g \in \mathcal{S}$. Thus $d(\mathcal{J}^n(f), \mathcal{H}) \to 0$, as $n \to \infty$. This implies the equality

$$\mathcal{H}(x) = \lim_{n \to \infty} k^n f\left(\frac{x}{k^n}\right)$$

for all $x \in \mathcal{A}$;

(3) $d(g, \mathcal{H}) \le \frac{1}{1-\mathcal{L}} d(g, \mathcal{J}(g))$, for all $g \in \mathcal{M}$. Therefore (3.4) shows us that

$$d(f, \mathcal{H}) \le \frac{1}{1-\mathcal{L}} d(f, \mathcal{J}(f)) \le \frac{\mathcal{L}}{2(1-\mathcal{L})}.$$

By this, we get the inequality (3.3).

It follows from (3.1) that

$$\begin{aligned} \|\varrho_\mu h(x_1,\cdots,x_p)\|_{\mathcal{B}} &= k^n \left\|\varrho_\mu f\left(\frac{x_1}{k^n},\cdots,\frac{x_p}{k^n}\right)\right\|_{\mathcal{B}} \le k^n\varphi\left(\frac{x_1}{k^n},\cdots,\frac{x_p}{k^n}\right) \\ &< \mathcal{L}^n\varphi\left(x_1,\cdots,x_p\right) \end{aligned}$$

for all $x_1,\cdots,x_p\in\mathcal{A}$, in which the right-hand side tends to zero as $n\to\infty$. Hence by Lemma 3.1, we deduce that $\mathcal{H}$ is $\mathbb{C}$-linear.

By (3.1) and (2.4), we obtain

$$\begin{aligned} &\left\|h(x^r x^{*s} x^t) - h(x)^r h(x)^{*s} h(x)^t\right\|_{\mathcal{B}} \\ &= k^{(r+s+t)n}\left\|f\left(\left(\frac{x}{k^n}\right)^r\left(\frac{x}{k^n}\right)^{*s}\left(\frac{x}{k^n}\right)^t\right) - f\left(\frac{x}{k^n}\right)^r f\left(\frac{x}{k^n}\right)^{*s} f\left(\frac{x}{k^n}\right)^t\right\|_{\mathcal{B}} \\ &\le k^{(r+s+t)n}\varphi\left(\frac{x}{k^n},\cdots,\frac{x}{k^n}\right) < \mathcal{L}^{(r+s+t)n}\varphi\left(x,\cdots,x\right) \end{aligned}$$

for all $x\in\mathcal{A}$. The right-hand side tends to zero as $n\to\infty$, and so the mapping $\mathcal{H}:\mathcal{A}\to\mathcal{B}$ is an (r,s,t)-J^*-homomorphism, as desired. □

Theorem 3.3. *Let $\varphi:\mathcal{A}^p\to[0,\infty)$ be a function with $\varphi(0,\cdots,0)=0$ and $p\neq 3$ such that there exists an $\mathcal{L}<1$ with*

$$\varphi(x_1,\cdots,x_p) < k\mathcal{L}\varphi\left(\frac{x_1}{k},\cdots,\frac{x_p}{k}\right) \tag{3.6}$$

for all $x_1,\cdots,x_p\in\mathcal{A}$, where $k=\frac{2}{p-1}$. Suppose that $f:\mathcal{A}\to\mathcal{B}$ is an odd mapping satisfying (3.2) and (2.4). Then there exists a unique (r,s,t)-J^*-homomorphism $\mathcal{H}:\mathcal{A}\to\mathcal{B}$ such that

$$\|f(x)-\mathcal{H}(x)\|_{\mathcal{B}} \le \frac{\mathcal{L}}{(1-\mathcal{L})(p-1)}\varphi\left(0,\frac{x}{k},\cdots,\frac{x}{k}\right) \tag{3.7}$$

for all $x\in\mathcal{A}$.

Proof. Let $\mathcal{S}$ be the defined set in the proof of Theorem 3.2. Consider the following generalized metric d:

$$d(g,h) = \inf_{x\in\mathcal{A}}\left\{\mathcal{C}\in\mathbb{R}^+ : \quad \|g(x)-h(x)\|_{\mathcal{B}} \le \mathcal{C}\varphi\left(0,\frac{x}{k},\cdots,\frac{x}{k}\right)\right\}.$$

It is easy to show that $(\mathcal{S},d)$ is complete (see the proof of [35, Lemma 2.1]). we define the linear mapping $\mathcal{J}:\mathcal{S}\to\mathcal{S}$ such that

$$\mathcal{J}\left(g(x)\right) := \frac{1}{k}g(kx)$$

for all $x\in\mathcal{A}$. By the same argument as in the proof of Theorem 3.2, we can obtain the mapping $\mathcal{H}:\mathcal{A}\to\mathcal{B}$, as the unique fixed point of $\mathcal{J}$ such that

$$\mathcal{H}(x) := \lim_{n\to\infty}\frac{1}{k^n}f(k^n x)$$

for all $x\in\mathcal{A}$. By (3.2) and (3.6), we have

$$\left\|f(x)-\frac{1}{k}f(kx)\right\|_{\mathcal{B}} \le \frac{1}{2}\varphi\left(0,x,\cdots,x\right) \le \frac{\mathcal{L}}{(p-1)}\varphi\left(0,\frac{x}{k},\cdots,\frac{x}{k}\right)$$

for all $x\in\mathcal{A}$. This means that $d(f,\mathcal{J}(f))\le\frac{\mathcal{L}}{(p-1)}$. Hence

$$d(f,\mathcal{H}) \le \frac{1}{1-\mathcal{L}}d\left(f,\mathcal{J}(f)\right) \le \frac{\mathcal{L}}{(1-\mathcal{L})(p-1)}$$

which implies that the inequality (3.7) holds.

The rest of the proof is similar to the proof of Theorem 3.2. □

Theorem 3.4. *Let $\varphi : \mathcal{A}^3 \to [0,\infty)$ be a function such that there exists an $\mathcal{L} < 1$ with*

$$\varphi(x,y,z) < \frac{\mathcal{L}}{2}\varphi(2x,2y,2z) \tag{3.8}$$

for all $x,y,z \in \mathcal{A}$. Suppose that $f : \mathcal{A} \to \mathcal{B}$ is an odd mapping satisfying

$$\begin{aligned}\Bigg\| f\left(\mu\frac{x+y+z}{2}\right) + f\left(\mu\frac{x+z-3y}{2}\right) + f\left(\mu\frac{x+y-3z}{2}\right)\\ + f\left(\mu\frac{y+z-x}{2}\right) - \mu f(x)\Bigg\|_{\mathcal{B}} \le \varphi(x,y,z),\end{aligned} \tag{3.9}$$

$$\|f(x^r x^{*s} x^t) - f(x)^r f(x)^{*s} f(x)^t\|_{\mathcal{B}} \le \varphi(x,x,x) \tag{3.10}$$

for all $\mu \in \mathbb{T}^1$ and all $x,y,z \in \mathcal{A}$. Then there exists a unique (r,s,t)-J^*-homomorphism $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ such that

$$\|f(x) - \mathcal{H}(x)\|_{\mathcal{B}} \le \frac{\mathcal{L}}{2(1-\mathcal{L})}\varphi\left(2x,0,0\right) \tag{3.11}$$

for all $x \in \mathcal{A}$.

Proof. Consider the defined set $\mathcal{S}$ in the proof of Theorem 3.2 and the following generalized metric d:

$$d(g,h) = \inf_{x\in\mathcal{A}} \left\{\mathcal{C} \in \mathbb{R}^+ : \quad \|g(x) - h(x)\|_{\mathcal{B}} \le \mathcal{C}\varphi\left(2x,0,0\right)\right\}.$$

Using the same method as in the proof of Theorem 3.2, we can get the mappings $\mathcal{J} : \mathcal{S} \to \mathcal{S}$ and $\mathcal{H} : \mathcal{A} \to \mathcal{B}$, with

$$\mathcal{J}\left(g(x)\right) := 2g\left(\frac{x}{2}\right), \qquad \mathcal{H}(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in \mathcal{A}$. By (3.8) and (3.9), we obtain

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\|_{\mathcal{B}} \le \varphi\left(x,0,0\right) < \frac{\mathcal{L}}{2}\varphi\left(2x,0,0\right)$$

for all $x \in \mathcal{A}$, which means $d(f,\mathcal{J}(f)) \le \frac{\mathcal{L}}{2}$. Hence $d(f,\mathcal{H}) \le \frac{\mathcal{L}}{2(1-\mathcal{L})}$. This implies that the inequality (3.11) holds. □

Theorem 3.5. *Let $\varphi : \mathcal{A}^3 \to [0,\infty)$ be a function such that there exists an $\mathcal{L} < 1$ with*

$$\varphi(x,y,z) < 2\mathcal{L}\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right)$$

for all $x,y,z \in \mathcal{A}$. Suppose that $f : \mathcal{A} \to \mathcal{B}$ is an odd mapping satisfying (3.9) and (3.10). Then there exists a unique (r,s,t)-J^*-homomorphism $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ such that

$$\|f(x) - \mathcal{H}(x)\|_{\mathcal{B}} \le \frac{\mathcal{L}}{(1-\mathcal{L})}\varphi\left(x,0,0\right)$$

for all $x \in \mathcal{A}$.

Proof. The proof is similar to the proof of Theorem 3.4. □

4. Hyers-Ulam stability of (r, s, t)-J^*-homomorphisms: direct method

In this section, by using the direct method, we prove the Hyers-Ulam stability of (r, s, t)-J^*-homomorphisms associated with the functional equation (0.2).

Theorem 4.1. *Let $\varphi : \mathcal{A}^p \to [0, \infty)$ be a function with $\varphi(0, \cdots, 0) = 0$ and $p \geq 4$. Denote by ϕ a function such that*

$$\phi(x_1, \cdots, x_p) := \sum_{n=0}^{\infty} k^{-(n+1)} \varphi\left(k^n x_1, \cdots, k^n x_p\right) < \infty, \tag{4.1}$$

$$\lim_{n\to\infty} k^{-(r+s+t)n} \varphi\left(k^n x, \cdots, k^n x\right) = 0 \tag{4.2}$$

for all $x, x_1, \cdots, x_p \in \mathcal{A}$, where $k = \frac{2}{p-1}$. Suppose that $f : \mathcal{A} \to \mathcal{B}$ is an odd mapping satisfying (3.2) *and* (2.4)*. Then there exists a unique (r, s, t)-J^*-homomorphism $h : \mathcal{A} \to \mathcal{B}$ such that*

$$\|f(x) - h(x)\|_{\mathcal{B}} \leq \frac{1}{p-1} \phi(0, x, \cdots, x) \tag{4.3}$$

for all $x \in \mathcal{A}$.

Proof. It follows from (3.2) that

$$\left\| \frac{1}{k} f(kx) - f(x) \right\|_{\mathcal{B}} \leq \frac{1}{2} \varphi(0, x, \cdots, x)$$

for all $x \in \mathcal{A}$. Using the induction method, we obtain

$$\left\| \frac{1}{k^n} f(k^n x) - f(x) \right\|_{\mathcal{B}} \leq \frac{1}{p-1} \sum_{s=0}^{n-1} k^{-(s+1)} \varphi(0, k^s x, \cdots, k^s x) \tag{4.4}$$

for all $n \geq 1$ and all $x \in \mathcal{A}$. Assume that m, l are positive integers with $m > l$. By (4.4), for $m - l > 0$ and $k^l x$, we have

$$\begin{aligned}
\left\| \frac{1}{k^m} f(k^m x) - \frac{1}{k^l} f(k^l x) \right\|_{\mathcal{B}} &= \frac{1}{k^l} \left\| \frac{1}{k^{m-l}} f(k^{m-l} k^l x) - f(k^l x) \right\|_{\mathcal{B}} \\
&\leq \frac{1}{p-1} \sum_{s=l}^{m-1} k^{-(s+1)} \varphi(0, k^s x, \cdots, k^s x) \\
&\leq \frac{1}{p-1} \sum_{s=l}^{\infty} k^{-(s+1)} \varphi(0, k^s x, \cdots, k^s x)
\end{aligned}$$

for all $x \in \mathcal{A}$. By (4.1), the right-hand side tends to zero as $l \to \infty$. Therefore the sequence $\{\frac{1}{k^n} f(k^n x)\}$ is Cauchy. Since $\mathcal{A}$ is a complete space, the sequence $\{\frac{1}{k^n} f(k^n x)\}$ is convergent and we can define for all $x \in \mathcal{A}$, the mapping $h : \mathcal{A} \to \mathcal{B}$ by

$$h(x) := \lim_{n\to\infty} \frac{1}{k^n} f(k^n x).$$

Passing the limit $n \to \infty$ in (4.4) and then by (4.1), we obtain (4.3).

It follows from (4.1) and (3.2) that

$$\begin{aligned}
\|\varrho_\mu h(x_1, \cdots, x_p)\|_{\mathcal{B}} &= \lim_{n\to\infty} \frac{1}{k^n} \left\| \varrho_\mu f\left(k^n x_1, \cdots, k^n x_p\right) \right\|_{\mathcal{B}} \\
&\leq \lim_{n\to\infty} \frac{1}{k^n} \varphi\left(k^n x_1 \cdots, k^n x_p\right) = 0
\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \cdots, x_p \in \mathcal{A}$. So by Lemma 3.1 we deduce that h is $\mathbb{C}$-linear.

By (4.2) and substituting x by $k^n x$ in (2.4), we obtain

$$\begin{aligned}
&\left\|h(x^r x^{*s} x^t) - h(x)^r h(x)^{*s} h(x)^t\right\|_{\mathcal{B}} \\
&\quad = \lim_{n\to\infty} \frac{1}{k^{(r+s+t)n}} \left\|f\left((k^n x)^r (k^n x)^{*s} (k^n x)^t\right) - f\left(k^n x\right)^r f\left(k^n x\right)^{*s} f\left(k^n x\right)^t\right\|_{\mathcal{B}} \\
&\quad \le \lim_{n\to\infty} \frac{1}{k^{(r+s+t)n}} \varphi\left(k^n x, \cdots, k^n x\right) = 0
\end{aligned}$$

for all $x \in \mathcal{A}$. Hence $h(x^r x^{*s} x^t) = h(x)^r h(x)^{*s} h(x)^t$ for all $x \in \mathcal{A}$.

Let $g : \mathcal{A} \to \mathcal{B}$ be another (r,s,t)-J^*-homomorphism satisfying (4.3). Then we have

$$\begin{aligned}
\|h(x) - g(x)\|_{\mathcal{B}} &\le \frac{1}{k^n}\|f(k^n x) - h(k^n x)\|_{\mathcal{B}} + \frac{1}{k^n}\|f(k^n x) - g(k^n x)\|_{\mathcal{B}} \\
&\le \frac{1}{k^n}\left(\frac{2}{p-1}\phi\left(0, k^n x, \cdots, k^n x\right)\right) \\
&= \frac{2}{p-1}\sum_{s=n}^{\infty} k^{-(s+1)}\varphi\left(0, k^s x, \cdots, k^s x\right)
\end{aligned}$$

for all $x \in \mathcal{A}$. By (4.1), the right-hand side tends to zero as $n \to \infty$, which means h is unique. $\square$

Theorem 4.2. *Let $\varphi : \mathcal{A}^p \to [0,\infty)$ be a function with $\varphi(0,\cdots,0) = 0$ and $p \ge 4$. Denote by ϕ a function such that*

$$\phi(x_1, \cdots, x_p) := \sum_{n=0}^{\infty} k^n \varphi\left(k^{-(n+1)}x_1, \cdots, k^{-(n+1)}x_p\right) < \infty \tag{4.5}$$

for all $x_1, \cdots, x_p \in \mathcal{A}$, where $k = \frac{2}{p-1}$. Suppose that $f : \mathcal{A} \to \mathcal{B}$ be an odd mapping satisfying (3.2) and (2.4). Then there exists a unique (r,s,t)-J^*-homomorphism $h : \mathcal{A} \to \mathcal{B}$ satisfying (4.3).

Proof. It follows from (3.2) that

$$\left\|kf\left(\frac{x}{k}\right) - f(x)\right\|_{\mathcal{B}} \le \frac{1}{p-1}\varphi\left(0, \frac{x}{k}\cdots, \frac{x}{k}\right)$$

for all $x \in \mathcal{A}$. By the same method which was done in the proof of Theorem 4.1, we can get the unique and $\mathbb{C}$-linear mapping $h(x) := \lim_{n\to\infty} k^n f\left(\frac{1}{k^n}x\right)$ satisfying (4.3). By (2.4), (4.5) and the fact that $k < 1$, we have

$$\begin{aligned}
&\left\|h(x^r x^{*s} x^t) - h(x)^r h(x)^{*s} h(x)^t\right\|_{\mathcal{B}} \\
&\quad = \lim_{n\to\infty} k^{(r+s+t)n} \left\|f\left(\left(\frac{x}{k^n}\right)^r \left(\frac{x}{k^n}\right)^{*s} \left(\frac{x}{k^n}\right)^t\right) - f\left(\frac{x}{k^n}\right)^r f\left(\frac{x}{k^n}\right)^{*s} f\left(\frac{x}{k^n}\right)^t\right\|_{\mathcal{B}} \\
&\quad \le \lim_{n\to\infty} k^{(r+s+t)n}\varphi\left(\frac{x}{k^n}, \cdots, \frac{x}{k^n}\right) \le \lim_{n\to\infty} k^n \varphi\left(\frac{x}{k^n}, \cdots, \frac{x}{k^n}\right) = 0
\end{aligned}$$

for all $x \in \mathcal{A}$. Hence $h(x^r x^{*s} x^t) = h(x)^r h(x)^{*s} h(x)^t$ for all $x \in \mathcal{A}$. $\square$

Corollary 4.3. *Let θ be a nonnegative real number and $q_1, \cdots, q_p$ be positive real numbers such that $q_1, \cdots, q_p > r+s+t$ or $q_1, \cdots, q_p < 1$. Let $f : \mathcal{A} \to \mathcal{B}$ be an odd mapping satisfying (2.5) and*

$$\|\varrho_\mu f(x_1, \cdots, x_p)\|_{\mathcal{B}} \le \theta(\|x_1\|_{\mathcal{A}}^{q_1} + \cdots + \|x_p\|_{\mathcal{A}}^{q_p})$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \cdots, x_p \in \mathcal{A}$. Then there exists a unique (r,s,t)-J^*-homomorphism $h : \mathcal{A} \to \mathcal{B}$ such that

$$\|f(x) - h(x)\|_{\mathcal{B}} \le \sum_{j=2}^{p} \frac{\theta \|x\|_{\mathcal{A}}^{q_j}}{2\,|1 - k^{q_j - 1}|}$$

for all $x \in \mathcal{A}$.

Proof. Defining $\varphi(x_1, \cdots, x_p) = \theta\left(\|x_1\|_{\mathcal{A}}^{q_1} + \cdots + \|x_p\|_{\mathcal{A}}^{q_p}\right)$ and applying Theorem 4.1 for the case $q_1, \cdots, q_p > r+s+t$, and Theorem 4.2 for the case $q_1, \cdots, q_p < 1$, we get the result. □

Theorem 4.4. *Let $\varphi : \mathcal{A}^2 \to [0, \infty)$ be a function with $\varphi(0,0) = 0$. Denote by ϕ a function such that*

$$\phi(x,y) := \sum_{n=0}^{\infty} 2^{-(n+1)} \varphi\left(2^n x, 2^n y\right) < \infty$$

for all $x, y \in \mathcal{A}$. Suppose that $f : \mathcal{A} \to \mathcal{B}$ is an odd mapping satisfying

$$\|f(\mu x + \mu y) + f(\mu x - 2\mu y) + f(\mu y - \mu x) - \mu f(x)\|_{\mathcal{B}} \le \varphi(x,y), \tag{4.6}$$

$$\|f(x^r x^{*s} x^t) - f(x)^r f(x)^{*s} f(x)^t\|_{\mathcal{B}} \le \varphi(x,x) \tag{4.7}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in \mathcal{A}$. Then there exists a unique (r,s,t)-J^*-homomorphism $h : \mathcal{A} \to \mathcal{B}$ such that

$$\|f(x) - h(x)\|_{\mathcal{B}} \le \phi(0,x) \tag{4.8}$$

for all $x \in \mathcal{A}$.

Proof. From (4.6), it follows that

$$\left\|\frac{1}{2} f(2x) - f(x)\right\|_{\mathcal{B}} \le \frac{1}{2}\varphi(0,x)$$

for all $x \in \mathcal{A}$. Using the same method as in the proof of Theorem 4.1, we conclude that the mapping $h(x) := \lim_{n\to\infty} \frac{1}{2^n} f\left(2^n x\right)$ is a unique (r,s,t)-J^*-homomorphism satisfying (4.8). □

Theorem 4.5. *Let $\varphi : \mathcal{A}^2 \to [0, \infty)$ be a function with $\varphi(0,0) = 0$. Denote by ϕ a function such that*

$$\phi(x,y) := \sum_{n=0}^{\infty} 2^n \varphi\left(2^{-(n+1)} x, 2^{-(n+1)} y\right) < \infty,$$

$$\lim_{n\to\infty} 2^{(r+s+t)n} \varphi\left(2^{-n} x, 2^{-n} x\right) = 0$$

for all $x, y \in \mathcal{A}$. Suppose that $f : \mathcal{A} \to \mathcal{B}$ is an odd mapping satisfying (4.6) and (4.7). Then there exists a unique (r,s,t)-J^*-homomorphism $h : \mathcal{A} \to \mathcal{B}$ satisfying (4.8).

Proof. The proof is similar to the proof of Theorem 4.4. □

Corollary 4.6. *Let θ be a nonnegative real number and q_1, q_2 be positive real numbers such that $q_1, q_2 < 1$ or $q_1, q_2 > r+s+t$. Let $f : \mathcal{A} \to \mathcal{B}$ be an odd mapping satisfying*

$$\begin{aligned}
\|f(\mu x + \mu y) + f(\mu x - 2\mu y) + f(\mu y - \mu x) - \mu f(x)\|_{\mathcal{B}} &\le \theta(\|x\|_{\mathcal{A}}^{q_1} + \|y\|_{\mathcal{A}}^{q_2}), \\
\|f(x^r x^{*s} x^t) - f(x)^r f(x)^{*s} f(x)^t\|_{\mathcal{B}} &\le \theta(\|x\|_{\mathcal{A}}^{q_1} + \|x\|_{\mathcal{A}}^{q_2})
\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in \mathcal{A}$. Then there exists a unique (r,s,t)-J^*-homomorphism $h : \mathcal{A} \to \mathcal{B}$ such that

$$\|f(x) - h(x)\|_{\mathcal{B}} \le \frac{\theta \|x\|_{\mathcal{A}}^{q_2}}{|2 - 2^{q_2}|}$$

for all $x \in \mathcal{A}$.

Proof. Defining $\varphi(x,y) = \theta\left(\|x\|_{\mathcal{A}}^{q_1} + \|y\|_{\mathcal{A}}^{q_2}\right)$ and applying Theorem 4.4 for the case $q_1, q_2 < 1$, and Theorem 4.5 for the case $q_1, q_2 > r+s+t$, we get the result. □

Theorem 4.7. *Let $\varphi : \mathcal{A}^3 \to [0,\infty)$ be a function. Denote by ϕ a function such that*

$$\phi(x,y,z) := \sum_{n=1}^{\infty} 2^{-n}\varphi\left(2^n x, 2^n y, 2^n z\right) < \infty$$

for all $x,y,z \in \mathcal{A}$. Suppose that $f : \mathcal{A} \to \mathcal{B}$ is an odd mapping satisfying (3.9) *and* (3.10)*. Then there exists a unique (r,s,t)-J^*-homomorphism $h : \mathcal{A} \to \mathcal{B}$ such that*

$$\|f(x) - h(x)\|_{\mathcal{B}} \le \phi(x,0,0) \tag{4.9}$$

for all $x \in \mathcal{A}$.

Proof. By (3.9), we get $\left\|\frac{1}{2}f(2x) - f(x)\right\|_{\mathcal{B}} \le \frac{1}{2}\varphi(2x,0,0)$ for all $x \in \mathcal{A}$. The same method as in the proof of Theorem 4.1, leads us to the unique (r,s,t)-J^*-homomorphism $h(x) := \lim_{n\to\infty}\frac{1}{2^n}f\left(2^n x\right)$satisfying (4.9). □

Theorem 4.8. *Let $\varphi : \mathcal{A}^3 \to [0,\infty)$ be a function. Denote by ϕ a function such that*

$$\phi(x,y,z) := \sum_{n=0}^{\infty} 2^{n}\varphi\left(2^{-n} x, 2^{-n} y, 2^{-n} z\right) < \infty,$$

$$\lim_{n\to\infty} 2^{(r+s+t)n}\varphi\left(2^{-n} x, 2^{-n} x, 2^{-n} x\right) = 0$$

for all $x,y,z \in \mathcal{A}$. Suppose that $f : \mathcal{A} \to \mathcal{B}$ is an odd mapping satisfying (3.9) *and* (3.10)*. Then there exists a unique (r,s,t)-J^*-homomorphism $h : \mathcal{A} \to \mathcal{B}$ satisfying* (4.9)*.*

Proof. The proof is similar to the proof of Theorem 4.7. □

Corollary 4.9. *Let θ be a nonnegative real number and q_1, q_2, q_3 be positive real numbers such that $q_1, q_2, q_3 < 1$ or $q_1, q_2, q_3 > r+s+t$. Let $f : \mathcal{A} \to \mathcal{B}$ be an odd mapping satisfying*

$$\begin{aligned}\Bigg\| f\left(\mu\frac{x+y+z}{2}\right) + f\left(\mu\frac{x+z-3y}{2}\right) + f\left(\mu\frac{x+y-3z}{2}\right)& \\ + f\left(\mu\frac{y+z-x}{2}\right) - \mu f(x)\Bigg\|_{\mathcal{B}} &\le \theta\left(\|x\|_{\mathcal{A}}^{q_1} + \|y\|_{\mathcal{A}}^{q_2} + \|z\|_{\mathcal{A}}^{q_3}\right), \\ \|f(x^r x^{*s} x^t) - f(x)^r f(x)^{*s} f(x)^t\|_{\mathcal{B}} &\le \theta\left(\|x\|_{\mathcal{A}}^{q_1} + \|x\|_{\mathcal{A}}^{q_2} + \|x\|_{\mathcal{A}}^{q_3}\right)\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x,y,z \in \mathcal{A}$. Then there exists a unique (r,s,t)-J^*-homomorphism $h : \mathcal{A} \to \mathcal{B}$ such that

$$\|f(x) - h(x)\|_{\mathcal{B}} \le \frac{2^{q_1}}{|2-2^{q_1}|}\theta\|x\|_{\mathcal{A}}^{q_1}$$

for all $x \in \mathcal{A}$.

Proof. Defining $\varphi(x,y,z) = \theta\left(\|x\|_{\mathcal{A}}^{q_1} + \|y\|_{\mathcal{A}}^{q_2} + \|z\|_{\mathcal{A}}^{q_3}\right)$ and applying Theorem 4.7 for the case $q_1, q_2, q_3 < 1$, and Theorem 4.8 for the case $q_1, q_2, q_3 > r+s+t$, we get the result. □

Remark 4.10. The obtained results in this paper, could be more remarkable and interesting. In other words, as a consequence including simpler and better results, one can set $q_1 = \cdots = q_p = q$, as well as $r = s = t = 1$ (or a fixed $n \in \mathbb{N}$) in all the statements. Furthermore, all the obtained results do also hold for (r,s,t)-J^*-derivations similarly. The reader can directly verify this point just with a little difference in details.

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Differential subordinations obtained by using a generalization of Marx-Strohhäcker theorem

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Abstract

In [1] and [6] Marx and Strohhäcker have proved that if $f \in \mathcal{A}$ is a convex function, then it has the property of starlikeness of order $\frac{1}{2}$. In [5, Theorem 9.5.6], P. T. Mocanu extended this result to the class $\mathcal{A}_2$ for a convex function of order $-\frac{1}{2}$. In this paper we extend the results proven by Marx and Strohhäcker and by P. T. Mocanu and we'll prove that, if the function $f \in \mathcal{A}_n$, $n \geq 3$, is a close-to-convex function, then it is starlike of order $\frac{1}{2}$.

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1 Introduction and preliminaries

Let U be the unit disc of the complex plane $U = \{z \in \mathbb{C} : \ |z| < 1\}$. Let $\mathcal{H}(U)$ be the class of holomorphic functions in U. Also, let $\mathcal{A}_n = \{f \in \mathcal{H}(U) : \ f(z) = z + a_{n+1}z^{n+1} + \dots, \ z \in U\}$, with $\mathcal{A}_1 = \mathcal{A}$.

Let $\mathcal{S} = \{f \in \mathcal{A} : \ f \text{ univalent in } U\}$ be the class of holomorphic and univalent functions in the open unit disc U, with conditions $f(0) = 0$, $f'(0) = 1$, that is the holomorphic and univalent functions with the following power series development $f(z) = z + a_2z^2 + \dots$, $z \in U$.

Denote by $\mathcal{K} = \left\{f \in \mathcal{A} : \ \mathrm{Re}\,\frac{zf''(z)}{f'(z)} + 1 > 0, \ z \in U\right\}$ the class of normalized convex functions in U and by $\mathcal{C} = \left\{f \in \mathcal{A} : \ \exists\, \varphi \in \mathcal{K}, \ \mathrm{Re}\,\frac{f'(z)}{\varphi'(z)} > 0, \ z \in U\right\}$ the class of normalized close-to-convex functions in U.

An equivalent formulation would involve the existence of a starlike function h (not necessarily normalized) such that $\mathrm{Re}\,\frac{zf'(z)}{h(z)} > 0$, $z \in U$. We consider $\mathcal{K}\left(-\frac{1}{2\gamma}\right) = \left\{f \in \mathcal{A}_n : \ \mathrm{Re}\,\frac{zf''(z)}{f'(z)} + 1 > -\frac{1}{2\gamma}, \ z \in U, \ \gamma \geq 1\right\}$.

Let $\mathcal{S}^* = \left\{f \in \mathcal{A} : \ \mathrm{Re}\,\frac{zf'(z)}{f(z)} > 0, \ z \in U\right\}$ denote the class of starlike functions in U, and
$\mathcal{S}^*(\alpha) = \left\{f \in \mathcal{A} : \ \mathrm{Re}\,\frac{zf'(z)}{f(z)} > \alpha, \ z \in U\right\}$, denote the class of starlike functions of order α, with $0 \leq \alpha < 1$.

In order to prove our original results, we use the following lemmas:

Lemma 1.1 *[2], [3], [4, Theorem 2.3.i, p. 35] Let* $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$, *satisfy the condition* $\mathrm{Re}\,\psi(is, t) \leq 0, \ z \in U$, *for* $s, t \in R$, $t \leq -\frac{n}{2}(1 + s^2)$. *If* $p(z) = 1 + p_nz^n + p_{n+1}z^{n+1} + \dots$ *satisfies* $\mathrm{Re}\,[p(z), zp'(z); z] > 0$, *then* $\mathrm{Re}\,p(z) > 0, \ z \in U$.

More general forms of this lemma can be found in [6].

Lemma 1.2 *[5, Theorem 4.6.3, p. 84] The function* $f \in \mathcal{A}$, *with* $f'(z) \neq 0$, $z \in U$, *is close-to-convex if and only if* $\int_{\theta_1}^{\theta_2} \mathrm{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] d\theta > -\pi$, $z = re^{i\theta}$, *for all* θ_1, θ_2, *with* $0 \leq \theta_1 < \theta_2 \leq 2\pi$ *and* $r \in (0, 1)$.

Definition 1.1 *[4, Definition 2.2.b, p. 21] We denote by* Q *the set of functions* q *that are analytic and injective on* $\overline{U} \setminus E(q)$, *where* $E(q) = \left\{\zeta \in \partial U : \ \lim\limits_{z \to \zeta} q(z) = \infty\right\}$ *and are such that* $q'(\zeta) \neq 0$, *for* $\zeta \in \partial U \setminus E(q)$. *The set* $E(q)$ *is called exception set. The subclass of* Q *for which* $f(0) = a$ *is denoted by* $Q(a)$.

Definition 1.2 *[4, Definition 2.3.a, p. 27] Let Ω be a set in $\mathbb{C}$, $q \in Q$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C} \times U \to \mathbb{C}$ that satisfy the admissibility condition*

$$\text{(A)} \qquad \psi(r, s, t) \notin \Omega$$

where $r = q(\zeta)$, $s = m\zeta q'(\zeta)$, $R\left(\frac{t}{s} + 1\right) \geq m\mathrm{Re}\left[\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right]$, $z \in U$, $\zeta \in \partial U \setminus E(q)$, $m \geq n$. We write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$.

In the special case when Ω is a simply connected domain, $\Omega \neq \mathbb{C}$, and h is a conformal mapping of U onto Ω, we denote this class by $\Psi_n[h, q]$.

If $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$, then the admissibility condition (A) reduces to

$$\text{(A}'\text{)} \qquad \psi(q(\zeta), m\zeta q'(\zeta); z) \notin \Omega,$$

where $z \in U$, $\zeta \in \partial U \setminus E(q)$ and $m \geq n$.

If $\psi : \mathbb{C} \times U \to \mathbb{C}$, then the admissibility condition (A) reduces to

$$\text{(A}''\text{)} \qquad \psi(q(\zeta); z) \notin \Omega$$

where $z \in U$ and $\zeta \in \partial U \setminus E(q)$.

Definition 1.3 *[4, p. 36] Let f and F be members of $\mathcal{H}(U)$. The function f is said to be subordinate to F, written $f \prec F$ or $f(z) \prec F(z)$, if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, and $f(0) = F(0)$ and $f(U) \subset F(U)$.*

Definition 1.4 *[4, p. 16] Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the (second-order) differential subordination*

(i) $\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z)$,

*then p is called a **solution** of the differential subordination. The univalent function q is called a **dominant of the solutions of the differential subordination**, or more simply a **dominant**, if $p \prec q$ for all p satisfying (i).*

*A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominant q of (i) is said to be the **best dominant** of (i). (Note that the best dominant is unique up to a rotation of U).*

*If we require the more restrictive condition $p \in [a, n]$, then p will be called an (q, n)-**solution**, q an (a, n)-**dominant**, and $\widetilde{q}$ the best (a, n)-**dominant**,*

Lemma 1.3 *[4, Theorem 2.3.c, p. 30] Let $\psi \in \Psi_n[h, q]$ with $q(0) = a$. If $p \in \mathcal{H}[a, n]$, $\psi(p(z), zp'(z), z^2p''(z); z)$ is analytic in U, and*

(ii) $\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z)$,

then $p(z) \prec q(z)$, $z \in U$.

Theorem 1.1 *[1, 6, Marx-Strohhacker] If $f \in \mathcal{A}$ and satisfy the condition $\mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > 0$, then*

(a) Re $\frac{zf'(z)}{f(z)} > \frac{1}{2}$ $\left[\text{ i.e., } f \in \mathcal{S}^*\left(\frac{1}{2}\right)\right]$ and

(b) Re $\frac{f(z)}{z} > \frac{1}{2}$, for $z \in U$.

In [5] has shown that the odd and convex functions of order $-\frac{1}{2}$ are starlike functions of order $\frac{1}{2}$.

Theorem 1.2 *[5, Marx-Strohhscker, Theorem 9.5.6, p. 218] If $f \in \mathcal{A}_2$ and satisfy the condition $\mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > -\frac{1}{2}$, then $\mathrm{Re}\,\frac{zf'(z)}{f(z)} > \frac{1}{2}$ $\left[\text{i.e., } f \in \mathcal{S}^*\left(\frac{1}{2}\right)\right]$, for $z \in U$.*

2 Main results

We'll extend the theorem Marx-Strohhäcker for the functions $f \in \mathcal{A}_n$, $n \geq 3$, which are close-to-convex functions.

Theorem 2.1 *Let $n \geq 3$, $\gamma \geq 1$, $f \in \mathcal{A}_n$, satisfy the condition*

$$\operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > -\frac{1}{2\gamma}, \tag{2.1}$$

then $\operatorname{Re} \frac{zf'(z)}{f(z)} > \frac{1}{2}$.

Proof. According to Lemma 1.2 we obtain

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \left[1 + \frac{zf''(z)}{f'(z)}\right] d\theta \geq \int_{\theta_1}^{\theta_2} -\frac{1}{2\gamma} d\theta = -\frac{1}{2\gamma} \int_{\theta_1}^{\theta_2} d\theta = -\frac{1}{2\gamma}(\theta_2 - \theta_1) > -\frac{2\pi}{2\gamma} = -\frac{\pi}{\gamma} > -\pi, \quad \lambda \geq 1. \tag{2.2}$$

From (2.2) we have $f \in \mathbb{C}$, hence it is univalent.

Let $p(z) = 2 \cdot \frac{zf'(z)}{f(z)} - 1$. Since $f \in \mathrm{A}_n$ and f is close-to-convex function (univalent), the function p is analytic in U and $p(0) = 1$.

A simple computation leads to

$$\frac{p(z)+1}{2} = \frac{zf'(z)}{f(z)}. \tag{2.3}$$

By differentiating (2.3), we obtain

$$\frac{zp'(z)}{p(z)+1} = 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}. \tag{2.4}$$

Using (2.3) in (2.4), we have

$$\frac{p(z)+1}{2} + \frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)}. \tag{2.5}$$

Using (2.1) in (2.5), we obtain $\operatorname{Re}\left[\frac{p(z)+1}{2} + \frac{zp'(z)}{p(z)+1}\right] > -\frac{1}{2\gamma}$, which is equivalent to

$$\operatorname{Re}\left[\frac{p(z)+1}{2} + \frac{zp'(z)}{p(z)+1} + \frac{1}{2\gamma}\right] > 0. \tag{2.6}$$

Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$, $\psi(p(z), zp'(z); z) = \frac{p(z)+1}{2} + \frac{zp'(z)}{p(z)+1} + \frac{1}{2\gamma}$, where $\psi(r,s) = \frac{r+1}{2} + \frac{s}{r+1} + \frac{1}{2\gamma}$. Then (2.6) is equivalent to $\operatorname{Re}\psi(p(z), zp'(z); z) > 0$, $z \in U$.

In order to prove Theorem 2.1, we use Lemma 1.1. For that we calculate $\operatorname{Re}\psi(is,t) = \operatorname{Re}\left(\frac{is+1}{2} + \frac{t}{1+is} + \frac{1}{2\gamma}\right)$ $= \operatorname{Re}\left(\frac{is+1}{2} + \frac{t(1-is)}{1+s^2} + \frac{1}{2\gamma}\right) = \frac{1}{2} + \frac{t}{1+s^2} + \frac{1}{2\gamma} \leq \frac{1}{2} - \frac{n(1+s^2)}{2(1+s^2)} + \frac{1}{2\gamma} = \frac{1-n}{2} + \frac{1}{2\gamma} = \frac{(1-n)\gamma+1}{2\gamma} \leq 0$. Since $n \geq 3$, $\gamma \geq 1$. Now, using Lemma 1.1, we get that $\operatorname{Re} p(z) > 0$, $z \in U$, i.e., $\operatorname{Re}\frac{zf'(z)}{f(z)} > \frac{1}{2}$, $z \in U$. ■

Remark 2.1 *Each of the four conditions in the Marx-Strohhäcker theorem can be rewritten in terms of subordination. This leads to the following equivalent form of the theorem.*

Theorem 2.2 *Let $n \geq 3$, $\gamma \geq 1$, $f \in \mathcal{A}_n$, satisfies the condition $\frac{zf''(z)}{f'(z)} + 1 \prec \frac{1-\left(\frac{1}{\gamma}+1\right)z}{1+z}$, then $\frac{zf'(z)}{f(z)} \prec \frac{1}{1+z}$.*

Theorem 2.3 *Let $n \geq 3$, $\gamma \geq 1$, $f \in \mathcal{A}_n$ satisfies the conditions*

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > -\frac{1}{2\gamma} \tag{2.7}$$

and

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \frac{1}{2} \tag{2.8}$$

then $\operatorname{Re}\frac{f(z)}{z} > \frac{1}{2}$, *for* $z \in U$.

Proof. In order to prove Theorem 2.1, we saw that, if $f \in \mathrm{A}_n$, $n \geq 3$ and satisfies the condition (2.1) or (2.7), then the function f is close-to-convex (univalent).

Let $p(z) = \frac{2f(z)}{z} - 1$. Since $f \in \mathcal{A}_n$, $n \geq 3$ and f is close-to-convex function (univalent) then the function p is analytic in U and $p(0) = 1$. A simple computation leads to

$$\frac{p(z)+1}{2} = \frac{f(z)}{z}. \tag{2.9}$$

By differentiating (2.9), we obtain

$$\frac{zp'(z)}{p(z)+1} = \frac{zf'(z)}{f(z)} - 1. \tag{2.10}$$

Using (2.8) in (2.10), we have

$$\operatorname{Re}\left(\frac{zp'(z)}{p(z)+1} + \frac{1}{2}\right) > 0, z \in U. \tag{2.11}$$

Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$, $\psi(p(z), zp'(z); z) = \frac{zp'(z)}{1+p(z)} + \frac{1}{2}$, where $\psi(r,s) = \frac{1}{2} + \frac{s}{1+r}$. Then (2.11) is equivalent to $\operatorname{Re}\psi(p(z), zp'(z); z) > 0$, $z \in U$.

In order to prove Theorem 2.1, we use Lemma 1.1. For that we calculate $\operatorname{Re}\psi(is,t) = \operatorname{Re}\left[\frac{1}{2} + \frac{t}{1+is}\right] = \operatorname{Re}\left[\frac{1}{2} + \frac{t(1-is)}{1+s^2}\right] = \frac{1}{2} + \frac{t}{1+s^2} \leq \frac{1}{2} - \frac{n(1+s^2)}{2(1+s^2)} = \frac{1-n}{2} < 0$, since $n \geq 3$. Therefore, by applying Lemma 1.1 we conclude that p satisfies $\operatorname{Re} p(z) > 0$. This is equivalent to $\operatorname{Re}\frac{f(z)}{z} > \frac{1}{2}$, $z \in U$. ∎

For $0 < \gamma < 1$, $n \geq 3$, Theorem 2.2 can be written as the following corollary.

Corollary 2.4 *Let $n \geq 3$, $0 < \gamma < 1$, $f \in \mathcal{A}_n$ satisfy the conditions* $\operatorname{Re}\left[\frac{zf''(z)}{f'(z)} + 1\right] > -\frac{\gamma}{2}$ *and* $\operatorname{Re}\frac{zf'(z)}{f(z)} > \frac{1}{2}$, *then* $\operatorname{Re}\frac{f(z)}{z} > \frac{1}{2}$, $z \in U$.

Theorem 2.5 *Let $n \geq 3$, $\gamma \geq 1$, $f \in \mathcal{A}_n$ satisfy differential subordination*

$$\frac{zf''(z)}{f'(z)} + 1 \prec \frac{1 - \left(\frac{1}{\gamma} + 1\right)z}{1+z}, \tag{2.12}$$

and

$$\frac{zf'(z)}{f(z)} \prec \frac{1}{1+z} \tag{2.13}$$

then $\frac{f(z)}{z} \prec \frac{1}{1+z}$, $z \in U$.

Proof. Consider

$$p(z) = \frac{2f(z)}{z} - 1. \tag{2.14}$$

Since $f \in \mathcal{A}_n$, and f is close-to-convex function (univalent) then the function p is analytic in U, and $p(0) = 1$.

By differentiating (2.14), we obtain

$$\frac{zp'(z)}{p(z)+1} + 1 = \frac{zf'(z)}{z}. \tag{2.15}$$

Using (2.13) in (2.15), we have

$$\frac{zp'(z)}{p(z)+1} + 1 \prec \frac{1}{1+z}. \tag{2.16}$$

Since $\operatorname{Re}\frac{1}{1+z} \geq \frac{1}{2}$, differential subordination (2.16) is equivalent to

$$\operatorname{Re}\left(\frac{zp'(z)}{p(z)+1} + \frac{1}{2}\right) > 0, \quad z \in U. \tag{2.17}$$

Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$, $\psi(p(z), zp'(z); z) = \frac{zp'(z)}{p(z)+1} + \frac{1}{2}$, then (2.17) becomes $\operatorname{Re}\psi(p(z), zp'(z); z) > 0$, $z \in U$.

In order to prove Theorem 2.5, we use Lemma 1.3. For that we calculate $\operatorname{Re}\psi(is,t) = \operatorname{Re}\left(\frac{t}{1+is} + \frac{1}{2}\right) = \operatorname{Re}\left[\frac{t(1-is)}{1+s^2} + \frac{1}{2}\right] = \frac{t}{1+s^2} + \frac{1}{2} \leq \frac{-n(1+s^2)}{2(1+s^2)} + \frac{1}{2} = \frac{1-n}{2} < 0$. Using Definition 1.2, we have $\psi \in \Psi_n[h,q]$. Therefore by Lemma 1.3, we conclude that $p(z) \prec q(z)$, i.e., $\frac{f(z)}{z} \prec \frac{1}{1+z}$, for $z \in U$. ∎

Theorem 2.6 *If $f \in \mathcal{A}_n$, $n \geq 3$, $\gamma \geq 1$ and satisfy the condition* $\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}+1\right) > -\frac{1}{2\gamma}$, *then* $\operatorname{Re}\sqrt{f'(z)} > \frac{1}{2}$, *for $z \in U$.*

Proof. Consider $p(z) = 2\sqrt{f'(z)} - 1$, $z \in U$. Since $f \in \mathcal{A}_n$, $n \geq 3$, and f is close-to-convex function (univalent) then the function p is analytic in U and $p(0) = 1$. A simple computation leads to

$$\frac{p(z)+1}{2} = \sqrt{f'(z)}. \tag{2.18}$$

By differentiating (2.18), we have $\frac{2zp'(z)}{1+p(z)} + 1 = \frac{zf''(z)}{f'(z)} + 1$. Using (2.1), we have

$$\operatorname{Re}\left[\frac{2zp'(z)}{1+p(z)} + 1 + \frac{1}{2\gamma}\right] > 0. \tag{2.19}$$

If we let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$, $\psi(p(z), zp'(z)) = \frac{2zp'(z)}{1+p(z)} + \frac{1+2\gamma}{2\gamma}$, then (2.19) becomes $\operatorname{Re}\psi(p(z), zp'(z)) > 0$.

In order to prove Theorem 2.6, we use Lemma 1.1. For that, we calculate $\operatorname{Re}\psi(is,t) = \operatorname{Re}\left(\frac{2t}{1+is} + \frac{1+2\gamma}{2\gamma}\right)$ $= \operatorname{Re}\left(\frac{2t(1-is)}{1+s^2} + \frac{1+2\gamma}{2\gamma}\right) = \frac{2t}{1+s^2} + \frac{1+2\gamma}{2\gamma} \leq \frac{-n(1+s^2)}{1+s^2} + \frac{1+2\gamma}{2\gamma} = \frac{-2\gamma n+1+2\gamma}{2\gamma} = \frac{2\gamma(1-n)+1}{2\gamma} \leq 0$, since $n \geq 3$, $\gamma \geq 1$.

Using Lemma 1.1, we have $\operatorname{Re} p(z) > 0$, i.e., $\operatorname{Re}\sqrt{f'(z)} > \frac{1}{2}$. ■

For $0 < \gamma < 1$, $n \geq 3$, Theorem 2.6 can be written as the following corollary.

Corollary 2.7 *If $f \in \mathcal{A}_n$, $n \geq 3$, $0 < \gamma < 1$, satisfy the condition* $\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}+1\right) > -\frac{\gamma}{2}$, *then* $\operatorname{Re}\sqrt{f'(z)} > \frac{1}{2}$, *for $z \in U$.*

In differential subordination language Theorem 2.6 can be written as

Theorem 2.8 *If $f \in \mathcal{A}_n$, $n \geq 3$, $\gamma \geq 1$, and satisfy the differential subordination*

$$\frac{zf''(z)}{f'(z)} + 1 \prec \frac{1-\left(\frac{1}{\gamma}+1\right)z}{1+z}, \tag{2.20}$$

then $\sqrt{f'(z)} \prec \frac{1}{1+z}$, *for $z \in U$.*

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A finite difference method for Burgers' equation in the unbounded domain using artificial boundary conditions *

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Abstract: This paper discusses the numerical solution of one-dimensional Burgers' equation in the infinite domain. The original problem is converted by Hopf-Cole transformation to the heat equation in the infinite domain, the latter is reduced to an equivalent problem in a finite computational domain with two artificial integral boundary conditions, a finite difference method is constructed for last problem by the method of reduction of order, and therefore the numerical solution of Burgers' equation is obtained. The method is proved and verified to be uniquely solvable, unconditionally stable and convergent with the order 2 in space and the order 3/2 in time for solving the heat equation as well as Burgers' equation in the computational domain.

Keywords: Burgers' equation; infinite domain; Hopf-Cole transformation; Artificial boundary condition; Finite difference method

1 Introduction

When an analytic solution is not available, or the analytic one is not suitable to be used, a numerical method is necessary for solving partial differential equations. Therefore, several kinds of exterior problems in the areas of heat transfer, fluid dynamics and other applications were solved numerically by using artificial boundary conditions [1-5].

The artificial boundary methods were established on bounded computational domains for various problems of heat equation on unbounded domains and the feasibility and effectiveness of the methods were shown by the numerical examples [6, 7]. Moreover, for the heat equation in

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a semi-unbounded domain $[-1,\infty)\times[0,\infty)$, by using an artificial integral boundary condition

$$u_x(0,t) = -\frac{1}{\sqrt{\pi}}\int_0^t \frac{u_\lambda(0,\lambda)}{\sqrt{t-\lambda}}d\lambda,$$

Sun and Wu [8] firstly proved that the finite difference scheme is uniquely solvable, unconditionally stable and convergent with the order 2 in space and the order 3/2 in time under an energy norm. Wu and Zhang [9] also obtained the high-order artificial boundary conditions for the heat equation in unbounded domains, but only proved that the reduced initial-boundary-value problems were stable.

Furthermore, Han, Wu and Xu [10] started to consider the nonlinear Burgers' equation in the unbounded domain as follows:

$$w_t + ww_x - \nu w_{xx} = F(x,t), \quad \forall (x,t)\in\mathbb{R}\times(0,T], \tag{1.1}$$

$$w(x,0) = f(x), \quad \forall x\in\mathbb{R}, \tag{1.2}$$

$$w(x,t)\to 0, \quad \text{when } |x|\to+\infty, \quad \forall t\in[0,T], \tag{1.3}$$

where $\nu = \frac{1}{Re}$, Re is the Reynolds number, and the given functions F and f are sufficiently smooth with compact supports $\text{supp}\{F(x,t)\}\subset[x_l,x_r]\times[0,T]$ and $\text{supp}\{f(x)\}\subset[x_l,x_r]$. They obtained nonlinear artificial boundary conditions, constructed a nonlinear difference method with no theoretical convergence analysis, and supported it by numerical examples. Recently, Sun and Wu [11] introduced a function transformation to reduce nonlinear Burgers' equation to a linear initial boundary value problem, deduced a linear finite difference scheme, and also proved that the finite difference scheme is uniquely solvable, unconditionally stable and convergent with the order 2 in space and 3/2 in time.

In this paper, we consider the problem (1.1)-(1.3) with $F\equiv 0$ and convert it into an initial value problem of heat equation by using Hopf-Cole transformation in the following. Let

$$\omega(x,t) = -\int_x^\infty w(y,t)dy, \quad \forall (x,t)\in\mathbb{R}\times(0,T],$$

we obtain

$$\omega_t + \frac{1}{2}\omega_x^2 - \nu\omega_{xx} = 0,$$

$$\omega(x,0) = -\int_x^\infty f(y)dy, \quad \forall x\in\mathbb{R},$$

$$\omega(x,t)\to 0, \quad \text{when } |x|\to+\infty, \quad \forall t\in[0,T].$$

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Let $u = \exp(-\omega/2\nu) - 1$, then we have the initial value problem of heat equation:

$$u_t - \nu u_{xx} = 0, \quad \forall (x,t) \in \mathbb{R} \times (0,T], \tag{1.4}$$

$$u(x,0) = \phi(x) := \exp\left(\frac{1}{2\nu}\int_x^{\infty} f(y)dy\right) - 1, \tag{1.5}$$

$$u(x,t) \to 0, \quad \text{when } |x| \to +\infty, \quad \forall t \in [0,T], \tag{1.6}$$

where the sufficiently smooth given function $\phi(x)$ has compact support $\text{supp}\{\phi(x)\} \subset [x_l, x_r]$.

By using artificial linear integral boundary conditions similar to that in [8], we reduce the problem (1.4)-(1.6) to a problem in the bounded computational domain:

$$u_t - \nu u_{xx} = 0, \quad \forall (x,t) \in [x_l, x_r] \times [0,T], \tag{1.7}$$

$$u(x,0) = \phi(x), \quad \forall x \in [x_l, x_r], \tag{1.8}$$

$$u_x(x_l,t) = \frac{1}{\sqrt{\pi\nu}}\int_0^t \frac{u_\lambda(x_l,\lambda)}{\sqrt{t-\lambda}}d\lambda, \quad \forall t \in [0,T], \tag{1.9}$$

$$u_x(x_r,t) = -\frac{1}{\sqrt{\pi\nu}}\int_0^t \frac{u_\lambda(x_r,\lambda)}{\sqrt{t-\lambda}}d\lambda, \quad \forall t \in [0,T]. \tag{1.10}$$

In section 2, we construct a finite difference scheme for solving the problem (1.7)-(1.10). Then a new solution of Burgers' equation is obtained and the difficulty for solving the nonlinear problem is avoided. In section 3, we prove that the finite difference scheme is uniquely solvable, unconditionally stable and convergent with the order 2 in space and 3/2 in time. In section 4, a numerical example confirms the stability and convergence of the finite difference method.

2 The construction of the difference scheme

In order to construct the finite difference method, the bounded computational domain is divided into an $M \times N$ uniform mesh. Let $h = (x_r - x_l)/M$, $x_i = x_l + ih$ for $0 \le i \le M$, $\tau = T/N$, $t_n = n\tau$ for $0 \le n \le N$, $r = \frac{\nu\tau}{h^2}$, and u_i^n be the numerical solution of $u(x,t)$ at (x_i, t_n). Introduce the notations:

$$u_{i-\frac{1}{2}}^n = \frac{1}{2}(u_i^n + u_{i-1}^n), \quad \delta_x u_{i-\frac{1}{2}}^n = \frac{1}{h}(u_i^n - u_{i-1}^n), \quad u_i^{n-\frac{1}{2}} = \frac{1}{2}(u_i^n + u_i^{n-1}),$$

$$\delta_t u_i^{n-\frac{1}{2}} = \frac{1}{\tau}(u_i^n - u_i^{n-1}), \quad \delta_x^2 u_i^n = \frac{1}{h^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n),$$

$$\|u^n\|_A = \sqrt{h\sum_{i=1}^{M}(u_{i-\frac{1}{2}}^n)^2}, \quad \|\delta_x u^n\| = \sqrt{h\sum_{i=1}^{M}(\delta_x u_{i-\frac{1}{2}}^n)^2}.$$

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Lemma 2.1 *Suppose $f(t) \in C^2[0, t_n]$, then*

$$|\int_0^{t_n} f'(t)\frac{dt}{\sqrt{t_n - t}} - \sum_{k=1}^{n}\frac{f(t_k) - f(t_{k-1})}{\tau}\int_{t_{k-1}}^{t_k}\frac{dt}{\sqrt{t_n - t}}| \le \frac{1}{12}(20\sqrt{2} - 23)\max_{0\le t\le t_n}|f''(t)|\tau^{\frac{3}{2}}.$$

Proof Lemma 2.1 is proved by using $\sqrt{t_n - t} - (\frac{t_k - t}{\tau}\sqrt{t_n - t_{k-1}} + \frac{t - t_{k-1}}{\tau}\sqrt{t_n - t_k}) = \frac{1}{8}(t_n - \xi_k)^{-\frac{3}{2}}(t - t_{k-1})(t_k - t)$ to correct (2.2) and thereupon (2.1) in [8], as corrected in [12]. □

By introducing a new variable $v = \frac{\partial u}{\partial x}$ to reduce the order of heat equation, the problem (1.7)-(1.10) is equivalent to the problem of first-order differential equations:

$$\frac{\partial u}{\partial x} = \nu\frac{\partial v}{\partial x}, \quad \forall(x,t) \in [x_l, x_r] \times [0, T], \tag{2.1}$$

$$v - \frac{\partial u}{\partial x} = 0, \quad \forall(x,t) \in [x_l, x_r] \times [0, T], \tag{2.2}$$

$$u(x, 0) = \phi(x), \quad x_l \le x \le x_r, \tag{2.3}$$

$$v(x_l, t) = \frac{1}{\sqrt{\pi\nu}}\int_0^t \frac{\partial u(x_l, \lambda)}{\partial\lambda}\frac{1}{\sqrt{t - \lambda}}d\lambda, \tag{2.4}$$

$$v(x_r, t) = -\frac{1}{\sqrt{\pi\nu}}\int_0^t \frac{\partial u(x_r, \lambda)}{\partial\lambda}\frac{1}{\sqrt{t - \lambda}}d\lambda. \tag{2.5}$$

Define the grid functions:

$$U_i^n = u(x_i, t_n), \quad V_i^n = v(x_i, t_n), \quad 0 \le i \le M, \quad n \ge 0.$$

Using Lemma 2.1, it follows from (2.5) that

$$\begin{aligned} V_M^n &= -\frac{1}{\sqrt{\pi\nu}}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\frac{\partial u(x_r, \lambda)}{\partial\lambda}\frac{d\lambda}{\sqrt{t_n - \lambda}} \\ &= -\frac{1}{\sqrt{\pi\nu}}\sum_{k=1}^{n}\frac{U_M^k - U_M^{k-1}}{\tau}\int_{t_{k-1}}^{t_k}\frac{d\lambda}{\sqrt{t_n - \lambda}} + O(\tau^{\frac{3}{2}}) \\ &= -\frac{2}{\sqrt{\pi\nu}}\sum_{k=1}^{n}(U_M^k - U_M^{k-1})a_{n-k} + O(\tau^{\frac{3}{2}}) \\ &= -\frac{2}{\sqrt{\pi\nu}}[a_0U_M^n - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})U_M^k - a_{n-1}U_M^0] + O(\tau^{\frac{3}{2}}), \quad n = 1, 2, \cdots. \end{aligned}$$

Therefore, we have

$$V_M^{n-\frac{1}{2}} = \frac{1}{2}(V_M^{n-1} + V_M^n) = -\frac{2}{\sqrt{\pi\nu}}[a_0U_M^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})U_M^{k-\frac{1}{2}} - a_{n-1}U_M^0] + O(\tau^{\frac{3}{2}}),$$

and similarly,

$$V_0^{n-\frac{1}{2}} = \frac{1}{2}(V_0^{n-1} + V_0^n) = \frac{2}{\sqrt{\pi\nu}}[a_0U_0^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})U_0^{k-\frac{1}{2}} - a_{n-1}U_0^0] + O(\tau^{\frac{3}{2}}).$$

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Using Taylor expansion, we have

$$\delta_t U_{i-\frac{1}{2}}^{n-\frac{1}{2}} - \nu \delta_x V_{i-\frac{1}{2}}^{n-\frac{1}{2}} = p_{i-\frac{1}{2}}^{n-\frac{1}{2}}, \quad 1 \le i \le M, \quad n \ge 1, \tag{2.6}$$

$$V_{i-\frac{1}{2}}^{n-\frac{1}{2}} - \delta_x U_{i-\frac{1}{2}}^{n-\frac{1}{2}} = q_{i-\frac{1}{2}}^{n-\frac{1}{2}}, \quad 1 \le i \le M, \quad n \ge 1, \tag{2.7}$$

$$U_i^0 = \phi(x_i), \quad 0 \le i \le M, \tag{2.8}$$

$$V_0^{n-\frac{1}{2}} = \frac{2}{\sqrt{\pi\nu}}[a_0 U_0^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k}) U_0^{k-\frac{1}{2}} - a_{n-1} U_0^0] + s^{n-\frac{1}{2}}, \quad n \ge 1, \tag{2.9}$$

$$V_M^{n-\frac{1}{2}} = -\frac{2}{\sqrt{\pi\nu}}[a_0 U_M^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k}) U_M^{k-\frac{1}{2}} - a_{n-1} U_M^0] + t^{n-\frac{1}{2}}, \quad n \ge 1, \tag{2.10}$$

where

$$|p_{i-\frac{1}{2}}^{n-\frac{1}{2}}| \le c(\tau^2 + h^2), \quad |q_{i-\frac{1}{2}}^{n-\frac{1}{2}}| \le c(\tau^2 + h^2), \quad 1 \le i \le M, \quad n \ge 1, \tag{2.11}$$

$$|t^{n-\frac{1}{2}}| \le c\tau^{\frac{3}{2}}, \quad |s^{n-\frac{1}{2}}| \le c\tau^{\frac{3}{2}}, \quad n \ge 1, \tag{2.12}$$

and c is a constant.

Thus, we construct a difference scheme for (2.1)-(2.5) in the following:

$$\delta_t u_{i-\frac{1}{2}}^{n-\frac{1}{2}} - \nu \delta_x v_{i-\frac{1}{2}}^{n-\frac{1}{2}} = 0, \quad 1 \le i \le M, \quad n \ge 1, \tag{2.13}$$

$$v_{i-\frac{1}{2}}^{n-\frac{1}{2}} - \delta_x u_{i-\frac{1}{2}}^{n-\frac{1}{2}} = 0, \quad 1 \le i \le M, \quad n \ge 1, \tag{2.14}$$

$$u_i^0 = \phi(x_i), \quad 0 \le i \le M, \tag{2.15}$$

$$v_0^{n-\frac{1}{2}} = \frac{2}{\sqrt{\pi\nu}}[a_0 u_0^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k}) u_0^{k-\frac{1}{2}} - a_{n-1} u_0^0], \quad n \ge 1, \tag{2.16}$$

$$v_M^{n-\frac{1}{2}} = -\frac{2}{\sqrt{\pi\nu}}[a_0 u_M^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k}) u_M^{k-\frac{1}{2}} - a_{n-1} u_M^0]. \quad n \ge 1, \tag{2.17}$$

Theorem 2.2 *The difference scheme (2.13)-(2.17) is equivalent to the following (2.18)-(2.22):*

$$u_i^0 = \phi(x_i), \quad 0 \le i \le M, \tag{2.18}$$

$$\frac{1}{2}(\delta_t u_{i-\frac{1}{2}}^{n-\frac{1}{2}} + \delta_t u_{i+\frac{1}{2}}^{n-\frac{1}{2}}) - \nu \delta_x^2 u_i^{n-\frac{1}{2}} = 0, \quad 1 \le i \le M-1, \quad n \ge 1, \tag{2.19}$$

$$\delta_t u_{\frac{1}{2}}^{n-\frac{1}{2}} + \frac{2\nu}{h}[\frac{2}{\sqrt{\pi\nu}}(a_0 u_0^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k}) u_0^{k-\frac{1}{2}} - a_{n-1} u_0^0) - \delta_x u_{\frac{1}{2}}^{n-\frac{1}{2}}], \quad n \ge 1, \tag{2.20}$$

$$\delta_t u_{M-\frac{1}{2}}^{n-\frac{1}{2}} + \frac{2\nu}{h}[\frac{2}{\sqrt{\pi\nu}}(a_0 u_M^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k}) u_0^{k-\frac{1}{2}} - a_{n-1} u_M^0) + \delta_x u_{M-\frac{1}{2}}^{n-\frac{1}{2}}], \quad n \ge 1, \tag{2.21}$$

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where

$$a_m = \frac{1}{\sqrt{t_{m+1}} + \sqrt{t_m}} = \frac{1}{\sqrt{\tau}(\sqrt{m+1} + \sqrt{m})}, \quad m = 0, 1, 2, \cdots. \tag{2.22}$$

Proof Multiplying (2.13) by $\frac{1}{2}h$ and using (2.14) we obtain

$$v_i^{n-\frac{1}{2}} = \delta_x u_{i-\frac{1}{2}}^{n-\frac{1}{2}} + \frac{h}{2\nu}\delta_t u_{i-\frac{1}{2}}^{n-\frac{1}{2}}, \quad 1 \le i \le M, \quad n \ge 1, \tag{2.23}$$

$$v_i^{n-\frac{1}{2}} = \delta_x u_{i+\frac{1}{2}}^{n-\frac{1}{2}} - \frac{h}{2\nu}\delta_t u_{i+\frac{1}{2}}^{n-\frac{1}{2}}, \quad 0 \le i \le M-1, \quad n \ge 1, \tag{2.24}$$

From (2.23) and (2.24) for i from 1 to $M-1$ we obtain

$$\delta_x u_{i-\frac{1}{2}}^{n-\frac{1}{2}} + \frac{h}{2\nu}\delta_t u_{i-\frac{1}{2}}^{n-\frac{1}{2}} = \delta_x u_{i+\frac{1}{2}}^{n-\frac{1}{2}} - \frac{h}{2\nu}\delta_t u_{i+\frac{1}{2}}^{n-\frac{1}{2}}, \quad 1 \le i \le M-1, \quad n \ge 1,$$

or

$$\frac{1}{2}(\delta_t u_{i-\frac{1}{2}}^{n-\frac{1}{2}} + \delta_t u_{i+\frac{1}{2}}^{n-\frac{1}{2}}) - \nu\delta_x^2 u_i^{n-\frac{1}{2}} = 0, \quad 1 \le i \le M-1, \quad n \ge 1,$$

which is (2.19).

When $i = 0$, from (2.16) and (2.24), we know that

$$\frac{2\sqrt{\nu}}{\sqrt{\pi}}[a_0 u_0^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})u_0^{k-\frac{1}{2}} - a_{n-1}u_0^0] = \nu\delta_x u_{\frac{1}{2}}^{n-\frac{1}{2}} - \frac{h}{2}\delta_t u_{\frac{1}{2}}^{n-\frac{1}{2}}.$$

Dividing by $h/2$ on the both sides we obtain (2.20).

Similarly, when $i = M$, from (2.17) and (2.23), we know that

$$-\frac{2\sqrt{\nu}}{\sqrt{\pi}}[a_0 u_M^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})u_M^{k-\frac{1}{2}} - a_{n-1}u_M^0] = \nu\delta_x u_{M-\frac{1}{2}}^{n-\frac{1}{2}} + \frac{h}{2}\delta_t u_{M-\frac{1}{2}}^{n-\frac{1}{2}},$$

Dividing by $h/2$ on the both sides we obtain (2.21). □

The difference scheme (2.18)-(2.21) can be sorted as the following:

$$(\frac{1}{2}-r)u_{i+1}^n + (1+2r)u_i^n + (\frac{1}{2}-r)u_{i-1}^n = (\frac{1}{2}+r)u_{i+1}^{n-1} + (1-2r)u_i^{n-1} + (\frac{1}{2}+r)u_{i-1}^{n-1}, 1 \le i \le M-1, \tag{2.25}$$

$$\begin{aligned}(1+2r+\frac{4\sqrt{r}}{\sqrt{\pi}})u_0^n + (1-2r)u_1^n &= (1-2r-\frac{4\sqrt{r}}{\sqrt{\pi}})u_0^{n-1} + (1+2r)u_1^{n-1} \\ &+ \frac{4\sqrt{r\tau}}{\sqrt{\pi}}\sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})(u_0^k + u_0^{k-1}) + \frac{8\sqrt{r\tau}}{\sqrt{\pi}}a_{n-1}u_0^0,\end{aligned} \tag{2.26}$$

$$\begin{aligned}(1+2r+\frac{4\sqrt{r}}{\sqrt{\pi}})u_M^n + (1-2r)u_{M-1}^n &= (1-2r-\frac{4\sqrt{r}}{\sqrt{\pi}})u_M^{n-1} + (1+2r)u_{M-1}^{n-1} \\ &+ \frac{4\sqrt{r\tau}}{\sqrt{\pi}}\sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})(u_M^k + u_M^{k-1}) + \frac{8\sqrt{r\tau}}{\sqrt{\pi}}a_{n-1}u_M^0.\end{aligned} \tag{2.27}$$

3 The error estimate of the difference scheme

Lemma 3.1 *For any $F = \{F_1, F_2, F_3, \cdots\}$, we have*

$$\sum_{l=1}^{n}[a_0F_l - \sum_{k=1}^{l-1}(a_{l-k-1} - a_{l-k})F_k]F_l \geq \frac{1}{2\sqrt{t_n}}\sum_{l=1}^{n}F_l^2, \quad n = 1, 2, \cdots,$$

where a_m is defined in (2.22).

Proof Let $b_m = a_{m-1} - a_m = \frac{1}{\sqrt{\tau}}(\frac{1}{\sqrt{m}+\sqrt{m-1}} - \frac{1}{\sqrt{m+1}+\sqrt{m}})$, $m \geq 1$, then $b_m > 0$, and

$$\begin{aligned}
&\sum_{l=1}^{n}[a_0F_l - \sum_{k=1}^{l-1}(a_{l-k-1} - a_{l-k})F_k]F_l \\
= &\sum_{l=1}^{n}a_0F_l^2 - \sum_{l=1}^{n}\sum_{m=1}^{l-1}(a_{m-1} - a_m)F_{l-m}F_l \\
\geq &\sum_{l=1}^{n}a_0F_l^2 - \frac{1}{2}\sum_{l=1}^{n}\sum_{m=1}^{l-1}b_m(F_{l-m}^2 + F_l^2) \\
= &\sum_{l=1}^{n}a_0F_l^2 - \frac{1}{2}\sum_{l=1}^{n}\sum_{m=1}^{l-1}b_{l-m}F_m^2 - \frac{1}{2}\sum_{l=1}^{n}\sum_{m=1}^{l-1}b_mF_l^2 \\
= &\sum_{l=1}^{n}a_0F_l^2 - \frac{1}{2}\sum_{m=1}^{n}\sum_{l=m+1}^{n}b_{l-m}F_m^2 - \frac{1}{2}\sum_{l=1}^{n}\sum_{m=1}^{l-1}b_mF_l^2 \\
\geq &\sum_{l=1}^{n}a_0F_l^2 - (\sum_{m=1}^{n-1}b_m)\sum_{l=1}^{n}F_l^2 \\
= &[\frac{1}{\sqrt{\tau}} - \frac{1}{\sqrt{\tau}}(1 - \frac{1}{\sqrt{n}+\sqrt{n-1}})]\sum_{l=1}^{n}F_l^2 \\
\geq &\frac{1}{2\sqrt{t_n}}\sum_{l=1}^{n}F_l^2. \quad \square
\end{aligned}$$

Lemma 3.2 *Suppose $\{u_i^n\}$ be the solution of*

$$\delta_t u_{i-\frac{1}{2}}^{n-\frac{1}{2}} - \nu\delta_x v_{i-\frac{1}{2}}^{n-\frac{1}{2}} = P_{i-\frac{1}{2}}^{n-\frac{1}{2}}, \quad 1 \leq i \leq M, \quad n \geq 1, \tag{3.1}$$

$$v_{i-\frac{1}{2}}^{n-\frac{1}{2}} - \delta_x u_{i-\frac{1}{2}}^{n-\frac{1}{2}} = Q_{i-\frac{1}{2}}^{n-\frac{1}{2}}, \quad 1 \leq i \leq M, \quad n \geq 1, \tag{3.2}$$

$$u_i^0 = \phi(x_i), \quad 0 \leq i \leq M, \tag{3.3}$$

$$v_0^{n-\frac{1}{2}} = \frac{2}{\sqrt{\pi\nu}}[a_0u_0^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})u_0^{k-\frac{1}{2}} - a_{n-1}u_0^0] + S^{n-\frac{1}{2}}, \quad n \geq 1, \tag{3.4}$$

$$v_M^{n-\frac{1}{2}} = -\frac{2}{\sqrt{\pi\nu}}[a_0u_M^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})u_M^{k-\frac{1}{2}} - a_{n-1}u_M^0] + T^{n-\frac{1}{2}}. \quad n \geq 1, \tag{3.5}$$

where $Supp\{\phi(x)\} \subset [x_0, x_M]$, then

$$\begin{aligned}\|u^n\|_A^2 \quad &\le \quad \exp(\tfrac{2T}{4-\tau}) \cdot \tfrac{1}{1-\frac{\tau}{4}}\{\|u^0\|_A^2 + \tfrac{\sqrt{\pi\nu t_n}}{2}\tau \textstyle\sum_{l=1}^{n}[(T^{l-\frac{1}{2}})^2 + (S^{l-\frac{1}{2}})^2] \\ &\quad +2\tau \textstyle\sum_{l=1}^{n}(\|P^{l-\frac{1}{2}}\|_A^2 + \|Q^{l-\frac{1}{2}}\|_A^2)\}, \quad n = 1, 2, \cdots .\end{aligned} \tag{3.6}$$

Proof Multiplying (3.1) by $2u_{i-\frac{1}{2}}^{n-\frac{1}{2}}$ and multiplying (3.2) by $2v_{i-\frac{1}{2}}^{n-\frac{1}{2}}$, then adding the results, we have

$$\begin{aligned}&\tfrac{1}{\tau}[(u_{i-\frac{1}{2}}^{n})^2 - (u_{i-\frac{1}{2}}^{n-1})^2] + 2(v_{i-\frac{1}{2}}^{n-\frac{1}{2}})^2 = \tfrac{2}{h}(u_i^{n-\frac{1}{2}}v_i^{n-\frac{1}{2}} - u_{i-1}^{n-\frac{1}{2}}v_{i-1}^{n-\frac{1}{2}}) + 2u_{i-\frac{1}{2}}^{n-\frac{1}{2}}P_{i-\frac{1}{2}}^{n-\frac{1}{2}} + 2v_{i-\frac{1}{2}}^{n-\frac{1}{2}}Q_{i-\frac{1}{2}}^{n-\frac{1}{2}} \\ &\le \tfrac{2}{h}(u_i^{n-\frac{1}{2}}v_i^{n-\frac{1}{2}} - u_{i-1}^{n-\frac{1}{2}}v_{i-1}^{n-\frac{1}{2}}) + \tfrac{1}{2}(u_{i-\frac{1}{2}}^{n-\frac{1}{2}})^2 + 2(P_{i-\frac{1}{2}}^{n-\frac{1}{2}})^2 + \tfrac{1}{2}(v_{i-\frac{1}{2}}^{n-\frac{1}{2}})^2 + 2(Q_{i-\frac{1}{2}}^{n-\frac{1}{2}})^2, \\ &\qquad 1 \le i \le M, n \ge 1.\end{aligned} \tag{3.7}$$

Multiplying the above inequality by τh and summing up for i from 1 to M, we obtain

$$\begin{aligned}(\|u^n\|_A^2 - \|u^{n-1}\|_A^2) + 2\tau\|v^{n-\frac{1}{2}}\|_A^2 \quad &\le \quad 2\tau(u_M^{n-\frac{1}{2}}v_M^{n-\frac{1}{2}} - u_0^{n-\frac{1}{2}}v_0^{n-\frac{1}{2}}) + \frac{\tau}{2}\|u^{n-\frac{1}{2}}\|_A^2 + \frac{\tau}{2}\|v^{n-\frac{1}{2}}\|_A^2 \\ &\quad +2\tau\|P^{n-\frac{1}{2}}\|_A^2 + 2\tau\|Q^{n-\frac{1}{2}}\|_A^2, \quad n \ge 1.\end{aligned} \tag{3.8}$$

Noticing $\frac{\tau}{2}\|u^{n-\frac{1}{2}}\|_A^2 \le \frac{\tau}{4}(\|u^n\|_A^2 + \|u^{n-1}\|_A^2)$, thus

$$\begin{aligned}\|u^l\|_A^2 - \|u^{l-1}\|_A^2 \quad &\le \quad 2\tau(u_M^{l-\frac{1}{2}}v_M^{l-\frac{1}{2}} - u_0^{l-\frac{1}{2}}v_0^{l-\frac{1}{2}}) + \frac{\tau}{4}(\|u^l\|_A^2 + \|u^{l-1}\|_A^2) \\ &\quad +2\tau\|P^{l-\frac{1}{2}}\|_A^2 + 2\tau\|Q^{l-\frac{1}{2}}\|_A^2, \quad l = 1, 2, \ldots, n.\end{aligned}$$

Summing up for l from 1 to n, we have

$$\begin{aligned}\|u^n\|_A^2 \quad &\le \quad \|u^0\|_A^2 + 2\tau\sum_{l=1}^{n}(u_M^{l-\frac{1}{2}}v_M^{l-\frac{1}{2}} - u_0^{l-\frac{1}{2}}v_0^{l-\frac{1}{2}}) \\ &\quad +\frac{\tau}{4}\|u^n\|_A^2 + \frac{\tau}{2}\sum_{l=0}^{n-1}\|u^l\|_A^2 + 2\tau\sum_{l=1}^{n}(\|P^{l-\frac{1}{2}}\|_A^2 + \|Q^{l-\frac{1}{2}}\|_A^2).\end{aligned}$$

Substituting (3.4) and (3.5) into the above inequality, and using Lemma 3.1, we have

$$\begin{aligned}\|u^n\|_A^2 \quad &\le \quad \frac{1}{1-\frac{\tau}{4}}[\|u^0\|_A^2 + 2\tau\sum_{l=1}^{n}(u_M^{l-\frac{1}{2}}v_M^{l-\frac{1}{2}} - u_0^{l-\frac{1}{2}}v_0^{l-\frac{1}{2}}) + 2\tau\sum_{l=1}^{n}(\|P^{l-\frac{1}{2}}\|_A^2 + \|Q^{l-\frac{1}{2}}\|_A^2) \\ &\quad +\frac{\tau}{2}\sum_{l=0}^{n-1}\|u^l\|_A^2]\end{aligned}$$

$$
\begin{aligned}
= \quad & \frac{1}{1-\frac{\tau}{4}}\|u^0\|_A^2 + \frac{2\tau}{1-\frac{\tau}{4}}\cdot(-\frac{2}{\sqrt{\pi\nu}})\sum_{l=1}^{n}[a_0 u_M^{l-\frac{1}{2}} - \sum_{k=1}^{l-1}(a_{l-k-1}-a_{l-k})u_M^{k-\frac{1}{2}}]u_M^{l-\frac{1}{2}} \\
& + \frac{2\tau}{1-\frac{\tau}{4}}\sum_{l=1}^{n} u_M^{l-\frac{1}{2}} T^{l-\frac{1}{2}} - \frac{2\tau}{1-\frac{\tau}{4}}\cdot(\frac{2}{\sqrt{\pi\nu}})\sum_{l=1}^{n}[a_0 u_0^{l-\frac{1}{2}} - \sum_{k=1}^{l-1}(a_{l-k-1}-a_{l-k})u_0^{k-\frac{1}{2}}]u_0^{l-\frac{1}{2}} \\
& + \frac{2\tau}{1-\frac{\tau}{4}}\sum_{l=1}^{n} u_0^{l-\frac{1}{2}} S^{l-\frac{1}{2}} + \frac{2\tau}{1-\frac{\tau}{4}}\sum_{l=1}^{n}(\|P^{l-\frac{1}{2}}\|_A^2 + \|Q^{l-\frac{1}{2}}\|_A^2) + \frac{2\tau}{4-\tau}\sum_{l=0}^{n-1}\|u^l\|_A^2 \\
\leq \quad & \frac{1}{1-\frac{\tau}{4}}\|u^0\|_A^2 - \frac{2\tau}{1-\frac{\tau}{4}}\cdot\frac{2}{\sqrt{\pi\nu}}\cdot\frac{1}{2\sqrt{t_n}}\sum_{l=1}^{n}(u_M^{l-\frac{1}{2}})^2 + \frac{\tau}{1-\frac{\tau}{4}}(\frac{2}{\sqrt{\pi\nu t_n}}\sum_{l=1}^{n}(u_M^{l-\frac{1}{2}})^2 \\
& + \frac{\sqrt{\pi\nu t_n}}{2}\sum_{l=1}^{n}(T^{l-\frac{1}{2}})^2) - \frac{2\tau}{1-\frac{\tau}{4}}\cdot\frac{2}{\sqrt{\pi\nu}}\cdot\frac{1}{2\sqrt{t_n}}\sum_{l=1}^{n}(u_0^{l-\frac{1}{2}})^2 \\
& + \frac{\tau}{1-\frac{\tau}{4}}(\frac{2}{\sqrt{\pi\nu t_n}}\sum_{l=1}^{n}(u_0^{l-\frac{1}{2}})^2 + \frac{\sqrt{\pi\nu t_n}}{2}\sum_{l=1}^{n}(S^{l-\frac{1}{2}})^2) \\
& + \frac{2\tau}{1-\frac{\tau}{4}}\sum_{l=1}^{n}(\|P^{l-\frac{1}{2}}\|_A^2 + \|Q^{l-\frac{1}{2}}\|_A^2) + \frac{2\tau}{4-\tau}\sum_{l=0}^{n-1}\|u^l\|_A^2 \\
\leq \quad & \frac{1}{1-\frac{\tau}{4}}\|u^0\|_A^2 + \frac{\tau}{1-\frac{\tau}{4}}\frac{\sqrt{\pi\nu t_n}}{2}\sum_{l=1}^{n}[(T^{l-\frac{1}{2}})^2 + (S^{l-\frac{1}{2}})^2] \\
& + \frac{2\tau}{1-\frac{\tau}{4}}\sum_{l=1}^{n}(\|P^{l-\frac{1}{2}}\|_A^2 + \|Q^{l-\frac{1}{2}}\|_A^2) + \frac{2\tau}{4-\tau}\sum_{l=0}^{n-1}\|u^l\|_A^2, \quad n=1,2,\cdots.
\end{aligned}
$$

Using Gronwall's lemma, we can obtain (3.6). □

Theorem 3.3 *The difference scheme (2.18)-(2.22) is uniquely solvable.*

Proof By Theorem 2.2, it suffices to prove that the difference scheme (2.13)-(2.17) is solvable uniquely. When initial value is homogeneous, by Lemma 3.2, we have $\|u^n\|_A^2 = 0, n = 1, 2, \cdots$. □

Theorem 3.4 *Let $\{u_i^n | 0 \leq i \leq M, n \geq 1\}$ be the solution of (2.18)-(2.22), then*

$$\|u^n\|_A^2 \leq \frac{\exp(\frac{2T}{4-\tau})}{1-\frac{\tau}{4}}\|u^0\|_A^2, \quad n=1,2,\cdots. \tag{3.9}$$

Proof From Theorem 2.2, it suffices to prove that (3.9) hold for the difference scheme (2.13)-(2.17). Therefore, (3.9) follows directly from Lemma 3.2. □

Theorem 3.5 *Suppose (1.4)-(1.6) have solution $u(x,t) \in C_{x,t}^{4,3}(\mathbb{R}\times[0,T])$. Let $\{u_i^n\}$ be the solution of (2.18)-(2.22), and let $\tilde{u}_i^n = U_i^n - u_i^n$, then*

$$\|\tilde{u}^n\|_A^2 \leq \frac{CT}{4-\tau}(\sqrt{\pi\nu T}+4)\exp(\frac{2T}{4-\tau})(\tau^{\frac{3}{2}}+h^2)^2, \quad n=1,2,\cdots,[T/\tau], \tag{3.10}$$

where C is a constant independent of τ and h.

Proof Subtracting (2.13)-(2.17) from (2.6)-(2.10), respectively, we obtain the error equations:

$$\delta_t \tilde{u}_{i-\frac{1}{2}}^{n-\frac{1}{2}} - \nu \delta_x \tilde{v}_{i-\frac{1}{2}}^{n-\frac{1}{2}} = p_{i-\frac{1}{2}}^{n-\frac{1}{2}}, \quad 1 \le i \le M, \quad n \ge 1, \tag{3.11}$$

$$\tilde{v}_{i-\frac{1}{2}}^{n-\frac{1}{2}} - \delta_x \tilde{u}_{i-\frac{1}{2}}^{n-\frac{1}{2}} = q_{i-\frac{1}{2}}^{n-\frac{1}{2}}, \quad 1 \le i \le M, \quad n \ge 1, \tag{3.12}$$

$$\tilde{u}_i^0 = 0, \quad 0 \le i \le M, \tag{3.13}$$

$$\tilde{v}_0^{n-\frac{1}{2}} = \frac{2}{\sqrt{\pi\nu}}[a_0 \tilde{u}_0^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})\tilde{u}_0^{k-\frac{1}{2}} - a_{n-1}\tilde{u}_0^0] + s^{n-\frac{1}{2}}, \quad n \ge 1, \tag{3.14}$$

$$\tilde{v}_M^{n-\frac{1}{2}} = -\frac{2}{\sqrt{\pi\nu}}[a_0 \tilde{u}_M^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})\tilde{u}_M^{k-\frac{1}{2}} - a_{n-1}\tilde{u}_M^0] + t^{n-\frac{1}{2}}, \quad n \ge 1. \tag{3.15}$$

Using Lemma 3.2 and applying (2.11), (2.12) and (3.13), we obtain

$$\begin{aligned} \|\tilde{u}^n\|_A^2 &= \exp(\frac{2T}{4-\tau}) \cdot \frac{1}{1-\frac{\tau}{4}} \cdot \{\|\tilde{u}^0\|_A^2 + \frac{\sqrt{\pi\nu t_n}}{2}\tau \sum_{l=1}^{n}[(t^{l-\frac{1}{2}})^2 + (s^{l-\frac{1}{2}})^2] \\ &\quad + 2\tau \sum_{l=1}^{n}(\|p^{l-\frac{1}{2}}\|^2 + \|q^{l-\frac{1}{2}}\|^2)\} \\ &\le \frac{CT}{4-\tau}(\sqrt{\pi\nu T} + 4)\exp(\frac{2T}{4-\tau})(\tau^{\frac{3}{2}} + h^2)^2, \quad n = 1, 2, \cdots, [T/\tau]. \quad \square \end{aligned}$$

Theorem 3.5 shows that the convergence order of (2.18)-(2.21) is 2 in space and 3/2 in time for the problem (1.7)-(1.10). Finally, the numerical solution of Burgers' equation is obtained by

$$w_i^n = -\frac{\nu}{h}\frac{u_{i+1}^n - u_{i-1}^n}{1 + u_i^n}, \tag{3.16}$$

which keeps the corresponding unique solvability, unconditional stability and convergence.

4 The numerical example

For the problem of Burgers' equation with an initial condition $f(x) = -\frac{8\nu x(x^2-9)}{(x^2-9)^2+1}$ in the support $[x_l, x_r] = [-3, 3]$, the exact solution is $w(x,t) = 2\nu \frac{\frac{1}{2\sqrt{\pi\nu t}}\int_{-3}^{3}\frac{x-\xi}{2\nu t}(\xi^2-9)^2\exp(-\frac{(x-\xi)^2}{4\nu t})d\xi}{1+\frac{1}{2\sqrt{\pi\nu t}}\int_{-3}^{3}(\xi^2-9)^2\exp(-\frac{(x-\xi)^2}{4\nu t})d\xi}$. The numerical solutions are obtained by the proposed scheme, then the convergence order w.r.t h is shown in Table 1, and the convergence order w.r.t τ is shown in Table 2.

Table 1. Convergence w.r.t. h of the problem for $T = 1$, $\nu = 0.1$, $\tau = 0.01$ and $\tau = h^{4/3}$ respectively.

M	N	L^∞-error	order	L^2-error	order	N	L^∞-error	order	L^2-error	order
50	100	2.2705e-3	—	2.0737e-3	—	9	3.1455e-3	—	2.5729e-3	—
100	100	6.0651e-4	1.9044	5.5643e-4	1.8979	22	7.6893e-4	2.0324	6.5174e-4	1.9810
200	100	1.6444e-4	1.8830	1.4962e-4	1.8949	54	1.8419e-4	2.0617	1.6620e-4	1.9714
400	100	5.0024e-5	1.7169	4.5653e-5	1.7125	137	4.5577e-5	2.0148	4.1607e-5	1.9980
800	100	3.0569e-5	0.7106	1.9714e-5	1.2115	345	1.1295e-5	2.0126	1.0393e-5	2.0012

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Table 2. Convergence w.r.t. τ of the problem for $T = 1$, $\nu = 0.1$, $h = 0.002$ and $h = \tau^{3/4}$ respectively.

N	M	L^∞-error	order	L^2-error	order	M	L^∞-error	order	L^2-error	order
20	3000	1.0398e-3	—	2.1342e-4	—	95	8.7265e-4	—	7.2610e-4	—
40	3000	3.6910e-4	1.4942	6.2138e-5	1.7801	159	2.9735e-4	1.5532	2.6197e-4	1.4708
80	3000	1.0386e-4	1.8294	1.8884e-5	1.7183	267	1.0258e-4	1.5354	9.3238e-5	1.4904
160	3000	2.6518e-5	1.9696	6.3713e-6	1.5675	450	3.5936e-5	1.5132	3.2868e-5	1.5042
320	3000	1.5322e-5	0.7914	2.6822e-6	1.2482	757	1.2623e-5	1.5094	1.1614e-5	1.5008

5 Conclusions

In this works, a new finite difference method for Burgers' equation in the unbounded domain is presented by (2.18), (2.25)-(2.27) and (3.16) succinctly. The inequality in Lemma 2.1 is slightly stronger than Lemma 1 in [8]. Lemma 3.2 is proved by using Gronwall's lemma, but for heat equation in the semi-infinite domain, similar Lemma 4 in [8], i.e. Lemma 3.2.4 in [12], was incorrectly proved by not using Gronwall's lemma, and the lemma can be modified and proved as Lemma 3.2. Finally, the proposed method is clearly proved and verified to be uniquely solvable, unconditionally stable and convergent with the order 2 in space and the order 3/2 in time to solve Burgers' equation in the unbounded domain.

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Barnes-type Peters polynomials associated with poly-Cauchy polynomials of the second kind

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Abstract

In this paper, by considering Barnes-type Peters polynomials of the second kind as well as poly-Cauchy polynomials of the second kind, we define and investigate the mixed-type polynomials of these polynomials. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

1

1 Introduction

In this paper, we consider the polynomials

$$\widehat{s}_n^{(k)}(x) = \widehat{s}_n^{(k)}(x|\lambda;\mu) = \widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)$$

called the Barnes-type Peters of the second kind and poly-Cauchy of the second kind mixed-type polynomials, whose generating function is given by

$$\prod_{j=1}^r \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j} \mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)(1+t)^x = \sum_{n=0}^\infty \widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\frac{t^n}{n!}, \tag{1}$$

where $\lambda_1,\dots,\lambda_r,\mu_1,\dots,\mu_r \in \mathbb{C}$ with $\lambda_1,\dots,\lambda_r \neq 0$. Here, $\mathrm{Lif}_k(x)$ ($k \in \mathbb{Z}$) is the polyfactorial function ([8]) defined by

$$\mathrm{Lif}_k(x) = \sum_{m=0}^\infty \frac{x^m}{m!(m+1)^k}.$$

When $x = 0$, $\widehat{s}_n^{(k)} = \widehat{s}_n^{(k)}(0) = \widehat{s}_n^{(k)}(0|\lambda;\mu) = \widehat{s}_n^{(k)}(0;\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)$ are called the the Barnes-type Peters of the second kind and poly-Cauchy of the second kind mixed-type numbers.

Recall that the Barnes-type Peters polynomials of the second kind, denoted by $\widehat{s}_n(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)$, are given by the generating function as

$$\prod_{j=1}^r \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j} (1+t)^x = \sum_{n=0}^\infty \widehat{s}_n(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\frac{t^n}{n!}.$$

If $r = 1$, then $\widehat{s}_n(x|\lambda;\mu)$ are the Peters polynomials of the second kind. Peters polynomials were mentioned in [12, p.128] and have been investigated in e.g. [7].

The poly-Cauchy polynomials of the second kind, denoted by $\widehat{c}_n^{(k)}(x)$ ([6, 9]), are given by the generating function as

$$\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)(1+t)^x = \sum_{n=0}^\infty \widehat{c}_n^{(k)}(x)\frac{t^n}{n!}.$$

The generalized Barnes-type Euler polynomials $E_n(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)$ are defined by the generating function

$$\prod_{j=1}^r \left(\frac{2}{1+e^{\lambda_j t}}\right)^{\mu_j} e^{xt} = \sum_{n=0}^\infty E_n(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\frac{t^n}{n!}.$$

If $\mu_1 = \cdots = \mu_r = 1$, then $E_n(x|\lambda_1,\dots,\lambda_r) = E_n(x|\lambda_1,\dots,\lambda_r;1,\dots,1)$ are called the Barnes-type Euler polynomials. If further $\lambda_1 = \cdots = \lambda_r = 1$, then $E_n^{(r)}(x) = E_n(x|1,\dots,1;1,\dots,1)$ are called the Euler polynomials of order r.

In this paper, by considering Barnes-type Peters polynomials of the second kind as well as poly-Cauchy polynomials of the second kind, we define and investigate the mixed-type polynomials of these polynomials. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

2 Umbral calculus

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable t:

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \middle| a_k \in \mathbb{C} \right\}. \tag{2}$$

Let $\mathbb{P} = \mathbb{C}[x]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L|p(x)\rangle$ is the action of the linear functional L on the polynomial $p(x)$, and we recall that the vector space operations on $\mathbb{P}^*$ are defined by $\langle L+M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle$, $\langle cL|p(x)\rangle = c\langle L|p(x)\rangle$, where c is a complex constant in $\mathbb{C}$. For $f(t) \in \mathcal{F}$, let us define the linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle = a_n, \quad (n \geq 0). \tag{3}$$

In particular,

$$\left\langle t^k|x^n\right\rangle = n!\delta_{n,k} \quad (n, k \geq 0), \tag{4}$$

where $\delta_{n,k}$ is the Kronecker's symbol.

For $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$, we have $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. That is, $L = f_L(t)$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the *umbral algebra* and the *umbral calculus* is the study of umbral algebra. The order $O\big(f(t)\big)$ of a power series $f(t)(\neq 0)$ is the smallest integer k for which the coefficient of t^k does not vanish. If $O\big(f(t)\big) = 1$, then $f(t)$ is called a *delta series*; if $O\big(f(t)\big) = 0$, then $f(t)$ is called an *invertible series*. For $f(t), g(t) \in \mathcal{F}$ with $O\big(f(t)\big) = 1$ and $O\big(g(t)\big) = 0$, there exists a unique sequence $s_n(x)$ ($\deg s_n(x) = n$) such that $\left\langle g(t)f(t)^k|s_n(x)\right\rangle = n!\delta_{n,k}$, for $n, k \geq 0$. Such a sequence $s_n(x)$ is called the *Sheffer sequence* for $\big(g(t), f(t)\big)$ which is denoted by $s_n(x) \sim \big(g(t), f(t)\big)$, (see [1, 4-12]).

For $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$\langle f(t)g(t)|p(x)\rangle = \langle f(t)|g(t)p(x)\rangle = \langle g(t)|f(t)p(x)\rangle \tag{5}$$

and

$$f(t) = \sum_{k=0}^{\infty} \left\langle f(t)|x^k\right\rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \left\langle t^k|p(x)\right\rangle \frac{x^k}{k!} \tag{6}$$

([12, Theorem 2.2.5]). Thus, by (6), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad \text{and} \quad e^{yt}p(x) = p(x+y). \tag{7}$$

Sheffer sequences are characterized in the generating function ([12, Theorem 2.3.4]).

Lemma 1 *The sequence $s_n(x)$ is Sheffer for $\big(g(t), f(t)\big)$ if and only if*

$$\frac{1}{g\big(\bar{f}(t)\big)} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{s_k(y)}{k!} t^k \quad (y \in \mathbb{C}),$$

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where $\bar{f}(t)$ is the compositional inverse of $f(t)$.

For $s_n(x) \sim \big(g(t), f(t)\big)$, we have the following equations ([12, Theorem 2.3.7, Theorem 2.3.5, Theorem 2.3.9]):

$$f(t)s_n(x) = ns_{n-1}(x) \quad (n \geq 1), \tag{8}$$

$$s_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \left\langle g\big(\bar{f}(t)\big)^{-1} \bar{f}(t)^j | x^n \right\rangle x^j, \tag{9}$$

$$s_n(x+y) = \sum_{j=0}^{n} \binom{n}{j} s_j(x) p_{n-j}(y)\,, \tag{10}$$

where $p_n(x) = g(t)s_n(x)$.

Assume that $p_n(x) \sim \big(1, f(t)\big)$ and $q_n(x) \sim \big(1, g(t)\big)$. Then the transfer formula ([12, Corollary 3.8.2]) is given by

$$q_n(x) = x \left(\frac{f(t)}{g(t)}\right)^n x^{-1} p_n(x) \quad (n \geq 1).$$

For $s_n(x) \sim \big(g(t), f(t)\big)$ and $r_n(x) \sim \big(h(t), l(t)\big)$, assume that

$$s_n(x) = \sum_{m=0}^{n} C_{n,m} r_m(x) \quad (n \geq 0)\,.$$

Then we have ([12, p.132])

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{h\big(\bar{f}(t)\big)}{g\big(\bar{f}(t)\big)} l\big(\bar{f}(t)\big)^m \middle| x^n \right\rangle . \tag{11}$$

3 Main results

From the definition (1), $\widehat{s}_n^{(k)}(x|\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r)$ is the Sheffer sequence for the pair

$$g(t) = \prod_{j=1}^{r} \left(\frac{1+e^{\lambda_j t}}{e^{\lambda_j t}}\right)^{\mu_j} \frac{1}{\mathrm{Lif}_k(-t)} \quad \text{and} \quad f(t) = e^t - 1.$$

So,

$$\widehat{s}_n^{(k)}(x|\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r) \sim \left(\prod_{j=1}^{r} \left(\frac{1+e^{\lambda_j t}}{e^{\lambda_j t}}\right)^{\mu_j} \frac{1}{\mathrm{Lif}_k(-t)}, e^t - 1\right) . \tag{12}$$

3.1 Explicit expressions

Let $(n)_j = n(n-1)\cdots(n-j+1)$ $(j \ge 1)$ with $(n)_0 = 1$. The (signed) Stirling numbers of the first kind $S_1(n,m)$ are defined by

$$(x)_n = \sum_{m=0}^{n} S_1(n,m)x^m\,.$$

Theorem 1 *Let $\lambda\mu = \sum_{j=1}^{r} \lambda_j\mu_j$. Then, we have*

$$\begin{aligned}
&\widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\
&= 2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}\sum_{l=0}^{m}\frac{(-1)^l\binom{m}{l}}{(l+1)^k}S_1(n,m)E_{m-l}(x+\lambda\mu|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) && (13)\\
&= \sum_{j=0}^{n}\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\widehat{s}_{n-l}^{(k)}x^j && (14)\\
&= \sum_{j=0}^{n}\sum_{l=0}^{n-j}\sum_{i=0}^{l}\binom{n}{l}\binom{l}{i}S_1(n-l,j)\widehat{c}_i^{(k)}\widehat{s}_{l-i}(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)x^j && (15)\\
&= \sum_{l=0}^{n}\binom{n}{l}\widehat{s}_{n-l}(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\widehat{c}_l^{(k)}(x)\,, && (16)\\
&= \sum_{l=0}^{n}\binom{n}{l}\widehat{s}_l(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\widehat{c}_{n-l}^{(k)}\,. && (17)
\end{aligned}$$

Proof. Since

$$\prod_{j=1}^{r}\left(\frac{1+e^{\lambda_j t}}{e^{\lambda_j t}}\right)^{\mu_j}\frac{1}{\mathrm{Lif}_k(-t)}\widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\sim(1,e^t-1) \tag{18}$$

and

$$(x)_n \sim (1,e^t-1)\,, \tag{19}$$

we have

$$
\begin{aligned}
&\widehat{s}_n^{(k)}(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)\\
&=\prod_{j=1}^{r}\left(\frac{e^{\lambda_j t}}{1+e^{\lambda_j t}}\right)^{\mu_j}\operatorname{Lif}_k(-t)(x)_n\\
&=\sum_{m=0}^{n}S_1(n,m)\prod_{j=1}^{r}\left(\frac{e^{\lambda_j t}}{1+e^{\lambda_j t}}\right)^{\mu_j}\operatorname{Lif}_k(-t)x^m\\
&=\sum_{m=0}^{n}S_1(n,m)\prod_{j=1}^{r}\left(\frac{e^{\lambda_j t}}{1+e^{\lambda_j t}}\right)^{\mu_j}\sum_{l=0}^{m}\frac{(-1)^l t^l}{l!(l+1)^k}x^m\\
&=\sum_{m=0}^{n}S_1(n,m)\sum_{l=0}^{m}\frac{(-1)^l\binom{m}{l}}{(l+1)^k}\prod_{j=1}^{r}\left(\frac{e^{\lambda_j t}}{1+e^{\lambda_j t}}\right)^{\mu_j}x^{m-l}\\
&=2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}S_1(n,m)\sum_{l=0}^{m}\frac{(-1)^l\binom{m}{l}}{(l+1)^k}e^{\lambda\mu t}\prod_{j=1}^{r}\left(\frac{2}{1+e^{\lambda_j t}}\right)^{\mu_j}x^{m-l}\\
&=2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}S_1(n,m)\sum_{l=0}^{m}\frac{(-1)^l\binom{m}{l}}{(l+1)^k}E_{m-l}(x+\lambda\mu|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)\,.
\end{aligned}
$$

So, we get (13).
By (9) with (12), we get

$$
\begin{aligned}
&\left\langle g\big(\bar f(t)\big)^{-1}\bar f(t)^j|x^n\right\rangle\\
&=\left\langle\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\operatorname{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^j\Big|x^n\right\rangle\\
&=\left\langle\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\operatorname{Lif}_k\big(-\ln(1+t)\big)\Big|j!\sum_{l=j}^{\infty}S_1(l,j)\frac{t^l}{l!}x^n\right\rangle\\
&=j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\left\langle\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\operatorname{Lif}_k\big(-\ln(1+t)\big)\Big|x^{n-l}\right\rangle\\
&=j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\left\langle\sum_{i=0}^{\infty}\widehat{s}_i^{(k)}\frac{t^i}{i!}\Big|x^{n-l}\right\rangle\\
&=j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\widehat{s}_{n-l}^{(k)}\,.
\end{aligned}
$$

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On the other hand,

$$\begin{aligned}
&\left\langle g\big(\bar{f}(t)\big)^{-1}\bar{f}(t)^j \middle| x^n\right\rangle \\
&= j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\middle|\mathrm{Lif}_k\big(-\ln(1+t)\big)x^{n-l}\right\rangle \\
&= j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\sum_{i=0}^{n-l}\binom{n-l}{i}\widehat{c}_i^{(k)}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\middle|x^{n-l-i}\right\rangle \\
&= j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\sum_{i=0}^{n-l}\binom{n-l}{i}\widehat{c}_i^{(k)}\left\langle \sum_{m=0}^{\infty}\widehat{s}_m(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\frac{t^m}{m!}\middle|x^{n-l-i}\right\rangle \\
&= j!\sum_{l=j}^{n}\sum_{i=0}^{n-l}\binom{n}{l}\binom{n-l}{i}S_1(l,j)\widehat{c}_i^{(k)}\widehat{s}_{n-l-i}(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\,.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\widehat{s}_n^{(k)}(x) &= \sum_{j=0}^{n}\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\widehat{s}_{n-l}^{(k)}x^j \\
&= \sum_{j=0}^{n}\sum_{l=0}^{n-j}\sum_{i=0}^{l}\binom{n}{l}\binom{l}{i}S_1(n-l,j)\widehat{c}_i^{(k)}\widehat{s}_{l-i}(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)x^j\,,
\end{aligned}$$

which are the identities (14) and (15).

7

Next,

$$\begin{aligned}
\widehat{s}_n^{(k)}(y|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) &= \left\langle \sum_{i=0}^{\infty} \widehat{s}_i^{(k)}(y|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\frac{t^i}{i!} \middle| x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} \mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)(1+t)^y \middle| x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} \middle| \mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)(1+t)^y x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} \middle| \sum_{l=0}^{\infty} \widehat{c}_l^{(k)}(y)\frac{t^l}{l!} x^n \right\rangle \\
&= \sum_{l=0}^{n} \binom{n}{l} \widehat{c}_l^{(k)}(y) \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} \middle| x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n} \binom{n}{l} \widehat{c}_l^{(k)}(y) \left\langle \sum_{i=0}^{\infty} \widehat{s}_i(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\frac{t^i}{i!} \middle| x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n} \binom{n}{l} \widehat{c}_l^{(k)}(y) \widehat{s}_{n-l}(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\,.
\end{aligned}$$

Thus, we obtain (16).
Finally, we obtain that

$$\begin{aligned}
\widehat{s}_n^{(k)}(y|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) &= \left\langle \sum_{i=0}^{\infty} \widehat{s}_i^{(k)}(y|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\frac{t^i}{i!} \middle| x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} \mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)(1+t)^y \middle| x^n \right\rangle \\
&= \left\langle \mathrm{Lif}_k\bigl(-\ln(1+t)\bigr) \middle| \prod_{j=1}^{r} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^y x^n \right\rangle \\
&= \left\langle \mathrm{Lif}_k\bigl(-\ln(1+t)\bigr) \middle| \sum_{l=0}^{\infty} \widehat{s}_l(y|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\frac{t^l}{l!} x^n \right\rangle \\
&= \sum_{l=0}^{n} \widehat{s}_l(y|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\binom{n}{l} \left\langle \mathrm{Lif}_k\bigl(-\ln(1+t)\bigr) \middle| x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n} \widehat{s}_l(y|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\binom{n}{l} \left\langle \sum_{i=0}^{\infty} \widehat{c}_i^{(k)} \frac{t^i}{i!} \middle| x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n} \binom{n}{l} \widehat{s}_l(y|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) \widehat{c}_{n-l}^{(k)}\,.
\end{aligned}$$

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Thus, we get the identity (17). ∎

3.2 Sheffer identity

Theorem 2

$$\widehat{s}_n^{(k)}(x+y|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)=\sum_{j=0}^{n}\binom{n}{j}\widehat{s}_j^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)(y)_{n-j}\,. \tag{20}$$

Proof. By (12) with

$$\begin{aligned}p_n(x)&=\prod_{j=1}^{r}\left(\frac{1+e^{\lambda_j t}}{e^{\lambda_j t}}\right)^{\mu_j}\frac{1}{\mathrm{Lif}_k(-t)}\widehat{s}_n(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\&=(x)_n\sim(1,e^t-1)\,,\end{aligned}$$

using (10), we have (20). ∎

3.3 Difference relations

Theorem 3

$$\begin{aligned}&\widehat{s}_n^{(k)}(x+1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)-\widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\&\qquad=n\widehat{s}_{n-1}^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\,.\end{aligned} \tag{21}$$

Proof. By (8) with (12), we get

$$(e^t-1)\widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)=n\widehat{s}_{n-1}^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\,.$$

By (7), we have (21). ∎

3.4 Recurrence

Theorem 4

$$\begin{aligned}
&\widehat{s}_{n+1}^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\
&= (x+\lambda\mu)\widehat{s}_n^{(k)}(x-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\
&\quad - 2^{-1-\sum_{j=1}^r \mu_j}\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{i=1}^{r} S_1(n,m)\frac{(-1)^{m-l}\binom{m}{l}}{(m-l+1)^k}\mu_i\lambda_i E_l(x+\lambda(\mu+e_i)-1|\lambda;\mu+e_i)\\
&\quad - 2^{-\sum_{j=1}^r \mu_j}\sum_{m=0}^{n}\sum_{l=0}^{m}\frac{(-1)^{m-l}\binom{m}{l}}{(m-l+2)^k}S_1(n,m)E_l(x+\lambda\mu-1|\lambda;\mu) \qquad (22)\\
&= (x+\mu\lambda)\widehat{s}_n^{(k)}(x-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\
&\quad - \frac{1}{2}\sum_{j=0}^{n}\sum_{l=0}^{n-j}\sum_{i=1}^{r}\binom{n}{l}\mu_i\lambda_i^{j+1}S_1(n-l,j)\widehat{s}_l^{(k)}E_j\left(\frac{x+\lambda_i-1}{\lambda_i}\right)\\
&\quad - 2^{-\sum_{j=1}^r \mu_j}\sum_{m=0}^{n}\sum_{l=0}^{m}\frac{(-1)^{m-l}\binom{m}{l}}{(m-l+2)^k}S_1(n,m)E_l(x+\lambda\mu-1|\lambda;\mu)\,, \qquad (23)
\end{aligned}$$

$\lambda\mu=\sum_{j=1}^r \lambda_i\mu_i$.

Remark. Comparing (22) and (23),

$$\begin{aligned}
&2^{-\sum_{j=1}^r \mu_j}\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{i=1}^{r}\frac{(-1)^{m-l}\binom{m}{l}}{(m-l+1)^k}\mu_i\lambda_i S_1(n,m)E_l\big(x+\lambda(\mu+e_i)-1|\lambda;\mu+e_i\big)\\
&=\sum_{j=0}^{n}\sum_{l=0}^{n-j}\sum_{i=1}^{r}\binom{n}{l}\mu_i\lambda_i^{j+1}S_1(n-l,j)\widehat{s}_l^{(k)}E_j\left(\frac{x+\lambda_i-1}{\lambda_i}\right)\,.
\end{aligned}$$

Proof. By applying

$$s_{n+1}(x)=\left(x-\frac{g'(t)}{g(t)}\right)\frac{1}{f'(t)}s_n(x) \qquad (24)$$

([12, Corollary 3.7.2]) with (12), we get

$$\begin{aligned}
&\widehat{s}_{n+1}^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\
&\qquad = x\widehat{s}_n^{(k)}(x-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) - e^{-t}\frac{g'(t)}{g(t)}\widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\,.
\end{aligned}$$

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Since

$$\begin{aligned}\frac{g'(t)}{g(t)} &= \big(\ln g(t)\big)' \\ &= \left(\sum_{i=1}^{r} \mu_i \ln(1+e^{\lambda_i t}) - \left(\sum_{i=1}^{r} \mu_i \lambda_i\right) t - \ln \mathrm{Lif}_k(-t)\right)' \\ &= \sum_{i=1}^{r} \frac{\mu_i \lambda_i e^{\lambda_i t}}{1+e^{\lambda_i t}} - \sum_{i=1}^{r} \mu_i \lambda_i + \frac{\mathrm{Lif}'_k(-t)}{\mathrm{Lif}_k(-t)},\end{aligned}$$

by (13), we have

$$\begin{aligned}&\frac{g'(t)}{g(t)} \widehat{s}_n^{(k)}(x) \\ &= \left(\sum_{i=1}^{r} \frac{\mu_i \lambda_i e^{\lambda_i t}}{1+e^{\lambda_i t}} - \sum_{i=1}^{r} \mu_i \lambda_i + \frac{\mathrm{Lif}'_k(-t)}{\mathrm{Lif}_k(-t)}\right) \widehat{s}_n^{(k)}(x) \\ &= 2^{-1-\sum_{j=1}^{r}\mu_j} \sum_{m=0}^{n} \sum_{l=0}^{m} S_1(n,m) \frac{(-1)^l \binom{m}{l}}{(l+1)^k} \sum_{i=1}^{r} \mu_i \lambda_i e^{(\lambda\mu+\lambda_i)t} \frac{2}{1+e^{\lambda_i t}} \prod_{j=1}^{r} \left(\frac{2}{1+e^{\lambda_j t}}\right)^{\mu_j} x^{m-l} \\ &\quad - \lambda\mu \widehat{s}_n^{(k)}(x) + \sum_{m=0}^{n} S_1(n,m) \prod_{j=1}^{r} \left(\frac{e^{\lambda_j t}}{1+e^{\lambda_j t}}\right)^{\mu_j} \mathrm{Lif}'_k(-t) x^m. \qquad (25)\end{aligned}$$

The first term in (25) is

$$2^{-1-\sum_{j=1}^{r}\mu_j} \sum_{m=0}^{n} \sum_{l=0}^{m} \sum_{i=1}^{r} S_1(n,m) \frac{(-1)^l \binom{m}{l}}{(l+1)^k} \mu_i \lambda_i E_{m-l}\big(x+\lambda(\mu+e_i)|\lambda;\mu+e_i\big),$$

where $\lambda = (\lambda_1, \dots, \lambda_r)$, $\mu = (\mu_1, \dots, \mu_r)$ and $e_i = (\underbrace{0,\dots,0}_{i-1}, 1, \underbrace{0,\dots,0}_{r-i})$ $(i = 1, 2, \dots, r)$.

Since

$$\mathrm{Lif}_{k-1}(-t) - \mathrm{Lif}_k(-t) = \left(\frac{1}{2^k} - \frac{1}{2^{k-1}}\right) t + \cdots,$$

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the third term in (25) is

$$
\begin{aligned}
&2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}S_1(n,m)e^{\lambda\mu t}\mathrm{Lif}_k'(-t)\prod_{j=1}^{r}\left(\frac{2}{1+e^{\lambda_j t}}\right)^{\mu_j}x^m\\
&=2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}S_1(n,m)e^{\lambda\mu t}\frac{\mathrm{Lif}_{k-1}(-t)-\mathrm{Lif}_k(-t)}{-t}E_m(x|\lambda;\mu)\\
&=-2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}S_1(n,m)e^{\lambda\mu t}\big(\mathrm{Lif}_{k-1}(-t)-\mathrm{Lif}_k(-t)\big)\frac{E_{m+1}(x|\lambda;\mu)}{m+1}\\
&=-2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}\frac{S_1(n,m)}{m+1}e^{\lambda\mu t}\\
&\quad\times\left(\sum_{l=0}^{m+1}\frac{(-1)^l t^l}{l!(l+1)^{k-1}}E_{m+1}(x|\lambda;\mu)-\sum_{l=0}^{m+1}\frac{(-1)^l t^l}{l!(l+1)^{k}}E_{m+1}(x|\lambda;\mu)\right)\\
&=-2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}\frac{S_1(n,m)}{m+1}e^{\lambda\mu t}\\
&\quad\times\left(\sum_{l=0}^{m+1}\frac{(-1)^l\binom{m+1}{l}}{(l+1)^{k-1}}E_{m+1-l}(x|\lambda;\mu)-\sum_{l=0}^{m+1}\frac{(-1)^l\binom{m+1}{l}}{(l+1)^{k}}E_{m+1-l}(x|\lambda;\mu)\right)\\
&=-2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}\frac{S_1(n,m)}{m+1}e^{\lambda\mu t}\sum_{l=1}^{m+1}\frac{(-1)^l\binom{m+1}{l}l}{(l+1)^{k}}E_{m+1-l}(x|\lambda;\mu)\\
&=2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}S_1(n,m)\sum_{l=0}^{m}\frac{(-1)^{m-l}\binom{m}{l}}{(m+2-l)^k}E_l(x+\lambda\mu|\lambda;\mu)\,.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&\widehat{s}_{n+1}^{(k)}(x)=(x+\lambda\mu)\widehat{s}_n^{(k)}(x-1)\\
&\quad-2^{-1-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{i=1}^{r}S_1(n,m)\frac{(-1)^{m-l}\binom{m}{l}}{(m+1-l)^k}\mu_i\lambda_i E_l\big(x+\lambda(\mu+e_i)-1|\lambda;\mu+e_i\big)\\
&\quad-2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}\sum_{l=0}^{m}\frac{(-1)^{m-l}\binom{m}{l}}{(m+2-l)^k}S_1(n,m)E_l(x+\lambda\mu-1|\lambda;\mu)\,,
\end{aligned}
$$

which is (22).

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On the other hand, by (14) with (22), we have

$$\begin{aligned}
&\frac{g'(t)}{g(t)}\widehat{s}_n^{(k)}(x)\\
&=\left(\sum_{i=1}^{r}\frac{\mu_i\lambda_i e^{\lambda_i t}}{1+e^{\lambda_i t}}-\sum_{i=1}^{r}\mu_i\lambda_i+\frac{\mathrm{Lif}_k'(-t)}{\mathrm{Lif}_k(-t)}\right)\widehat{s}_n^{(k)}(x)\\
&=\frac{1}{2}\sum_{i=1}^{r}\mu_i\lambda_i e^{\lambda_i t}\frac{2}{1+e^{\lambda_i t}}\sum_{j=0}^{n}\sum_{l=0}^{n-j}\binom{n}{l}S_1(n-l,j)\widehat{s}_l^{(k)}x^j\\
&\quad-\mu\lambda\widehat{s}_n^{(k)}(x)+2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}\sum_{l=0}^{m}\frac{(-1)^{m-l}\binom{m}{l}}{(m+2-l)^k}S_1(n,m)E_l(x+\lambda\mu|\lambda;\mu)\,. \qquad (26)
\end{aligned}$$

The first term in (26) is

$$\begin{aligned}
&\frac{1}{2}\sum_{j=0}^{n}\sum_{l=0}^{n-j}\binom{n}{l}S_1(n-l,j)\widehat{s}_l^{(k)}\sum_{i=1}^{r}\mu_i\lambda_i e^{\lambda_i t}\frac{2}{1+e^{\lambda_i t}}x^j\\
&=\frac{1}{2}\sum_{j=0}^{n}\sum_{l=0}^{n-j}\binom{n}{l}S_1(n-l,j)\widehat{s}_l^{(k)}\sum_{i=1}^{r}\mu_i\lambda_i e^{\lambda_i t}\lambda_i^j E_j\left(\frac{x}{\lambda_i}\right)\\
&=\frac{1}{2}\sum_{j=0}^{n}\sum_{l=0}^{n-j}\binom{n}{l}S_1(n-l,j)\widehat{s}_l^{(k)}\sum_{i=1}^{r}\mu_i\lambda_i^{j+1}E_j\left(\frac{x+\lambda_i}{\lambda_i}\right)\,.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\widehat{s}_{n+1}^{(k)}(x)=(x+\mu\lambda)\widehat{s}_n^{(k)}(x-1)\\
&-\frac{1}{2}\sum_{j=0}^{n}\sum_{l=0}^{n-j}\sum_{i=1}^{r}\binom{n}{l}\mu_i\lambda_i^{j+1}S_1(n-l,j)\widehat{s}_l^{(k)}E_j\left(\frac{x+\lambda_i-1}{\lambda_i}\right)\\
&-2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}\sum_{l=0}^{m}\frac{(-1)^{m-l}\binom{m}{l}}{(m+2-l)^k}S_1(n,m)E_l(x+\lambda\mu-1|\lambda;\mu)\,.
\end{aligned}$$

which is (23). ∎

3.5 Differentiation

Theorem 5

$$\frac{d}{dx}\widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)=n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}\widehat{s}_l^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\,. \qquad (27)$$

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Proof. We shall use

$$\frac{d}{dx}\widehat{s}_n(x) = \sum_{l=0}^{n-1}\binom{n}{l}\left\langle \bar{f}(t)|x^{n-l}\right\rangle \widehat{s}_l(x)$$

(*Cf.* [12, Theorem 2.3.12]). Since

$$\begin{aligned}\left\langle \bar{f}(t)|x^{n-l}\right\rangle &= \left\langle \ln(1+t)|x^{n-l}\right\rangle\\ &= \left\langle \sum_{m=1}^{\infty}\frac{(-1)^{m-1}t^m}{m}\Big|x^{n-l}\right\rangle\\ &= \sum_{m=1}^{n-l}\frac{(-1)^{m-1}}{m}\left\langle t^m|x^{n-l}\right\rangle\\ &= \sum_{m=1}^{n-l}\frac{(-1)^{m-1}}{m}(n-l)!\delta_{m,n-l}\\ &= (-1)^{n-l-1}(n-l-1)!\,,\end{aligned}$$

with (12), we have

$$\begin{aligned}&\frac{d}{dx}\widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\ &= \sum_{l=0}^{n-1}\binom{n}{l}(-1)^{n-l-1}(n-l-1)!\widehat{s}_l^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\ &= n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}\widehat{s}_l^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\,,\end{aligned}$$

which is the identity (27). ∎

3.6 A more relation

The classical Cauchy numbers c_n of the first kind are defined by

$$\frac{t}{\ln(1+t)} = \sum_{n=0}^{\infty}c_n\frac{t^n}{n!}$$

(see e.g. [3, 8]).

Theorem 6

$$\begin{aligned}&\widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\ &= x\widehat{s}_{n-1}^{(k)}(x-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) + \frac{1}{n}\sum_{l=1}^{n}\binom{n}{l}c_{n-l}\big(\widehat{s}_l^{(k-1)}(x-1) - \widehat{s}_l^{(k)}(x-1)\big)\\ &\quad + \sum_{i=1}^{r}\mu_i\lambda_i\widehat{s}_{n-1}^{(k)}(x-\lambda_i-1|\lambda;\mu+e_i)\,. \end{aligned} \tag{28}$$

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Proof. For $n \geq 1$, we have

$$\begin{aligned}
&\widehat{s}_n^{(k)}(y|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\
&=\left\langle \sum_{l=0}^{\infty}\widehat{s}_l^{(k)}(y|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\frac{t^l}{l!}\middle| x^n\right\rangle\\
&=\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y\middle| x^n\right\rangle\\
&=\left\langle \partial_t\left(\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y\right)\middle| x^{n-1}\right\rangle\\
&=\left\langle \left(\partial_t\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y\middle| x^{n-1}\right\rangle\\
&\quad+\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\Big(\partial_t\mathrm{Lif}_k\big(-\ln(1+t)\big)\Big)(1+t)^y\middle| x^{n-1}\right\rangle\\
&\quad+\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\big(-\ln(1+t)\big)\left(\partial_t(1+t)^y\right)\middle| x^{n-1}\right\rangle .
\end{aligned}$$

The third term is

$$\begin{aligned}
&y\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{y-1}\middle| x^{n-1}\right\rangle\\
&=y\widehat{s}_{n-1}^{(k)}(y-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\,.
\end{aligned}$$

Since

$$\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)-\mathrm{Lif}_k\big(-\ln(1+t)\big)=\left(\frac{1}{2^k}-\frac{1}{2^{k-1}}\right)t+\cdots,$$

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the second term is

$$\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j} \frac{\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)-\mathrm{Lif}_k\big(-\ln(1+t)\big)}{(1+t)\ln(1+t)}(1+t)^y \Big| x^{n-1}\right\rangle$$

$$= \Bigg\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}$$

$$\frac{\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)-\mathrm{Lif}_k\big(-\ln(1+t)\big)}{t}(1+t)^{y-1}\Big| \frac{t}{\ln(1+t)}x^{n-1}\Bigg\rangle$$

$$= \Bigg\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}$$

$$\frac{\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)-\mathrm{Lif}_k\big(-\ln(1+t)\big)}{t}(1+t)^{y-1}\Big| \sum_{l=0}^{\infty} c_l\frac{t^l}{l!}x^{n-1}\Bigg\rangle$$

$$= \sum_{l=0}^{n-1}\binom{n-1}{l}c_l \Bigg\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}$$

$$(1+t)^{y-1}\Big| \frac{\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)-\mathrm{Lif}_k\big(-\ln(1+t)\big)}{t}x^{n-1-l}\Bigg\rangle$$

$$= \sum_{l=0}^{n-1}\frac{1}{n-l}\binom{n-1}{l}c_l \Bigg\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}$$

$$(1+t)^{y-1}\Big| \left(\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)-\mathrm{Lif}_k\big(-\ln(1+t)\big)\right)x^{n-l}\Bigg\rangle$$

$$= \sum_{l=0}^{n-1}\frac{1}{n-l}\binom{n}{l}c_l \Bigg(\Bigg\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)(1+t)^{y-1}\Big| x^{n-l}\Bigg\rangle$$

$$- \Bigg\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_{k}\big(-\ln(1+t)\big)(1+t)^{y-1}\Big| x^{n-l}\Bigg\rangle\Bigg)$$

$$= \frac{1}{n}\sum_{l=0}^{n-1}\binom{n}{l}c_l\big(\widehat{s}_{n-l}^{(k-1)}(y-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)-\widehat{s}_{n-l}^{(k)}(y-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\big)$$

$$= \frac{1}{n}\sum_{l=1}^{n}\binom{n}{l}c_{n-l}\big(\widehat{s}_{l}^{(k-1)}(y-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)-\widehat{s}_{l}^{(k)}(y-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\big).$$

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Since

$$\begin{aligned}&\partial_t \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\\&=\sum_{i=1}^{r}\mu_i\lambda_i(1+t)^{-\lambda_i-1}\frac{(1+t)^{\lambda_i}}{(1+(1+t)\lambda_i}\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j},\end{aligned}$$

the first term is

$$\begin{aligned}&\sum_{i=1}^{r}\mu_i\lambda_i\left\langle\frac{(1+t)^{\lambda_i}}{(1+(1+t)\lambda_i}\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{y-\lambda_i-1}|x^{n-1}\right\rangle\\&=\sum_{i=1}^{r}\mu_i\lambda_i\widehat{s}_{n-1}^{(k)}(y-\lambda_i-1|\lambda;\mu+e_i)\,.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}&\widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\&=x\widehat{s}_{n-1}^{(k)}(x-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)+\frac{1}{n}\sum_{l=1}^{n}\binom{n}{l}c_{n-l}\big(\widehat{s}_l^{(k-1)}(x-1)-\widehat{s}_l^{(k)}(x-1)\big)\\&\quad+\sum_{i=1}^{r}\mu_i\lambda_i\widehat{s}_{n-1}^{(k)}(x-\lambda_i-1|\lambda;\mu+e_i)\,,\end{aligned}$$

which is the identity (28). ∎

3.7 A relation including the Stirling numbers of the first kind

Theorem 7 *For $n-1\ge m\ge 1$, we have*

$$\begin{aligned}&m\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\widehat{s}_l^{(k)}(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\&=(m-1)\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{s}_l^{(k)}(-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\&\quad+\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{s}_l^{(k-1)}(-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\&\quad+m\sum_{l=0}^{n-m-1}\sum_{i=1}^{r}\binom{n-1}{l}S_1(n-l-1,m)\mu_i\lambda_i\widehat{s}_l^{(k)}(-\lambda_i-1|\lambda;\mu+e_i)\,.\end{aligned}\tag{29}$$

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Proof. We shall compute

$$\left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} \mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m \Big| x^n \right\rangle$$

in two different ways. On the one hand, it is equal to

$$\begin{aligned}
&\left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} \mathrm{Lif}_k\big(-\ln(1+t)\big) \Big| \big(\ln(1+t)\big)^m x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} \mathrm{Lif}_k\big(-\ln(1+t)\big) \Big| m! \sum_{l=0}^{\infty} S_1(l,m) \frac{t^l}{l!} x^n \right\rangle \\
&= m! \sum_{l=m}^{n} \binom{n}{l} S_1(l,m) \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} \mathrm{Lif}_k\big(-\ln(1+t)\big) \Big| x^{n-l} \right\rangle \\
&= m! \sum_{l=m}^{n} \binom{n}{l} S_1(l,m) \left\langle \sum_{i=0}^{\infty} \widehat{s}_i^{(k)}(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) \frac{t^i}{i!} \Big| x^{n-l} \right\rangle \\
&= m! \sum_{l=m}^{n} \binom{n}{l} S_1(l,m) \widehat{s}_{n-l}^{(k)}(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) \\
&= m! \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l,m) \widehat{s}_l^{(k)}(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) .
\end{aligned}$$

On the other hand, it is equal to

$$\begin{aligned}
&\left\langle \partial_t \left(\prod_{j=1}^{r} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} \mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m \right) \Big| x^{n-1} \right\rangle \\
&= \left\langle \left(\partial_t \prod_{j=1}^{r} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} \right) \mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m \Big| x^{n-1} \right\rangle \\
&\quad + \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} \big(\partial_t \mathrm{Lif}_k\big(-\ln(1+t)\big)\big) \big(\ln(1+t)\big)^m \Big| x^{n-1} \right\rangle \\
&\quad + \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} \mathrm{Lif}_k\big(-\ln(1+t)\big) \big(\partial_t \big(\ln(1+t)\big)^m\big) \Big| x^{n-1} \right\rangle . \qquad (30)
\end{aligned}$$

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The third term of (30) is equal to

$$m\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1}\Big|\big(\ln(1+t)\big)^{m-1}x^{n-1}\right\rangle$$

$$=m\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1}\Big|\right.$$

$$\left.(m-1)!\sum_{l=m-1}^{\infty}S_1(l,m-1)\frac{t^l}{l!}x^{n-1}\right\rangle$$

$$=m!\sum_{l=m-1}^{n-1}\binom{n-1}{l}S_1(l,m-1)$$

$$\times\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1}\Big|x^{n-1-l}\right\rangle$$

$$=m!\sum_{l=m-1}^{n-1}\binom{n-1}{l}S_1(l,m-1)\widehat{s}_{n-1-l}^{(k)}(-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)$$

$$=m!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{s}_{l}^{(k)}(-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\,.$$

The second term of (30) is equal to

$$\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\left(\frac{\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)-\mathrm{Lif}_k\big(-\ln(1+t)\big)}{(1+t)\ln(1+t)}\right)\big(\ln(1+t)\big)^{m}\Big|x^{n-1}\right\rangle$$

$$=\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)(1+t)^{-1}\Big|\big(\ln(1+t)\big)^{m-1}x^{n-1}\right\rangle$$

$$-\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_{k}\big(-\ln(1+t)\big)(1+t)^{-1}\Big|\big(\ln(1+t)\big)^{m-1}x^{n-1}\right\rangle$$

$$=(m-1)!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{s}_{l}^{(k-1)}(-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)$$

$$-(m-1)!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{s}_{l}^{(k)}(-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\,.$$

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The first term of (30) is equal to

$$\begin{aligned}
&\left\langle \left(\partial_t \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m \middle| x^{n-1}\right\rangle \\
&= \sum_{i=1}^{r}\mu_i\lambda_i\left\langle \frac{(1+t)^{\lambda_i}}{1+(1+t)^{\lambda_i}}\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\right. \\
&\qquad\qquad \left.\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-\lambda_i-1}\middle|\big(\ln(1+t)\big)^m x^{n-1}\right\rangle \\
&= \sum_{i=1}^{r}\mu_i\lambda_i\left\langle \frac{(1+t)^{\lambda_i}}{1+(1+t)^{\lambda_i}}\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\right. \\
&\qquad\qquad \left.\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-\lambda_i-1}\middle| m!\sum_{l=m}^{\infty}S_1(l,m)\frac{t^l}{l!}x^{n-1}\right\rangle \\
&= m!\sum_{i=1}^{r}\mu_i\lambda_i\sum_{l=m}^{n-1}\binom{n-1}{l}S_1(l,m) \\
&\quad\times\left\langle \frac{(1+t)^{\lambda_i}}{1+(1+t)^{\lambda_i}}\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-\lambda_i-1}\middle| x^{n-1-l}\right\rangle \\
&= m!\sum_{i=1}^{r}\mu_i\lambda_i\sum_{l=m}^{n-1}\binom{n-1}{l}S_1(l,m)\widehat{s}_{n-1-l}^{(k)}(-\lambda_i-1|\lambda;\mu+e_i) \\
&= m!\sum_{l=0}^{n-m-1}\sum_{i=1}^{r}\binom{n-1}{l}S_1(n-1-l,m)\mu_i\lambda_i\widehat{s}_l^{(k)}(-\lambda_i-1|\lambda;\mu+e_i)\,.
\end{aligned}$$

Therefore, we get, for $n-1\ge m\ge 1$,

$$\begin{aligned}
&m!\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\widehat{s}_l^{(k)}(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) \\
&= m!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{s}_l^{(k)}(-1) \\
&\quad + (m-1)!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{s}_l^{(k-1)}(-1) \\
&\quad - (m-1)!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{s}_l^{(k)}(-1) \\
&\quad + m!\sum_{l=0}^{n-m-1}\sum_{i=1}^{r}\binom{n-1}{l}S_1(n-l-1,m)\mu_i\lambda_i\widehat{s}_l^{(k)}(-\lambda_i-1|\lambda;\mu+e_i)\,.
\end{aligned}$$

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Dividing both sides by $(m-1)!$, we obtain, for $n-1 \geq m \geq 1$,

$$\begin{aligned}
& m\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\widehat{s}_l^{(k)}(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) \\
&= (m-1)\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{s}_l^{(k)}(-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) \\
&\quad + \sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{s}_l^{(k-1)}(-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) \\
&\quad + m\sum_{l=0}^{n-m-1}\sum_{i=1}^{r}\binom{n-1}{l}S_1(n-l-1,m)\mu_i\lambda_i\widehat{s}_l^{(k)}(-\lambda_i-1|\lambda;\mu+e_i)\,.
\end{aligned}$$

Thus, we get (29). ∎

3.8 A relation with the falling factorials

Theorem 8

$$\widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) = \sum_{m=0}^{n}\binom{n}{m}\widehat{s}_{n-m}^{(k)}(x)_m\,. \tag{31}$$

Proof. For (12) and (19), assume that $\widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) = \sum_{m=0}^{n} C_{n,m}(x)_m$. By (11), we have

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \frac{1}{\prod_{j=1}^{r}\left(\frac{1+e^{\lambda_j\ln(1+t)}}{e^{\lambda_j\ln(1+t)}}\right)^{\mu_j}}\mathrm{Lif}_k\big(-\ln(1+t)\big)t^m\Big|x^n\right\rangle \\
&= \frac{1}{m!}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\big(-\ln(1+t)\big)\Big|t^m x^n\right\rangle \\
&= \binom{n}{m}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\big(-\ln(1+t)\big)\Big|x^{n-m}\right\rangle \\
&= \binom{n}{m}\widehat{s}_{n-m}^{(k)}\,.
\end{aligned}$$

Thus, we get the identity (31). ∎

3.9 A relation with higher-order Frobenius-Euler polynomials

For $\alpha \in \mathbb{C}$ with $\alpha \neq 1$, the Frobenius-Euler polynomials of order r, $H_n^{(r)}(x|\alpha)$ are defined by the generating function

$$\left(\frac{1-\alpha}{e^t-\alpha}\right)^r e^{xt} = \sum_{n=0}^{\infty}H_n^{(r)}(x|\alpha)\frac{t^n}{n!}$$

(see e.g. [10]).

Theorem 9

$$\begin{aligned}\widehat{s}_n^{(k)}(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)&=\sum_{m=0}^{n}\left(\sum_{j=0}^{n-m}\sum_{l=0}^{n-m-j}\binom{s}{j}\binom{n-j}{l}(n)_j\right.\\&\left.\times(1-\alpha)^{-j}S_1(n-j-l,m)\widehat{s}_l^{(k)}\right)H_m^{(s)}(x|\alpha)\,.\end{aligned}\tag{32}$$

Proof. For (12) and

$$H_n^{(s)}(x|\alpha)\sim\left(\left(\frac{e^t-\alpha}{1-\alpha}\right)^s,t\right)\,,\tag{33}$$

assume that $\widehat{s}_n^{(k)}(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)=\sum_{m=0}^n C_{n,m}H_m^{(s)}(x|\alpha)$. By (11), similarly to the proof of (29), we have

$$\begin{aligned}C_{n,m}&=\frac{1}{m!}\left\langle\frac{\left(\frac{e^{\ln(1+t)}-\alpha}{1-\alpha}\right)^s}{\prod_{j=1}^r\left(\frac{1+e^{\lambda_j\ln(1+t)}}{e^{\lambda_j\ln(1+t)}}\right)^{\mu_j}}\mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m\Big|x^n\right\rangle\\&=\frac{1}{m!(1-\alpha)^s}\left\langle\prod_{j=1}^r\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m(1-\alpha+t)^s\Big|x^n\right\rangle\\&=\frac{1}{m!(1-\alpha)^s}\\&\quad\times\left\langle\prod_{j=1}^r\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m\Big|\sum_{i=0}^{\min\{s,n\}}\binom{s}{i}(1-\alpha)^{s-i}t^ix^n\right\rangle\\&=\frac{1}{m!(1-\alpha)^s}\sum_{i=0}^{n-m}\binom{s}{i}(1-\alpha)^{s-i}(n)_i\\&\quad\times\left\langle\prod_{j=1}^r\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m\Big|x^{n-i}\right\rangle\\&=\frac{1}{m!(1-\alpha)^s}\sum_{i=0}^{n-m}\binom{s}{i}(1-\alpha)^{s-i}(n)_i\sum_{l=0}^{n-m-i}m!\binom{n-i}{l}S_1(n-i-l,m)\widehat{s}_l^{(k)}\\&=\sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{s}{i}\binom{n-i}{l}(n)_i(1-\alpha)^{-i}S_1(n-i-l,m)\widehat{s}_l^{(k)}\,.\end{aligned}$$

Thus, we get the identity (32). ∎

3.10 A relation with higher-order Bernoulli polynomials

Bernoulli polynomials $\mathfrak{B}_n^{(r)}(x)$ of order r are defined by

$$\left(\frac{t}{e^t-1}\right)^r e^{xt}=\sum_{n=0}^\infty\frac{\mathfrak{B}_n^{(r)}(x)}{n!}t^n$$

(see e.g. [12, Section 2.2]). In addition, Cauchy numbers of the first kind $\mathfrak{C}_n^{(r)}$ of order r are defined by

$$\left(\frac{t}{\ln(1+t)}\right)^r=\sum_{n=0}^\infty\frac{\mathfrak{C}_n^{(r)}}{n!}t^n$$

(see e.g. [2, (2.1)], [11, (6)]).

Theorem 10

$$\begin{aligned}&\widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\&=\sum_{m=0}^n\left(\sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{n}{i}\binom{n-i}{l}\mathfrak{C}_i^{(s)}S_1(n-i-l,m)\widehat{s}_l^{(k)}\right)\mathfrak{B}_m^{(s)}(x)\,.\end{aligned}\tag{34}$$

Proof. For (12) and

$$\mathfrak{B}_n^{(s)}(x)\sim\left(\left(\frac{e^t-1}{t}\right)^s,t\right),\tag{35}$$

assume that $\widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)=\sum_{m=0}^n C_{n,m}\mathfrak{B}_m^{(s)}(x)$. By (11), similarly to the proof of (29), we have

$$\begin{aligned}C_{n,m}&=\frac{1}{m!}\left\langle\frac{\left(\frac{e^{\ln(1+t)}-1}{\ln(1+t)}\right)^s}{\prod_{j=1}^r\left(\frac{1+e^{\lambda_j\ln(1+t)}}{e^{\lambda_j\ln(1+t)}}\right)^{\mu_j}}\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)\bigl(\ln(1+t)\bigr)^m\Big|x^n\right\rangle\\
&=\frac{1}{m!}\left\langle\prod_{j=1}^r\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)\bigl(\ln(1+t)\bigr)^m\Big|\left(\frac{t}{\ln(1+t)}\right)^s x^n\right\rangle\\
&=\frac{1}{m!}\left\langle\prod_{j=1}^r\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)\bigl(\ln(1+t)\bigr)^m\Big|\sum_{i=0}^\infty\mathfrak{C}_i^{(s)}\frac{t^i}{i!}x^n\right\rangle\\
&=\frac{1}{m!}\sum_{i=0}^{n-m}\mathfrak{C}_i^{(s)}\binom{n}{i}\left\langle\prod_{j=1}^r\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)\bigl(\ln(1+t)\bigr)^m\Big|x^{n-i}\right\rangle\\
&=\frac{1}{m!}\sum_{i=0}^{n-m}\mathfrak{C}_i^{(s)}\binom{n}{i}\sum_{l=0}^{n-m-i}m!\binom{n-i}{l}S_1(n-i-l,m)\widehat{s}_l^{(k)}\\
&=\sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{n}{i}\binom{n-i}{l}\mathfrak{C}_i^{(s)}S_1(n-i-l,m)\widehat{s}_l^{(k)}\,.\end{aligned}$$

Thus, we get the identity (34). ∎

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On the solution for a system of two rational difference equations

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Abstract: This paper is concerned with the dynamical behavior and the expression of the solution for a system of two rational difference equations

$$x_{n+1} = \frac{x_{n-3}}{A + x_{n-3}y_{n-1}}, \quad y_{n+1} = \frac{y_{n-3}}{B + y_{n-3}x_{n-1}}, \quad n = 0, 1, \cdots,$$

where the parameters A, B and the initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}, y_0$ are positive real numbers.

Keywords: difference equations; expression of solutions; recursive sequences, equilibrium point; asymptotical stability.

1. Introduction

Rational difference equations that are one of the most important and practical classes of nonlinear difference equations have applications in various scientific branches such as biology, ecology, physiology, physics, engineering and economics, etc [1-4]. Although difference equations are very simple in form, it is extremely difficult to understand thoroughly the behaviors of their solutions. So recently there has been an increasing interest in the study of qualitative analysis of rational difference equation and systems of difference equations [5-7]. In particular, Papaschinopoulos and Schinas [8] studied the oscillatory behavior, the boundedness of the solutions, and the global asymptotic stability of the positive equilibrium of the system of two nonlinear difference equations

$$x_{n+1}=A+\frac{y_n}{x_{n-p}},\quad y_{n+1}=A+\frac{x_n}{y_{n-q}},\quad n=0,1,\cdots, \tag{1.1}$$

where p,q are positive integers. Clark and Kulenovic [9, 10] investigated the global stability properties and asymptotic behavior of solutions of the recursive sequences

$$x_{n+1}=\frac{x_n}{a+cy_n},\qquad y_{n+1}=\frac{y_n}{b+dx_n},\qquad n=0,1,\cdots. \tag{1.2}$$

where $a,b,c,d\in(0,\infty)$ and the initial conditions x_0 and y_0 are arbitrary nonnegative numbers. The periodicity of the positive solutions of the system of rational difference equations

$$x_{n+1}=\frac{1}{y_n},\qquad y_{n+1}=\frac{y_n}{x_{n-1}y_{n-1}},\qquad n=0,1,\cdots, \tag{1.3}$$

was studied by Cinar in [11]. Yalcinkaya [12] has obtained the sufficient conditions for the global asymptotic stability of the system of two nonlinear difference equations

$$x_{n+1}=\frac{x_n+y_{n-1}}{x_ny_{n-1}-1},\qquad y_{n+1}=\frac{y_n+x_{n-1}}{y_nx_{n-1}-1},\qquad n=0,1,\cdots. \tag{1.4}$$

More recently, Din et al. [13] studied the equilibrium points, local asymptotic stability of an equilibrium point, instability of equilibrium points, periodicity behavior of positive solutions, and global character of an equilibrium point of the following fourth-order system of rational difference equations

$$x_{n+1}=\frac{\alpha x_{n-3}}{\beta+\gamma y_ny_{n-1}y_{n-2}y_{n-3}},\ y_{n+1}=\frac{\alpha_1 y_{n-3}}{\beta_1+\gamma_1 x_nx_{n-1}x_{n-2}x_{n-3}},\ n=0,1,\cdots. \tag{1.5}$$

In [14], Elsayed deals with the form of the solutions of the following rational difference system

$$x_{n+1}=\frac{x_{n-1}}{\pm 1+x_{n-1}y_n},\quad y_{n+1}=\frac{y_{n-1}}{\mp 1+y_{n-1}x_n},\quad n=0,1,\cdots, \tag{1.6}$$

with nonzero real number initial conditions. Other related results on the difference equation can be found in references [15-28] and references therein.

Based on the above results, we are mainly interested in study the asymptotic behavior and the expression of the solution for the following nonlinear rational difference equations

$$x_{n+1}=\frac{x_{n-3}}{A+x_{n-3}y_{n-1}},\quad y_{n+1}=\frac{y_{n-3}}{B+y_{n-3}x_{n-1}},\quad n=0,1,\cdots, \tag{1.7}$$

where the parameters A,B and the initial conditions $x_{-3},x_{-2},x_{-1},x_0,y_{-3},y_{-2},y_{-1},y_0$ are positive real numbers.

This paper proceeds as follows. In Section 2, we introduce some definitions and preliminary results. The main results and their proofs are given in Section 3.

2. Preliminaries and notations

In this section we prepare some materials used throughout this paper, namely notations, the basic definitions and preliminary results. We refer to the monographs of Kocic et al. [5, 29, 30].

Lemma 2.1 Let I_x, I_y be some intervals of real numbers and $f: I_x^4 \times I_y^4 \to I_x$, $g: I_x^4 \times I_y^4 \to I_y$ be continuously differentiable functions. Then for every initial conditions $(x_i, y_i) \in I_x \times I_y, (i = -3, -2, -1, 0)$, the system of difference equations

$$\begin{cases} x_{n+1} = f(x_n, x_{n-1}, x_{n-2}, x_{n-3}, y_n, y_{n-1}, y_{n-2}, y_{n-3}), \\ y_{n+1} = g(x_n, x_{n-1}, x_{n-2}, x_{n-3}, y_n, y_{n-1}, y_{n-2}, y_{n-3}), \end{cases} \quad n = 0, 1, 2, \cdots, \tag{2.1}$$

has a unique solution $\{(x_n, y_n)\}_{n=-3}^{\infty}$.

Definition 2.1 A point $(\bar{x}, \bar{y}) \in I_x \times I_y$ is called an equilibrium point of system (2.1) if

$$\bar{x} = f(\bar{x}, \bar{x}, \bar{x}, \bar{x}, \bar{y}, \bar{y}, \bar{y}, \bar{y}), \ \bar{y} = g(\bar{x}.\bar{x}, \bar{x}, \bar{x}, \bar{y}, \bar{y}, \bar{y}, \bar{y}).$$

That is, $(x_n, y_n) = (\bar{x}, \bar{y})$ for all $n \geq 1$ when the initial conditions

$$(x_0, x_{-1}, x_{-2}, x_{-3}, y_0, y_{-1}, y_{-2}, y_{-3}) = (\bar{x}, \bar{x}, \bar{x}, \bar{x}, \bar{y}, \bar{y}, \bar{y}, \bar{y}).$$

Definition 2.2 Let $(\bar{x}, \bar{y})$ be an equilibrium point of system (2.1). Then

(1) The equilibrium $(\bar{x}, \bar{y})$ of system (2.1) is said to be stable relative to $I_x \times I_y$ if for every $\varepsilon > 0$, there exits $\delta > 0$ such that for any initial conditions $(x_i, y_i) \in I_x \times I_y$ $(i = -3, -2, -1, 0)$, with $\sum_{i=-3}^{0} |x_i - \bar{x}| < \delta$, $\sum_{i=-3}^{0} |y_i - \bar{y}| < \delta$ implies $|x_n - \bar{x}| < \varepsilon$, $|y_n - \bar{y}| < \varepsilon$.

(2) The equilibrium $(\bar{x}, \bar{y})$ of system (2.1) is called an attractor relative to $I_x \times I_y$ if for all $(x_i, y_i) \in I_x \times I_y$ $(i = -3, -2, -1, 0)$, $\lim_{n \to \infty} x_n = \bar{x}$ and $\lim_{n \to \infty} y_n = \bar{y}$ hold.

(3) The equilibrium $(\bar{x}, \bar{y})$ of system (2.1) is called asymptotically stable relative to $I_x \times I_y$ if it is stable and an attractor.

(4) The equilibrium $(\bar{x}, \bar{y})$ of system (2.1) is called unstable if it is not stable.

Definition 2.3 Let $(\bar{x}, \bar{y})$ be an equilibrium point of the system (2.1), and f and g are continuously differentiable functions at $(\bar{x}, \bar{y})$. The linearized system of system (2.1) about

the equilibrium point $(\bar{x}, \bar{y})$ is

$$X_{n+1} = F(X_n) = F_J X_n$$

where $X_n = (x_n, x_{n-1}, x_{n-2}, x_{n-3}, y_n, y_{n-1}, y_{n-2}, y_{n-3})^T$ and F_J is a Jacobian matrix of the system (2.1) about the equilibrium point $(\bar{x}, \bar{y})$.

Lemma 2.2 Assume that $X_{n+1} = F(X_n)$, $n = 0, 1, \cdots$, is a system of difference equations and $\bar{X}$ is the equilibrium point of this system i.e., $F(\bar{X}) = \bar{X}$. If all eigenvalues of the Jacobian matrix F_J about $\bar{X}$ lie inside the open unit disk $|\lambda| < 1$, then $\bar{X}$ is locally asymptotically stable. If one of them has a modulus greater than one, then $\bar{X}$ is unstable.

3. Main results and their proofs

It is obviously, if $A > 1, B \neq 1 \text{ or } B > 1, A \neq 1$, then $(0,0)$ is the unique equilibrium point of the system (1.7).

Theorem 3.1 Let $\{x_n, y_n\}_{n=-3}^{\infty}$ be positive solutions of system (1.7), then for all $k \geq 0$,

$$(1)\quad 0 \leq x_n \leq \begin{cases} \dfrac{x_{-3}}{A^{k+1}}, & n = 4k+1, \\ \dfrac{x_{-2}}{A^{k+1}}, & n = 4k+2, \\ \dfrac{x_{-1}}{A^{k+1}}, & n = 4k+3, \\ \dfrac{x_0}{A^{k+1}}, & n = 4k+4. \end{cases} \qquad (2)\quad 0 \leq y_n \leq \begin{cases} \dfrac{y_{-3}}{B^{k+1}}, & n = 4k+1, \\ \dfrac{y_{-2}}{B^{k+1}}, & n = 4k+2, \\ \dfrac{y_{-1}}{B^{k+1}}, & n = 4k+3, \\ \dfrac{y_0}{B^{k+1}}, & n = 4k+4. \end{cases} \tag{3.1}$$

Proof. This assertion is true for $k = 0$, Assume that it is true for $k = m$, then for $k = m+1$, we have

$$x_n = \begin{cases} x_{4(m+1)+1} \leq \dfrac{x_{4(m+1)-3}}{A} = \dfrac{x_{4m+1}}{A} \leq \dfrac{1}{A}\dfrac{x_{-3}}{A^{m+1}} = \dfrac{x_{-3}}{A^{(m+1)+1}}, & n = 4(m+1)+1; \\ x_{4(m+1)+2} \leq \dfrac{x_{4(m+1)+1-3}}{A} = \dfrac{x_{4m+2}}{A} \leq \dfrac{1}{A}\dfrac{x_{-2}}{A^{m+1}} = \dfrac{x_{-2}}{A^{(m+1)+1}}, & n = 4(m+1)+2, \\ x_{4(m+1)+3} \leq \dfrac{x_{4(m+1)+2-3}}{A} = \dfrac{x_{4m+3}}{A} \leq \dfrac{1}{A}\dfrac{x_{-1}}{A^{m+1}} = \dfrac{x_{-1}}{A^{(m+1)+1}}, & n = 4(m+1)+3, \\ x_{4(m+1)+4} \leq \dfrac{x_{4(m+1)+3-3}}{A} = \dfrac{x_{4m+4}}{A} \leq \dfrac{1}{A}\dfrac{x_0}{A^{m+1}} = \dfrac{x_0}{A^{(m+1)+1}}, & n = 4(m+1)+4. \end{cases}$$

$$y_n=\begin{cases} y_{4(m+1)+1}\le \dfrac{y_{4(m+1)-3}}{B}=\dfrac{y_{4m+1}}{B}\le \dfrac{1}{B}\dfrac{y_{-3}}{B^{m+1}}=\dfrac{y_{-3}}{B^{(m+1)+1}}, & n=4(m+1)+1;\\ y_{4(m+1)+2}\le \dfrac{y_{4(m+1)+1-3}}{B}=\dfrac{y_{4m+2}}{B}\le \dfrac{1}{B}\dfrac{y_{-2}}{B^{m+1}}=\dfrac{y_{-2}}{B^{(m+1)+1}}, & n=4(m+1)+2,\\ y_{4(m+1)+3}\le \dfrac{y_{4(m+1)+2-3}}{B}=\dfrac{y_{4m+3}}{B}\le \dfrac{1}{B}\dfrac{y_{-1}}{B^{m+1}}=\dfrac{y_{-1}}{B^{(m+1)+1}}, & n=4(m+1)+3,\\ y_{4(m+1)+4}\le \dfrac{y_{4(m+1)+3-3}}{B}=\dfrac{y_{4m+4}}{B}\le \dfrac{1}{B}\dfrac{y_{0}}{B^{m+1}}=\dfrac{y_{0}}{B^{(m+1)+1}}, & n=4(m+1)+4. \end{cases}$$

This completes our inductive proof.

Corollary 3.1 If $A>1, B>1$, then by Theorem 3.1 $\{(x_n, y_n)\}_{n=-3}^{\infty}$ the solutions of the system (1.7) exponentially converges to the equilibrium point (0, 0).

Theorem 3.2 For the equilibrium point (0, 0) of the system (1.7), the following results hold:

(1) If $A>1, B>1$, then the equilibrium point (0,0) of the system (1.7) is locally asymptotically stable.

(2) If $A<1$ or $B<1$, then the equilibrium point (0,0) of the system (1.7) is unstable.

Proof. We can easily obtain that the linearized system of (1.7) about the equilibrium point (0, 0) is

$$\varphi_{n+1}=D\varphi_n \tag{3.2}$$

where

$$\varphi_n=\begin{bmatrix} x_n\\ x_{n-1}\\ x_{n-2}\\ x_{n-3}\\ y_n\\ y_{n-1}\\ y_{n-2}\\ y_{n-3}\end{bmatrix},\qquad D=\begin{bmatrix} 0&0&0&\frac{1}{A}&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&\frac{1}{B}\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0 \end{bmatrix}$$

the characteristic equation of (3.2) is

$$f(\lambda)=(\lambda^4-\frac{1}{A})(\lambda^4-\frac{1}{B})=0. \tag{3.3}$$

(1) If $A>1, B>1$, then we have $|\frac{1}{A}|<1, |\frac{1}{B}|<1$, this shows that all the roots of characteristic equation (3.3) lie inside unit disk. So the unique equilibrium (0, 0) is

locally asymptotically stable.

(2) It is easy to see that if $A<1$ or $B<1$, then there exists at least one root λ of the characteristic equation (3.3) such that $|\lambda|>1$. Thus, the equilibrium (0, 0) of the system (1.7) is unstable when $A<1$ or $B<1$.

By Corollary 3.1 and Theorem 3.2, we have the following result.

Corollary 3.2 If $A>1, B>1$, then the equilibrium point (0, 0) is globally asymptotically stable.

Theorem 3.3 If $A=B=1$, then every solution of the system (1.7) is bounded when the initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}$ and y_0 are positive real numbers.

Proof. It follows from Eq. (1.7) that

$$x_{n+1}=\frac{x_{n-3}}{1+x_{n-3}y_{n-1}}\le x_{n-3},\ y_{n+1}=\frac{y_{n-3}}{1+y_{n-3}x_{n-1}}\le y_{n-3}.$$

Then the subsequences

$$\{x_{4n-3}\}_{n=0}^{\infty},\ \{x_{4n-2}\}_{n=0}^{\infty},\ \{x_{4n-1}\}_{n=0}^{\infty},\ \{x_{4n}\}_{n=0}^{\infty}$$

are decreasing and so are bounded from above by $M=\max\{x_{-3}, x_{-2}, x_{-1}, x_0\}$, also, the subsequences

$$\{y_{4n-3}\}_{n=0}^{\infty},\ \{y_{4n-2}\}_{n=0}^{\infty},\ \{y_{4n-1}\}_{n=0}^{\infty},\ \{y_{4n}\}_{n=0}^{\infty}$$

are decreasing and so are bounded from above by $m=\max\{y_{-3}, y_{-2}, y_{-1}, y_0\}$. Hence, every solution of the system (1.7) is bounded for any positive initial conditions.

In next section, we study the expressions of the solutions for the systems (1.7) with the parameters $A=B$.

Theorem 3.4 If $A=B$, suppose that $\{(x_n, y_n)\}_{n=-3}^{\infty}$ are solutions of the system (1.7). Also, assume that $x_{-3}, x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}$ and y_0 are arbitrary positive numbers and let $x_{-3}=a, x_{-2}=b, x_{-1}=c, x_0=d, y_{-3}=e, y_{-2}=f, y_{-1}=g, y_0=h$. Then

$$\begin{aligned}
x_{4n-3}&=a\prod_{i=0}^{n-1}\frac{A^{2i}+A^{2i-1}ag+\cdots+Aag+ag}{A^{2i+1}+A^{2i}ag+A^{2i-1}ag+\cdots+Aag+ag},\\
x_{4n-2}&=b\prod_{i=0}^{n-1}\frac{A^{2i}+A^{2i-1}bh+\cdots+Abh+bh}{A^{2i+1}+A^{2i}bh+A^{2i-1}bh+\cdots+Abh+bh},\\
x_{4n-1}&=c\prod_{i=0}^{n-1}\frac{A^{2i+1}+A^{2i}ce+\cdots+Ace+ce}{A^{2i+2}+A^{2i+1}ce+A^{2i}ce+\cdots+Ace+ce},\\
x_{4n}&=d\prod_{i=0}^{n-1}\frac{A^{2i+1}+A^{2i}df+\cdots+Adf+df}{A^{2i+2}+A^{2i+1}df+A^{2i}df+\cdots+Adf+df},
\end{aligned}\tag{3.4}$$

$$\begin{aligned}
y_{4n-3} &= e\prod_{i=0}^{n-1}\frac{A^{2i}+A^{2i-1}ce+\cdots+Ace+ce}{A^{2i+1}+A^{2i}ce+A^{2i-1}ce+\cdots+Ace+ce},\\
y_{4n-2} &= f\prod_{i=0}^{n-1}\frac{A^{2i}+A^{2i-1}df+\cdots+Adf+df}{A^{2i+1}+A^{2i}df+A^{2i-1}df+\cdots+Adf+df},\\
y_{4n-1} &= g\prod_{i=0}^{n-1}\frac{A^{2i+1}+A^{2i}ag+\cdots+Aag+ag}{A^{2i+2}+A^{2i+1}ag+A^{2i}ag+\cdots+Aag+ag},\\
y_{4n} &= h\prod_{i=0}^{n-1}\frac{A^{2i+1}+A^{2i}bh+\cdots+Abh+bh}{A^{2i+2}+A^{2i+1}bh+A^{2i}bh+\cdots+Abh+bh},
\end{aligned} \tag{3.5}$$

where $n=1, 2, \cdots$.

Proof. If $A=B$, then the system (1.7) is reduced to

$$x_{n+1}=\frac{x_{n-3}}{A+x_{n-3}y_{n-1}}, \quad y_{n+1}=\frac{y_{n-3}}{A+y_{n-3}x_{n-1}}. \tag{3.6}$$

It is easy to prove that Eqs. (3.4) and (3.5) hold for $n=1$. Now suppose that $k\in N, k>1$ and that Eqs. (3.4) and (3.5) hold for $n=k-1$. That is,

$$\begin{aligned}
x_{4k-7} &= a\prod_{i=0}^{k-2}\frac{A^{2i}+A^{2i-1}ag+\cdots+Aag+ag}{A^{2i+1}+A^{2i}ag+A^{2i-1}ag+\cdots+Aag+ag},\\
x_{4k-6} &= b\prod_{i=0}^{k-2}\frac{A^{2i}+A^{2i-1}bh+\cdots+Abh+bh}{A^{2i+1}+A^{2i}bh+A^{2i-1}bh+\cdots+Abh+bh},\\
x_{4k-5} &= c\prod_{i=0}^{k-2}\frac{A^{2i+1}+A^{2i}ce+\cdots+Ace+ce}{A^{2i+2}+A^{2i+1}ce+A^{2i}ce+\cdots+Ace+ce},\\
x_{4k-4} &= d\prod_{i=0}^{k-2}\frac{A^{2i+1}+A^{2i}df+\cdots+Adf+df}{A^{2i+2}+A^{2i+1}df+A^{2i}df+\cdots+Adf+df},\\
y_{4k-7} &= e\prod_{i=0}^{k-2}\frac{A^{2i}+A^{2i-1}ce+\cdots+Ace+ce}{A^{2i+1}+A^{2i}ce+A^{2i-1}ce+\cdots+Ace+ce},\\
y_{4k-6} &= f\prod_{i=0}^{k-2}\frac{A^{2i}+A^{2i-1}df+\cdots+Adf+df}{A^{2i+1}+A^{2i}df+A^{2i-1}df+\cdots+Adf+df},\\
y_{4k-5} &= g\prod_{i=0}^{k-2}\frac{A^{2i+1}+A^{2i}ag+\cdots+Aag+ag}{A^{2i+2}+A^{2i+1}ag+A^{2i}ag+\cdots+Aag+ag},\\
y_{4k-4} &= h\prod_{i=0}^{k-2}\frac{A^{2i+1}+A^{2i}bh+\cdots+Abh+bh}{A^{2i+2}+A^{2i+1}bh+A^{2i}bh+\cdots+Abh+bh}.
\end{aligned}$$

Then, it follows from Eq. (3.6) and our assumptions that

$$
\begin{aligned}
x_{4k-3} &= \frac{x_{4k-7}}{A + x_{4k-7} y_{4k-5}} \\
&= \frac{a\prod_{i=0}^{k-2} \frac{A^{2i} + A^{2i-1}ag + \cdots + Aag + ag}{A^{2i+1} + A^{2i}ag + A^{2i-1}ag + \cdots + Aag + ag}}{A + a\prod_{i=0}^{k-2} \frac{A^{2i} + A^{2i-1}ag + \cdots + Aag + ag}{A^{2i+1} + A^{2i}ag + A^{2i-1}ag + \cdots + Aag + ag} g\prod_{i=0}^{k-2} \frac{A^{2i+1} + A^{2i}ag + \cdots + Aag + ag}{A^{2i+2} + A^{2i+1}ag + A^{2i}ag + \cdots + Aag + ag}} \\
&= \frac{a\prod_{i=0}^{k-2} \frac{A^{2i} + A^{2i-1}ag + \cdots + Aag + ag}{A^{2i+1} + A^{2i}ag + A^{2i-1}ag + \cdots + Aag + ag}}{A + ag \frac{1}{A^{2k-2} + A^{2k-1}ag + \cdots + Aag + ag}} \\
&= a\prod_{i=0}^{k-2} \frac{A^{2i} + A^{2i-1}ag + \cdots + Aag + ag}{A^{2i+1} + A^{2i}ag + A^{2i-1}ag + \cdots + Aag + ag} \left(\frac{A^{2k-2} + A^{2k-1}ag + \cdots + Aag + ag}{A^{2k-1} + A^{2k-2}ag + \cdots + Aag + ag} \right) \\
&= a\prod_{i=0}^{k-1} \frac{A^{2i} + A^{2i-1}ag + \cdots + Aag + ag}{A^{2i+1} + A^{2i}ag + A^{2i-1}ag + \cdots + Aag + ag}.
\end{aligned}
$$

That is

$$
x_{4k-3} = a\prod_{i=0}^{k-1} \frac{A^{2i} + A^{2i-1}ag + \cdots + Aag + ag}{A^{2i+1} + A^{2i}ag + A^{2i-1}ag + \cdots + Aag + ag}.
$$

In addition to, by Eq. (3.6) and our assumptions one has

$$
\begin{aligned}
y_{4k-3} &= \frac{y_{4k-7}}{A + y_{4k-7} x_{4k-5}} \\
&= \frac{e\prod_{i=0}^{k-2} \frac{A^{2i} + A^{2i-1}ce + \cdots + Ace + ce}{A^{2i+1} + A^{2i}ce + A^{2i-1}ce + \cdots + Ace + ce}}{A + e\prod_{i=0}^{k-2} \frac{A^{2i} + A^{2i-1}ce + \cdots + Ace + ce}{A^{2i+1} + A^{2i}ce + A^{2i-1}ce + \cdots + Ace + ce} c\prod_{i=0}^{k-2} \frac{A^{2i+1} + A^{2i}ce + \cdots + Ace + ce}{A^{2i+2} + A^{2i+1}ce + A^{2i}ce + \cdots + Ace + ce}} \\
&= \frac{e\prod_{i=0}^{k-2} \frac{A^{2i} + A^{2i-1}ce + \cdots + Ace + ce}{A^{2i+1} + A^{2i}ce + A^{2i-1}ce + \cdots + Ace + ce}}{A + ce \frac{1}{A^{2n-2} + A^{2n-1}ce + \cdots + Ace + ce}} \\
&= e\prod_{i=0}^{k-2} \frac{A^{2i} + A^{2i-1}ce + \cdots + Ace + ce}{A^{2i+1} + A^{2i}ce + A^{2i-1}ce + \cdots + Ace + ce} \left(\frac{A^{2k-2} + A^{2k-1}ce + \cdots + Ace + ce}{A^{2k-1} + A^{2k-2}ce + \cdots + Ace + ce} \right) \\
&= e\prod_{i=0}^{k-1} \frac{A^{2i} + A^{2i-1}ce + \cdots + Ace + ce}{A^{2i+1} + A^{2i}ce + A^{2i-1}ce + \cdots + Ace + ce}.
\end{aligned}
$$

That is,

$$y_{4k-3} = e\prod_{i=0}^{k-1} \frac{A^{2i} + A^{2i-1}ce + \cdots + Ace + ce}{A^{2i+1} + A^{2i}ce + A^{2i-1}ce + \cdots + Ace + ce}.$$

Similarly, one can prove

$$x_{4k-2} = b\prod_{i=0}^{k-1} \frac{A^{2i} + A^{2i-1}bh + \cdots + Abh + bh}{A^{2i+1} + A^{2i}bh + A^{2i-1}bh + \cdots + Abh + bh},$$

$$x_{4k-1} = c\prod_{i=0}^{k-1} \frac{A^{2i+1} + A^{2i}ce + \cdots + Ace + ce}{A^{2i+2} + A^{2i+1}ce + A^{2i}ce + \cdots + Ace + ce},$$

$$x_{4k} = d\prod_{i=0}^{k-1} \frac{A^{2i+1} + A^{2i}df + \cdots + Adf + df}{A^{2i+2} + A^{2i+1}df + A^{2i}df + \cdots + Adf + df},$$

$$y_{4k-2} = f\prod_{i=0}^{k-1} \frac{A^{2i} + A^{2i-1}df + \cdots + Adf + df}{A^{2i+1} + A^{2i}df + A^{2i-1}df + \cdots + Adf + df},$$

$$y_{4k-1} = g\prod_{i=0}^{k-1} \frac{A^{2i+1} + A^{2i}ag + \cdots + Aag + ag}{A^{2i+2} + A^{2i+1}ag + A^{2i}ag + \cdots + Aag + ag},$$

$$y_{4k} = h\prod_{i=0}^{k-1} \frac{A^{2i+1} + A^{2i}bh + \cdots + Abh + bh}{A^{2i+2} + A^{2i+1}bh + A^{2i}bh + \cdots + Abh + bh}.$$

Hence, Eqs. (3.4) and (3.5) hold for $n = k$. The proof is complete according to the mathematical induction.

Corollary 3.3 If $A = B = 1$, suppose that $\{(x_n, y_n)\}_{n=-3}^{\infty}$ are solutions of the system (1.7). Also, assume that $x_{-3}, x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}$ and y_0 are arbitrary positive numbers and let $x_{-3} = a, x_{-2} = b, x_{-1} = c, x_0 = d, y_{-3} = e, y_{-2} = f, y_{-1} = g, y_0 = h$, then one has

$$x_{4n-3} = a\prod_{i=0}^{n-1} \frac{1+2iag}{1+(2i+1)ag}, \quad x_{4n-2} = b\prod_{i=0}^{n-1} \frac{1+2ibh}{1+(2i+1)bh},$$

$$x_{4n-1} = c\prod_{i=0}^{n-1} \frac{1+(2i+1)ce}{1+(2i+2)ce}, \quad x_{4n} = d\prod_{i=0}^{n-1} \frac{1+(2i+1)df}{1+(2i+2)df},$$

$$y_{4n-3} = e\prod_{i=0}^{n-1} \frac{1+2ice}{1+(2i+1)ce}, \quad y_{4n-2} = f\prod_{i=0}^{n-1} \frac{1+2idf}{1+(2i+1)df},$$

$$y_{4n-1} = g\prod_{i=0}^{n-1} \frac{1+(2i+1)ag}{1+(2i+2)ag}, \quad y_{4n} = h\prod_{i=0}^{n-1} \frac{1+(2i+1)bh}{1+(2i+2)bh},$$

where $n = 1, 2, \cdots$.

4. Conclusions

It is obvious that the system of two rational difference equations (1.7) is the extension of the models in [9, 10, 13, 14]. In this paper, we investigated the globally asymptotically stable of the equilibrium point $(0,0)$ for the difference equation (1.7) with the parameters $A > 1, B > 1$, and the unstable of the equilibrium point $(0,0)$ with the parameter $A < 1$ or $B < 1$ using

linearization method. Moreover, the expressions of solutions of the system (1.7) with the parameters $A=B$ are obtained according to the mathematical induction. This paper presents the use of a variational iteration method and mathematical induction for systems of nonlinear difference equations. This technique is a powerful tool for solving various difference equations and can also be applied to other nonlinear differential equations in mathematical physics. In addition, the sufficient conditions that we obtained are very simple, which provide flexibility for the application and analysis of nonlinear difference equation. In addition, the system can be used to analyze and describe the pier buffering isolation system.

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Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

On Distributions of Discrete Order Statistics

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Abstract. In this study, the joint distributions of order statistics of *innid* discrete random variables are expressed in the form of an integral. Then, the results related to *pf* and *df* are given.

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Key words and phrases: Order statistics, discrete random variable, probability function, distribution function.

1. Introduction

The joint probability density function(*pdf*) and marginal *pdf* of order statistics of independent but not necessarily identically distributed(*innid*) random variables was derived by Vaughan and Venables[22] by means of permanents. In addition, Balakrishnan[3], and Bapat and Beg[8] obtained the joint *pdf* and distribution function(*df*) of order statistics of *innid* random variables by means of permanents. In the first of two papers, Balasubramanian et al.[5] obtained the distribution of single order statistic in terms of distribution functions of the minimum and maximum order statistics of some subsets of $\{X_1, X_2, ..., X_n\}$ where X_i's are *innid* random variables. Later, Balasubramanian et al.[6] generalized their previous results[5] to the case of the joint distribution function of several order statistics. Recurrence relationships among the distribution functions of order statistics arising from *innid* random variables were obtained by Cao and West[10]. Using multinomial arguments, the *pdf* of $X_{r:n+1}$ $(1 \le r \le n+1)$ was obtained by Childs and Balakrishnan[11] by adding another independent random variable to the original n variables $X_1, X_2, ..., X_n$. Also, Balasubramanian et

al.[7] established the identities satisfied by distributions of order statistics from non-independent non-identical variables through operator methods based on the difference and differential operators. In a paper published in 1991, Beg[9] obtained several recurrence relations and identities for product moments of order statistics of *innid* random variables using permanents. Recently, Cramer et al.[13] derived the expressions for the distribution and density functions by Ryser's method and the distributions of maxima and minima based on permanents.

A multivariate generalization of classical order statistics for random samples from a continuous multivariate distribution was defined by Corley[12]. Guilbaud[17] expressed the probability of the functions of *innid* random vectors as a linear combination of probabilities of the functions of independent and identically distributed(*iid*) random vectors and thus also for order statistics of random variables. Expressions for generalized joint densities of order statistics of *iid* random variables in terms of Radon-Nikodym derivatives with respect to product measures based on *df* were derived by Goldie and Maller[16]. Several identities and recurrence relations for *pdf* and *df* of order statistics of *iid* random variables were established by numerous authors including Arnold et al.[1], Balasubramanian and Beg[4], David[14], and Reiss[21]. Furthermore, Arnold et al.[1], David[14], Gan and Bain[15], and Khatri[18] obtained the probability function(*pf*) and *df* of order statistics of *iid* random variables from a discrete parent. Balakrishnan[2] showed that several relations and identities that have been derived for order statistics from continuous distributions also hold for the discrete case. In a paper published in 1986, Nagaraja[19] explored the behavior of higher order conditional probabilities of order statistics in a attempt to understand the structure of discrete order statistics. Later, Nagaraja[20] considered some results on order statistics of a random sample taken from a discrete population.

In general, the distribution theory for order statistics is complex when the parent distribution is discrete. In this study, the joint distributions of p order statistics of *innid* discrete random variables are obtained as an p fold integral.

As far as we know, these approaches have not been considered in the framework of order statistics from *innid* discrete random variables.

From now on, the subscripts and superscripts are defined in the first place in which they are used and these definitions will be valid unless they are redefined.

Let $X_1, X_2, ..., X_n$ be *innid* discrete random variables and $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ be the order statistics obtained by arranging the n X_i's in increasing order of magnitude. Let F_i and f_i be *df* and *pf* of X_i (i =1, 2,…, n), respectively. For notational convenience we write $\sum\limits_{z_1, z_2, ..., z_p}$, $\sum\limits_{m_p, k_p ..., m_1, k_1}$, $\int$ and $\int\limits_V$ instead of $\sum\limits_{z_1=0}^{x_1} \sum\limits_{z_2=z_1}^{x_2} \sum\limits_{z_3=z_2}^{x_3} ... \sum\limits_{z_p=z_{p-1}}^{x_p}$,

$$\sum_{m_p=0}^{n-r_p} \sum_{k_p=0}^{r_p-1-r_{p-1}} ... \sum_{m_2=0}^{r_3-1-r_2} \sum_{k_2=0}^{r_2-1-r_1} \sum_{m_1=0}^{r_2-1-r_1} \sum_{k_1=0}^{r_1-1} , \int_{F_{i_{r_1}}(x_1-)}^{F_{i_{r_1}}(x_1)} \int_{F_{i_{r_2}}(x_2-)}^{F_{i_{r_2}}(x_2)} ... \int_{F_{i_{r_p}}(x_p-)}^{F_{i_{r_p}}(x_p)} \text{ and } \int_0^{F_{i_{r_1}}(x_1)} \int_{v_{i_{r_1}}^{(1)}}^{F_{i_{r_2}}(x_2)} ... \int_{v_{i_{r_{(p-1)}}}^{(p-1)}}^{F_{i_{r_p}}(x_p)} \text{ in}$$

the expressions below, respectively $(x_i = 0, 1, 2, \ldots)$ $(z_0 = 0)$.

2. Theorems for distribution and probability functions

In this section, the theorems related to *pf* and *df* of $X_{r_1:n}, X_{r_2:n}, ..., X_{r_p:n}$ ($1 \leq r_1 < r_2 < ... < r_p \leq n$, p=1, 2,…, n) will be given. We will now express the following theorem for the joint *pf* of order statistics of *innid* discrete random variables.

Theorem 2.1.

$$f_{r_1, r_2, ..., r_p:n}(x_1, x_2, ..., x_p) = \mathrm{D} \sum_P \int \left(\prod_{w=1}^{p+1} \prod_{l=r_{w-1}+1}^{r_w-1} [v_{i_l}^{(w)} - v_{i_l}^{(w-1)}] \right) \prod_{w=1}^{p} dv_{i_{r_w}}^{(w)} , \tag{2.1}$$

where $x_1 < x_2 < ... < x_p$, $\sum_P$ denotes the sum over all $n!$ permutations $(i_1, i_2, ..., i_n)$ of (1,2,…,n), $\mathrm{D} = \prod_{w=1}^{p+1} [(r_w - r_{w-1} - 1)!]^{-1}$, $r_0 = 0$, $r_{p+1} = n+1$, $v_{i_l}^{(0)} = 0$, $v_{i_l}^{(p+1)} = 1$ and $v_{i_l}^{(w)} = [v_{i_{r_w}}^{(w)} - F_{i_{r_w}}(x_w -)] \dfrac{f_{i_l}(x_w)}{f_{i_{r_w}}(x_w)} + F_{i_l}(x_w -)$.

Proof. Consider the event

$$\{X_{r_1:n} = x_1, X_{r_2:n} = x_2, ..., X_{r_p:n} = x_p\}.$$

The above event can be realized mutually exclusive as follows: $r_1 - 1 - k_1$ observations are less than x_1, $k_w + 1 + m_w$ (w=1, 2,…, p) observations are equal to x_w, $r_\xi - 1 - k_\xi - m_{\xi-1} - r_{\xi-1}$ ($\xi = 2, 3, ..., p$) observations are in interval $(x_{\xi-1}, x_\xi)$ and $n - m_p - r_p$ observation exceed x_p. The probability function of the above event can be written as

$$f_{r_1, r_2, ..., r_p:n}(x_1, x_2, ..., x_p) = P\{X_{r_1:n} = x_1, X_{r_2:n} = x_2, ..., X_{r_p:n} = x_p\}. \tag{2.2}$$

(2.2) can be expressed as

$$f_{r_1, r_2, ..., r_p:n}(x_1, x_2, ..., x_p)$$

$$= \sum_{m_p, k_p ..., m_1, k_1} \mathrm{C} \sum_P \left(\prod_{w=1}^{p+1} \prod_{l=r_{w-1}+m_{w-1}+1}^{r_w - 1 - k_w} [F_{i_l}(x_w -) - F_{i_l}(x_{w-1})] \right) \prod_{w=1}^{p} \prod_{j=r_w - k_w}^{r_w + m_w} f_{i_j}(x_w), \tag{2.3}$$

where $\mathrm{C} = \left(\prod_{w=1}^{p+1} [(r_w - 1 - k_w - m_{w-1} - r_{w-1})!]^{-1} \right) \prod_{w=1}^{p} [(k_w + 1 + m_w)!]^{-1}$, $m_0 = 0$, $k_{p+1} = 0$, $F_{i_l}(x_0) = 0$, $F_{i_l}(x_{p+1} -) = 1$, $F_{i_l}(x_w -) = P(X_{i_l} < x_w)$ and $m_{w-1} + k_w \le r_w - r_{w-1} - 1$ (w=1, 2,…, p+1).

(2.3) can be written as

$$f_{r_1, r_2, ..., r_p:n}(x_1, x_2, ..., x_p) = \sum_{m_p, k_p ..., m_1, k_1} \mathrm{C} \sum_P \left(\prod_{w=1}^{p+1} \prod_{l=r_{w-1}+m_{w-1}+1}^{r_w - 1 - k_w} [F_{i_l}(x_w -) - F_{i_l}(x_{w-1})] \right)$$

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$$\cdot\prod_{w=1}^{p}\frac{(k_w+1+m_w)!}{k_w!\,m_w!}\left(\prod_{j=r_w-k_w}^{r_w-1}f_{i_j}(x_w)\right)f_{i_{r_w}}(x_w)\left(\prod_{j=r_w+1}^{r_w+m_w}f_{i_j}(x_w)\right)\int_0^1 y_w^{k_w}(1-y_w)^{m_w}\,dy_w\ . \qquad (2.4)$$

Also, (2.4) can be clearly written as

$$f_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p)=\sum_{m_p,k_p\ldots,m_1,k_1}\sum_{P}\int_0^1\int_0^1\ldots\int_0^1$$

$$\cdot\frac{1}{(r_1-1-k_1)!k_1!}F_{i_1}(x_1-)F_{i_2}(x_1-)\ldots F_{i_{r_1-1-k_1}}(x_1-)y_1f_{i_{r_1-k_1}}(x_1)y_1f_{i_{r_1-k_1+1}}(x_1)\ldots y_1f_{i_{r_1-1}}(x_1)dy_1f_{i_{r_1}}(x_1)$$

$$\cdot\frac{1}{(r_2-r_1-m_1-k_2-1)!m_1!k_2!}(1-y_1)f_{i_{r_1+1}}(x_1)(1-y_1)f_{i_{r_1+2}}(x_1)\ldots(1-y_1)f_{i_{r_1+m_1}}(x_1)[F_{i_{r_1+m_1+1}}(x_2-)-F_{i_{r_1+m_1+1}}(x_1)]\ldots$$

$$\cdot[F_{i_{r_2-k_2-1}}(x_2-)-F_{i_{r_2-k_2-1}}(x_1)]\,y_2f_{i_{r_2-k_2}}(x_2)\,y_2f_{i_{r_2-k_2+1}}(x_2)\ldots y_2f_{i_{r_2-1}}(x_2)\,dy_2f_{i_{r_2}}(x_2)\ldots$$

$$\cdot\frac{1}{(r_p-r_{p-1}-m_{p-1}-k_p-1)!m_{p-1}!k_p!}(1-y_{p-1})f_{i_{r_{p-1}+1}}(x_{p-1})(1-y_{p-1})f_{i_{r_{p-1}+2}}(x_{p-1})\ldots(1-y_{p-1})f_{i_{r_{p-1}+m_{p-1}}}(x_{p-1})$$

$$\cdot[F_{i_{r_{p-1}+m_{p-1}+1}}(x_{r_p}-)-F_{i_{r_{p-1}+m_{p-1}+1}}(x_{r_{p-1}})]\ldots[F_{i_{r_p-k_p-1}}(x_{r_p}-)-F_{i_{r_p-k_p-1}}(x_{r_{p-1}})]$$

$$\cdot y_pf_{i_{r_p-k_p}}(x_p)y_pf_{i_{r_p-k_p+1}}(x_p)\ldots y_pf_{i_{r_p-1}}(x_p)dy_pf_{i_{r_p}}(x_p)$$

$$\cdot\frac{1}{(n-r_p-m_p)!m_p!}(1-y_p)f_{i_{r_p+1}}(x_p)(1-y_p)f_{i_{r_p+2}}(x_p)\ldots(1-y_p)f_{i_{r_p+m_p}}(x_p)[1-F_{i_{r_p+m_p+1}}(x_p)]\ldots[1-F_{i_n}(x_p)].$$

The following expression can be written from the last identity.

$$f_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p)=\sum_{m_p,k_p\ldots,m_1,k_1}\sum_{P}\int_0^1\int_0^1\ldots\int_0^1\left\{\prod_{w=1}^{p+1}\frac{1}{(r_w-1-k_w-m_{w-1}-r_{w-1})!\;m_{w-1}!\,k_w!}\right.$$

$$\cdot\left(\prod_{\ell_1=r_{w-1}+1}^{r_{w-1}+m_{w-1}}(1-y_{w-1})f_{i_{\ell_1}}(x_{w-1})\right)\left(\prod_{\ell_2=r_{w-1}+m_{w-1}+1}^{r_w-1-k_w}[F_{i_{\ell_2}}(x_w-)-F_{i_{\ell_2}}(x_{w-1})]\right)$$

$$\left.\cdot\left(\prod_{\ell_3=r_w-k_w}^{r_w-1}y_wf_{i_{\ell_3}}(x_w)\right)\right\}\prod_{w=1}^{p}f_{i_{r_w}}(x_w)dy_w\ . \qquad (2.5)$$

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In (2.5), if $v_{i_j}^{(w)} = y_w f_{i_j}(x_w) + F_{i_j}(x_w-)$, the following identity is obtained.

$$f_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p) = \sum_{m_p,k_p\ldots,m_1,k_1} \sum_P \int_{F_{i_{r_1}}(x_1-)}^{F_{i_{r_1}}(x_1)} \int_{F_{i_{r_2}}(x_2-)}^{F_{i_{r_2}}(x_2)} \cdots \int_{F_{i_{r_p}}(x_p-)}^{F_{i_{r_p}}(x_p)} \left\{ \prod_{w=1}^{p+1} \frac{1}{(r_w-1-k_w-m_{w-1}-r_{w-1})!\; m_{w-1}!\, k_w!} \right.$$

$$\cdot \left(\prod_{\ell_1=r_{w-1}+1}^{r_{w-1}+m_{w-1}} [F_{i_{\ell_1}}(x_{w-1}) - v_{i_{\ell_1}}^{(w-1)}] \right) \left(\prod_{\ell_2=r_{w-1}+m_{w-1}+1}^{r_w-1-k_w} [F_{i_{\ell_2}}(x_w-) - F_{i_{\ell_2}}(x_{w-1})] \right)$$

$$\left. \cdot \left(\prod_{\ell_3=r_w-k_w}^{r_w-1} [v_{i_{\ell_3}}^{(w)} - F_{i_{\ell_3}}(x_w-)] \right) \right\} \prod_{w=1}^{p} dv_{i_{r_w}}^{(w)}. \tag{2.6}$$

By considering

$$\sum_{\tau=0}^{n} \sum_{\xi=0}^{n} \sum_P \frac{1}{\xi!(n-\tau-\xi)!\,\tau!} \left(\prod_{\ell_1=1}^{\xi} G_{i_{\ell_1}}^{(1)}(x) \right) \left(\prod_{\ell_2=\xi+1}^{n-\tau} G_{i_{\ell_2}}^{(2)}(x) \right) \prod_{\ell_3=n-\tau+1}^{n} G_{i_{\ell_3}}^{(3)}(x)$$

$$= \frac{1}{n!} \sum_P \prod_{l=1}^{n} [G_{i_l}^{(1)}(x) + G_{i_l}^{(2)}(x) + G_{i_l}^{(3)}(x)], \tag{2.7}$$

where $\tau + \xi \le n$ and using (2.7) for each m_{w-1} and k_w in (2.6), we get

$$f_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p)$$

$$= \left(\prod_{w=1}^{p+1} \frac{1}{(r_w - r_{w-1} - 1)!} \right) \sum_P \int \left(\prod_{w=1}^{p+1} \prod_{l=r_{w-1}+1}^{r_w-1} [F_{i_l}(x_w-) - F_{i_l}(x_{w-1}) + v_{i_l}^{(w)} - F_{i_l}(x_w-) + F_{i_l}(x_{w-1}) - v_{i_l}^{(w-1)}] \right) \prod_{w=1}^{p} dv_{i_{r_w}}^{(w)}.$$

Thus, the proof is completed.

Specially, in Theorem 2.1, by taking $p=2$, $n=3$, $r_1=1$, $r_2=2$, $v_{i_3}^{(2)} = [v_{i_2}^{(2)} - F_{i_2}(x_2-)] \dfrac{f_{i_3}(x_2)}{f_{i_2}(x_2)} + F_{i_3}(x_2-)$ and for $x_1 < x_2$,

$$f_{1,2:3}(x_1,x_2) = \sum_P \int_{F_{i_1}(x_1-)}^{F_{i_1}(x_1)} \int_{F_{i_2}(x_2-)}^{F_{i_2}(x_2)} [1 - v_{i_3}^{(2)}]\, dv_{i_2}^{(2)} dv_{i_1}^{(1)}$$

$$=\sum_P f_{i_1}(x_1)\left\{f_{i_2}(x_2)-[\frac{1}{2}f_{i_2}(x_2)\left[F_{i_2}(x_2)+F_{i_2}(x_2-)\right]-f_{i_2}(x_2)F_{i_2}(x_2-)]\frac{f_{i_3}(x_2)}{f_{i_2}(x_2)}-f_{i_2}(x_2)F_{i_3}(x_2-)\right\}$$

$$=f_1(x_1)\left\{f_2(x_2)+\frac{1}{2}f_3(x_2)F_2(x_2-)-\frac{1}{2}f_3(x_2)F_2(x_2)-f_2(x_2)F_3(x_2-)\right\}$$

$$+f_1(x_1)\left\{f_3(x_2)+\frac{1}{2}f_2(x_2)F_3(x_2-)-\frac{1}{2}f_2(x_2)F_3(x_2)-f_3(x_2)F_2(x_2-)\right\}$$

$$+f_2(x_1)\left\{f_3(x_2)+\frac{1}{2}f_1(x_2)F_3(x_2-)-\frac{1}{2}f_1(x_2)F_3(x_2)-f_3(x_2)F_1(x_2-)\right\}$$

$$+f_2(x_1)\left\{f_1(x_2)+\frac{1}{2}f_3(x_2)F_1(x_2-)-\frac{1}{2}f_3(x_2)F_1(x_2)-f_1(x_2)F_3(x_2-)\right\}$$

$$+f_3(x_1)\left\{f_1(x_2)+\frac{1}{2}f_2(x_2)F_1(x_2-)-\frac{1}{2}f_2(x_2)F_1(x_2)-f_1(x_2)F_2(x_2-)\right\}$$

$$+f_3(x_1)\left\{f_2(x_2)+\frac{1}{2}f_1(x_2)F_2(x_2-)-\frac{1}{2}f_1(x_2)F_2(x_2)-f_2(x_2)F_1(x_2-)\right\}.$$

Morever, the above identity in the *iid* case can be expressed as

$$f_{1,2:3}(x_1,x_2)=6f(x_1)f(x_2)-6f(x_1)f(x_2)F(x_2)+3f(x_1)f^2(x_2)\ .$$

This result is obtained, if $i=1$, $j=2$ and $n=3$ in equation (6) in [18].

In case $x_1<x_2<...<x_p$, $v_{i_{r_1}}^{(1)}\le v_{i_{r_2}}^{(2)}\le...\le v_{i_{r_p}}^{(p)}$ is automatically satisfied because of

$$F_{i_{r_1}}\left(x_1-\right)\le v_{i_{r_1}}^{(1)}\le F_{i_{r_1}}\left(x_1\right),\ F_{i_{r_2}}\left(x_2-\right)\le v_{i_{r_2}}^{(2)}\le F_{i_{r_2}}\left(x_2\right),\dots,F_{i_{r_p}}\left(x_p-\right)\le v_{i_{r_p}}^{(p)}\le F_{i_{r_p}}\left(x_p\right).$$

Also, in case $x_1=x_2=...=x_p=x$, the integration region is over

$$F_{i_{r_1}}\left(x-\right)\le v_{i_{r_1}}^{(1)}\le v_{i_{r_2}}^{(2)}\le...\le v_{i_{r_p}}^{(p)}\le F_{i_{r_p}}\left(x\right),\ F_{i_{r_1}}\left(x-\right)\le v_{i_{r_1}}^{(1)}\le F_{i_{r_1}}\left(x\right),$$

$$F_{i_{r_2}}\left(x-\right)\le v_{i_{r_2}}^{(2)}\le F_{i_{r_2}}\left(x\right),\dots,F_{i_{r_p}}\left(x-\right)\le v_{i_{r_p}}^{(p)}\le F_{i_{r_p}}\left(x\right)\ .$$

So, if $x_1 \le x_2 \le \ldots \le x_p$, it should be written $\int\int\ldots\int$ instead of $\int_{F_{i_{r_1}}(x_1-)}^{F_{i_{r_1}}(x_1)} \int_{F_{i_{r_2}}(x_2-)}^{F_{i_{r_2}}(x_2)} \ldots \int_{F_{i_{r_p}}(x_p-)}^{F_{i_{r_p}}(x_p)}$

in (2.1), where $\int\int\ldots\int$ is to be carried out over the region: $v_{i_{r_1}}^{(1)} \le v_{i_{r_2}}^{(2)} \le \ldots \le v_{i_{r_p}}^{(p)}$,

$F_{i_{r_1}}(x_1-) \le v_{i_{r_1}}^{(1)} \le F_{i_{r_1}}(x_1)$, $F_{i_{r_2}}(x_2-) \le v_{i_{r_2}}^{(2)} \le F_{i_{r_2}}(x_2), \ldots, F_{i_{r_p}}(x_p-) \le v_{i_{r_p}}^{(p)} \le F_{i_{r_p}}(x_p)$.

The proof was given only in case $x_1 < x_2 < \ldots < x_p$, the proof for case $x_1 \le x_2 \le \ldots \le x_p$ is omitted.

Specially, in Theorem 2.1, by taking $p=2$, $n=3$, $r_1=1$, $r_2=2$, $v_{i_3}^{(2)} = [v_{i_2}^{(2)} - F_{i_2}(x_2-)]\dfrac{f_{i_3}(x_2)}{f_{i_2}(x_2)} + F_{i_3}(x_2-)$ and for $x_1 = x_2 = x$,

$$f_{1,2:3}(x,x) = \sum_P \int_{F_{i_1}(x-)}^{F_{i_1}(x)} \int_{v_{i_1}^{(1)}}^{F_{i_2}(x)} [1 - v_{i_3}^{(2)}]\, dv_{i_2}^{(2)} dv_{i_1}^{(1)}$$

$$= \sum_P \Bigg\{ F_{i_2}(x) f_{i_1}(x) - \frac{1}{2}\left[F_{i_1}(x) + F_{i_1}(x-)\right] f_{i_1}(x) - \frac{1}{2} F_{i_2}^2(x) f_{i_1}(x) \frac{f_{i_3}(x)}{f_{i_2}(x)} + \frac{1}{6}\left[F_{i_1}^3(x) - F_{i_1}^3(x-)\right] \frac{f_{i_3}(x)}{f_{i_2}(x)}$$
$$+ F_{i_2}(x) F_{i_2}(x-) \frac{f_{i_1}(x) f_{i_3}(x)}{f_{i_2}(x)} - \frac{1}{2}\left[F_{i_1}(x) + F_{i_1}(x-)\right] f_{i_1}(x) F_{i_2}(x-) \frac{f_{i_3}(x)}{f_{i_2}(x)} - F_{i_2}(x) F_{i_3}(x-) f_{i_1}(x)$$
$$+ \frac{1}{2}\left[F_{i_1}(x) + F_{i_1}(x-)\right] f_{i_1}(x) F_{i_3}(x-) \Bigg\}$$

$$= \Bigg\{ F_2(x) f_1(x) - \frac{1}{2}\left[F_1(x) + F_1(x-)\right] f_1(x) - \frac{1}{2} F_2^2(x) f_1(x) \frac{f_3(x)}{f_2(x)} + \frac{1}{6}\left[F_1^3(x) - F_1^3(x-)\right] \frac{f_3(x)}{f_2(x)}$$
$$+ F_2(x) F_2(x-) \frac{f_1(x) f_3(x)}{f_2(x)} - \frac{1}{2}\left[F_1(x) + F_1(x-)\right] f_1(x) F_2(x-) \frac{f_3(x)}{f_2(x)} - F_2(x) F_3(x-) f_1(x)$$
$$+ \frac{1}{2}\left[F_1(x) + F_1(x-)\right] f_1(x) F_3(x-) \Bigg\}$$

$$+\left\{F_3(x)f_1(x)-\frac{1}{2}\left[F_1(x)+F_1(x-)\right]f_1(x)-\frac{1}{2}F_3^2(x)f_1(x)\frac{f_2(x)}{f_3(x)}+\frac{1}{6}\left[F_1^3(x)-F_1^3(x-)\right]\frac{f_2(x)}{f_3(x)}\right.$$
$$+F_3(x)F_3(x-)\frac{f_1(x)f_2(x)}{f_3(x)}-\frac{1}{2}\left[F_1(x)+F_1(x-)\right]f_1(x)F_3(x-)\frac{f_2(x)}{f_3(x)}-F_3(x)F_2(x-)f_1(x)$$
$$\left.+\frac{1}{2}\left[F_1(x)+F_1(x-)\right]f_1(x)F_2(x-)\right\}$$

$$+\left\{F_1(x)f_2(x)-\frac{1}{2}\left[F_2(x)+F_2(x-)\right]f_2(x)-\frac{1}{2}F_1^2(x)f_2(x)\frac{f_3(x)}{f_1(x)}+\frac{1}{6}\left[F_2^3(x)-F_2^3(x-)\right]\frac{f_3(x)}{f_1(x)}\right.$$
$$+F_1(x)F_1(x-)\frac{f_2(x)f_3(x)}{f_1(x)}-\frac{1}{2}\left[F_2(x)+F_2(x-)\right]f_2(x)F_1(x-)\frac{f_3(x)}{f_1(x)}-F_1(x)F_3(x-)f_2(x)$$
$$\left.+\frac{1}{2}\left[F_2(x)+F_2(x-)\right]f_2(x)F_3(x-)\right\}$$

$$+\left\{F_3(x)f_2(x)-\frac{1}{2}\left[F_2(x)+F_2(x-)\right]f_2(x)-\frac{1}{2}F_3^2(x)f_2(x)\frac{f_1(x)}{f_3(x)}+\frac{1}{6}\left[F_2^3(x)-F_2^3(x-)\right]\frac{f_1(x)}{f_3(x)}\right.$$
$$+F_3(x)F_3(x-)\frac{f_2(x)f_1(x)}{f_3(x)}-\frac{1}{2}\left[F_2(x)+F_2(x-)\right]f_2(x)F_3(x-)\frac{f_1(x)}{f_3(x)}-F_3(x)F_1(x-)f_2(x)$$
$$\left.+\frac{1}{2}\left[F_2(x)+F_2(x-)\right]f_2(x)F_1(x-)\right\}$$

$$+\left\{F_2(x)f_3(x)-\frac{1}{2}\left[F_3(x)+F_3(x-)\right]f_3(x)-\frac{1}{2}F_2^2(x)f_3(x)\frac{f_1(x)}{f_2(x)}+\frac{1}{6}\left[F_3^3(x)-F_3^3(x-)\right]\frac{f_1(x)}{f_2(x)}\right.$$
$$+F_2(x)F_2(x-)\frac{f_3(x)f_1(x)}{f_2(x)}-\frac{1}{2}\left[F_3(x)+F_3(x-)\right]f_3(x)F_2(x-)\frac{f_1(x)}{f_2(x)}-F_2(x)F_1(x-)f_3(x)$$
$$\left.+\frac{1}{2}\left[F_3(x)+F_3(x-)\right]f_3(x)F_1(x-)\right\}$$

$$+\left\{F_1(x)f_3(x)-\frac{1}{2}\left[F_3(x)+F_3(x-)\right]f_3(x)-\frac{1}{2}F_1^2(x)f_3(x)\frac{f_2(x)}{f_1(x)}+\frac{1}{6}\left[F_3^3(x)-F_3^3(x-)\right]\frac{f_2(x)}{f_1(x)}\right.$$
$$+F_1(x)F_1(x-)\frac{f_3(x)f_2(x)}{f_1(x)}-\frac{1}{2}\left[F_3(x)+F_3(x-)\right]f_3(x)F_1(x-)\frac{f_2(x)}{f_1(x)}-F_1(x)F_2(x-)f_3(x)$$
$$\left.+\frac{1}{2}\left[F_3(x)+F_3(x-)\right]f_3(x)F_2(x-)\right\}.$$

Morever, the above identity in the *iid* case can be expressed as

$$
\begin{aligned}
&= 6F(x)f(x) - 3[F(x)+F(x-)]f(x) - 3F^2(x)f(x) + [F^3(x)-F^3(x-)] + 6F(x)F(x-)f(x) \\
&-3[F(x)+F(x-)]F(x-)f(x) - 6F(x)F(x-)f(x) + 3[F(x)+F(x-)]f(x)F(x-) \\
&= 6F(x)f(x) - 3F(x)f(x) - 3F(x-)f(x) - 3F^2(x)f(x) + F^3(x) - F^3(x-) \\
&= 3f^2(x) - 3F^2(x)f(x) + f(x)[3F^2(x) - 3F(x)f(x) + f^2(x)] \\
&= f^3(x) + 3f^2(x)[1-F(x)].
\end{aligned}
$$

This result is obtained, if $r=1$, $s=2$ and $n=3$ in equation (2.4.3) in [14].

We will now express the following theorem to obtain the joint *df* of order statistics of *innid* discrete random variables.

Theorem 2.2.

$$F_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p) = \mathrm{D}\sum_P \int_V \left(\prod_{w=1}^{p+1} \prod_{l=r_{w-1}+1}^{r_w-1} [v_{i_l}^{(w)} - v_{i_l}^{(w-1)}] \right) \prod_{w=1}^{p} dv_{i_{r_w}}^{(w)} . \tag{2.8}$$

Proof. We have

$$F_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p) = \sum_{z_1,z_2,\ldots,z_p} f_{r_1,r_2,\ldots,r_p:n}(z_1,z_2,\ldots,z_p) . \tag{2.9}$$

Using (2.1) in (2.9), (2.8) is obtained.

3. Results for distribution and probability functions

In this section, the results related to *pf* and *df* of $X_{r_1:n}, X_{r_2:n}, \ldots, X_{r_p:n}$ will be given. We will express the following result for *pf* of the *r*th order statistic of *innid* discrete random variables.

Result 3.1.

$$f_{r_1:n}(x_1) = \frac{1}{(r_1-1)!(n-r_1)!} \sum_P \int_{F_{i_{r_1}}(x_1-)}^{F_{i_{r_1}}(x_1)} \left(\prod_{l=1}^{r_1-1} v_{i_l}^{(1)} \right) \left(\prod_{l=r_1+1}^{n} [1-v_{i_l}^{(1)}] \right) dv_{i_{r_1}}^{(1)} . \tag{3.1}$$

Proof. In (2.1), if $p=1$, (3.1) is obtained.

In Result 3.2 and Result 3.3, the *pf* 's of minimum and maximum order statistics of *innid* discrete random variables are given, respectively.

Result 3.2.

$$f_{1:n}(x_1) = \frac{1}{(n-1)!} \sum_P \int_{F_{i_1}(x_1-)}^{F_{i_1}(x_1)} \left(\prod_{l=2}^{n} [1 - v_{i_l}^{(1)}] \right) dv_{i_1}^{(1)}. \tag{3.2}$$

Proof. Putting $r_1 = 1$ in (3.1), one will get (3.2).

Result 3.3.

$$f_{n:n}(x_1) = \frac{1}{(n-1)!} \sum_P \int_{F_{i_n}(x_1-)}^{F_{i_n}(x_1)} \left(\prod_{l=1}^{n-1} v_{i_l}^{(1)} \right) dv_{i_n}^{(1)}. \tag{3.3}$$

Proof. On taking $r_1 = n$ in (3.1), one will get (3.3).

In the following result, we will give the joint *pf* of $X_{1:n}, X_{2:n}, ..., X_{p:n}$.

Result 3.4. If $x_1 \le x_2 \le ... \le x_p$,

$$f_{1,2,...,p:n}(x_1, x_2, ..., x_p) = \frac{1}{(n-p)!} \sum_P \int\int...\int \left(\prod_{l=p+1}^{n} [1 - v_{i_l}^{(p)}] \right) \prod_{w=1}^{p} dv_{i_w}^{(w)}, \tag{3.4}$$

where $\int\int...\int$ is to be carried out over the region: $v_{i_1}^{(1)} \le v_{i_2}^{(2)} \le ... \le v_{i_p}^{(p)}$,

$F_{i_1}(x_1-) \le v_{i_1}^{(1)} \le F_{i_1}(x_1)$, $F_{i_2}(x_2-) \le v_{i_2}^{(2)} \le F_{i_2}(x_2)$, $\ldots$, $F_{i_p}(x_p-) \le v_{i_p}^{(p)} \le F_{i_p}(x_p)$.

Proof. On taking $r_w = w$ for $w = 1, 2, ..., p$ and $\int\int...\int$ instead of $\int$ in (2.1), one will get (3.4).

We will now give three results for the *df* of single order statistic of *innid* discrete random variables.

Result 3.5.

$$F_{r_1:n}(x_1)=\frac{1}{(r_1-1)!(n-r_1)!}\sum_P \int_0^{F_{i_{r_1}}(x_1)}\left(\prod_{l=1}^{r_1-1} v_{i_l}^{(1)}\right)\left(\prod_{l=r_1+1}^{n}[1-v_{i_l}^{(1)}]\right)dv_{i_{r_1}}^{(1)}. \quad (3.5)$$

Proof. On taking $p=1$ in (2.8), one will get (3.5).

Result 3.6.

$$F_{1:n}(x_1)=\frac{1}{(n-1)!}\sum_P \int_0^{F_{i_1}(x_1)}\left(\prod_{l=2}^{n}[1-v_{i_l}^{(1)}]\right)dv_{i_1}^{(1)}. \quad (3.6)$$

Proof. Putting $r_1=1$ in (3.5), one will get (3.6).

Result 3.7.

$$F_{n:n}(x_1)=\frac{1}{(n-1)!}\sum_P \int_0^{F_{i_n}(x_1)}\left(\prod_{l=1}^{n-1} v_{i_l}^{(1)}\right)dv_{i_n}^{(1)}. \quad (3.7)$$

Proof. On taking $r_1=n$ in (3.5), one will get (3.7).

Specially, in (3.7), by taking n=2 and $v_{i_1}^{(1)}=[v_{i_2}^{(1)}-F_{i_2}(x_1-)]\dfrac{f_{i_1}(x_1)}{f_{i_2}(x_1)}+F_{i_1}(x_1-)$, the following identity is obtained.

$$\begin{aligned}
F_{2:2}(x_1) &= \sum_P \int_0^{F_{i_2}(x_1)} v_{i_1}^{(1)}dv_{i_2}^{(1)} \\
&=\sum_P\left[\left(\frac{\left(v_{i_2}^{(1)}\right)^2}{2}-v_{i_2}^{(1)}F_{i_2}(x_1-)\right)\frac{f_{i_1}(x_1)}{f_{i_2}(x_1)}+v_{i_2}^{(1)}F_{i_1}(x_1-)\right]_0^{F_{i_2}(x_1)} \\
&=\sum_P\left\{\left(\frac{F_{i_2}^2(x_1)}{2}-F_{i_2}(x_1)F_{i_2}(x_1-)\right)\frac{f_{i_1}(x_1)}{f_{i_2}(x_1)}+F_{i_2}(x_1)F_{i_1}(x_1-)\right\} \\
&=\left[\frac{F_2^2(x_1)}{2}-F_2(x_1)F_2(x_1-)\right]\frac{f_1(x_1)}{f_2(x_1)}+F_2(x_1)F_1(x_1-) \\
&\quad+\left[\frac{F_1^2(x_1)}{2}-F_1(x_1)F_1(x_1-)\right]\frac{f_2(x_1)}{f_1(x_1)}+F_1(x_1)F_2(x_1-).
\end{aligned}$$

Morever, the above identity for *iid* case can be expressed as

$$F_{2:2}(x_1) = F^2(x_1).$$

Also, the above identity for $x_1 = 1$ can be written as

$$F_{2:2}(1) = F^2(1)$$
$$= \left[f(0) + f(1)\right]^2.$$

In the following result, we will give the joint *df* of $X_{1:n}, X_{2:n}, ..., X_{p:n}$.

Result 3.8.

$$F_{1,2,...,p:n}(x_1, x_2, ..., x_p) = \frac{1}{(n-p)!} \sum_P \int_0^{F_{i_1}(x_1)} \int_{v_{i_1}^{(1)}}^{F_{i_2}(x_2)} \cdots \int_{v_{i_{p-1}}^{(p-1)}}^{F_{i_p}(x_p)} \left(\prod_{l=p+1}^{n} [1 - v_{i_l}^{(p)}] \right) \prod_{w=1}^{p} dv_{i_w}^{(w)}. \qquad (3.8)$$

Proof. On considering $r_w = w$ for $w = 1, 2, ..., p$ from (2.8), one will get (3.8).

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Journal Article

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Book

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

Contribution to a Book

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

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A Recurrent Neural Fuzzy Network

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Abstract

Besides the feedforward neural networks, there are the recurrent networks, where the impulses can be transmitted in both directions due to some reaction connections in these networks. Recurrent neural networks are linear or nonlinear dynamic systems. The dynamic behavior presented by the recurrent neural networks can be described both in continuous time, by differential equations and at discrete times by the recurrence relations (difference equations). The distinction between recurrent (or dynamic) neural networks and static neural networks is due to recurrent connections both between the layers of neurons of these networks and within the same layer, too. The aim of this paper is to describe a Recurrent Fuzzy Neural Network (RFNN) model, whose learning algorithm is based on the Improved Particle Swarm Optimization (IPSO) method. Each particle (candidate solution), which is moving permanently includes the parameters of the membership function and the weights of the recurrent neural-fuzzy network; initially, their values are randomly generated. The RFNN presented in this paper is unlike the others variants of RFNN models, by the number of the evolution directions that they use: in this paper, we update the velocity and the position of all particles along three dimensions, while in [8] are used two dimensions.

Keywords: recurrent networks; Improved Particle Swarm Optimization method; fuzzy rules; Wavelet Neural Network; feedback weight; delayed operator.

1 Introduction

Neural network (NN) is one of the important components in Artificial Intelligence (AI). NN architectures used in modelling of the nervous systems can be classified into three categories, each with a different philosophy: feedforward, recurrent (feedback), self-organizing map. Neural networks (NNs) are used in many different application domains in order to solve various information processing problems. For several years now, neural network models have enjoyed wide popularity [4], being applied to problems of regression, classification, computational science, computer vision, data processing and time series analysis.

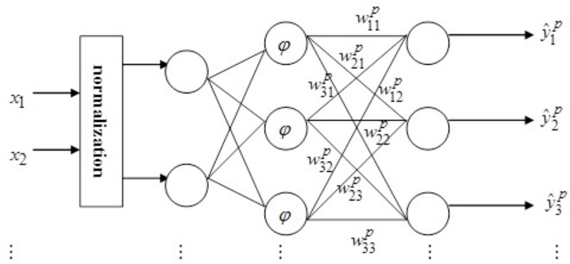

Figure 1: Schematic diagram of the WNN.

The main drawback of the feedforward neural networks is that the updating of the weights can fall [17] in a local minimum. An other major drawback of the feedforward neural networks consists in the fact that their application domain is limited to static problems due their inherent feedforward structure.

Since recurrent networks incorporate feedback, they have powerful representation capability and can [17] successfully overcome disadvantages of feedforward networks. This feedback implies that the network has [12] local memory characteristics that is able to store activity patterns and present those patterns to the network more than once, allowing the layer with feedback connections to use its own past activation in its preceding behavior.

The Recurrent Neural Network (RNN) has the feedforward and feedback connections contrasted which provides it with nonlinear mapping capacity and dynamical characteristics, so it can be used [22] to simulate dynamical system and solve dynamic problems. Different architectures can be created [12] by adding recurrent connections at different points in the basic feedforward architecture.

Recently some researchers have proposed several recurrent neuro- fuzzy networks. Kumar et al. 2004 compares the traditional feedforward approach of RNNs to forecast monthly river flows. Lin & Hsu, 2007 has proposed [10] a recurrent wavelet-based neuro- fuzzy system with the reinforcement hybrid evolutionary learning algorithm for solving various control problems. Carcano et al., 2008 has simulated [3] daily river flows for water resource purposes using the Jordan Recurrent Neural Network. Maraqua et al., 2012 has proposed [12] the use of a recurrent network architecture as a classification engine for automatic Arabic Sign Language recognition system. Šter, 2013 has introduced [18] an extended architecture of recurrent neural networks (called *Selective Recurrent Neural Network*) for dealing with long term dependencies.

1.1 Wavelet Neural Networks

Neural networks employing wavelet neurons are referred to as Wavelet Neural Networks(WNNs) [10]; they are characterized by weights and wavelet bases.

Lin & Chin, 2004 was proposed a Recurrent Neural Fuzzy Network (RNFN) where each fuzzy rule corresponding to a WNN (see Figure 1) consists (see [11], [8]) of single-scaling wavelets. The shape and position of the wavelet bases are shown [11] in Figure 2.

An ordinary wavelet neural network model is often used to normalize input vectors in the interval [0, 1]. The functions $\phi_{a.b}(x_i)$ are used to input vectors

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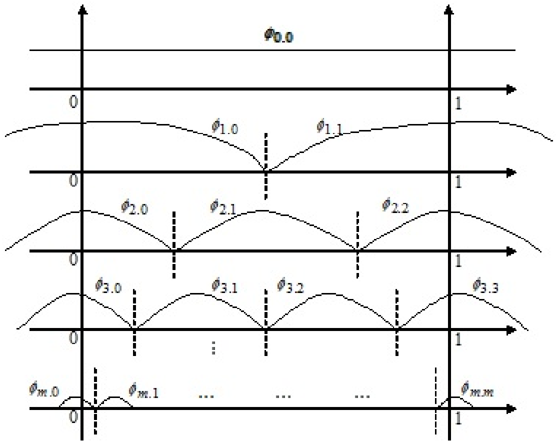

Figure 2: Wavelet bases are over-complete and compactly supported.

to fire up the wavelet interval; a such value is given in the following equation, which gives the shape of the M wavelet bases $\phi_{0.0},\ \phi_{1.0},\dots,\phi_{m.m}$:

$$\begin{cases} \phi(x_i)=\cos(x_i),\ -0.5\leq x_i\leq 0.5 \\ 0 \text{ otherwise}, \qquad \phi_{a.b}(x_i)=\cos(ax_i-b), \end{cases} \tag{1}$$

$b=\overline{1,a},\ a=\overline{1,m}$, b being a shifting parameter and a meaning a scaling parameter corresponding to the maximum value of b.

A crisp value $\varphi_{a.b}$ can be obtained as follows:

$$\varphi_{a.b}=\frac{\sum_{j=1}^{n}\phi_{a.b}(x_i)}{|X|}, \tag{2}$$

where $|X|$ represents the number of input dimensions and n is the dimension of the input vector to the model.

1.2 Z- transform

The Z- transform is [20] the discrete- time counterpart of the Laplace transform. The Z- transform can be considered to be an extension of the discrete- time Fourier transform as the Laplace transform can be considered an extension of the Fourier transform.

The *bilateral* Z- transform of a discrete- time sequence $x(n)$ is:

$$Z\{x(n)\}=X(z)=\sum_{n=-\infty}^{\infty}x(n)z^{-n}. \tag{3}$$

For causal sequences ($n\geq 0$) the Z- transform becomes:

$$Z\{x(n)\}=X(z)=\sum_{n=0}^{\infty}x(n)z^{-n}. \tag{4}$$

The equation (4) is called the *unilateral* Z- transform; it exists only if the power series from its expression converges.

3

There are several methods for computing the inverse Z- transform, namely the sequence $x(n)$, given $X(z)$:

1. using the *inversion integral*:

$$x(n) = \frac{1}{2\pi j} \oint_{\Gamma} X(z) z^{n-1} dz, \tag{5}$$

   where $\oint_{\Gamma}$ means the integration along the closed contour Γ in the counterclockwise closed contour in the region of convergence of $X(z)$;

2. by a power series expansion: expressing $X(z)$ in a power series in z^{-1}, $x(n)$ can be achieved by identifying it with the coefficient of z^{-n} in the power series expansion;

3. by partial fraction expansion: for a rational functions, can be obtained a partial fraction expansion of $X(z)$ over its poles and the table of Z-transform helps to identify the sequences corresponding to the terms in that partial fraction expansion.

1.3 Application of Genetic Algorithms

The specialists think that the Genetic Algorithms are a computational intelligence application as well as the expert systems, fuzzy systems, neural networks, the intelligent agents, hybrid intelligent systems, electronic voice.

The genetic algorithms are some adaptive techniques of heuristic search, based on the genetic and selection natural principles, enunciated by Darwin (the best adapted will survive). The mechanism is similar to the evolutionary biological process. This process has a feature through that only the species which one adapt better to the environment are capable to survive and to develop into generations, while that those less adapted fail to survive and they disappear in time, as a result of the natural selection. The main notions that allow the analogy between the solution of the search problems and the natural evolution are:

1. *Population.* A population consists in some individuals (*chromosomes*) that have to live in an environment to which they must adapt.

2. *Fitness.* Each of the population individuals is adapted more or less to the environment. The fitness is a measure of the degree of adaptation to the environment.

3. *Chromosome.* It is a ordered set of elements, named *genes*, whose values establish the individual features.

4. *Generation.* A stage in a population evolution. If we see evolution as an iterative process in which a population turns to another population, then the generation is an iteration in this process.

5. *Selection.* The process of natural selection has the survival of individuals with a high environmental fitness (high fitness) as effect.

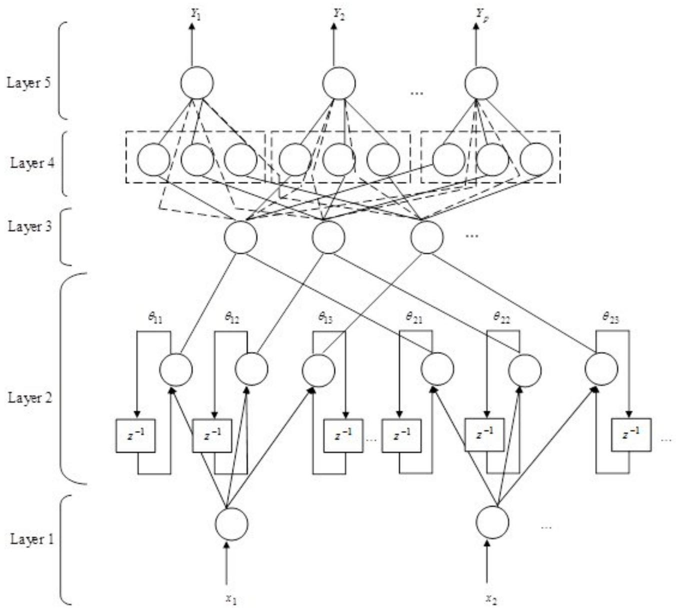

Figure 3: The RNFN architecture.

6. *Reproduction.* It is the process through which one passes from one generation to another. The individuals of the new generation inherit some features from their precursors (parents) but they can also get some new features as a result of some processes of mutation that have a random character. In the case when in the reproduction process at least two parents occur, the inherited features of the survivor (son) are obtained by combining (crossover) of the parent features.

The remainder of the paper is organized as follows. In Section 2 is discussed and analyzed the RNFN. We follow with the learning algorithm of the recurrent model in Section 3. We conclude in Section 4.

2 RNFN Architecture

The network construction is based on fuzzy rules, each corresponding to a Wavelet Neural Network (WNN).

The figure Figure 3 illustrates the RNFN model, whose training algorithm is based on Improved Particle Swarm Optimization (IPSO) method.

The nodes from the first layer constitute some input nodes; hence they only pass the input signal to the next layer, namely:

$$O_i^{(1)} = x_i^{(1)}. \tag{6}$$

The neurons in the second layer act as a membership function, meaning that they determine how an input value belongs to a fuzzy set. The following Gaussian function is chosen as the membership function:

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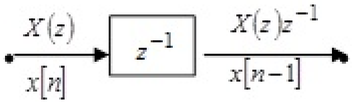

Figure 4: Delayed cell.

$$O_{ij}^{(2)} = \mathrm{e}^{-\frac{\left(I_{ij}^{(2)}-m_{ij}\right)^2}{\sigma_{ij}^2}}, \tag{7}$$

where:

- m_{ij} and σ_{ij} are the mean and standard deviation, respectively;
- $I_{ij}^{(2)}$ denotes the input of this layer for the discrete time scan:

$$I_{ij}^{(2)} = O_i^{(2)} + O_{ij}^{(f)}, \tag{8}$$

where

$$O_{ij}^{(f)} = O_{ij}^{(2)}(t-1)\theta_{ij}. \tag{9}$$

The inputs of this layer contain the terms of memory $O_{ij}^{(2)}(t-1)$, that store network information at a previous time; this information, which is an additional input of the network will be reintroduced at the entrance of the second layer.

The weight θ_{ij} constitutes the feedback weight of the network and z^{-1} signifies the delayed operator.

Figure 4 represents [14] a delayed cell, $X(z)$ being the Z- transform of the signal $x[n]$.

The neurons of the third layer achieve the product operation of their input signals:

$$O_j^3 = \prod_{i=1}^{n} O_{ij}^{(2)} = \prod_{i=1}^{n} \mathrm{e}^{-\frac{\left(I_{ij}^{(2)}-m_{ij}\right)^2}{\sigma_{ij}^2}}, \tag{10}$$

where n is the number of external dimensions.

The neurons of the fourth layer receive both the output of a WNN, denoted $\hat{y}_j$ and of a neuron from the third layer, namely O_j^3. The mathematical function of each node j is:

$$O_j^4 = \hat{y}_j^p \cdot O_j^3, \tag{11}$$

$\hat{y}_j^p$ being the local output of the WNN for the output y_p and the j-th rule:

$$\hat{y}_j^p = \sum_{k=1}^{M} w_{jk}^p \varphi_{a.b}, \tag{12}$$

with $\varphi_{a.b}$ from (2), where:

- $M = m+1$ denotes the number of wavelet bases, which equals the number of existing fuzzy rules in the considered model,

6

- the link w^p_{jk} is the output action strength associated with in the p output, j-th rule and k-th $\varphi_{a.b}$.

The fifth layer acts as a defuzzifier namely it provides the nonfuzzy outputs y_p of the fuzzy recurrent neural network:

$$y_p = \frac{1}{1 + \mathrm{e}^{-\lambda\cdot\frac{\sum_{j=1}^{M} O_j^4}{\sum_{j=1}^{M} O_j^3}}} = \frac{1}{1 + \mathrm{e}^{-\lambda\cdot\frac{\sum_{j=1}^{M} \hat{y}_j^p \cdot O_j^3}{\sum_{j=1}^{M} O_j^3}}}, \tag{13}$$

namely:

$$y_p = \frac{1}{1 + \mathrm{e}^{-\lambda\cdot\frac{\sum_{j=1}^{M}(w^p_{j1}\varphi_{1.1}+w^p_{j2}\varphi_{2.1}+\ldots+w^p_{jM}\varphi_{m.m})\cdot O_j^3}{\sum_{j=1}^{M} O_j^3}}},\ \lambda \in \Re. \tag{14}$$

3 Learning Algorithm of RNFN

The training algorithm of the network is based on the Improved optimization method Particle Swarm Optimization (IPSO). The new optimization algorithm called the IPSO enhances the traditional PSO (Particle Swarm Optimization) to enable it to obtain optimal solution capability.

We assume that each particle includes the mean, deviation and weight variables of the RNFN, being d- dimensional.

The following parameters will be determined by the learning procedure:

- the position vector $X_i = (x_{i1}, x_{i2}, \ldots, x_{id})$,
  and respectively
- the velocity vector $V_i = (v_{i1}, v_{i2}, \ldots, v_{id})$

of the i- th particle in the N-dimensional search space.

We denote by:

- $P_i = (P_{i1}, P_{i2}, \ldots, P_{id})$ the best position of each particle,
- $P_g = (P_{g1}, P_{g2}, \ldots, P_{gd})$ the fittest particle found so far,

according to an user-defined fitness function.

The steps of the learning procedure are:

Step 1 (*Individual initialization*). Set the initial values for every particle like being random values.

Step 2 (*Evaluate fitness*). Evaluate each particle in a swarm, by defining the fitness function:

$$f_i = \frac{1}{Y}, \tag{15}$$

where

$$Y = \sqrt{\frac{1}{N}\sum_{p=1}^{N}(y_p - \overline{y}_p)^2}, \tag{16}$$

- N represents the number of input data,
- $y_p,\ p = \overline{1, N}$ are the model outputs,
- $\overline{y}_p,\ p = \overline{1, N}$ constitute the desired outputs.

After a generation of learning, we achieve the following fifth best particles, ordered according to their fitness: *unimportant*, *rather unimportant*, *moderately important*, *rather important*, *very important* particles.

The input (preferred) particles are:

1. *unimportant* particle

$$C_u = (C_{u1}, C_{u2}, \ldots, C_{ud}),$$

   with the fitness F_u;

2. *rather unimportant* particle

$$C_r = (C_{r1}, C_{r2}, \ldots, C_{rd}),$$

   with the fitness F_r;

3. *moderately important* particle

$$C_m = (C_{m1}, C_{m2}, \ldots, C_{md}),$$

   with the fitness F_m;

4. *rather important* particle

$$C_R = (C_{R1}, C_{R2}, \ldots, C_{Rd}),$$

   with the fitness F_R;

5. *very important* particle

$$C_v = (C_{v1}, C_{v2}, \ldots, C_{vd}),$$

   with the fitness F_v.

The membership functions of the fuzzy terms *unimportant*, *rather unimportant*, *moderately important*, *rather important*, and respectively *very important* can be represented as fuzzy numbers in Figure 5,

being defined in the following relations:

$$\mu_{unimportant} = \begin{cases} 1 - \frac{x}{0.25},\ \text{if } 0 \leq x \leq 0.25, \\ 0,\ \text{otherwise.} \end{cases} \tag{17}$$

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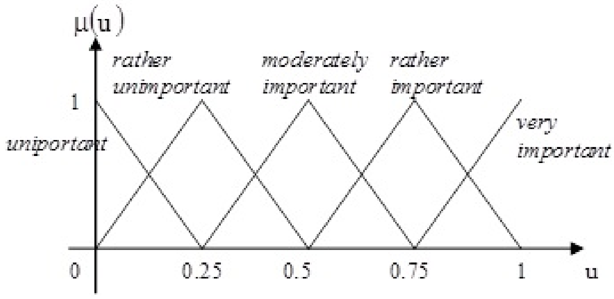

Figure 5: The membership functions of *importance*.

$$\mu_{rather\ unimportant} = \begin{cases} \frac{x}{0.25} \text{ if } 0 \leq x \leq 0.25, \\ 1 - \frac{x-0.25}{0.25} \text{ if } 0.25 < x \leq 0.5, \end{cases} \tag{18}$$

$$\mu_{moderately\ important} = \begin{cases} \frac{x-0.25}{0.25}, \text{ if } 0.25 \leq x \leq 0.5, \\ 1 - \frac{x-0.5}{0.25}, \text{ if } 0.5 < x \leq 0.75, \end{cases} \tag{19}$$

$$\mu_{rather\ important} = \begin{cases} \frac{x-0.5}{0.25}, \text{ if } 0.5 \leq x \leq 0.75, \\ 1 - \frac{x-0.75}{0.25}, \text{ if } 0.75 < x \leq 1, \end{cases} \tag{20}$$

$$\mu_{very\ important} = \begin{cases} \frac{x-0.75}{0.25}, \text{ if } 0.75 \leq x \leq 1, \\ 0, \text{ otherwise.} \end{cases} \tag{21}$$

The output (created) particle is *output particle*:

$$C_o = (C_{o1}, C_{o2}, \ldots, C_{od}),$$

with the fitness F_o.

Step 3 (*Improve the capability of finding the global solution* (ICFGS)). Set: $D_1 = D_2 = D_3 = 1$ the magnitudes of the three evolution directions, $T_s = 1$ the initial index of the ICFGS, the number N_L of the ICFGS loop, the fifth particles with the best fitness values from the local best swarm to C_u, C_r, C_m, C_R, C_v.

Use a special equation to update the: *unimportant* particle, *rather unimportant* particle, *moderately important* particle and *rather important* particle to generate the migrant individuals, based on the best individual, $X_i = (x_{i1}, \ldots, x_{id})$ in the aim of improving the fitness value [8]:

$$x_{id} = \begin{cases} x_{id} + \rho(x_{id}^L - x_{id}), \text{ if } r_1 < \frac{x_{id} - x_{id}^L}{x_{id}^L - x_{id}^U} \\ x_{id} + \rho(x_{id}^U - x_{id}) \text{ otherwise,} \end{cases} \tag{22}$$

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where ρ and r_1 are random numbers in the range of $[0, 1]$ and L, U meaning "lower" and "upper".

Compute C_o:

$$C_{oj} = C_{uj} + D_1(C_{uj} - C_{rj}) + \\ + D_2(C_{uj} - C_{mj}) + D_3(C_{uj} - C_{Rj}) \tag{23}$$

Evaluate the new fitness F_o corresponding to the newly created output particle C_o.

Update the *unimportant* particle C_u, the *rather unimportant* particle C_r, *moderately important* particle C_m, *rather important* particle C_R and the *very important* particle C_v as follows:

(1) If $F_o > F_v$ then

$$\begin{cases} C_v = C_o \\ C_R = C_v \\ C_m = C_R \\ C_r = C_m \\ C_u = C_r. \end{cases}$$

(2) Else if $F_o > F_R$ and $F_o < F_v$ then

$$\begin{cases} C_R = C_o \\ C_m = C_R \\ C_r = C_m \\ C_u = C_r. \end{cases}$$

(3) Else if $F_o > F_m$ and $F_o < F_R$ then

$$\begin{cases} C_m = C_o \\ C_r = C_m \\ C_u = C_r. \end{cases}$$

(4) Else if $F_o > F_r$ and $F_o < F_m$ then

$$\begin{cases} C_r = C_o \\ C_u = C_r. \end{cases}$$

(5) Else if $F_o > F_u$ and $F_o < F_r$ then

$$C_u = C_o.$$

(6) Else if $F_o = F_u = F_r = F_m = F_R = F_v$
then

$$C_o = C_o + N_r \ (N_r \in [0, 1]).$$

(7) Else if $F_o <= F_u$ then it will decrease the moving velocity:

$$\begin{cases} D_1 = -0.5D_1 \\ D_2 = -0.5D_2 \\ D_3 = -0.5D_3 \end{cases}$$

to obtain a good fitness.

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The random number N_r is added at the statement (23) to prevent the learning algorithm from falling into a local optimum.

Test If *Step 3* isn't finished then $T_s = T_s + 1$; else update the global best: if the fitness value of the new particle is higher than that of the global best, then the global best will also be replaced with the particle.

Step 4 (*Update the velocity and the position*). Update the velocity and the position of all particles along each dimension using the equations:

$$v_{id}^{k+1} = \omega \cdot v_{id}^{k} + c_1 \cdot \mathrm{rand}(\cdot)(P_{id} - x_{id}^{k}) + c_2 \cdot \mathrm{rand}(\cdot)(P_{gd} - x_{id}^{k}) \tag{24}$$

$$x_{id}^{k+1} = x_{id}^{k} + v_{id}^{k+1}, \tag{25}$$

where: w is the coefficient of the inertia term; c_1 and c_2 are called the cognitive term and the society term, respectively; the function $\mathrm{rand}(\cdot)$ yields uniformly distributed random numbers in $[0, 1]$.

The second term from (24) known as the cognitive component, represents the personal thinking of each particle, which encourages the particles to move toward their own best positions. The third term from (24) called the social component represents the collaborative effect of the particles, in finding the global optimal solution.

4 Conclusion

Dynamic or Recurrent Neural Networks (RNNs) are unlike from static neural networks since they include feedback or recurrent connections between the network layers and within the layer itself.

The learning algorithm of the Recurrent Neural Fuzzy Network (RNFN) model presented in this paper is based on the Improved Particle Swarm Optimization (IPSO) method, which is similar to evolutionary algorithms, but requires less computational bookkeeping and generally fewer lines of code. The new optimization algorithm called the IPSO enhances the traditional PSO (Particle Swarm Optimization) to enable it to obtain optimal solution capability.

The RFNN presented in this paper is unlike the others variants of RFNN models, by the number of the evolution directions that they use: in this paper, we update the velocity and the position of all particles along three dimensions.

The network construction is based on fuzzy rules, each corresponding to a WNN (Wavelet Neural Network).

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Qualitative Behavior of some Rational Difference Equations

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Abstract

We obtain in this paper the analytical forms of the solutions for the following difference equations

$$x_{n+1} = \frac{x_{n-1}x_{n-4}}{x_{n-2}(\pm 1 \pm x_{n-1}x_{n-4})}, \quad n = 0, 1, ...,$$

where the initial conditions are arbitrary real numbers. Also, we study the dynamics behavior of the solutions of the considered equations.

Keywords: difference equations, recursive sequences, stability, periodic solution.
Mathematics Subject Classification: 39A10

1 Introduction

The behavior of the solutions of the difference equations has been investigated by many authors, see for examples: Agarwal et al. [1] investigated the global stability, periodicity character and gave the solution of some special cases of the difference equations

$$x_{n+1} = ax_n + \frac{bx_n x_{n-3}}{cx_{n-2} + dx_{n-3}}.$$

1

Cinar [2] investigated the solutions of the following difference equation

$$x_{n+1} = \frac{ax_{n-1}}{1+bx_n x_{n-1}}.$$

Elabbasy et al. [3] investigated the global attractivity of the equilibrium point and the asymptotic behavior of the solutions of the following difference equation and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{ax_{n-l}x_{n-k}}{bx_{n-p}-cx_{n-q}}.$$

Elsayed [8] deal with some properties of the solutions of the difference equation

$$x_{n+1} = ax_n + \frac{bx_n}{cx_n-dx_{n-1}},$$

and obtained the form of the solution of special case of this difference equation

Karatas et al. [11] get the form of the solution of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1+x_{n-2}x_{n-5}}.$$

In [14] Wang et al. investigated the global attractivity of the equilibrium point, and the asymptotic behavior of the solutions of the following difference equation

$$x_{n+1} = \frac{\sum_{i=1}^{s} A_{k_i}x_{n-k_i}}{B_0+\sum_{j=1}^{t} B_{l_j}x_{n-l_j}}.$$

In [15] Yalçınkaya studied the behavior of the following difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}.$$

Zayed et al. [17] studied a qualitative behavior of the rational recursive sequence

$$x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+Bx_n+Cx_{n-1}}.$$

Other related results on rational difference equations can be found in refs. [4-16].

In this paper, we study the existence of the analytical solutions for the following difference equations

$$x_{n+1} = \frac{x_{n-1}x_{n-4}}{x_{n-2}(\pm 1 \pm x_{n-1}x_{n-4})}, \quad n = 0, 1, ..., \tag{1}$$

where the initial conditions are arbitrary real numbers. Also, we study the global behavior of the solutions.

The following theorem will be useful in our current study.

Theorem A [13]: Assume that $p_i \in R,\ i = 1, 2, ..., k$ and $k \in \{0, 1, 2, ...\}$. Then

$$\sum_{i=1}^{k} |p_i| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \quad n = 0, 1, ...$$

In the following we investigate the behavior of the solutions for some different cases of Eq.(1).

2 On the Equation $x_{n+1} = \frac{x_{n-1}x_{n-4}}{x_{n-2}(1+x_{n-1}x_{n-4})}$

In this section we give a specific form of the solution of the equation

$$x_{n+1} = \frac{x_{n-1}x_{n-4}}{x_{n-2}(1+x_{n-1}x_{n-4})}, \quad n = 0, 1, ..., \tag{2}$$

where the initial values are arbitrary positive real numbers.

Theorem 1 *Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of Eq.(2). Then for $n = 0, 1, ...$*

$$\begin{aligned}
x_{6n-2} &= \tfrac{x_{-2}x_0^n x_{-3}^n}{x_{-1}^n x_{-4}^n} \prod_{i=0}^{n-1} \left(\tfrac{(1+(3i+1)x_{-1}x_{-4})}{(1+(3i+2)x_0x_{-3})} \right), \ x_{6n-1} = \tfrac{x_{-1}^{n+1}x_{-4}^n}{x_0^n x_{-3}^n} \prod_{i=0}^{n-1} \left(\tfrac{(1+(3i+1)x_0x_{-3})}{(1+(3i+3)x_{-1}x_{-4})} \right), \\
x_{6n} &= \tfrac{x_0^{n+1}x_{-3}^n}{x_{-1}^n x_{-4}^n} \prod_{i=0}^{n-1} \left(\tfrac{(1+(3i+2)x_{-1}x_{-4})}{(1+(3i+3)x_0x_{-3})} \right), \\
x_{6n+1} &= \tfrac{x_{-1}^{n+1}x_{-4}^{n+1}}{x_{-2}x_0^n x_{-3}^n(1+x_{-1}x_{-4})} \prod_{i=0}^{n-1} \left(\tfrac{(1+(3i+2)x_0x_{-3})}{(1+(3i+4)x_{-1}x_{-4})} \right), \\
x_{6n+2} &= \tfrac{x_0^{n+1}x_{-3}^{n+1}}{x_{-1}^{n+1}x_{-4}^n(1+x_0x_{-3})} \prod_{i=0}^{n-1} \left(\tfrac{(1+(3i+3)x_{-1}x_{-4})}{(1+(3i+4)x_0x_{-3})} \right), \\
x_{6n+3} &= \tfrac{x_{-1}^{n+1}x_{-4}^{n+1}}{x_0^{n+1}x_{-3}^n(1+2x_{-1}x_{-4})} \prod_{i=0}^{n-1} \left(\tfrac{(1+(3i+3)x_0x_{-3})}{(1+(3i+5)x_{-1}x_{-4})} \right).
\end{aligned}$$

Proof: For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is;

$$\begin{aligned}
x_{6n-8} &= \tfrac{x_{-2}x_0^{n-1}x_{-3}^{n-1}}{x_{-1}^{n-1}x_{-4}^{n-1}} \prod_{i=0}^{n-2} \tfrac{(1+(3i+1)x_{-1}x_{-4})}{(1+(3i+2)x_0x_{-3})}, \ x_{6n-7} = \tfrac{x_{-1}^n x_{-4}^{n-1}}{x_0^{n-1}x_{-3}^{n-1}} \prod_{i=0}^{n-2} \tfrac{(1+(3i+1)x_0x_{-3})}{(1+(3i+3)x_{-1}x_{-4})}, \\
x_{6n-6} &= \tfrac{x_0^n x_{-3}^{n-1}}{x_{-1}^{n-1}x_{-4}^{n-1}} \prod_{i=0}^{n-2} \tfrac{1+(3i+2)x_{-1}x_{-4}}{1+(3i+3)x_0x_{-3}}, \ x_{6n-5} = \tfrac{x_{-1}^n x_{-4}^n}{x_{-2}x_0^{n-1}x_{-3}^{n-1}(1+x_{-1}x_{-4})} \prod_{i=0}^{n-2} \tfrac{1+(3i+2)x_0x_{-3}}{1+(3i+4)x_{-1}x_{-4}}, \\
x_{6n-4} &= \tfrac{x_0^n x_{-3}^n}{x_{-1}^n x_{-4}^{n-1}(1+x_0x_{-3})} \prod_{i=0}^{n-2} \left(\tfrac{(1+(3i+3)x_{-1}x_{-4})}{(1+(3i+4)x_0x_{-3})} \right), \\
x_{6n-3} &= \tfrac{x_{-1}^n x_{-4}^n}{x_0^n x_{-3}^{n-1}(1+2x_{-1}x_{-4})} \prod_{i=0}^{n-2} \left(\tfrac{(1+(3i+3)x_0x_{-3})}{(1+(3i+5)x_{-1}x_{-4})} \right).
\end{aligned}$$

Now, it follows from Eq.(2) that

$$x_{6n-2} = \tfrac{x_{6n-4}x_{6n-7}}{x_{6n-5}(1+x_{6n-4}x_{6n-7})}$$

$$= \frac{\frac{x_0^n x_{-3}^n}{x_{-1}^n x_{-4}^{n-1}(1+x_0x_{-3})} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+3)x_{-1}x_{-4})}{(1+(3i+4)x_0x_{-3})}\right) \frac{x_{-1}^n x_{-4}^{n-1}}{x_0^{n-1}x_{-3}^{n-1}} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+1)x_0x_{-3})}{(1+(3i+3)x_{-1}x_{-4})}\right)}{\left(\frac{x_{-1}^n x_{-4}^n}{x_{-2}x_0^{n-1}x_{-3}^{n-1}(1+x_{-1}x_{-4})} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+2)x_0x_{-3})}{(1+(3i+4)x_{-1}x_{-4})}\right)\right) \left(1+\frac{x_0^n x_{-3}^n}{x_{-1}^n x_{-4}^{n-1}(1+x_0x_{-3})} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+3)x_{-1}x_{-4})}{(1+(3i+4)x_0x_{-3})}\right) \frac{x_{-1}^n x_{-4}^{n-1}}{x_0^{n-1}x_{-3}^{n-1}} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+1)x_0x_{-3})}{(1+(3i+3)x_{-1}x_{-4})}\right)\right)}$$

$$= \frac{\frac{x_0x_{-3}}{(1+x_0x_{-3})} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+1)x_0x_{-3})}{(1+(3i+4)x_0x_{-3})}\right)}{\left(\frac{x_{-1}^n x_{-4}^n}{x_{-2}x_0^{n-1}x_{-3}^{n-1}(1+x_{-1}x_{-4})} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+2)x_0x_{-3})}{(1+(3i+4)x_{-1}x_{-4})}\right)\right)\left(1+\frac{x_0x_{-3}}{(1+x_0x_{-3})} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+1)x_0x_{-3})}{(1+(3i+4)x_0x_{-3})}\right)\right)}$$

$$= \left(\frac{x_{-2}x_0^{n-1}x_{-3}^{n-1}(1+x_{-1}x_{-4})}{x_{-1}^n x_{-4}^n}\right) \prod_{i=0}^{n-2}\left(\frac{(1+(3i+4)x_{-1}x_{-4})}{(1+(3i+2)x_0x_{-3})}\right) \frac{\frac{x_0x_{-3}}{(1+(3n-2)x_0x_{-3})}}{\left(1+\frac{x_0x_{-3}}{(1+(3n-2)x_0x_{-3})}\right)}$$

$$= \left(\frac{x_{-2}x_0^n x_{-3}^n(1+x_{-1}x_{-4})}{x_{-1}^n x_{-4}^n}\right) \prod_{i=0}^{n-2}\left(\frac{(1+(3i+4)x_{-1}x_{-4})}{(1+(3i+2)x_0x_{-3})}\right) \frac{1}{(1+(3n-1)x_0x_{-3})}.$$

Hence, we have

$$x_{6n-2} = \frac{x_{-2}x_0^n x_{-3}^n}{x_{-1}^n x_{-4}^n} \prod_{i=0}^{n-1}\left(\frac{(1+(3i+1)x_{-1}x_{-4})}{(1+(3i+2)x_0x_{-3})}\right).$$

Similarly

$$x_{6n-1} = \frac{x_{6n-3}x_{6n-6}}{x_{6n-4}(1+x_{6n-3}x_{6n-6})}$$

$$= \frac{\frac{x_{-1}^n x_{-4}^n}{x_0^n x_{-3}^{n-1}(1+2x_{-1}x_{-4})} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+3)x_0x_{-3})}{(1+(3i+5)x_{-1}x_{-4})}\right) \frac{x_0^n x_{-3}^{n-1}}{x_{-1}^{n-1}x_{-4}^{n-1}} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+2)x_{-1}x_{-4})}{(1+(3i+3)x_0x_{-3})}\right)}{\left(\frac{x_0^n x_{-3}^n}{x_{-1}^n x_{-4}^{n-1}(1+x_0x_{-3})} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+3)x_{-1}x_{-4})}{(1+(3i+4)x_0x_{-3})}\right)\right)\left(1+\frac{x_{-1}^n x_{-4}^n}{x_0^n x_{-3}^{n-1}(1+2x_{-1}x_{-4})} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+3)x_0x_{-3})}{(1+(3i+5)x_{-1}x_{-4})}\right) \frac{x_0^n x_{-3}^{n-1}}{x_{-1}^{n-1}x_{-4}^{n-1}} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+2)x_{-1}x_{-4})}{(1+(3i+3)x_0x_{-3})}\right)\right)}$$

$$= \frac{\frac{x_{-1}x_{-4}}{(1+2x_{-1}x_{-4})} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+2)x_{-1}x_{-4})}{(1+(3i+5)x_{-1}x_{-4})}\right)}{\left(\frac{x_0^n x_{-3}^n}{x_{-1}^n x_{-4}^{n-1}(1+x_0x_{-3})} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+3)x_{-1}x_{-4})}{(1+(3i+4)x_0x_{-3})}\right)\right)\left(1+\frac{x_{-1}x_{-4}}{(1+2x_{-1}x_{-4})} \prod_{i=0}^{n-2}\left(\frac{(1+(3i+2)x_{-1}x_{-4})}{(1+(3i+5)x_{-1}x_{-4})}\right)\right)}$$

$$= \left(\frac{x_{-1}^n x_{-4}^{n-1}(1+x_0x_{-3})}{x_0^n x_{-3}^n}\right) \prod_{i=0}^{n-2}\left(\frac{(1+(3i+4)x_0x_{-3})}{(1+(3i+3)x_{-1}x_{-4})}\right) \frac{\frac{x_{-1}x_{-4}}{(1+(3n-1)x_{-1}x_{-4})}}{\left(1+\frac{x_{-1}x_{-4}}{(1+(3n-1)x_{-1}x_{-4})}\right)}$$

$$= \left(\frac{x_{-1}^n x_{-4}^{n-1}(1+x_0x_{-3})}{x_0^n x_{-3}^n}\right) \prod_{i=0}^{n-2}\left(\frac{(1+(3i+4)x_0x_{-3})}{(1+(3i+3)x_{-1}x_{-4})}\right) \frac{x_{-1}x_{-4}}{(1+(3n)x_{-1}x_{-4})}.$$

Hence, we have

$$x_{6n-1} = \frac{x_{-1}^{n+1}x_{-4}^n}{x_0^n x_{-3}^n} \prod_{i=0}^{n-1}\left(\frac{(1+(3i+1)x_0x_{-3})}{(1+(3i+3)x_{-1}x_{-4})}\right).$$

Similarly, we can easily obtain the other relations. Thus, the proof is completed.

4

Theorem 2 *Eq.(2) has* $\overline{x}=0$ *as a unique equilibrium point and it is unstable.*

Proof: For the equilibrium points of Eq.(2), set

$$\overline{x}=\frac{\overline{x}^2}{\overline{x}\left(1+\overline{x}^2\right)}.$$

Then

$$\overline{x}^2\left(1+\overline{x}^2\right)=\overline{x}^2, \quad \Rightarrow \quad \overline{x}^2\left(1+\overline{x}^2-1\right)=0, \ \Rightarrow \quad \overline{x}^4=0.$$

Thus the equilibrium point of Eq.(2) is $\overline{x}=0$.
Let $f:(0,\infty)^3 \longrightarrow (0,\infty)$ be a function defined by

$$f(t,u,v,w)=\tfrac{vw}{u(1+vw)}.$$

Thus the linearized equation of Eq.(2) about the equilibrium point $\overline{x}$ is given by

$$y_{n+1}=\sum_{i=0}^{4}\tfrac{\partial f(\overline{x},\overline{x},\overline{x},\overline{x})}{\partial x_{n-i}}.$$

The proof follows by Theorem A.
Numerical examples

For confirming the results of this section, we consider some numerical examples which represent different types of solutions to Eq.(2).
Example 1. Consider Eq.(2) with $x_{-4}=0.21,\ x_{-3}=2,\ x_{-2}=0.5,\ x_{-1}=7,\ x_0=0.3$. See Fig. 1.
Example 2. Consider Eq.(2) with $x_{-4}=9,\ x_{-3}=2,\ x_{-2}=6,\ x_{-1}=7,\ x_0=3$. See Fig. 2.

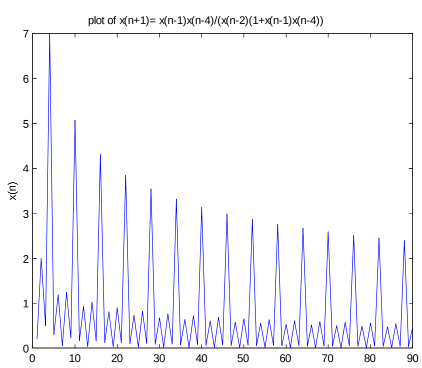

Figure 1.

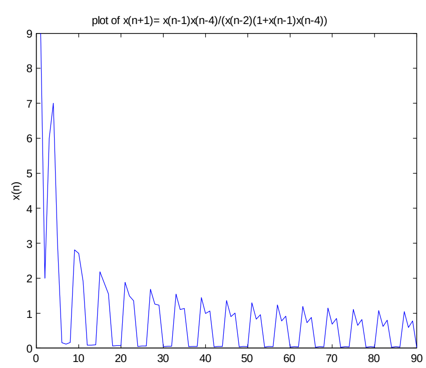

Figure 2.

3 On the Equation $x_{n+1} = \frac{x_{n-1}x_{n-4}}{x_{n-2}(-1+x_{n-1}x_{n-4})}$

In this section we obtain the solution of the difference equation

$$x_{n+1} = \frac{x_{n-1}x_{n-4}}{x_{n-2}(-1+x_{n-1}x_{n-4})}, \quad n = 0, 1, ..., \tag{3}$$

where the initial values are arbitrary non zero real numbers with $x_{-1}x_{-4} \neq 1,\ x_{-3}x_0 \neq 1$.

Theorem 3 *Every solution $\{x_n\}_{n=-4}^{\infty}$ of Eq.(3) has the form*

$$\begin{aligned}
x_{12n-4} &= \frac{x_0^{2n}x_{-3}^{2n}}{x_{-1}^{2n}x_{-4}^{2n-1}}\frac{(-1+x_{-1}x_{-4})^n}{(-1+x_{-3}x_0)^n}, & x_{12n-3} &= \frac{x_{-1}^{2n}x_{-4}^{2n}}{x_0^{2n}x_{-3}^{2n-1}}\frac{(-1+x_{-3}x_0)^n}{(-1+x_{-1}x_{-4})^n},\\
x_{12n-2} &= \frac{x_{-2}x_0^{2n}x_{-3}^{2n}}{x_{-1}^{2n}x_{-4}^{2n}}\frac{(-1+x_{-1}x_{-4})^n}{(-1+x_{-3}x_0)^n}, & x_{12n-1} &= \frac{x_{-1}^{2n+1}x_{-4}^{2n}}{x_0^{2n}x_{-3}^{2n}}\frac{(-1+x_{-3}x_0)^n}{(-1+x_{-1}x_{-4})^n},\\
x_{12n} &= \frac{x_0^{2n+1}x_{-3}^{2n}}{x_{-1}^{2n}x_{-4}^{2n}}\frac{(-1+x_{-1}x_{-4})^n}{(-1+x_{-3}x_0)^n}, & x_{12n+1} &= \frac{x_{-1}^{2n+1}x_{-4}^{2n+1}}{x_{-2}x_0^{2n}x_{-3}^{2n}}\frac{(-1+x_{-3}x_0)^n}{(-1+x_{-1}x_{-4})^{n+1}},\\
x_{12n+2} &= \frac{x_0^{2n+1}x_{-3}^{2n+1}}{x_{-1}^{2n+1}x_{-4}^{2n}}\frac{(-1+x_{-1}x_{-4})^n}{(-1+x_{-3}x_0)^{n+1}}, & x_{12n+3} &= \frac{x_{-1}^{2n+1}x_{-4}^{2n+1}}{x_0^{2n+1}x_{-3}^{2n}}\frac{(-1+x_{-3}x_0)^n}{(-1+x_{-1}x_{-4})^n},\\
x_{12n+4} &= \frac{x_{-2}x_0^{2n+1}x_{-3}^{2n+1}}{x_{-1}^{2n+1}x_{-4}^{2n+1}}\frac{(-1+x_{-1}x_{-4})^{n+1}}{(-1+x_{-3}x_0)^n}, & x_{12n+5} &= \frac{x_{-1}^{2n+2}x_{-4}^{2n+1}}{x_0^{2n+1}x_{-3}^{2n+1}}\frac{(-1+x_{-3}x_0)^{n+1}}{(-1+x_{-1}x_{-4})^{n+1}},\\
x_{12n+6} &= \frac{x_0^{2n+2}x_{-3}^{2n+1}}{x_{-1}^{2n+1}x_{-4}^{2n+1}}\frac{(-1+x_{-1}x_{-4})^n}{(-1+x_{-3}x_0)^{n+1}}, & x_{12n+7} &= \frac{x_{-1}^{2n+2}x_{-4}^{2n+2}}{x_{-2}x_0^{2n+1}x_{-3}^{2n+1}}\frac{(-1+x_{-3}x_0)^n}{(-1+x_{-1}x_{-4})^{n+1}}.
\end{aligned}$$

Proof: For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1$. That is;

$$\begin{aligned}
x_{12n-16} &= \frac{x_0^{2n-2}x_{-3}^{2n-2}}{x_{-1}^{2n-2}x_{-4}^{2n-3}}\frac{(-1+x_{-1}x_{-4})^{n-1}}{(-1+x_{-3}x_0)^{n-1}}, & x_{12n-15} &= \frac{x_{-1}^{2n-2}x_{-4}^{2n-2}}{x_0^{2n-2}x_{-3}^{2n-3}}\frac{(-1+x_{-3}x_0)^{n-1}}{(-1+x_{-1}x_{-4})^{n-1}},\\
x_{12n-14} &= \frac{x_{-2}x_0^{2n-2}x_{-3}^{2n-2}}{x_{-1}^{2n-2}x_{-4}^{2n-2}}\frac{(-1+x_{-1}x_{-4})^{n-1}}{(-1+x_{-3}x_0)^{n-1}}, & x_{12n-13} &= \frac{x_{-1}^{2n-1}x_{-4}^{2n-2}}{x_0^{2n-2}x_{-3}^{2n-2}}\frac{(-1+x_{-3}x_0)^{n-1}}{(-1+x_{-1}x_{-4})^{n-1}},\\
x_{12n-12} &= \frac{x_0^{2n-1}x_{-3}^{2n-2}}{x_{-1}^{2n-2}x_{-4}^{2n-2}}\frac{(-1+x_{-1}x_{-4})^{n-1}}{(-1+x_{-3}x_0)^{n-1}}, & x_{12n-11} &= \frac{x_{-1}^{2n-1}x_{-4}^{2n-1}}{x_{-2}x_0^{2n-2}x_{-3}^{2n-2}}\frac{(-1+x_{-3}x_0)^{n-1}}{(-1+x_{-1}x_{-4})^n},\\
x_{12n-10} &= \frac{x_0^{2n-1}x_{-3}^{2n-1}}{x_{-1}^{2n-1}x_{-4}^{2n-2}}\frac{(-1+x_{-1}x_{-4})^{n-1}}{(-1+x_{-3}x_0)^n}, & x_{12n-9} &= \frac{x_{-1}^{2n-1}x_{-4}^{2n-1}}{x_0^{2n-1}x_{-3}^{2n-2}}\frac{(-1+x_{-3}x_0)^{n-1}}{(-1+x_{-1}x_{-4})^{n-1}},\\
x_{12n-8} &= \frac{x_{-2}x_0^{2n-1}x_{-3}^{2n-1}}{x_{-1}^{2n-1}x_{-4}^{2n-1}}\frac{(-1+x_{-1}x_{-4})^n}{(-1+x_{-3}x_0)^{n-1}}, & x_{12n-7} &= \frac{x_{-1}^{2n}x_{-4}^{2n-1}}{x_0^{2n-1}x_{-3}^{2n-1}}\frac{(-1+x_{-3}x_0)^n}{(-1+x_{-1}x_{-4})^n},\\
x_{12n-6} &= \frac{x_0^{2n}x_{-3}^{2n-1}}{x_{-1}^{2n-1}x_{-4}^{2n-1}}\frac{(-1+x_{-1}x_{-4})^{n-1}}{(-1+x_{-3}x_0)^n}, & x_{12n-5} &= \frac{x_{-1}^{2n}x_{-4}^{2n}}{x_{-2}x_0^{2n-1}x_{-3}^{2n-1}}\frac{(-1+x_{-3}x_0)^{n-1}}{(-1+x_{-1}x_{-4})^n}.
\end{aligned}$$

Now, it follows from Eq.(3) that

$$\begin{aligned}
x_{12n-4} &= \frac{x_{12n-6}x_{12n-9}}{x_{12n-7}(-1+x_{12n-6}x_{12n-9})}\\
&= \frac{\left(\frac{x_0^{2n}x_{-3}^{2n-1}}{x_{-1}^{2n-1}x_{-4}^{2n-1}}\frac{(-1+x_{-1}x_{-4})^{n-1}}{(-1+x_{-3}x_0)^n}\right)\left(\frac{x_{-1}^{2n-1}x_{-4}^{2n-1}}{x_0^{2n-1}x_{-3}^{2n-2}}\frac{(-1+x_{-3}x_0)^{n-1}}{(-1+x_{-1}x_{-4})^{n-1}}\right)}{\frac{x_{-1}^{2n}x_{-4}^{2n-1}}{x_0^{2n-1}x_{-3}^{2n-1}}\frac{(-1+x_{-3}x_0)^n}{(-1+x_{-1}x_{-4})^n}\left(-1+\frac{x_0^{2n}x_{-3}^{2n-1}}{x_{-1}^{2n-1}x_{-4}^{2n-1}}\frac{(-1+x_{-1}x_{-4})^{n-1}}{(-1+x_{-3}x_0)^n}\frac{x_{-1}^{2n-1}x_{-4}^{2n-1}}{x_0^{2n-1}x_{-3}^{2n-2}}\frac{(-1+x_{-3}x_0)^{n-1}}{(-1+x_{-1}x_{-4})^{n-1}}\right)}\\
&= \frac{\left(\frac{x_0x_{-3}}{(-1+x_{-3}x_0)}\right)}{\left(\frac{x_{-1}^{2n}x_{-4}^{2n-1}}{x_0^{2n-1}x_{-3}^{2n-1}}\frac{(-1+x_{-3}x_0)^n}{(-1+x_{-1}x_{-4})^n}\right)\left(-1+\left(\frac{x_0x_{-3}}{(-1+x_{-3}x_0)}\right)\right)} = \frac{x_0^{2n}x_{-3}^{2n}}{x_{-1}^{2n}x_{-4}^{2n-1}}\frac{(-1+x_{-1}x_{-4})^n}{(-1+x_{-3}x_0)^n},
\end{aligned}$$

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$$
\begin{aligned}
x_{12n-3} &= \frac{x_{12n-5}x_{12n-8}}{x_{12n-6}(-1+x_{12n-5}x_{12n-8})} \\
&= \frac{\frac{x_{-1}^{2n}x_{-4}^{2n}}{x_{-2}x_0^{2n-1}x_{-3}^{2n-1}}\frac{(-1+x_{-3}x_0)^{n-1}}{(-1+x_{-1}x_{-4})^{n}}\frac{x_{-2}x_0^{2n-1}x_{-3}^{2n-1}}{x_{-1}^{2n-1}x_{-4}^{2n-1}}\frac{(-1+x_{-1}x_{-4})^{n}}{(-1+x_{-3}x_0)^{n-1}}}{\frac{x_0^{2n}x_{-3}^{2n-1}(-1+x_{-1}x_{-4})^{n-1}}{x_{-1}^{2n-1}x_{-4}^{2n-1}(-1+x_{-3}x_0)^{n}}\left(-1+\frac{x_{-1}^{2n}x_{-4}^{2n}(-1+x_{-3}x_0)^{n-1}}{x_{-2}x_0^{2n-1}x_{-3}^{2n-1}(-1+x_{-1}x_{-4})^{n}}\frac{x_{-2}x_0^{2n-1}x_{-3}^{2n-1}(-1+x_{-1}x_{-4})^{n}}{x_{-1}^{2n-1}x_{-4}^{2n-1}(-1+x_{-3}x_0)^{n-1}}\right)} \\
&= \frac{x_{-1}^{2n-1}x_{-4}^{2n-1}}{x_0^{2n}x_{-3}^{2n-1}}\frac{(-1+x_{-3}x_0)^{n}}{(-1+x_{-1}x_{-4})^{n-1}}\frac{x_{-1}x_{-4}}{(-1+x_{-1}x_{-4})} = \frac{x_{-1}^{2n}x_{-4}^{2n}}{x_0^{2n}x_{-3}^{2n-1}}\frac{(-1+x_{-3}x_0)^{n}}{(-1+x_{-1}x_{-4})^{n}},
\end{aligned}
$$

Similarly, we can easily obtain the other relations. Thus, the proof is completed.

Theorem 4 *Eq.(3) has three equilibrium points which are $\overline{x}=0$ and $\overline{x}=\pm\sqrt{2}$and all of them are unstable.*

Proof: The proof is similar to Theorem 2 and will be omitted.
Lemma 1. It is easy to see that every solution of Eq.(3) is unbounded except in the case $x_{-3}x_0 = x_{-1}x_{-4}$.

Theorem 5 *Eq.(3) has a periodic solution of period twelve iff $x_{-3}x_0 = x_{-1}x_{-4}$. Moreover the periodic solution has the following form*

$$
\left\{ \begin{array}{c} x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0, \frac{x_{-1}x_{-4}}{x_{-2}(-1+x_{-1}x_{-4})}, \frac{x_0x_{-3}}{x_{-1}(-1+x_{-3}x_0)}, x_{-3}, x_{-2}\left(-1+x_{-1}x_{-4}\right), \\ x_{-1}, \frac{x_0}{(-1+x_{-3}x_0)}, \frac{x_{-1}x_{-4}}{x_{-2}(-1+x_{-1}x_{-4})}, x_{-4}, x_{-3}, \ldots \end{array} \right\}.
$$

Proof: First suppose that there exists a prime period twelve solution of Eq.(3) of the following form

$$
\left\{ \begin{array}{c} x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0, \frac{x_{-1}x_{-4}}{x_{-2}(-1+x_{-1}x_{-4})}, \frac{x_0x_{-3}}{x_{-1}(-1+x_{-3}x_0)}, x_{-3}, x_{-2}\left(-1+x_{-1}x_{-4}\right), \\ x_{-1}, \frac{x_0}{(-1+x_{-3}x_0)}, \frac{x_{-1}x_{-4}}{x_{-2}(-1+x_{-1}x_{-4})}, x_{-4}, x_{-3}, \ldots \end{array} \right\}.
$$

Then we see from Theorem 3 that

$$
\begin{aligned}
x_{12n-4} &= \frac{x_0^{2n}x_{-3}^{2n}}{x_{-1}^{2n}x_{-4}^{2n-1}}\frac{(-1+x_{-1}x_{-4})^{n}}{(-1+x_{-3}x_0)^{n}} = x_{-4}, \quad x_{12n-3} = \frac{x_{-1}^{2n}x_{-4}^{2n}}{x_0^{2n}x_{-3}^{2n-1}}\frac{(-1+x_{-3}x_0)^{n}}{(-1+x_{-1}x_{-4})^{n}} = x_{-3}, \\
x_{12n-2} &= \frac{x_{-2}x_0^{2n}x_{-3}^{2n}}{x_{-1}^{2n}x_{-4}^{2n}}\frac{(-1+x_{-1}x_{-4})^{n}}{(-1+x_{-3}x_0)^{n}} = x_{-2}, \quad x_{12n-1} = \frac{x_{-1}^{2n+1}x_{-4}^{2n}}{x_0^{2n}x_{-3}^{2n}}\frac{(-1+x_{-3}x_0)^{n}}{(-1+x_{-1}x_{-4})^{n}} = x_{-1}, \\
x_{12n} &= \frac{x_0^{2n+1}x_{-3}^{2n}(-1+x_{-1}x_{-4})^{n}}{x_{-1}^{2n}x_{-4}^{2n}(-1+x_{-3}x_0)^{n}} = x_0, x_{12n+1} = \frac{x_{-1}^{2n+1}x_{-4}^{2n+1}(-1+x_{-3}x_0)^{n}}{x_{-2}x_0^{2n}x_{-3}^{2n}(-1+x_{-1}x_{-4})^{n+1}} = \frac{x_{-1}x_{-4}}{x_{-2}(-1+x_{-1}x_{-4})}, \\
x_{12n+2} &= \frac{x_0^{2n+1}x_{-3}^{2n+1}}{x_{-1}^{2n+1}x_{-4}^{2n}}\frac{(-1+x_{-1}x_{-4})^{n}}{(-1+x_{-3}x_0)^{n+1}} = \frac{x_0x_{-3}}{x_{-1}(-1+x_{-3}x_0)}, \\
x_{12n+3} &= \frac{x_{-1}^{2n+1}x_{-4}^{2n+1}}{x_0^{2n+1}x_{-3}^{2n}}\frac{(-1+x_{-3}x_0)^{n}}{(-1+x_{-1}x_{-4})^{n}} = x_{-3}, \\
x_{12n+4} &= \frac{x_{-2}x_0^{2n+1}x_{-3}^{2n+1}}{x_{-1}^{2n+1}x_{-4}^{2n+1}}\frac{(-1+x_{-1}x_{-4})^{n+1}}{(-1+x_{-3}x_0)^{n}} = x_{-2}\left(-1+x_{-1}x_{-4}\right), \\
x_{12n+5} &= \frac{x_{-1}^{2n+2}x_{-4}^{2n+1}}{x_0^{2n+1}x_{-3}^{2n+1}}\frac{(-1+x_{-3}x_0)^{n+1}}{(-1+x_{-1}x_{-4})^{n+1}} = x_{-1}, \\
x_{12n+6} &= \frac{x_0^{2n+2}x_{-3}^{2n+1}}{x_{-1}^{2n+1}x_{-4}^{2n+1}}\frac{(-1+x_{-1}x_{-4})^{n}}{(-1+x_{-3}x_0)^{n+1}} = \frac{x_0}{(-1+x_{-3}x_0)}, \\
x_{12n+7} &= \frac{x_{-1}^{2n+2}x_{-4}^{2n+2}}{x_{-2}x_0^{2n+1}x_{-3}^{2n+1}}\frac{(-1+x_{-3}x_0)^{n}}{(-1+x_{-1}x_{-4})^{n+1}} = \frac{x_{-1}x_{-4}}{x_{-2}(-1+x_{-1}x_{-4})}.
\end{aligned}
$$

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Then we get $(-1+x_{-3}x_0)=(-1+x_{-1}x_{-4})$.
Second assume that $(-1+x_{-3}x_0)=(-1+x_{-1}x_{-4})$. Then we see from the form of the solution of Eq.(3) that

$$\begin{aligned}
x_{12n-4} &= x_{-4}, \quad x_{12n-3}=x_{-3}, \quad x_{12n-2}=x_{-2}, \quad x_{12n-1}=x_{-1}, \quad x_{12n}=x_0,\\
x_{12n+1} &= \frac{x_{-1}x_{-4}}{x_{-2}(-1+x_{-1}x_{-4})}, \quad x_{12n+2}=\frac{x_0x_{-3}}{x_{-1}(-1+x_{-3}x_0)}, \quad x_{12n+3}=\frac{x_{-1}x_{-4}}{x_0}=x_{-3},\\
x_{12n+4} &= x_{-2}\left(-1+x_{-1}x_{-4}\right), \ x_{12n+5}=x_{-1},\\
x_{12n+6} &= \frac{x_0}{-1+x_{-3}x_0}, \ x_{12n+7}=\frac{x_{-1}x_{-4}}{x_{-2}(-1+x_{-1}x_{-4})}.
\end{aligned}$$

Thus we have a periodic solution of period twelve and the proof is complete.

Theorem 6 *Eq.(3) has a periodic solution of period six iff* $x_{-1}x_{-4}=x_{-3}x_0=2$ *and has the form* $\left\{x_{-4},x_{-3},x_{-2},x_{-1},x_0,\frac{2}{x_{-2}},x_{-4},x_{-3},x_{-2},...\right\}$.

Proof: The proof is consequently from the previous Theorems and will be omitted.
In the following we present some figures illustrate the behavior of the solutions of Eq.(3) under some different initial values.

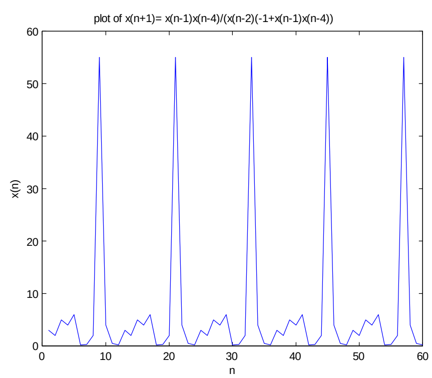

Figure 3.
$x_{-4}=3, x_{-3}=2, x_{-2}=5,$
$x_{-1}=4, \ x_0=6.$

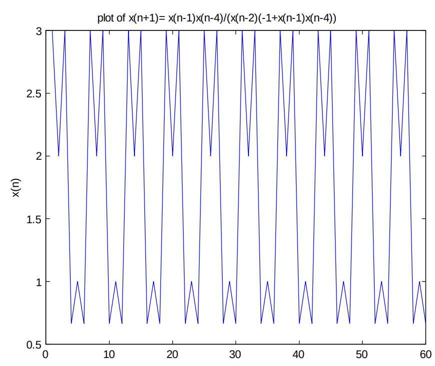

Figure 4.
$x_{-4}=3, x_{-3}=2, x_{-2}=3,$
$x_{-1}=2/3, \ x_0=1.$

The following cases can be treated similarly.

4 On the Equation $x_{n+1}=\frac{x_{n-1}x_{n-4}}{x_{n-2}(1-x_{n-1}x_{n-4})}$

In this section we get the solution of the third following equation

$$x_{n+1}=\frac{x_{n-1}x_{n-4}}{x_{n-2}(1-x_{n-1}x_{n-4})}, \quad n=0,1,..., \tag{4}$$

where the initial values are arbitrary positive real numbers.

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Theorem 7 *Assume that $\{x_n\}_{n=-4}^{\infty}$ be a solution of Eq.(4). Then for $n = 0, 1, ...$*

$$x_{6n-2} = \frac{x_{-2}x_0^n x_{-3}^n}{x_{-1}^n x_{-4}^n}\prod_{i=0}^{n-1}\left(\frac{(1-(3i+1)x_{-1}x_{-4})}{(1-(3i+2)x_0x_{-3})}\right), \quad x_{6n-1} = \frac{x_{-1}^{n+1}x_{-4}^n}{x_0^n x_{-3}^n}\prod_{i=0}^{n-1}\left(\frac{(1-(3i+1)x_0x_{-3})}{(1-(3i+3)x_{-1}x_{-4})}\right),$$

$$x_{6n} = \frac{x_0^{n+1}x_{-3}^n}{x_{-1}^n x_{-4}^n}\prod_{i=0}^{n-1}\left(\frac{1-(3i+2)x_{-1}x_{-4}}{1-(3i+3)x_0x_{-3}}\right), \quad x_{6n+1} = \frac{x_{-1}^{n+1}x_{-4}^{n+1}}{x_{-2}x_0^n x_{-3}^n(1-x_{-1}x_{-4})}\prod_{i=0}^{n-1}\left(\frac{1-(3i+2)x_0x_{-3}}{1-(3i+4)x_{-1}x_{-4}}\right),$$

$$x_{6n+2} = \frac{x_0^{n+1}x_{-3}^{n+1}}{x_{-1}^{n+1}x_{-4}^n(1-x_0x_{-3})}\prod_{i=0}^{n-1}\left(\frac{(1-(3i+3)x_{-1}x_{-4})}{(1-(3i+4)x_0x_{-3})}\right),$$

$$x_{6n+3} = \frac{x_{-1}^{n+1}x_{-4}^{n+1}}{x_0^{n+1}x_{-3}^n(1-2x_{-1}x_{-4})}\prod_{i=0}^{n-1}\left(\frac{(1-(3i+3)x_0x_{-3})}{(1-(3i+5)x_{-1}x_{-4})}\right).$$

Theorem 8 *Eq.(4) has the unique equilibrium point $\overline{x} = 0$ and it is unstable.*

Example 3. Consider Eq.(4) with $x_{-4} = 3,\ x_{-3} = 5,\ x_{-2} = 2,\ x_{-1} = 2/3,\ x_0 = 0.4$. See Fig. 5.

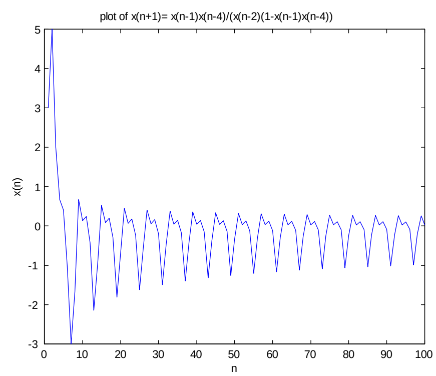

Figure 5.

5 On the Equation $x_{n+1} = \frac{x_{n-1}x_{n-4}}{x_{n-2}(-1-x_{n-1}x_{n-4})}$

Here we obtain the analytical form of the solutions of the equation

$$x_{n+1} = \frac{x_{n-1}x_{n-4}}{x_{n-2}(-1-x_{n-1}x_{n-4})}, \quad n = 0, 1, ..., \tag{5}$$

where the initial values are arbitrary non zero real numbers with $x_{-1}x_{-4} \neq -1,\ x_{-3}x_0 \neq -1$.

Theorem 9 *Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of Eq.(5). Then for $n = 0, 1, 2, ...$ the solution of Eq.(5) is given by*

$$x_{12n-4} = \frac{x_0^{2n}x_{-3}^{2n}}{x_{-1}^{2n}x_{-4}^{2n-1}}\frac{(-1-x_{-1}x_{-4})^n}{(-1-x_{-3}x_0)^n}, \qquad x_{12n-3} = \frac{x_{-1}^{2n}x_{-4}^{2n}}{x_0^{2n}x_{-3}^{2n-1}}\frac{(-1-x_{-3}x_0)^n}{(-1-x_{-1}x_{-4})^n},$$

$$x_{12n-2} = \frac{x_{-2}x_0^{2n}x_{-3}^{2n}}{x_{-1}^{2n}x_{-4}^{2n}}\frac{(-1-x_{-1}x_{-4})^n}{(-1-x_{-3}x_0)^n}, \qquad x_{12n-1} = \frac{x_{-1}^{2n+1}x_{-4}^{2n}}{x_0^{2n}x_{-3}^{2n}}\frac{(-1-x_{-3}x_0)^n}{(-1-x_{-1}x_{-4})^n},$$

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$$\begin{aligned}
x_{12n} &= \frac{x_0^{2n+1}x_{-3}^{2n}}{x_{-1}^{2n}x_{-4}^{2n}}\frac{(-1-x_{-1}x_{-4})^n}{(-1-x_{-3}x_0)^n}, \qquad x_{12n+1} = \frac{x_{-1}^{2n+1}x_{-4}^{2n+1}}{x_{-2}x_0^{2n}x_{-3}^{2n}}\frac{(-1-x_{-3}x_0)^n}{(-1-x_{-1}x_{-4})^{n+1}},\\
x_{12n+2} &= \frac{x_0^{2n+1}x_{-3}^{2n+1}}{x_{-1}^{2n+1}x_{-4}^{2n}}\frac{(-1-x_{-1}x_{-4})^n}{(-1-x_{-3}x_0)^{n+1}}, \quad x_{12n+3} = \frac{x_{-1}^{2n+1}x_{-4}^{2n+1}}{x_0^{2n+1}x_{-3}^{2n}}\frac{(-1-x_{-3}x_0)^n}{(-1-x_{-1}x_{-4})^n},\\
x_{12n+4} &= \frac{x_{-2}x_0^{2n+1}x_{-3}^{2n+1}}{x_{-1}^{2n+1}x_{-4}^{2n+1}}\frac{(-1-x_{-1}x_{-4})^{n+1}}{(-1-x_{-3}x_0)^n}, \ x_{12n+5} = \frac{x_{-1}^{2n+2}x_{-4}^{2n+1}}{x_0^{2n+1}x_{-3}^{2n+1}}\frac{(-1-x_{-3}x_0)^{n+1}}{(-1-x_{-1}x_{-4})^{n+1}},\\
x_{12n+6} &= \frac{x_0^{2n+2}x_{-3}^{2n+1}}{x_{-1}^{2n+1}x_{-4}^{2n+1}}\frac{(-1-x_{-1}x_{-4})^n}{(-1-x_{-3}x_0)^{n+1}}, \quad x_{12n+7} = \frac{x_{-1}^{2n+2}x_{-4}^{2n+2}}{x_{-2}x_0^{2n+1}x_{-3}^{2n+1}}\frac{(-1-x_{-3}x_0)^n}{(-1-x_{-1}x_{-4})^{n+1}}.
\end{aligned}$$

Theorem 10 *Eq.(5) has $\overline{x}=0$ as a unique equilibrium point which is unstable.*

Lemma 2. It is easy to see that every solution of Eq.(5) is unbounded except in the case $x_{-3}x_0 = x_{-1}x_{-4}$.

Theorem 11 *Eq.(5) has a periodic solution of period twelve iff $x_{-3}x_0 = x_{-1}x_{-4}$. Moreover the periodic solution has the form* $\{x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0, \frac{x_{-1}x_{-4}}{x_{-2}(-1-x_{-1}x_{-4})},$

$\frac{x_0x_{-3}}{x_{-1}(-1-x_{-3}x_0)}, x_{-3}, x_{-2}\left(-1-x_{-1}x_{-4}\right), x_{-1}, \frac{x_0}{-1-x_{-3}x_0}, \frac{x_{-1}x_{-4}}{x_{-2}(-1-x_{-1}x_{-4})}, x_{-4}, ...\}$

Theorem 12 *Eq.(5) has a periodic solution of period six iff $x_{-3}x_0 = x_{-1}x_{-4} = -2$ and will be taken the form* $\{x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0, \frac{-2}{x_{-2}}, x_{-4}, ...\}$.

Example 4. Fig. 6 below shows the behavior of the solution of Eq.(5) whenever $x_{-4} = 3,\ x_{-3} = 5,\ x_{-2} = -7,\ x_{-1} = 4,\ x_0 = 2.$

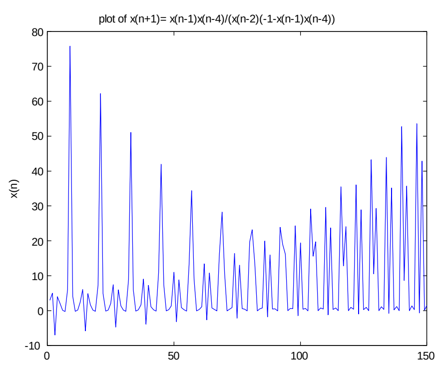

Figure 6.

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Worse-Case Conditional Value-at-Risk for Asymmetrically Distributed Asset Scenarios Returns

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Abstract: Many studies have reported empirical evidence of asymmetries in asset return distributions. Meanwhile, optimal solutions to the Conditional Value-at-Risk (CVaR) minimization are highly susceptible to estimation error of the risk measure because the estimate depends on only a small portion of sampled scenarios. In this paper, based on the robust optimization techniques Chen et al.(2007)[19], we propose a computationally tractable worst-case Conditional Value-at-Risk (CVaR). In the situation, the sampled scenario returns are generated by a factor model with some asymmetric affine uncertainty set. The remarkable characteristic of the new method is that the robust optimization model retains the complexity of original portfolio optimization problem, i.e., the robust counterpart problem is still a linear programming problem. Moreover, it takes into consideration asymmetries in the distributions of scenarios returns used for defining CVaR. We present some numerical experiments with simulated and real market data to illustrate the behavior of the robust optimization model.

Keywords: Portfolio optimization, Conditional value at risk(CVaR), Robust optimization, Linear programming(LP).

1. Introduction

Portfolio optimization problem is an attractive and important research topic since the pioneering Markowitz work on optimal portfolio selection [1]. It is now well known that while mean-variance optimization is appropriate for symmetrically distributed portfolio returns, it results in unsatisfactory asset allocations when returns are asymmetrically distributed, or when downside risk is more weighted than upside risk.

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Since the middle of 1990s, Value-at-Risk (VaR, [4]), a new measure of downside risk, has become popular in financial risk management. It has even been recommended as a standard on banking supervision by the Basel Committee. However, Critics have pointed out numerous shortcomings of VaR [5]. On the other hand, Conditional Value-at-Risk (CVaR), defined as the mean of the tail distribution exceeding VaR, has attracted much attention in recent years. As a measure of risk, CVaR exhibits some better properties than VaR. First, it can deal with the asymmetric distribution of asset return better than mean-variance analysis, especially for assets with returns that are heavy-tailed. Secondly, minimizing CVaR usually results in solving a convex programming problem, such as a linear programming problem, which allows the decision maker to deal with a large-scale portfolio problem efficiently [6, 7]. Finally, Artzner et al.[5] demonstrate that CVaR is a coherent measure of risk, which has been widely accepted as a benchmark to evaluate risk measures. All these stimulate the application of CVaR in practice, and CVaR is getting more and more popular in financial management.

In fact, it is noted that in the process of portfolio selection, the original data brought to the model are not always accurate, i.e., it may be subject to some errors. Thus the result may be influenced by perturbations in the parameters. As pointed out by Black and Litterman [8], in the classical mean-variance model, the portfolio decision is very sensitive to the mean and the covariance matrix, especially to the mean. Chopra and Ziemba [9] showed that small changes in the input parameters can result in large changes in the optimal portfolio allocation. Thus, the modeling risk arises due to the uncertainty of the underlying probability distribution.

Being aware of the importance of robustness in recent years, researchers from both finance and operations research have paid increasing attentions to the robust version of portfolio selection problems. Lobo and Boyd (2000)[10], Goldfarb and Iyengar (2003)[11] studied the robust portfolio problem under the mean-variance framework. Instead of assuming precise information on the mean and the covariance matrix of asset returns, they introduced some types of uncertainties, such as polyhedral uncertainty, box uncertainty and ellipsoidal uncertainty, in the parameters in determining the mean and the covariance matrix, and they then transformed the problem into semidefinite programs(SDP) or second-order cone programs(SOCP), which can be efficiently solved by interior-point algorithms developed in recent years. Halldórsson and Tütüncu (2004) [12] applied their interior-point method for saddle-point problems to the robust mean-variance portfolio selection under the box uncertainty of the elements in the mean vector and the covariance matrix. El Ghaoui, Oks and Oustry (2003)[13] investigated the robust portfolio optimization problem using worst-case VaR, where only the first- and second-moment information on the distribution is available. Several formulations corresponding to various structures of partial information have been extensively exploited to derive the resulting portfolio selection problems in a form of a semidefinite program(SDP). Natarajan, Pachamanova, and Sim, (2008) [14] proposed a computationally tractable approximation method for minimizing the VaR of a portfolio based on robust optimization techniques in Chen et al.(2007)[19]. The method results in the optimization of a modified VaR

measure, Asymmetry-Robust VaR, that takes into consideration asymmetries in the distributions of returns and is coherent. Zhu and Fukushima (2009)[15] further investigated the worst-case CVaR risk measure with several structures of uncertainty in the underlying distribution. They focus on the uncertainty in the probability distribution used for defining CVaR. Such a modeling is called distributionally robust modeling. It is true that the probability estimation itself is under uncertainty and we cannot know the true one. However, it is not easy to imagine what form of uncertainty set is proper for the probability measure. In this sense, employing the uncertainty of probability distribution may not provide investors with a satisfactory solution.

On the other hand, since the estimate of CVaR is computed by using only an upper tail part of the loss distribution, a large number of samples are required for assuring the statistical reliability of the estimate. Especially when CVaR is employed as the objective of a portfolio optimization, a much larger number of samples are required for ensuring the accuracy of the optimal portfolio. In practice, however, the number of samples which is available for the estimation is limited, and the estimated CVaR and the resulting optimal portfolio may contain considerable estimation error.

Meanwhile, many studies have reported empirical evidence of asymmetries and large kurtosis in asset return distributions. Empirically, however, there is evidence that both short- and long-horizon stock returns can be skewed and highly leptokurtic (Fama 1976 [22], Duffee 2002 [23]). Furthermore, the returns of portfolios involving derivatives or credit risky assets can have extremely left-skewed distributions (Schönbucher 2000 [24]). More recently, Ang and Chen (2002)[25] find that the asymmetries in the data reject the null hypothesis of multivariate normal distributions. Conine and Tamarkin (1981) [26] also claim that though diversification can change skewness exposure, the remaining idiosyncratic skewness is relevant in asset pricing and thus portfolio optimization under asymmetric distribution is a significant topic for research.

In this paper, we further study the Worse-Case Conditional Value-at-Risk by supposing the sampled scenario returns are generated by a factor model with some asymmetric affine uncertainty set in order to Mitigate the fragility of CVaR-based portfolio optimization problem. Motivated by the works in Chen et al.(2007)[19], we provide a computationally tractable robust optimization method for minimizing the Worse-Case CVaR of a portfolio. Moreover, it takes into consideration asymmetries in the distributions of returns used for defining CVaR.

Notations: Throughout this paper, we use boldface letter such as $\boldsymbol{x}$ for vector to distinguish it from scalar x.

2. Conditional value-at-risk (CVaR)

The conditional value-at-risk (CVaR) has gained growing popularity in financial risk management due to the coherence property and tractability in its optimization.

Let $f(\boldsymbol{x}, \boldsymbol{y})$ be the loss associated with the decision vector $\boldsymbol{x}$, to be chosen from a certain subset X of R^n, and the random vector $\boldsymbol{y}$ in R^m. For convenience, the underling probability of $\boldsymbol{y}$ will be

assumed to have a density function $p(.)$.

The probability of $f(\boldsymbol{x}, \boldsymbol{y})$ not exceeding a threshold α is then given by

$$\Psi(\boldsymbol{x}, \alpha) = \int_{f(\boldsymbol{x},\boldsymbol{y})\leq\alpha} p(\boldsymbol{y})d\boldsymbol{y}. \tag{2.1}$$

As a function of α for fixed $\boldsymbol{x}$, $\Psi(\boldsymbol{x}, \alpha)$ is the cumulative distribution function for the loss associated with $\boldsymbol{x}$.

For a confidence level β and a fixed $\boldsymbol{x} \in \mathrm{X}$ the value-at-risk, denoted by $\mathrm{VaR}_\beta(\boldsymbol{x})$ is defined as

$$\mathrm{VaR}_\beta(\boldsymbol{x}) = \min\{\alpha \in \mathrm{R} : \Psi(\boldsymbol{x}, \alpha) \geq \beta\}. \tag{2.2}$$

The conditional value-at-risk, denoted by $\mathrm{CVaR}_\beta(\boldsymbol{x})$, is defined as the expected value of the loss that exceeds $\mathrm{VaR}_\beta(\boldsymbol{x})$, that is,

$$\mathrm{CVaR}_\beta(\boldsymbol{x}) = (1-\beta)^{-1} \int_{f(\boldsymbol{x},\boldsymbol{y})\geq \mathrm{VaR}_\beta(\boldsymbol{x})} f(\boldsymbol{x}, \boldsymbol{y})p(\boldsymbol{y})d\boldsymbol{y}. \tag{2.3}$$

The CVaR is a coherent risk measure [5]. We note that the problem involved $\mathrm{CVaR}_\beta(\boldsymbol{x})$ is difficult to proceed due to its convoluted and implicit version. Rockafellar and Uryasev made a remarkable contribution in [6] by introducing a simpler auxiliary function F_β on $\mathrm{X} \times \mathrm{R}$, defined by

$$F_\beta(\boldsymbol{x}, \alpha) = \alpha + (1-\beta)^{-1} \int_{\boldsymbol{y}\in R^m} [f(\boldsymbol{x}, \boldsymbol{y}) - \alpha]^+ p(\boldsymbol{y})d\boldsymbol{y}, \tag{2.4}$$

In practice, the probability density function $p(\boldsymbol{y})$ is often not available, or is very difficult to estimate. Instead, we might have T different scenarios $\mathrm{Y} = (\boldsymbol{y}_{[1]}, \boldsymbol{y}_{[2]}, \ldots, \boldsymbol{y}_{[T]})$ that are sampled from the probability distribution or that have been obtained from computer simulations. Evaluating the auxiliary function $\tilde{F}_\beta(\boldsymbol{x}, \alpha)$ using the scenarios Y, we have

$$\tilde{F}_\beta(\boldsymbol{x}, \alpha) = \alpha + (1-\beta)^{-1} \sum_{t=1}^{T} \pi_t [f(\boldsymbol{x}, \boldsymbol{y}_{[t]}) - \alpha]^+, \tag{2.5}$$

where $\boldsymbol{y}_{[t]}$ denotes the tth sample (the subscript $[t]$ is used to distinguish a vector from a scalar) generated by simple random sampling with respect to $\boldsymbol{x}$ according to its density function $p(.)$, and T denotes the number of samples, where π_t are probabilities of scenarios $\boldsymbol{y}_{[t]}$. If π_t is equal to T^{-1} for all t, then (2.5) reduces to

$$\tilde{F}_\alpha(\boldsymbol{x}, \alpha) = \alpha + \frac{1}{T(1-\beta)} \sum_{t=1}^{T} [f(\boldsymbol{x}, \boldsymbol{y}_{[t]}) - \alpha]^+. \tag{2.6}$$

Obviously, $\tilde{F}_\alpha(\boldsymbol{x}, \alpha)$ is convex and piecewise linear with respect to α. Further, $\tilde{F}_\alpha(\boldsymbol{x}, \alpha)$ is convex for $(\boldsymbol{x}, \alpha)$ if $f(\boldsymbol{x}, \boldsymbol{y})$ is convex (see Theorem 2 in [6]). Replacing $[f(\boldsymbol{x}, \boldsymbol{y}_{[t]}) - \alpha]^+$ by the auxiliary variables d_t along with appropriate constraints, we obtain the equivalent optimization problem

$$\begin{aligned}
\min_{(\boldsymbol{x},\boldsymbol{d},\alpha)\in \mathrm{R}^n\times \mathrm{R}^T\times \mathrm{R}} \quad & \alpha + \frac{1}{T(1-\beta)} \sum_{t=1}^{T} d_t, \\
s.t. \quad & \boldsymbol{x} \in \mathrm{X} \\
& d_t \geq f(\boldsymbol{x}, \boldsymbol{y}_{[t]}) - \alpha, \qquad t = 1, \ldots, T, \\
& \boldsymbol{d} \geq 0.
\end{aligned} \tag{2.7}$$

Generally, the loss and return functions of portfolio allocation are chosen by:

$$f(\boldsymbol{x},\boldsymbol{y}) = -\boldsymbol{x}^T\boldsymbol{y}, \quad \mathrm{R}_p(x) = \mathrm{E}_p[\boldsymbol{x}^T\boldsymbol{y}] = \boldsymbol{x}^T\mathrm{E}_p[\boldsymbol{y}] = \boldsymbol{x}^T\boldsymbol{r}, \tag{2.8}$$

in which $\boldsymbol{y}$ is the vector of the assets' return, $\boldsymbol{r}$ is the vector of the expected assets' return, and $\boldsymbol{x}^T\boldsymbol{r}$ is the mean return of the portfolio. Hence, adding an auxiliary variable $\theta \in R$, the minimization model of CVaR (2.9) becomes the following linear programming (LP) problem with variables $(\boldsymbol{x}, \boldsymbol{d}, \alpha, \theta) \in \mathrm{R}^n \times \mathrm{R}^T \times \mathrm{R} \times \mathrm{R}$.

$$\begin{array}{ll} \min & \theta \\ s.t. & \boldsymbol{x} \in \mathrm{X} \\ & \alpha + \dfrac{1}{T(1-\beta)}\displaystyle\sum_{t=1}^{T} d_t \le \theta, \\ & d_t \ge -\boldsymbol{x}^T\boldsymbol{y}_{[t]} - \alpha, \qquad t = 1,\ldots,T, \\ & \boldsymbol{d} \ge 0. \end{array} \tag{2.9}$$

Portfolio optimization tries to find an optimal trade-off between the risk and the return according to the investor's preference. Thus, the portfolio selection problem using CVaR as a risk measure can be represented as

$$\min_{\boldsymbol{x}\in\mathrm{X}} \quad \mathrm{CVaR}_\beta(\boldsymbol{x})$$

where X denotes the constraint on the portfolio position, which usually includes the budget constraint and no short sales constraint

$$\boldsymbol{x}^T\mathbf{1} = 1, \quad \boldsymbol{x} \ge 0. \tag{2.10}$$

Let μ be the smallest expected return of the portfolio required by the investor. From (2.8), this return requirement can be represented as

$$\boldsymbol{x}^T\boldsymbol{r} \ge \mu. \tag{2.11}$$

Therefore, the feasible decision set of portfolios can be denoted as

$$\mathrm{X} = \{\boldsymbol{x} | \quad \boldsymbol{x}^T\mathbf{1} = 1, \quad \boldsymbol{x} \ge 0, \quad \boldsymbol{x}^T\boldsymbol{r} \ge \mu\}. \tag{2.12}$$

From (2.9) and 2.12, the mean-CVaR Portfolio optimization can be be written as the following linear program

$$\begin{array}{ll} \min & \theta \\ s.t. & \alpha + \dfrac{1}{T(1-\beta)}\displaystyle\sum_{t=1}^{T} d_t \le \theta, \\ & d_t \ge -\boldsymbol{x}^T\boldsymbol{y}_{[t]} - \alpha, \qquad t = 1,\ldots,T, \\ & \boldsymbol{d} \ge 0. \\ & \boldsymbol{x}^T\mathbf{1} = 1, \quad \boldsymbol{x} \ge 0, \quad \boldsymbol{x}^T\boldsymbol{r} \ge \mu. \end{array} \tag{2.13}$$

3. Worst-Case Conditional value-at-risk (CVaR)

However, optimal solutions to the CVaR minimization are highly susceptible to estimation error of the risk measure because the estimate depends on only a small portion of sampled scenarios, for example $\mathrm{Y} = (\boldsymbol{y}_{[1]}, \boldsymbol{y}_{[2]}, \ldots, \boldsymbol{y}_{[T]})$.

A practical way to alleviate the effect of such a perturbation is to employ a statistical model. For example, Konno, Waki and Yuuki (2002) replace the observed returns $\mathrm{Y} = (\boldsymbol{y}_{[1]}, \boldsymbol{y}_{[2]}, \ldots, \boldsymbol{y}_{[T]})$ in 2.6 with values estimated by a regression approach. Based on the robust optimization techniques in Chen et al.(2007)[19], we suppose that future asset returns $\tilde{r}$ are generated by the following factor model

$$\boldsymbol{r} = \boldsymbol{r}^0 + \Delta \boldsymbol{r}\boldsymbol{z}, \boldsymbol{z} \in C \tag{3.1}$$

in which $\boldsymbol{r}^0$ is a vector of expected returns, and $\Delta\boldsymbol{r}$ is a matrix of factor loadings. The factors $\boldsymbol{z}$ are stochastically independent with following support set

$$C = \Big\{\boldsymbol{z} : \exists \boldsymbol{v},\ \boldsymbol{w} \in R_+^N, \boldsymbol{z} = \boldsymbol{v} - \boldsymbol{w}, \|\mathbf{P}^{-1}\mathbf{v} + \mathbf{Q}^{-1}\mathbf{w}\| \leq \Omega\Big\}, \tag{3.2}$$

and $\mathbf{P} = \mathrm{diag}(p_1, \ldots, p_N)$, $\mathbf{Q} = \mathrm{diag}(q_1, \ldots, q_N)$. The parameters $p_j > 0$ and $q_j > 0$ are the "forward"and the "backward" deviations of random variable $z_j, j = 1, \ldots, N$, respectively. The uncertainty set C is convex, and its size is controlled by Ω. Intuitively speaking, the uncertain factors $\boldsymbol{z}$ are decomposed into two random variables: $\boldsymbol{v} = \max\{\boldsymbol{z}, 0\}$ and $\boldsymbol{w} = \max\{-\boldsymbol{z}, 0\}$, so that $\boldsymbol{z} = \boldsymbol{v} - \boldsymbol{w}$. The multipliers $\frac{1}{p_j}$ and $\frac{1}{q_j}$ normalize the effective perturbation contributed by both $\boldsymbol{v}$ and $\boldsymbol{w}$ such that the norm of the aggregated values falls within the budget of uncertainty. Therefore, considered sampling error of the samples, we present the Sample-based Worst-Case CVaR, its mathematical definition is as follows:

$$\mathrm{WSCVaR}_\beta(\boldsymbol{x}) = \sup_{(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_T)\in S_\Omega} \mathrm{CVaR}_\beta(\boldsymbol{x}),) \tag{3.3}$$

where

$$S_\Omega = \Big\{\boldsymbol{r}_t : \boldsymbol{r}_t = \boldsymbol{r}_t^0 + \Delta\boldsymbol{r}_t\boldsymbol{z}_t, \boldsymbol{z}_t \in C_t)\Big\}, \tag{3.4}$$

$$C_t = \Big\{\boldsymbol{z}_t : \exists \boldsymbol{v},\ \boldsymbol{w} \in R_+^N, \boldsymbol{z}_t = \boldsymbol{v}_t - \boldsymbol{w}_t, \|\mathbf{P}_t^{-1}\mathbf{v}_t + \mathbf{Q}_t^{-1}\mathbf{w}_t\| \leq \Omega\Big\}. \tag{3.5}$$

Next, we prove the WSCVaR 3.3 is a coherent risk measure.

Theorem 3.1 *If* $(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_T) \in S_\Omega$, *then WSCVaR is a coherent risk measure.*

Proof. Letting $\rho(\boldsymbol{x}) = \mathrm{CVaR}_\beta(\boldsymbol{x}), \rho_w(\boldsymbol{x}) = \mathrm{WSCVaR}_\beta(\boldsymbol{x})$, we have

$$\rho_w(\boldsymbol{x}) = \sup_{(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_T)\in S_\Omega} \rho(\boldsymbol{x}).$$

As $\mathrm{CVaR}_\beta(\boldsymbol{x})$ is a coherent risk measure, so $\rho(\boldsymbol{x})$ satisfies four axioms of Coherent risk measure.

In what following, we prove $\rho_w(\boldsymbol{x})$ also satisfies four axioms of Coherent risk measure.

- Monotonicity: if $\boldsymbol{x} < \boldsymbol{y}$, then $\rho(\boldsymbol{x}) < \rho(\boldsymbol{y})$. Therefore

$$\rho_w(\boldsymbol{x}) = \sup_{(\boldsymbol{r}_1,\dots,\boldsymbol{r}_T)\in S_\Omega} \rho(\boldsymbol{x}) < \sup_{(\boldsymbol{r}_1,\dots,\boldsymbol{r}_T)\in S_\Omega} \rho(\boldsymbol{y}) = \rho_w(\boldsymbol{y});$$

- subadditivity: for all $\boldsymbol{x}, \boldsymbol{y}$, we have

$$\rho_w(\boldsymbol{x}+\boldsymbol{y}) = \sup_{(\boldsymbol{r}_1,\dots,\boldsymbol{r}_T)\in S_\Omega} \rho(\boldsymbol{x}+\boldsymbol{y}) \leq \sup_{(\boldsymbol{r}_1,\dots,\boldsymbol{r}_T)\in S_\Omega} [\rho(\boldsymbol{x}) + \rho(\boldsymbol{y})] = \rho_w(\boldsymbol{x}) + \rho_w(\boldsymbol{y});$$

- positive homogeneity: for any $\lambda > 0$, we have

$$\rho_w(\lambda\boldsymbol{x}) = \sup_{(\boldsymbol{r}_1,\dots,\boldsymbol{r}_T)\in S_\Omega} \rho(\lambda\boldsymbol{x}) = \lambda \sup_{(\boldsymbol{r}_1,\dots,\boldsymbol{r}_T)\in S_\Omega} \rho(\boldsymbol{x}) = \lambda\rho_w(\boldsymbol{x});$$

- translation invariance: for any constant $a \in R$, we have

$$\rho_w(\boldsymbol{x}+a) = \sup_{(\boldsymbol{r}_1,\dots,\boldsymbol{r}_T)\in S_\Omega} \rho(\boldsymbol{x}+a) = \sup_{(\boldsymbol{r}_1,\dots,\boldsymbol{r}_T)\in S_\Omega} \rho(\boldsymbol{x}) + a = \rho_w(\boldsymbol{x}) + a.$$

Therefore, the theorem is true.

Chen, Sim and Sun [19] stated the uncertainty set S_Ω is convex, and its size is determined by Ω. Therefore, S_Ω is a compact convex set. Let $f(\boldsymbol{x}, \boldsymbol{y}) = -\boldsymbol{x}^T\boldsymbol{r}$ be the loss associated with the decision vector $\boldsymbol{x}$, to be chosen from a certain subset X of R^n, and the random vector $\boldsymbol{r}$ in R^m. So, from 2.6, WSCVaR can be converted to the following form:

$$\mathrm{WSCVaR}_\beta(\boldsymbol{x}) = \max_{(\boldsymbol{r}_1,\dots,\boldsymbol{r}_T)\in S_\Omega} \min\Big\{\alpha + \frac{1}{T(1-\beta)}\sum_{t=1}^{T}\max\{-\boldsymbol{r}_t^T\boldsymbol{x} - \alpha, 0\}\Big\}. \tag{3.6}$$

Next, we will show the WSCVaR enjoys an important nature, in the process the dual-norm $\|\boldsymbol{u}\|^*$, (see Bertsimas and Sim [18]) is required. It is defined as:

$$\|\boldsymbol{u}\|^* = \max_{\{\|\boldsymbol{x}\|\leq 1\}} \boldsymbol{u}^T\boldsymbol{x}.$$

Theorem 3.2 *If $(\boldsymbol{r}_1,\dots,\boldsymbol{r}_T) \in S_\Omega$, we have*

$$WSCVaR_\beta(\boldsymbol{x}) = CVaR_\beta(\boldsymbol{x}) + \frac{\Omega}{T(1-\beta)}\sum_{t=1}^{T}\|\boldsymbol{u}_t\|^*. \tag{3.7}$$

Proof. From 3.6, we can obtain

$$\begin{aligned}
\mathrm{WSCVaR}_\beta(\boldsymbol{x}) &= \max_{(\boldsymbol{r}_1,\dots,\boldsymbol{r}_T)\in S_\Omega} \min\Big\{\alpha + \frac{1}{T(1-\beta)}\sum_{t=1}^{T}\max\{-\boldsymbol{r}_t^T\boldsymbol{x} - \alpha, 0\}\Big\} \\
&= \max_{\boldsymbol{z}_t\in C_t} \min\Big\{\alpha + \frac{1}{T(1-\beta)}\sum_{t=1}^{T}\max\{-(\boldsymbol{r}_t^0)^T\boldsymbol{x} - (\Delta\boldsymbol{r}_t\boldsymbol{z}_t)^T\boldsymbol{x} - \alpha, 0\}\Big\} \\
&= \mathrm{CVaR}_\beta(\boldsymbol{x}) + \max_{\boldsymbol{z}_t\in C_t} \max\Big\{\frac{1}{T(1-\beta)}\sum_{t=1}^{T}(\Delta\boldsymbol{r}_t\boldsymbol{z}_t)^T\boldsymbol{x}\Big\} \\
&= \mathrm{CVaR}_\beta(\boldsymbol{x}) + \frac{1}{T(1-\beta)}\sum_{t=1}^{T}\max_{\boldsymbol{z}_t\in C_t}\Big\{\boldsymbol{z}_t^T\boldsymbol{y}_t\Big\}, \boldsymbol{y}_t = \Delta\boldsymbol{r}_t^T\boldsymbol{x}.
\end{aligned}$$

Observe that

$$\begin{aligned}
& \max_{\{\boldsymbol{z}_t \in C_t\}} \boldsymbol{z}_t^T \boldsymbol{y}_t \\
= & \max_{\{\mathbf{v}_t, \mathbf{w}_t \in R_+^N : \|\mathbf{P}_t^{-1}\mathbf{v}_t + \mathbf{Q}_t^{-1}\mathbf{w}_t\| \le \Omega\}} (\boldsymbol{v}_t - \boldsymbol{w}_t)^T \boldsymbol{y}_t \\
= & \max_{\{\mathbf{v}_t, \mathbf{w}_t \in R_+^N : \|\boldsymbol{v}_t + \boldsymbol{w}_t\| \le \Omega\}} (\mathbf{P}_t \boldsymbol{y}_t)^T \boldsymbol{v}_t - (\mathbf{Q}_t \boldsymbol{y}_t)^T \boldsymbol{w}_t \\
= & \Omega \|\mathbf{u}_t\|^*
\end{aligned}$$

where $\mathbf{u}_t = \max\{\mathbf{P}_t \boldsymbol{y}_t, -\mathbf{Q}_t \boldsymbol{y}_t, 0\} = \max\{\mathbf{P}_t \boldsymbol{y}_t, -\mathbf{Q}_t \boldsymbol{y}_t\}$

Note: Theorem 3.7 indicates that the WSCVaR can be seen as the original CVaR plus a regular item. It is easy to know that $\mathrm{CVaR}_\beta(\boldsymbol{x}) \le \mathrm{WSCVaR}_\beta(\boldsymbol{x})$. Obvious, WSCVaR is more cautious than the original CVaR.

4. Computing WSCVaR and its application in portfolio management

By the Chen, Sim and Sun [19] Theorem 2 and Theorem 3.2, adding an auxiliary variable $h_t \in R, t = 1, 2, \ldots, T$, the WSCVaR (3.7) can be transformed into the following form

$$\begin{aligned}
\min \quad & \alpha + \frac{1}{T(1-\beta)} \sum_{t=1}^{T} d_t + + \frac{\Omega}{T(1-\beta)} \sum_{t=1}^{T} h_t, \\
s.t. \quad & \|\boldsymbol{u}_t\|^* \le h_t, t = 1, 2, \ldots T, \\
& \mathbf{u}_t \ge -\mathbf{P}_t \Delta \boldsymbol{r}_t^T \boldsymbol{x}, t = 1, 2, \ldots T, \\
& \mathbf{u}_t \ge \mathbf{Q}_t \Delta \boldsymbol{r}_t^T \boldsymbol{x}, t = 1, 2, \ldots T, \\
& d_t \ge (\boldsymbol{r}_t^0)^T \boldsymbol{x} - \alpha, \qquad t = 1, \ldots, T, \\
& \boldsymbol{d} \ge 0.
\end{aligned} \tag{4.1}$$

The complete formulation and complexity class of the robust counterpart depends on the representation of the dual norm constraint, $\|\boldsymbol{u}_t\|^* \le h_t, t = 1, 2, \ldots T$. Table 1 lists the common choices of norms, the representation of their dual norms which is come from reference [18](See page 14, Table 2).

Table 1: Representation of the dual norm for $u \ge 0$.

Norms	$\|t\|$	$\|u\|^* \le h$
l_2	$\|t\|_2$	$\|u\|_2 \le h$
l_1	$\|t\|_1$	$u_j \le h, \forall j = \{1, \ldots, N\}$
l_∞	$\|t\|_\infty$	$\sum_{j=1}^{N} u_j \le h$
$l_1 \bigcap l_\infty$	$\max\{\frac{1}{\Omega}\|t\|_1, \|t\|_\infty\}$	$\Omega\delta + \sum_{j=1}^{N} v_j \le h; v_j + \delta \ge u_j, \forall j \in N; \delta \in R_+, \boldsymbol{v} \in R_+^N$

In [18], Bertsimas and Sim discussed the nature and size of the proposed robust conic problem. In terms of keeping the model linear and simplicity in size, the l_1 norm also is an attractive choice. In this paper, we adopt l_1 norm. So under l_1 norm, the constraints $\|\boldsymbol{u}_t\|^* \leq h_t, t = 1, 2, \ldots T$ in (4.1) is equivalent to

$$u_t^j \leq h_t, \forall j = \{1, \ldots, N\}, t = 1, 2, \ldots T. \tag{4.2}$$

Hence, the resulting problem (4.2) is still a linear constraint.

For the constraint term $\mathbf{u}_t \geq -\mathbf{P}_t \Delta \boldsymbol{r}_t^T \boldsymbol{x}, t = 1, 2, \ldots T$ in (4.1), as discussed in [18], when all the data entries of the problem have independent random perturbation, we can further reduce the size of the robust model. In this article, we assume that the dimension of $\boldsymbol{x}$ and $\mathbf{u}$ is identical (n=N), that is, z_t^j in (3.4)is the independent random variable associated with the j-th data element, and $\Delta \boldsymbol{r}_j$ contains mostly zeros except at the entries corresponding to the data element, such as $\Delta \boldsymbol{r}_t^j = (0, \ldots, 0, \Delta r_t^j, 0, \ldots, 0)^T$. Then $u_t^j \geq -p_t^j (\Delta \boldsymbol{r}_t^j)^T \boldsymbol{x}$ will reduce to $u_t^j \geq -p_t^j \Delta r_t^j \cdot x^j$. Then, the constraint term $\mathbf{u}_t \geq -\mathbf{P}_t \Delta \boldsymbol{r}_t^T \boldsymbol{x}, t = 1, 2, \ldots T$ in (4.1) can be transformed into the following form

$$u_t^j \geq -p_t^j \Delta r_t^j \cdot x^j, j = 1, \ldots, n, t = 1, 2, \ldots T. \tag{4.3}$$

Based on investor preferences, portfolio optimization try to find the balance between risk and return. Therefore, the WSCVaR-based portfolio problem can be expressed as

$$\min_{\boldsymbol{x} \in \mathrm{X}} \quad \mathrm{WSCVaR}_\beta(\boldsymbol{x}),$$

where X denotes the constraint on the portfolio position, which usually includes the budget constraint, no short sales constraint, and the return requirement. Therefore, the feasible decision set of portfolios can be denoted as

$$\mathrm{X} = \{\boldsymbol{x} | \quad \boldsymbol{x}^T \mathbf{1} = 1, \quad \boldsymbol{x} \geq 0, \quad \boldsymbol{x}^T \boldsymbol{r} \geq \mu\}. \tag{4.4}$$

From (4.1) (4.2) and (4.3), adding an auxiliary variable $\theta \in R$, the AWCVaR-based robust portfolio selection problem can be written as the following linear programming problem with variables $(\boldsymbol{x}, \boldsymbol{d}, \boldsymbol{u}_t, h_t, \theta, \alpha)$

$$\begin{array}{ll} \min & \theta \\ s.t. & \alpha + \dfrac{1}{T(1-\beta)} \displaystyle\sum_{t=1}^{T} d_t + + \dfrac{\Omega}{T(1-\beta)} \displaystyle\sum_{t=1}^{T} h_t \leq \theta, \\ & u_t^j \leq h_t, \forall j = \{1, \ldots, N\}, t = 1, 2, \ldots T, \\ & u_t^j \geq -p_t^j \Delta r_t^j \cdot x^j, j = 1, \ldots, n, t = 1, 2, \ldots T, \\ & u_t^j \geq q_t^j \Delta r_t^j \cdot x^j, j = 1, \ldots, n, t = 1, 2, \ldots T, \\ & d_t \geq (\boldsymbol{r}_t^0)^T \boldsymbol{x} - \alpha, \qquad t = 1, \ldots, T, \\ & \boldsymbol{d} \geq 0, \boldsymbol{u}_t \geq 0, \\ & \boldsymbol{x}^T \mathbf{1} = 1, \quad \boldsymbol{x} \geq 0, \quad \boldsymbol{x}^T \boldsymbol{r} \geq \mu. \end{array} \tag{4.5}$$

5. Computational Experiments

We compare the performance of minimizing-portfolio WSCVaR under our approach with the initial CVaR method [6]. Firstly, we use simulated asset returns and show that our WSCVaR approach performs well for negatively-skewed returns. Secondly, we compare initial CVaR method and the robust portfolio optimization methods by employing a widely available data set of Hedge Funds returns, from http://www.hedgeindex.com.

In our numerical experiments, the methods have the following meanings:

- "CVaR" stands for the initial mean-CVaR Portfolio optimization model (2.13)[6];
- "WSCVaR" stands for the robust mean-WSCVaR Portfolio optimization model (4.5).

We utilize Matlab2012 to solve models CVaR and WSCVaR, which are linear programming problems.

5.1. Experiments with Simulated Data

Consider a portfolio of $n = 20$ assets with uncertain returns $\tilde{r}_i^t, i = 1, \ldots, n, t = 1, \ldots, T$. Each return $\tilde{r}_i^t$ is determined by a simple single factor model $\tilde{r}_i^t = \hat{r}_i^t + \tilde{z}(\omega_i^t)$, where $\hat{r}_i^t = 1$. The factors $\tilde{z}^t(\omega_i)$ are independent and distributed as follows:

$$\tilde{z}(\omega_i^t) = \begin{cases} \frac{\sqrt{\omega_i^t(1-\omega_i^t)}}{\omega_i^t}, & \text{with probability} \quad \omega_i^t, \\ -\frac{\sqrt{\omega_i^t(1-\omega_i^t)}}{1-\omega_i^t}, & \text{with probability} \quad 1-\omega_i^t. \end{cases} \tag{5.1}$$

Note that the mean and the standard deviation of $\tilde{z}(\omega_i^t)$ are the same for all $\omega_i^t \in (0, 1)$ - they are 0 and 1, respectively. However, the degree of symmetry of $\tilde{z}^t(\omega_i^t)$ can be different. Higher values for $\omega_i^t(e.g., \omega_i^t = 0.9)$ result in larger negative skew. We generate values for ω_i^t as follows:

$$\omega_i^t = \frac{1}{2}\Big(1 + \frac{i}{N+t}\Big), i = 1, \ldots, n, t = 1, \ldots, T. \tag{5.2}$$

Therefore, the return distributions for stocks with high index numbers in the portfolio are more negatively skewed than those for stocks with low index numbers.

We use exact values for the parameters in the CVaR and WSCVaR optimization problems. These parameters include the standard deviation and average returns for the CVaR approaches, and the backward and forward deviations for the WSCVaR approach are set to $p_j^t = 1.5, q_j^t = 2$. Δr_j^t is set to the vector of standard deviation of asset returns estimated by the T samples. We use a training set of 1,000 simulated returns from the above distributions that is $T = 1000$. The optimal portfolio allocations resulting from the five approximate CVaR optimization approaches for $\beta = 1\%$ are shown in Figure 1.

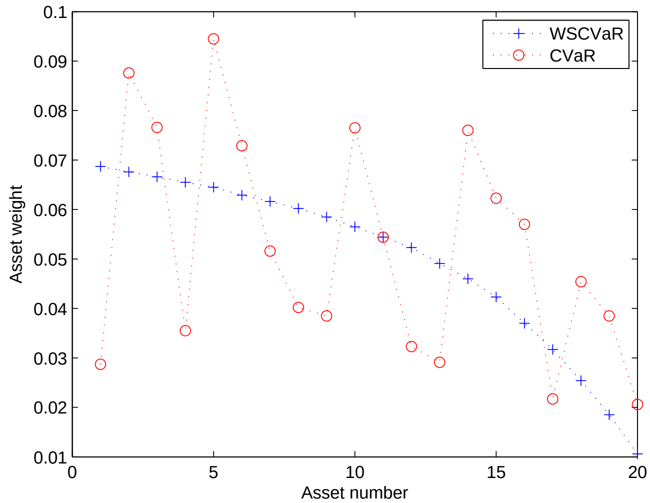

Figure 1-Optimal portfolio weights (as proportions) for assets numbered 1 through 20 resulting from different optimization formulations.

The behavior of the CVaR approach is erratic. In fact, the optimal weights for the portfolios found by the CVaR approach vary widely from sample to sample. WSCVaR is able to detect the asymmetry in the distributions, and allocates less in assets with more negatively skewed return distributions (those with high index numbers).

5.2. Experiments with Hedge Funds

We select 12 Credit Suisse/Tremont Hedge Fund Indices (listed in Table 2) as the candidates for constructing hedge fund portfolios. Monthly returns of these indices, from January 1994 to December 2012 (240 samples in total) are used as the data set, which can be freely downloaded from http://www.hedgeindex.com.

Table 2: Credit Suisse/Tremont Hedge Fund Indices

1	Convertible Arbitrage
2	Dedicated Short Bias
3	Emerging Markets
4	Equity Market Neutral
5	Event Driven
6	Distressed
7	Multi-Strategy
8	Risk Arbitrage
9	Fixed Income Arbitrage
10	Global Macro
11	Long/Short Equity
12	Managed Futures

To construct an optimal portfolio with an accuracy to certain degree, we need to generate adequate scenarios with the given 240 samples. A question we face first in scenario generation is which distribution the asset returns follow. Statistic test shows that most of the distributions of returns of these hedge fund indices are skewed and exhibit a high kurtosis. Thus, the returns should not be modeled by a normal distribution. Table 3 shows the means and standard deviations of these 12 asset returns within three different but overlapped time periods. Each of these three time periods covers 100 months. The beginning and the end dates for each time period are specified in Table 3. We find that, for most assets, there exist remarkable differences among three periods for both the mean and the standard deviation, especially for the mean. For example, the mean of asset 4 during the time period of 1/31/1994-4/30/2002 is 15 times of that during the time period of 6/30/2002-9/30/2010.

Table 3: Mean and standard deviation of asset returns within different time periods

Time	1/31/1994-4/30/2002		8/31/1997-11/30/2005		6/30/2002-9/30/2010	
Asset	Mean	Std	Mean	Std	Mean	Std
1	0.0084	0.0143	0.0066	0.0147	0.0046	0.0254
2	0.0005	0.0534	-0.0003	0.0534	-0.0036	0.0454
3	0.0061	0.0554	0.0048	0.0448	0.0091	0.0296
4	**0.0090**	0.0094	0.0076	0.0070	**0.0006**	0.0424
5	0.0094	0.0178	0.0080	0.0178	0.0072	0.0175
6	0.0110	0.0202	0.0092	0.0193	0.0070	0.0182
7	0.0086	0.0193	0.0073	0.0192	0.0074	0.0184
8	0.0078	0.0130	0.0056	0.0134	0.0041	0.0109
9	0.0057	0.0117	0.0039	0.0117	0.0029	0.0214
10	0.0117	0.0381	0.0088	0.0270	0.0087	0.0160
11	0.0107	0.0342	0.0096	0.0320	0.0062	0.0226
12	0.0038	0.0332	0.0062	0.0358	0.0075	0.0347

Since the distribution of asset returns is unknown, we adopt a distribution free method to generate scenarios given in Topaloglou et al. (2002)[20] and Zhu et al. (2013)[16].

We use back test method to check the performances of the robust approaches and the traditional approach in portfolio management, and the initial wealth is set at 1. Firstly, asset returns of the first N=162 (from 1/31/1994 to 7/31/2007) months are used to generate T=500 scenarios. Portfolio optimization models of the CVaR, and WSCVaR are then, respectively, solved to generate the traditional and the robust portfolio strategies. In month N+1, the two portfolios are constructed according to the derived strategies. At the beginning of month N+2, the scenarios are reproduced using the data from month 2 up to month N+1. The portfolio models are then re-solved, respectively, using the updated scenarios to generate new portfolio strategies for month N+1. The above procedure repeats until the end of the data set.

In this experiments, we also use exact values for the parameters in the CVaR, WSCVaR optimization problems. These parameters include the standard deviation and average returns for the CVaR, and the backward and forward deviations for the WSCVaR approach are set to

$p_t^j = 1.5, q_t^j = 2$. Δr_t^j is set to the vector of standard deviation of asset returns estimated by the $i-th$ T samples.

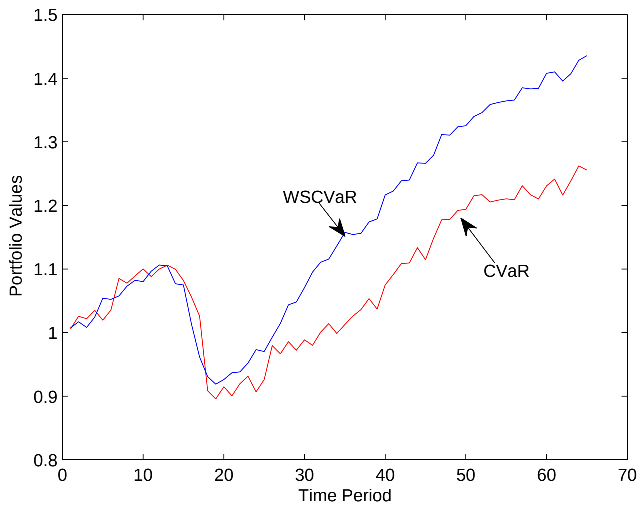

Figure 2-Portfolio Values for Out-of-Sample Observations When a Simple Buy-and-Hold Strategy is Employed

From Figure 2, we can see the optimal portfolio allocation based on the WSCVaR approach tends to result in stable returns, whereas, for example, the behavior of the optimal portfolio obtained with the CVaR approach is some erratic. In addition, the portfolio Values for generated by the WSCVaR model is better than the initial CVaR model at the end of investment period. But, during the gradually declining period from June to October, 2008, robust portfolio strategies perform better than the traditional ones in most cases.

6. Conclusion

With an asymmetric affine uncertainty set based on the factor model, which is often employed in practice for estimating the asset return distribution, we propose a computationally tractable robust optimization method for minimizing the Worse-Case CVaR of a portfolio. The remarkable characteristic of the new method is that the robust optimization model retains the complexity of original portfolio optimization problem, i.e., the robust counterpart problem is still a linear programming problem. Specially in the new method, we incorporate information about asymmetries in the distributions of uncertainties. We present some numerical experiments with simulated and real market data to illustrate the behavior of robust optimization model.

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A NOTE ON THE INTERVAL-VALUED SIMILARITY MEASURE AND THE INTERVAL-VALUED DISTANCE MEASURE INDUCED BY THE CHOQUET INTEGRAL WITH RESPECT TO AN INTERVAL-VALUED CAPACITY

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Abstract. In this paper, we introduce an interval-valued capacity which is motivated by the goal to represent reasonable capacity and to define the Choquet integral with respect to an interval-valued capacity. We also investigate some properties of the Choquet integral with respect to an interval-valued capacity on the space of fuzzy sets and discuss their applications, for examples, interval-valued similarity measure and interval-valued distance measure induced by the Choquet integral with respect to an interval-valued capacity.

1. Introduction

The theory of fuzzy sets defined by Zadeh (1965) has been researching many new approaches and theories, for examples, entropy, similarity measures, distance measures, Choquet integrals, fuzzy sets, and intuitionistic fuzzy sets which are applied to theories treating reasonability and uncertainty. Note that measuring the similarity between fuzzy sets is important in pattern recognition research and decision making.

Balopoulos-Hatzimichailidis-Papadopoulos [2], Fan-Ma-Xie [5], Hong-Lee [6], Li-Sheng [13], Liu [11], Turksen [22], Wang-Li [23], Wei-Chen [25], Xu-Xia [26], Zeng-Li [27], Zeng-Guo [28], and Zhang-Zhang-Mei [29] have studied some properties and applications of similarity measures, entropy, and distance measures on interval-valued fuzzy sets (or fuzzy set), and Choquet [3], Murofushi-Sugeno [15,16], and Narukawa-Murofushi-Sugeno [18,19] have studied the theory of fuzzy measures(or capacity) and Choquet integrals. Couso-Montes-Gil [4], Jang [12], Murofushi-Sugen0-Suzaki [17], Pedrycz-Yang-Ha [20], and Wang [24] have studied various convergence properties of the Choquet integral with respect to a capacity.

By using interval-valued functions, we have studied the Choquet integral with respect to a fuzzy measure of interval-valued functions which are able to better handle the representation of decision making and information theory (see [7-11]). Recently, we studied some convergence properties of the Choquet integral with respect to an interval-valued capacity functional (see

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[12]). Main purpose of this paper is to provide some applications of the Choquet integral with respect to an interval-valued capacity on the space of all fuzzy sets.

In section 2, we define an interval-valued similarity measure and an interval-valued distance measure, and discuss some basic properties of them. In section 3, we define an interval-valued capacity and the Choquet integral with respect to an interval-valued capacity of a fuzzy set, and discuss some properties of them. In section 4, we prove that an interval-valued mapping induced by the Choquet integral with respect to a continuous from below interval-valued capacity is an interval-valued similarity measure on the space of fuzzy sets, and discuss their applications, for examples, the interval-valued similarity measure and the interval-valued distance measure. In section 5, we discuss various convergence properties of the interval-valued distance measure induced by the Choquet integral with respect to an interval-valued capacity. In section 6, we give a brief summary results and some conclusions.

2. Choquet integrals and interval-valued similarity measures

In this section, we consider the Choquet integral with respect to a capacity and discuss their properties. Let $[0,1]$ be the unit interval in the set of real numbers and Ω be a σ-algebra on a set X.

Definition 2.1. ([14-17]) (1) A real-valued set function $\mu : \Omega \longrightarrow [0,1]$ is called a capacity if it satisfies the following properties:

(i) $\mu(\emptyset) = 0$ and $\mu(X) = 1$, and

(ii) $\mu(E_1) \leq \mu(E_2)$ whenever $E_1, E_2 \in \Omega$ and $E_1 \subset E_2$.

(2) A capacity μ is said to be continuous from below if for each increasing sequence $\{E_n\} \subset \Omega$, $\mu(\cup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mu(E_n)$.

(3) A capacity μ is said to be continuous from above if for each decreasing sequence $\{E_n\} \subset \Omega$, $\mu(\cap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mu(E_n)$.

(4) A capacity μ is said to be continuous if it is continuous from above and continuous from below.

(5) A capacity μ is said to be subadditive if $\mu(E_1 \cup E_2) \leq \mu(E_1) + \mu(E_2)$ whenever $E_1, E_2 \in \Omega$ and $E_1 \cap E_2 = \emptyset$.

We consider the Choquet integral with respect to a capacity which was introduced by Murofushi at el ([15-17]). Throughout this paper, we assume that the membership function of a fuzzy set A is a measurable function η_A from X to $[0,1]$.

Definition 2.2. ([14-17]) (1) The Choquet integral with respect to a capacity μ of a fuzzy set A is defined by

$$(C)\int A d\mu = \int_0^1 \mu_{\eta_A}(r)dr \tag{1}$$

where $\mu_{\eta_A}(r) = \mu(\{x \in X | \eta_A(x) > r\})$ for all $r \in [0,1]$ and the integral on the right-hand side is the Lebesgue integral of μ_{η_A}.

(2) A fuzzy set A is said to be μ-integrable if the Choquet integral of A on X exists.

We note that if A, B are fuzzy sets on X, then $A \leq B$ means $\eta_A(x) \leq \eta_B(x)$ for all $x \in X$ and that $\eta_{A\vee B}(x) = \eta_A(x) \vee \eta_B(x)$ and $\eta_{A\wedge B}(x) = \eta_A(x) \wedge \eta_B(x)$ for all $x \in X$.

Theorem 2.1. ([14-17]) *Let A and B be μ-integrable fuzzy sets.*

(1) *If $A \leq B$, then $(C)\int A d\mu \leq (C)\int B d\mu$.*

(2) *If $E_1, E_2 \in \Omega$ and $E_1 \subset E_2$, then $(C)\int_{E_1} A d\mu \leq (C)\int_{E_2} A d\mu$.*

(3) *If we define* $\eta_{A\vee B} = \eta_A(x) \vee \eta_B(x)$ *and* $\eta_{A\wedge B}(x) = \eta_A(x) \wedge \eta_B(x)$ *for all* $x \in X$*, then*

$$(C)\int A \vee B d\mu \geq (C)\int A d\mu \vee (C)\int B d\mu,$$

and

$$(C)\int A \wedge B d\mu \leq (C)\int A d\mu \wedge (C)\int B d\mu.$$

Let $[[0,1]]$ is the set of all closed intervals in $[0,1]$ as follows:

$$[[0,1]] = \{\bar{a} = [a^-, a^+] | a^-, a^+ \in [0,1] \text{ and } a^- \leq a^+\}.$$

For any $a \in [0,1]$, we define $a = [a,a]$. Obviously, $a \in [[0,1]]$(see [7-13, 21-223, 25, 27-29]).

Definition 2.3. Let I be an index set. If $\bar{a} = [a^-, a^+], \bar{b} = [b^-, b^+], \bar{a}_n = [a_n^-, a_n^+] \in [[0,1]]$ for all $n \in \mathbb{N}$ and $k \in [0,1]$, then we define arithmetic, minimum, maximum, order, and inclusion operations as follows:
(1) $k\bar{a} = [ka^-, ka^+]$,
(2) $\bar{a}\bar{b} = [a^-b^-, a^+b^+]$,
(3) $\bar{a} \wedge \bar{b} = [a^- \wedge b^-, a^+ \wedge b^+]$,
(4) $\bar{a} \vee \bar{b} = [a^- \vee b^-, a^+ \vee b^+]$,
(5) $\bar{a} \leq \bar{b}$ if and only if $a^- \leq b^-$ and $a^+ \leq b^+$,
(6) $\bar{a} < \bar{b}$ if and only if $\bar{a} \leq \bar{b}$ and $\bar{a} \neq \bar{b}$,
(7) $\bar{a} \subset \bar{b}$ if and only if $b^- \leq a^-$ and $a^+ \leq b^+$,
(8) $1 - \bar{a} = [1 - a^+, 1 - a^-]$,
(9) $\sup_{n\in I} \bar{a}_n = [\sup_{n\in I} a_n^-, \sup_{n\in I} a_n^+]$, and
(10) $\inf_{n\in I} \bar{a}_n = [\inf_{n\in I} a_n^-, \inf_{n\in I} a_n^+]$.

Theorem 2.2. *For* $\bar{a}, \bar{b}, \bar{c} \in [[0,1]]$*, we have*
(1) *idempotent law:* $\bar{a} \wedge \bar{a} = \bar{a}$ *and* $\bar{a} \vee \bar{a} = \bar{a}$*,*
(2) *commutative law:* $\bar{a} \wedge \bar{b} = \bar{b} \wedge \bar{a}$ *and* $\bar{a} \vee \bar{b} = \bar{b} \vee \bar{a}$*,*
(3) *associative law:* $(\bar{a} \wedge \bar{b}) \wedge \bar{c} = \bar{a} \wedge (\bar{b} \wedge \bar{c})$ *and* $(\bar{a} \vee \bar{b}) \vee \bar{c} = \bar{a} \vee (\bar{b} \vee \bar{c})$*,*
(4) *absorptive law:* $\bar{a} \wedge (\bar{a} \vee \bar{b}) = \bar{a} \vee (\bar{a} \wedge \bar{b}) = \bar{a}$*, and*
(5) *distributive law:* $\bar{a} \wedge (\bar{b} \vee \bar{c}) = (\bar{a} \wedge \bar{b}) \vee (\bar{a} \wedge \bar{c})$ *and* $\bar{a} \vee (\bar{b} \wedge \bar{c}) = (\bar{a} \vee \bar{b}) \wedge (\bar{a} \vee \bar{c})$*.*

Let $\mathfrak{F}(X)$ be the family of all fuzzy sets A of X with the membership measurable function $\eta_A : X \to [0,1]$. Recall that for $A, B \in \mathfrak{F}(X)$, $A \equiv B$ means $\mu(\{x \in X | \eta_A(x) \neq \eta_B(x)\}) = 0$, where μ is a capacity on X. We introduce the definitions of similarity measures and distance measures on $\mathfrak{F}(X)$, and some characterizations of them(see [2,5,6,14,26-29]).

Definition 2.4. (1) A real-valued function $s : \mathfrak{F}(X) \times \mathfrak{F}(X) \longrightarrow [0,1]$ is called a similarity measure if it satisfies the following properties:
(i) $s(A, A^c) = 0$ if A is a crisp set,
(ii) for $A, B \in \mathfrak{F}(X)$, $s(A,B) = 1$ if and only if $A \equiv B$,
(iii) for $A, B \in \mathfrak{F}(X)$, $s(A,B) = s(B,A)$, and
(iv) if $A, B, C \in \mathfrak{F}(X)$ and $A \leq B \leq C$, then $s(A,C) \leq s(A,B)$ and $s(A,C) \leq s(B,C)$.
(2) A real-valued function $d : \mathfrak{F}(X) \times \mathfrak{F}(X) \longrightarrow [0,1]$ is called a distance measure if it satisfies the following properties:
(i) $d(A, A^c) = 1$ if A is a crisp set,
(ii) for $A, B \in \mathfrak{F}(X)$, $d(A,B) = 0$ if and only if $A \equiv B$,
(iii) for $A, B \in \mathfrak{F}(X)$, $d(A,B) = d(B,A)$, and
(iv) if $A, B, C \in \mathfrak{F}(X)$ and $A \leq B \leq C$, then $d(A,C) \geq d(A,B)$ and $d(A,C) \geq d(B,C)$.

It is easy to see that if s is a similarity measure and we define $l_1 = 1 - s$, then l_1 is a distance measure and that if d is a distance measure and we define $l_2 = 1 - d$, then l_2 is a similarity measure.

Definition 2.5. (1) An interval-valued function $S = [s^-, s^+] : \mathfrak{F}(X) \times \mathfrak{F}(X) \longrightarrow [[0, 1]]$ is called an interval-valued similarity measure if it satisfies the following properties:
(i) $S(A, A^c) = 0$ if A is a crisp set,
(ii) for $A, B \in \mathfrak{F}(X)$, $S(A, B) = 1$ if and only if $A \equiv B$,
(iii) for $A, B \in \mathfrak{F}(X)$, $S(A, B) = S(B, A)$, and
(iv) if $A, B, C \in \mathfrak{F}(X)$ and $A \leq B \leq C$, then $S(A, C) \leq S(A, B)$ and $S(A, C) \leq S(B, C)$.
(2) An interval-valued function $D = [d^-, d^+] : \mathfrak{F}(X) \times \mathfrak{F}(X) \longrightarrow [[0, 1]]$ is called a distance measure if it satisfies the following properties:
(i) $D(A, A^c) = 1$ if A is a crisp set,
(ii) for $A, B \in \mathfrak{F}(X)$, $D(A, B) = 0$ if and only if $A \equiv B$,
(iii) for $A, B \in \mathfrak{F}(X)$, $D(A, B) = D(B, A)$, and
(iv) if $A, B, C \in \mathfrak{F}(X)$ and $A \leq B \leq C$, then $D(A, C) \geq D(A, B)$ and $D(A, C) \geq D(B, C)$.

By the definitions of an interval-valued similarity measure and an interval-valued distance measure, we can obtain the following theorem.

Theorem 2.3. (1) *An interval-valued function $S = [s^-, s^+]$ is an interval-valued similarity measure if and only if real-valued functions s^- and s^+ are real-valued similarity measures, and $0 \leq s^- \leq s^+ \leq 1$.*

(2) *An interval-valued function $D = [d^-, d^+]$ is an interval-valued distance measure if and only if real-valued functions d^- and d^+ are real-valued distance measures, and $0 \leq d^- \leq d^+ \leq 1$.*

(3) *If S is an interval-valued similarity measure and we define $H = 1 - S = [1 - s^+, 1 - s^-]$, then H is an interval-valued distance measure.*

(4) *If D is an interval-valued distance measure and we define $L = 1 - D = [1 - d^+, 1 - d^-]$, then L is an interval-valued similarity measure.*

Proof. (1) ($\Longrightarrow$) Suppose that S is an interval-valued similarity measure. If A is a crisp set, then

$$0 = S(A, A^c) = [s^-(A, A^c), s^+(A, A^c)].$$

Thus $s^-(A, A^c) = 0$ and $s^+(A, A^c) = 0$. Since $S(A, B) = S(B, A)$ for all $A, B \in \mathfrak{F}(X)$,

$$s^-(A, B) = s^-(B, A) \text{ and } s^+(A, B) = S^+(B, A).$$

Let $A, B, C \in \mathfrak{F}(X)$ and $A \leq B \leq C$. Then we have

$$S(A, C) \leq S(A, B) \text{ and } S(A, C) \leq S(B, C).$$

Thus, we have

$$s^-(A, C) \leq s^-(A, B) \text{ and } s^-(A, C) \leq s^-(B, C),$$

and

$$s^+(A, C) \leq s^+(A, B) \text{ and } s^+(A, C) \leq s^+(B, C),$$

Therefore, we obtain that s^- and s^+ are real-valued similarity measures and $0 \leq s^- \leq s^+ \leq 1$.

($\Longleftarrow$) The proof is similar to the proof of ($\Longrightarrow$).

(2) The proof is similar to the proof of (1).

(3) Let S be an interval-valued similarity measure and we define $H = 1-S = [1-s^+, 1-s^-]$. If A is a crisp set, then $S(A, A^c) = 0$. Thus, $H(A, A^c) = 1 - S(A, A^c) = 1 - 1 = 0$. Let $A, B \in \mathfrak{F}(X)$. Then, $A \equiv B$ if and only if $S(A, B) = 1$, that is, $H(A, B) = 1 - S(A, B) = 0$. If $A, B \in \mathfrak{F}(X)$, then $S(A, B) = S(B, A)$. Then,

$$H(A, B) = 1 - S(A, B) = 1 - S(B, A) = H(B, A).$$

If $A, B, C \in \mathfrak{F}(X)$ and $A \leq B \leq C$, then

$$S(A,C) \leq S(A,B) \text{ and } S(A,C) \leq S(B,C).$$

Thus, we have

$$H(A,C) = 1 - S(A,C) \geq 1 - S(A,B) = H(A,B)$$

and

$$H(A,C) = 1 - S(A,C) \geq 1 - S(B,C) = H(B,C).$$

Therefore, H is an interval-valued distance measure.

(4) The proof is similar to the proof of (3).

3. The Choquet integral with respect to an interval-valued capacity

In this section, we define an interval-valued capacity and the Choquet integral with respect to an interval-valued capacity of a fuzzy set. Note that a mapping $d_H : [[0,1]] \times [[0,1]] \longrightarrow [0,\infty)$ is the Hausdorff metric defined by

$$d_H(A,B) = \max\left\{\sup_{x\in A}\inf_{y\in B}|x-y|, \sup_{y\in B}\inf_{x\in A}|x-y|\right\} \tag{2}$$

for all $A, B \in [[0,1]]$, and $([[0,1]], d_H)$ is a metric space. By the definition of the Hausdorff metric, it is easy to see that for any $\bar{a} = [a^-, a^+], \bar{b} = [b^-, b^+] \in [[0,1]]$, we have

$$d_H(\bar{a}, \bar{b}) = \max\left\{|a^- - b^-|, |a^+ - b^+|\right\}. \tag{3}$$

We recall that for any $\{\bar{a}_n\} \subset [[0,1]]$ and $\bar{a} \in [[0,1]]$,

$$d_H - \lim_{n\to\infty} \bar{a}_n = \bar{a} \text{ means } \lim_{n\to\infty} d_H(\bar{a}_n, \bar{a}) = 0. \tag{4}$$

We define an interval-valued capacity $\bar{\mu} = [\mu^-, \mu^+] : \Omega \longrightarrow [[0,1]]$ on a measurable space (X, Ω) as follows:

Definition 3.1. (1) An interval-valued set function $\bar{\mu} : \Omega \longrightarrow [[0,1]]$ is called an interval-valued capacity if it satisfies the following properties:

(i) $\bar{\mu}(\emptyset) = 0$ and $\bar{\mu}(X) = 1$, and

(ii) $\bar{\mu}(E_1) \leq \bar{\mu}(E_2)$ whenever $E_1, E_2 \in \Omega$ and $E_1 \subset E_2$.

(2) An interval-valued capacity $\bar{\mu}$ is said to be continuous from above if for each increasing sequence $\{E_n\} \subset \Omega$, $\bar{\mu}(\cup_{n=1}^{\infty} E_n) = d_H - \lim_{n\to\infty} \bar{\mu}(E_n)$.

(3) An interval-valued capacity $\bar{\mu}$ is said to be continuous from below if for each decreasing sequence $\{E_n\} \subset \Omega$, $\bar{\mu}(\cap_{n=1}^{\infty} E_n) = d_H - \lim_{n\to\infty} \bar{\mu}(E_n)$.

(4) An interval-valued capacity $\bar{\mu}$ is said to be continuous if it is continuous from above and continuous from below.

(5) An interval-valued capacity $\bar{\mu}$ is said to be subadditive if $\bar{\mu}(E_1 \cup E_2) \leq \bar{\mu}(E_1) + \bar{\mu}(E_2)$, whenever $E_1, E_2 \in \Omega$ and $E_1 \cap E_2 = \emptyset$.

It is easy to see that for each increasing sequence $\{E_n\} \subset \Omega$ with $E = \cup_{n=1}^{\infty} E_n$,

$$\lim_{n\to\infty} d_H(\bar{\mu}(E_n), \bar{\mu}(E)) = 0 \text{ if and only if } \lim_{n\to\infty} \mu^-(E_n) = \mu^-(E) \text{ and } \lim_{n\to\infty} \mu^+(E_n) = \mu^+(E), \tag{5}$$

and for each decreasing sequence $\{E_n\} \subset \Omega$ with $F = \cap_{n=1}^{\infty} E_n$,

$$\lim_{n\to\infty} d_H(\bar{\mu}(E_n), \bar{\mu}(F)) = 0 \text{ if and only if } \lim_{n\to\infty} \mu^-(E_n) = \mu^-(F) \text{ and } \lim_{n\to\infty} \mu^+(E_n) = \mu^+(F). \tag{6}$$

By (5) and (6), we can directly derive the following theorem.

Theorem 3.1. (1) *An interval-valued set function $\bar{\mu} = [\mu^-, \mu^+] : \Omega \longrightarrow [[0,1]]$ is an interval-valued capacity if and only if μ^- and μ^+ are capacities and $\mu^- \leq \mu^+$.*

(2) *An interval-valued capacity $\bar{\mu} = [\mu^-, \mu^+]$ is continuous from below if and only if μ^- and μ^+ are continuous from below and $\mu^- \leq \mu^+$.*

(3) *An interval-valued capacity $\bar{\mu} = [\mu^-, \mu^+]$ is continuous from above if and only if μ^- and μ^+ are continuous from above and $\mu^- \leq \mu^+$.*

(4) *An interval-valued capacity $\bar{\mu} = [\mu^-, \mu^+]$ is continuous if and only if μ^- and μ^+ are continuous and $\mu^- \leq \mu^+$.*

(5) *An interval-valued capacity $\bar{\mu} = [\mu^-, \mu^+]$ is subadditive if and only if μ^- and μ^+ are subadditive and $\mu^- \leq \mu^+$.*

Recall that if $([0,1], \mathfrak{M}, m)$ is the Lebesgue measure space and $C([0,1])$ is the family of all closed subsets of I, then the Aumann integral of a closed set-valued function $G : [0,1] \longrightarrow C([0,1])$ is defined by

$$(A)\int Gdm = \left\{\int gdm |\ g \in S(G)\right\}, \tag{7}$$

where $S(G)$ is the set of all integrable selections of G, that is,

$$S(G) = \{g : [0,1] \longrightarrow [0,1] | \int gdm < \infty \text{ and } g(r) \in G(r)\ m - a.e.\}. \tag{8}$$

We note that $m - a.e.$ means almost everywhere in the Lebesgue measure m (see[1,16]). Then, we introduce the following theorems which are used to define the Choquet integral with respect to an interval-valued capacity of a fuzzy set.

Theorem 3.2. *(*[13, Lemma 2.1]*) If a closed set-valued function $G : [0,1] \longrightarrow C([0,1])$ is $\mathfrak{M}$-measurable, then $(A)\int Gdm$ is convex in $[0,1]$.*

Theorem 3.3. *(*[13, Lemma 2.2]*) If a closed set-valued function $G : [0,1] \longrightarrow C([0,1])$ is $\mathfrak{M}$-measurable and integrably bounded, that is, there exists a integrable function $\varphi : [0,1] \longrightarrow [0,1]$ such that*

$$\sup_{x \in G(r)} x \leq \varphi(r) \qquad \text{for } r \in [0,1], \tag{9}$$

then $(A)\int Gdm$ is nonempty compact convex in $[0,1]$.

From Theorem 3.3, we can see that $(A)\int Gdm$ is a nonempty bounded and closed subset in $[0,1]$ under the same assumption of G. Thus, we obtain the following corollary (see [12,13,21]).

Corollary 3.4. *If an interval-valued function $G = [g^-, g^+] : I \longrightarrow [[0,1]]$ is $\mathfrak{M}$-measurable and integrably bounded, then $g^-, g^+ \in S(F)$ and*

$$(A)\int Gdm = \left[\int g^- dm, \int g^+ dm\right], \tag{10}$$

where the integrals on the right-hand side are the Lebesgue integral with respect to m.

We write $\int gdm = \int_0^1 g(r)dm(r)$ for all measurable functions g. By using an interval-valued capacity, we define the Choquet integral with respect to an interval-valued capacity of a fuzzy set A.

Definition 3.2. (1) The Choquet integral with respect to an interval-valued capacity $\bar{\mu}$ of a fuzzy set $A \in \mathfrak{F}$ is defined by

$$(C)\int Ad\bar{\mu} = (A)\int_0^1 \bar{\mu}_A(r)dr, \tag{11}$$

where η_A is the membership measurable function of A, $\bar{\mu}_A(r) = \bar{\mu}(\{x \in X | \eta_A(x) > r\})$ for all $r \in [0, 1]$, and the integral on the right-hand side is the Aumann integral in (7).

(2) A fuzzy set $A \in \mathfrak{F}$ is said to be $\bar{\mu}$-integrable if $(C) \int A d\bar{\mu} \in [[0, 1]]$.

Note that if an interval-valued capacity $\bar{\mu}$ is continuous from below and $A \in \mathfrak{F}(X)$, then $\bar{\mu}_A : I \longrightarrow [[0, 1]]$ is continuous from below on $[0, 1]$. Thus, we obtain that $\bar{\mu}_A$ is $\mathfrak{M}$-measurable and integrably bounded on $[0, 1]$. Thus, by Definition 3.2 and Corollary 3.4, we can easily obtain the following theorem.

Theorem 3.5. *If an interval-valued capacity $\bar{\mu}$ is continuous from below and $A \in \mathfrak{F}$, then we have*

$$(C) \int A d\bar{\mu} = \left[(C) \int A d\mu^-, (C) \int A d\mu^+ \right], \tag{12}$$

where the integrals on the right-hand side are Choquet integrals.

Proof. By Definition 3.2 and Corollary 3.4, we can derive

$$\begin{aligned} (C) \int A d\bar{\mu} &= (A) \int_0^1 \bar{\mu}_A(r) dr \\ &= (A) \int_0^1 [\mu_A^-(r), \mu_A^+(r)] dr \\ &= \left[\int_0^1 \mu_A^-(r) dr, \int_0^1 \mu_A^+(r) dr \right] \\ &= \left[(C) \int A d\mu^-, (C) \int A d\mu^+ \right]. \end{aligned}$$

By Theorem 3.5, we can easily obtain the following basic properties of the Choquet integrals with respect to a continuous from below interval-valued capacity of a fuzzy set.

Theorem 3.6. *Let (X, Ω) be a measurable space. Assume that an interval-valued $\bar{\mu}$ is continuous from below.*

(1) *If $A, B \in \mathfrak{F}(X)$ and $A \leq B$, then*

$$(C) \int A d\bar{\mu} \leq (C) \int B d\bar{\mu}.$$

(2) *If $A, B \in \mathfrak{F}(X)$ and we define $\eta_{(A \vee B)}(x) = \eta_A(x) \vee \eta_B(x)$ for all $x \in X$, then*

$$(C) \int A \vee B d\bar{\mu} \geq (C) \int A d\bar{\mu} \vee (C) \int A d\bar{\mu}.$$

(3) *If $A, B \in \mathfrak{F}(X)$ and we define $\eta_{(A \wedge B)}(x) = \eta_A(x) \wedge \eta_B(x)$ for all $x \in X$, then*

$$(C) \int A \wedge B d\bar{\mu} \leq (C) \int A d\bar{\mu} \wedge (C) \int A d\bar{\mu}.$$

4. Interval-valued similarity measures induced by the Choquet integral

In this section, we discuss some applications of the Choquet integral with respect to a continuous from below interval-valued capacity of a fuzzy set.

Theorem 4.1. *Assume that an interval-valued $\bar{\mu}$ is continuous from below and $\bar{\mu}(X) = \bar{\{\mu\}}(X) = 1$. If we define an interval-valued function $S_{\bar{\mu}} : \mathfrak{F} \times \mathfrak{F} \longrightarrow [[0, 1]]$ as following*

$$S_{\bar{\mu}}(A, B) = 1 - (C) \int |\eta_A - \eta_B| d\bar{\mu} \tag{13}$$

for all $A, B \in \mathfrak{F}(X)$, then $S_{\bar{\mu}}$ is an interval-valued similarity measure.

Proof. (i) If A is a crisp measurable set, then the membership measurable function η_A of a fuzzy set A is defined by

$$\eta_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in A^c = I \setminus A. \end{cases}$$

We note that if the membership measurable function η_{A^c} of the complement of a fuzzy set A, then

$$\eta_{A^c}(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in A^c = I \setminus A. \end{cases}$$

Thus, we have $|\eta_A(x) - \eta_{A^c}(x)| = 1$ for all $x \in X$. Therefore, we have

$$\begin{aligned} S_{\bar{\mu}}(A, A^c) &= 1 - (C)\int |\eta_A - \eta_{A^c}| d\bar{\mu} \\ &= 1 - \int_0^1 \bar{\mu}(\{x \in X|\ |\eta_A(x) - \eta_{A^c}(x)| > r\})dr \\ &= 1 - \int_0^1 \bar{\mu}(X)dr = 0. \end{aligned}$$

(ii) If $A \equiv B$, then $\eta_A = \eta_B\ \bar{\mu} - a.e.$ on X. Thus, we have

$$\begin{aligned} S_{\bar{\mu}}(A, B) &= 1 - (C)\int |\eta_A - \eta_B| d\bar{\mu} \\ &= 1 - \int_0^1 \bar{\mu}(\{x \in X|\ |\eta_A(x) - \eta_B(x)| > r\})dr \\ &= 1 - \int_0^1 \bar{\mu}(\emptyset)dr = 1. \end{aligned}$$

If $S_{\bar{\mu}}(A, B) = 1$, then

$$\int_0^1 \bar{\mu}(\{x \in X|\ |\eta_A(x) - \eta_B(x)| > r\})dr = 0.$$

Then, it is easy to see that

$$\bar{\mu}(\{x \in X|\ |\eta_A(x) - \eta_B(x)| > r\}) = 0\ m - a.e. \text{ on } I. \tag{14}$$

From (14), we have

$$\bar{\mu}(\{x \in X|\ |\eta_A(x) - \eta_B(x)| \neq 0\}) = 0,$$

that is, $\eta_A = \eta_B\ \bar{\mu} - a.e.$ on X and hence $A \equiv B$.

(iii) If $A, B \in \mathfrak{F}(X)$, then we have

$$\begin{aligned} S_{\bar{\mu}}(A, B) &= 1 - (C)\int |\eta_A - \eta_B| d\bar{\mu} \\ &= 1 - (C)\int |\eta_B - \eta_A| d\bar{\mu} = S_{\bar{\mu}}(B, A). \end{aligned}$$

(iv) If $A, B, C \in \mathfrak{F}(X)$ and $A \leq B \leq C$, then $\eta_A \leq \eta_B \leq \eta_C$. Thus, we have

$$|\eta_A(x) - \eta_B(x)| \leq |\eta_A(x) - \eta_C(x)| \text{ and } |\eta_B(x) - \eta_C(x)| \leq |\eta_A(x) - \eta_C(x)|, \tag{15}$$

for all $x \in X$. By (15) and Theorem 2.2 (1), we have

$$\begin{aligned} S_{\bar{\mu}}(A, C) &= 1 - (C)\int |\eta_A - \eta_C| d\bar{\mu} \\ &\leq 1 - (C)\int |\eta_A - \eta_B| d\bar{\mu} = S_{\bar{\mu}}(A, B), \end{aligned}$$

and

$$\begin{aligned} S_{\bar{\mu}}(A,C) &= 1-(C)\int|\eta_A-\eta_C|d\bar{\mu} \\ &\leq 1-(C)\int|\eta_B-\eta_C|d\bar{\mu} = S_{\bar{\mu}}(B,C), \end{aligned}$$

By (i),(ii),(iii), and (iv), we see that $S_{\bar{\mu}}$ is an interval-valued similarity measure.

By Theorem 4.1 and Theorem 2.3(3), we can easily obtain the following corollary.

Corollary 4.2. *Assume that an interval-valued $\bar{\mu}$ is continuous from below and $\bar{\mu}(X) = \bar{\{}\mu\}(X) = 1$. If we define an interval-valued function $D_{\bar{\mu}} = 1 - S_{\bar{\mu}} = (C)\int|\eta_A - \eta_B|d\bar{\mu}$ for all $A, B \in \mathfrak{F}(X)$, then $D_{\bar{\mu}}$ is an interval-valued distance measure.*

In order to illustrate the proposed similarity measure are reasonable, we give the following example.

Example 4.1. Let $X = \{x_1, x_2, x_3\}$ and $\Omega = \wp(X)$ be the power set of X. Suppose that $\bar{\mu} : \Omega \longrightarrow [[0,1]]$ is defined by

$$\bar{\mu}(E) = [\mu^-(E), \mu^+(E)], \tag{16}$$

where $m(E)$ is the cardinality of $E \in \Omega$, $\mu^-(E) = \left(\frac{m(E)}{m(X)}\right)^2$, and $\mu^+(E) = \frac{m(E)}{m(X)}$. Since X is a finite set, clearly, we see that $\bar{\mu}$ is a continuous from below interval-valued capacity on a measurable space (X, Ω) and $\bar{\mu}(X) = \bar{\{}\mu\}(X) = 1.$. The three patterns are denoted as follows:

$$A_1 = \{(x_1, 0.3), (x_2, 0.2), (x_3, 0.1)\},$$
$$A_2 = \{(x_1, 0.2), (x_2, 0.2), (x_3, 0.2)\}, \text{ and}$$
$$A_3 = \{(x_1, 0.4), (x_2, 0.4), (x_3, 0.4)\}.$$

Assume that a sample $B = \{(x_1, 0.3), (x_2, 0.2), (x_3, 0.1)\}$ is given. In order to interpret the measure of similarity of B with these patterns, we calculate the proposed interval-valued similarity measure $S_{\bar{\mu}}$ as follows:

$$S_{\bar{\mu}}(A_1, B) = 1 - \sum_{i=1}^{3}(|\eta_{A_1}(x_{(i)}) - \eta_B(x_{(i)})|)(\bar{\mu}(A_{(i)}) = 1, \tag{17}$$

$$S_{\bar{\mu}}(A_2, B) = 1 - \sum_{i=1}^{3}(|\eta_{A_2}(x_{(i)}) - \eta_B(x_{(i)})|)(\bar{\mu}(A_{(i)}) = \left[\frac{14}{15}, \frac{43}{45}\right], \text{ and} \tag{18}$$

$$S_{\bar{\mu}}(A_3, B) = 1 - \sum_{i=1}^{3}(|\eta_{A_3}(x_{(i)}) - \eta_B(x_{(i)})|)(\bar{\mu}(A_{(i)}) = \left[\frac{4}{5}, \frac{38}{45}\right]. \tag{19}$$

By (17), (18), and (19), we interpret that B is equal(or, absolutely similar) to A_1 and B is more similar to A_2 than similar to A_3.

Example 4.2. Let $X = \{x_1, x_2, x_3\}$ and $\Omega = \wp(X)$ be the power set of X. Suppose that $\bar{\nu} : \Omega \longrightarrow [I]$ is defined by

$$\bar{\nu}(E) = [\nu^-(E), \nu^+(E)], \tag{20}$$

where $m(E)$ is the cardinality of $E \in \Omega$, $\nu^-(E) = \left(\frac{m(E)}{m(X)}\right)^3$, and $\nu^+(E) = \left(\frac{m(E)}{m(X)}\right)^2$.

The three patterns are denoted as follows:

$$A_1 = \{(x_1, 0.3), (x_2, 0.2), (x_3, 0.1)\},$$
$$A_2 = \{(x_1, 0.2), (x_2, 0.2), (x_3, 0.2)\}, \text{ and}$$
$$A_3 = \{(x_1, 0.4), (x_2, 0.4), (x_3, 0.4)\}.$$

Assume that a sample $B = \{(x_1, 0.3), (x_2, 0.2), (x_3, 0.1)\}$ is given. In order to interpret the measure of similarity of B with these patterns, we calculate the proposed interval-valued similarity measure $S_{\bar{\nu}}$ as follows:

$$S_{\bar{\nu}}(A_1, B) = 1,\ S_{\bar{\nu}}(A_2, B) = \left[\frac{43}{45}, \frac{131}{135}\right],\ \text{and}\ S_{\bar{\nu}}(A_3, B) = \left[\frac{38}{45}, \frac{13}{15}\right]. \tag{21}$$

Thus, we can see that there is an interpretation of the notions of these patterns under two different interval-valued capacity $\bar{\mu}$ and $\bar{\nu}$ as follows:

$$S_{\bar{\mu}}(A_1, B) = 1 = S_{\bar{\nu}}(A_1, B),$$

$$S_{\bar{\mu}}(A_2, B) = \left[\frac{14}{15}, \frac{43}{45}\right] < \left[\frac{43}{45}, \frac{131}{135}\right] = S_{\bar{\nu}}(A_2, B),\ \text{and}$$

$$S_{\bar{\mu}}(A_3, B) = \left[\frac{4}{5}, \frac{38}{45}\right] < \left[\frac{38}{45}, \frac{13}{15}\right] = S_{\bar{\nu}}(A_3, B).$$

Therefore, this means that $\bar{\nu}$ has more positive sense than $\bar{\mu}$.

5. Convergence in the interval-valued distance measure

Throughout this section, we assume that $\bar{\mu} = [\mu^-, \mu^+]$ is continuous from below. At first, we introduce uniformly μ-integrability and convergence in the interval-valued distance measure on $\mathfrak{F}(X)$.

Definition 5.1. ([26]) Let μ be a capacity on a measurable space (X, Ω), $\{A_n\}$ be a sequence of fuzzy sets and A be a fuzzy set.

(1) A sequence $\{A_n\}$ converges to A almost everywhere on X if there exist a null set $N \in \Omega$ with $\mu(N) = 0$ such that

$$\eta_A(x) = \lim_{n\to\infty} \eta_{A_n}(x), \quad \text{for all } x \in N^c. \tag{22}$$

(2) A sequence $\{A_n\}$ converges in the distance measure d_μ to A if

$$\lim_{n\to\infty} d_\mu(\eta_{A_n}, \eta_A) = 0, \tag{23}$$

where $d_\mu(\eta_{A_n}, \eta_A) = (C)\int |\eta_{A_n}(x) - \eta_A(x)| d\mu$ for all $n \in \mathbb{N}$.

Remark that convergence in the distance measure $d\mu$ is equal to convergence in μ-mean(see [4])

Definition 5.2. ([4]) Let μ be a capacity on a measurable space (X, Ω) and $I \subset \mathbb{N}$ be an index set. A class $\{A_n\}_{n\in I}$ of fuzzy sets is said to be uniform μ-integrable if

(i) $\sup_{n\in I} d_\mu(A_n, 0) < \infty$, (24)

(ii) $\forall\varepsilon > 0,\ \exists\delta(\varepsilon) > 0$ such that $d_{E,\mu}(A_n, 0) < \varepsilon$ if $E \in \Omega$ and $\mu(E) < \delta(\varepsilon)$, (25)

where $d_{E,\mu}(A_n, 0) = (C)\int_E |\eta_{A_n}| d\mu$ for all $n \in \mathbb{N}$.

We also introduce various convergence properties of the Choquet integral on $\mathfrak{F}(X)$ as follows:

Theorem 5.1. ([4]) *Let a capacity μ be subadditive and $\{A_n\}$ a sequence of fuzzy sets in $\mathfrak{F}(X)$. Then $\{A_n\}$ is an uniformly μ-integrable if and only if*

$$\lim_{a\to\infty} \sup_{n\in\mathbb{N}} d_{[|\eta_{A_n}|>a],\mu}(A_n, 0) = 0. \tag{26}$$

Theorem 5.2. ([4]) *Let a capacity μ be subadditive and a sequence $\{A_n\}$ of fuzzy sets in $\mathfrak{F}(X)$ converges to a fuzzy set A in $\mathfrak{F}(X)$ μ-almost everywhere on X and $A_n \leq B$ for some μ-integrable fuzzy set B, then we have*

(1) *A_n and A are μ-integrable for all $n \in \mathbb{N}$, and*

(2) *$\{A_n\}$ converges to A in the distance measure d_μ, that is,*

$$\lim_{n\to\infty} d_\mu(A_n, 0) = 0. \tag{27}$$

We assume that an interval-valued capacity $\bar{\mu}(X) = [\mu^-, \mu^+]$ is continuous from below. Then we define convergence in the interval-valued distance measure $D_{\bar{\mu}}$ and uniform $\bar{\mu}$-integrability on $\mathfrak{f}(X)$. It is easy to see that

$$D_{\bar{\mu}}(A, B) = [d_{\mu^-}(A, B), d_{\mu^+}(A, B)], \text{ for all } A, B \in \mathfrak{F}(X). \tag{28}$$

Definition 5.3. Let $I \subset \mathbb{N}$ be an index set.

(1) A sequence $\{A_n\}$ converges in the interval-valued distance measure $D_{\bar{\mu}}$ to A if

$$d_H - \lim_{n\to\infty} D_{\bar{\mu}}(A_n, A) = 0, \tag{29}$$

where

$$d_H - \lim_{n\to\infty} D_{\bar{\mu}}(A_n, A) = \lim_{n\to\infty} d_H\{D_{\bar{\mu}}(A_n, A), 0\}$$

and

$$D_{\bar{\mu}}(A_n, A) = [d_{\mu^-}(A_n, A), d_{\mu^-}(A_n, A)]$$

for all $n \in \mathbb{N}$.

(2) A class $\{A_n\}_{n\in I}$ of fuzzy sets in $\mathfrak{F}(X)$ is said to be $\bar{\mu}$-integrable if

(i) $$\sup_{n\in I} D_{\bar{\mu}}(A_n, 0) < \infty, \tag{30}$$

(ii) $$\forall \varepsilon > 0,\ \exists \delta(\varepsilon) > 0 \text{ such that } D_{E,\bar{\mu}}(A_n, 0) < \varepsilon \text{ if } E \in \Omega \text{ and } \bar{\mu}(E) < \delta(\varepsilon), \tag{31}$$

where $D_{E,\bar{\mu}}(A_n, 0) = (C)\int_E |\eta_{A_n}| d\bar{\mu}$ for all $n \in \mathbb{N}$.

By (3), it is easy to see that (29) holds if and only if

$$\lim_{n\to\infty} \max\{d_{\mu^-}(A_n, A), d_{\mu^+}(A_n, A)\}) = 0, \tag{32}$$

By Definition 5.1 and Definition 5.3, we obtain various convergence properties of the interval-valued distance measure $D_{\bar{\mu}}$ as follows:

Theorem 5.3. *Let $I \subset \mathbb{N}$ be an index set.*

(1) *A class $\{A_n\}_{n\in I}$ is uniformly $\bar{\mu}$-integrable if and only if it is uniformly μ^--integrable and uniformly μ^+-integrable, and $\mu^- \leq \mu^+$.*

(2) *A sequence $\{A_n\}$ of fuzzy sets in $\mathfrak{F}(X)$ converges to a fuzzy set $A \in \mathfrak{F}(X)$ in the interval-valued distance measure $D_{\bar{\mu}}$ if and only if $\{A_n\}$ converges to A in the distance measures d_{μ^-} and d_{μ^+}, and $d_{\mu^-} \leq d_{\mu^+}$.*

Proof. (1) Let $\{A_n\}$ be a sequence of fuzzy sets in $\mathfrak{F}(X)$. If $\{A_n\}$ converges to A in the interval-valued distance measure $D_{\bar{\mu}}$, then, by (12) and (29),

$$\begin{aligned}\lim_{n\to\infty} d_{\mu^-}(A_n, A) &\leq \lim_{n\to\infty}(\max\{d_{\mu^-}(A_n, A), d_{\mu^+}(A_n, A)\}) \\ &= \lim_{n\to\infty} d_H(D_{\bar{\mu}}(A_n, A), 0) = 0.\end{aligned} \tag{33}$$

As in the same method with (33), we obtain

$$\lim_{n\to\infty} d_{\mu^+}(A_n, A) = 0. \tag{34}$$

Thus, by (33) and (34), $\{A_n\}$ converges to A in the distance measure d_{μ^-} and d_{μ^+}.

Conversely, if we take an interval-valued distance measure $\bar{\mu} = [\mu^-, \mu^+]$, then, similarly, we can obtain the converse result.

(2) Suppose that $\{A_n\}_{n\in I}$ is uniformly $\bar{\mu}$-integrable and $\bar{\mu}$ is continuous from below. By (12) and Definition2.3 (9), we have

$$\begin{aligned}\sup_{n\in I} D_{\bar{\mu}}(A_n, 0) &= \sup_{n\in I}[d_{\mu^-}(A_n, 0), d_{\mu^+}(A_n, 0)] \\ &= [\sup_{n\in I} d_{\mu^-}(A_n, 0), \sup_{n\in I} d_{\mu^+}(A_n, 0)] < \infty, \qquad (35)\end{aligned}$$

and for arbitrary $\varepsilon > 0$ and $E \in \Omega$, there exists $\delta(\varepsilon) > 0$ such that

$$\begin{aligned}\sup_{n\in I} D_{E,\bar{\mu}}(A_n, 0) &= \sup_{n\in I}[d_{E,\mu^-}(A_n, 0), d_{E,\mu^+}(A_n, 0)] \\ &= [\sup_{n\in I} d_{E,\mu^-}(A_n, 0), \sup_{n\in I} d_{E,\mu^+}(A_n, 0)] < \varepsilon, \qquad (36)\end{aligned}$$

if $\bar{\mu} < \delta(\varepsilon)$. By (35) and (36), $\{A_n\}$ converges to A in the distance measures d_{μ^-} and d_{μ^+}, nd $d_{\mu^-} \le d_{\mu^+}$.

Conversely, if we take an interval-valued distance measure $\bar{\mu} = [\mu^-, \mu^+]$, then, similarly, we can obtain the converse result.

Theorem 5.4. *Let an interval-valued capacity $\bar{\mu}$ be subadditive and $\{A_n\}$ a sequence of fuzzy sets in $\mathfrak{F}(X)$. Then, $\{A_n\}$ is an uniformly $\bar{\mu}$-integrable if and only if*

$$\lim_{a\to\infty} \sup_{n\in\mathbb{N}} D_{[|\eta_{A_n}|>a],\bar{\mu}}(A_n, 0) = 0. \qquad (37)$$

Proof. Since an interval-valued capacity $\bar{\mu} = [\mu^-, \mu^+]$ is subadditive, by Theorem 3.1(5), μ^- and μ^+ are subadditive. From Theorem 5.3 (1), $\{A_n\}$ is an uniformly $\bar{\mu}$-integrable if and only if $\{A_n\}$ is an uniformly μ^--integrable and an uniformly μ^+-integrable. Thus, by Theorem 5.1, $\{A_n\}$ is an uniformly μ^--integrable if and only if

$$\lim_{a\to\infty} \sup_{n\in\mathbb{N}} d_{[|\eta_{A_n}|>a],\mu^-}(A_n, 0) = 0 \qquad (38)$$

and $\{A_n\}$ is an uniformly μ^+-integrable if and only if

$$\lim_{a\to\infty} \sup_{n\in\mathbb{N}} d_{[|\eta_{A_n}|>a],\mu^+}(A_n, 0) = 0 \qquad (39)$$

By (38) and (39), and (12), we have

$$\begin{aligned}&\lim_{a\to\infty} \sup_{n\in\mathbb{N}} d_H(D_{|\eta_{A_n}|>a],\bar{\mu}}(A_n, 0), 0) \\ = &\lim_{a\to\infty} \sup_{n\in\mathbb{N}} \max\{d_{[|\eta_{A_n}|>a],\mu^-}(A_n, 0), d_{[|\eta_{A_n}|>a],\mu^+}(A_n, 0)\} = 0. \qquad (40)\end{aligned}$$

Conversely, by the similar method of the above proof, we can obtain the converse result.

Lemma 5.5. *Assume that an interval-valued capacity $\bar{\mu} = [\mu^-, \mu^+]$ is continuous from below. Then $\{A_n\}$ is $\bar{\mu}$-integrable if and only if $\{A_n\}$ is μ^--integrable and μ^+-integrable*

Proof. The proof is trivial.

Theorem 5.6. *Let an interval-valued capacity μ be subadditive. If a sequence $\{A_n\}$ of fuzzy sets in $\mathfrak{F}(X)$ converges to a fuzzy set A in $\mathfrak{F}(X)$ μ-almost everywhere on X and $A_n \le B$ for some $\bar{\mu}$-integrable fuzzy set B, then we have*

(1) *A_n and A are $\bar{\mu}$-integrable for all $n \in \mathbb{N}$, and*

(2) *$\{A_n\}$ converges to A in the interval-valued distance measure $D_{\bar{\mu}}$, that is,*

$$d_H - \lim_{n\to\infty} D_{\bar{\mu}}(A_n, 0) = 0. \qquad (41)$$

Proof. Since B is $\bar{\mu}$-integrable fuzzy set and $A_n \leq B$, by Theorem 5.3 (1), we have
(i)A_n and A are μ^--integrable and μ^--integrable for all $n \in \mathbb{N}$, and
(ii) $\{A_n\}$ converges to A in the distance measure d_{μ^-} and in the distance measure d_{μ^+}.
Thus, by Lemma 5.5 and Theorem 5.3 (1) and (12), we obtain
(1) A_n and A are $\bar{\mu}$-integrable for all $n \in \mathbb{N}$, and
(2) $\{A_n\}$ converges to A in the interval-valued distance measure $D_{\bar{\mu}}$, that is,

$$d_H - \lim_{n\to\infty} D_{\bar{\mu}}(A_n, 0) = 0. \tag{42}$$

6. Conclusions

In this paper, we define the concept of interval-valued capacity which means reasonable capacity. By using Aumann integral of integrably bounded interval-valued functions in Corollary 3.4, we consider the Choquet integral with respect to a continuous interval-valued capacity of a fuzzy set.

From Definitions 2.3, 3.1, 3.2 and Theorems 3.5, 3.6, we discuss interval-valued similarity measures induced by the Choquet integral with respect to a continuous interval-valued capacity on $\mathfrak{F}(X)$. By Examples 4.1 and 4.2, it is possible that we interpret the interval-valued measure of similarity of a sample with the three patterns. From Definitions 5.1, 5.2, 5.3, and Theorems 5.3, 5.4, and 5.6, we can provide the concept of convergence in the interval-valued distance measure and discuss various convergence properties of the interval-valued distance3 measure on the space of fuzzy sets for the Choquet integral.

In the future, by using these results of this paper, we can develop various problems and models for representing uncertain similarity measures and uncertain distance measures in pattern recognition research, information theory, decision making, and fuzzy risk analysis, etc.

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n-JORDAN $*$-DERIVATIONS ON INDUCED FUZZY C^*-ALGEBRAS

GANG LU, YANDUO WANG, AND PENGYU YE

Abstract. Using the fixed point alternative theorem, we investigate the Hyers-Ulam stability of of n-Jordan $*$-derivations on induced fuzzy C^*-algebras associated with the following functional equation $f(y-x)+f(x-z)+f(3x-y+z)=f(3x)$.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [43] concerning the stability of group homomorphisms. Hyers [22] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [38] for linear mappings by considering an unbounded Cauchy difference. Those results have been recently complemented in [9]. A generalization of the Aoki and Rassias theorem was obtained by Găvruta [21], who used a more general function controlling the possibly unbounded Cauchy difference in the spirit of Rassias' approach. The stability problems for several functional equations or inequalities have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [8, 15], [23]–[31], [39]–[41]).

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.1 (see [14, 18]). *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty$, for all $n \ge n_0$;
(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $Y = \{y \in X | d(J^{n_0} x, y) < \infty\}$;
(4) $d(y, y^*) \le \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

By using the fixed point method, the stability problems of several functional equations have been extensively investigated by a number of authors (see [10, 13, 14, 17, 19, 28, 33, 34, 37, 46]).

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1

In 1984, Katsaras [27] defined a fuzzy norm on a linear space and at the same year Wu and Fang [44] also introduced a notion of fuzzy normed space and gave the generalization of the Kolmogoroff normalized theorem for fuzzy topological linear space. In [7], Biswas defined and studied fuzzy inner product spaces in linear space. Since then some mathematicians have defined fuzzy metrics and norms on a linear space from various points of view [6, 20, 30, 42, 45]. In 1994, Cheng and Mordeson introduced a definition of fuzzy norm on a linear space in such a manner that the corresponding induced fuzzy metric is of Kramosil and Michalek type [29]. In 2003, Bag and Samanta [6] modified the definition of Cheng and Mordeson [16] by removing a regular condition. They also established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy norms (see [3]). Following [2], we give the employing notion of a fuzzy norm.

Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0,1]$(the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $a, b \in \mathbb{R}$:
(N_1) $N(x,a) = 0$ for $a \leq 0$;
(N_2) $x = 0$ if and only if $N(x,a) = 1$ for all $a > 0$;
(N_3) $N(ax,b) = N(x, \frac{b}{|a|})$ if $a \neq 0$;
(N_4) $N(x+y, a+b) \geq min\{N(x,a), N(y,b)\}$;
(N_5) $N(x,.)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{a\to\infty} N(x,a) = 1$;
(N_6) For $x \neq 0$, $N(x,.)$ is (upper semi) continuous on $\mathbb{R}$.
The pair (X,N) is called a fuzzy normed linear space. One may regard $N(x,a)$ as the truth value of the statement the norm of x is less than or equal to the real number a'.

Definition 1.2. Let (X,N) be a fuzzy normed linear space. Let x_n be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, a) = 1$ for all $a > 0$. In that case, x is called the limit of the sequence x_n and we denote it by $N\text{-}lim_{n\to\infty}\, x_n = x$.

Definition 1.3. A sequence x_n in X is called *Cauchy* if for each $\epsilon > 0$ and each $a > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, a) > 1 - \epsilon$.

It is known that every convergent sequence in fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector space X, Y is continuous at point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X(see [2])

Definition 1.4. [36] Let X be a $*$-algebra and (X,N) a fuzzy normed space.

(1) The fuzzy normed space (X,N) is called a fuzzy normed $*$-algebra if

$$N(xy, st) \geq N(x,s) \cdot N(y,t) \quad \text{and} \quad N(x^*, t) = N(x,t).$$

(2) A complete fuzzy normed $*$-algebra is called a *fuzzy Banach $*$-algebra*.

Example 1.5. *Let $(X, \|.\|)$ be a normed $*$-algebras. Let*

$$N(x,a) = \begin{cases} \frac{a}{a+\|x\|}, & a > 0\ , x \in X, \\ 0, & a \leq 0, x \in X \end{cases}$$

Then $N(x,t)$ is a fuzzy norm on X and $(X, N(x,t))$ is a fuzzy normed $*$-algebra.

Definition 1.6. Let $(X, \|\cdot\|)$ be a C^*-algebra and N a fuzzy norm on X.

(1) The fuzzy normed $*$-algebra (X, N) is called an induced fuzzy normed $*$-algebra.
(2) The fuzzy Banach $*$-algebra (X, N) is called an induced fuzzy C^*-algebra.

Definition 1.7. Let $(X, \|\cdot\|)$ be an induced fuzzy normed $*$-algebra. Then a $\mathbb{C}$-linear mapping $D : (X, N) \to (X, N)$ is called a *fuzzy n-Jordan $*$-derivation* if

$$\begin{aligned} D(x^n) &= D(x)x^{n-1} + xD(x)x^{n-2} + \cdots + x^{n-2}D(x)x + x^{n-1}D(x), \\ D(x^*) &= D(x)^* \end{aligned}$$

for all $x \in X$.

Throughout this paper, assume that (X, N) is an induced fuzzy C^*-algebra.

2. Main results

Lemma 2.1. *Let (Z, N) be a fuzzy normed vector space and $f : X \to Z$ be a mapping such that*

$$N\left(f(y-x) + f(x-z) + f(3x-y+z), t\right) \geq N\left(f(3x), \frac{t}{2}\right) \tag{2.1}$$

for all $x, y, z \in X$ and all $t > 0$. Then f is additive.

Proof. Letting $x = y = z = 0$ in (2.1), we get

$$N(3f(0), t) = N\left(f(0), \frac{t}{3}\right) \geq N\left(f(0), \frac{t}{2}\right)$$

for all $t > 0$. By (N_5) and (N_6), $N(f(0), t) = 1$ for all $t > 0$. It follows from (N_2) that $f(0) = 0$.

Letting $x = z = 0$ in (2.1), we get

$$N(f(y) + f(0) + f(-y), t) \geq N\left(f(0), \frac{t}{2}\right) = 1$$

for all $t > 0$. It follows from (N_2) that $f(-y) + f(y) = 0$ for all $y \in X$. Thus

$$f(-y) = -f(y)$$

for all $y \in X$.

Letting $x = 0$ and replacing z by $-z$ in (2.1), we get

$$N(f(y) + f(z) + f(-y-z), t) \geq N\left(f(0), \frac{t}{2}\right) = 1$$

for all $t > 0$. It follows from (N_2) that

$$f(y) + f(z) + f(-y-z) = 0$$

for all $y, z \in X$. Thus

$$f(y+z) = f(y) + f(z)$$

for all $y, z \in X$, as desired. □

Theorem 2.2. *Let $\phi : X^3 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\phi\left(\frac{x}{3},\frac{y}{3},\frac{z}{3}\right) \leq \frac{L}{3}\phi(x,y,z) \tag{2.2}$$

for all $x, y, z \in X$. Let $f : X \to X$ be a mapping such that

$$\begin{aligned} &N\left(f\left(\mu(y-x)\right)+f\left(\mu(x-z)\right)+f\left(\mu(3x-y+z)\right)-\mu f\left(3x\right),t\right)\\ &\geq \frac{t}{t+\phi(x,y,z)}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} &N\big(f(w^n)-f(w)w^{n-1}-wf(w)w^{n-2}-\cdots-w^{n-2}f(w)w-w^{n-1}f(w)\\ &+f(v^*)-f(v)^*,t\big) \geq \frac{t}{t+\phi(w,v,0)} \end{aligned} \tag{2.4}$$

for all $x, y, z, w, v \in X$, all $t > 0$ and all $\mu \in \mathbb{T}^1 := \{c \in \mathbb{C} : |c| = 1\}$. Then the limit $A(x) = N - \lim_{n\to\infty} 3^n f\left(\frac{x}{3^n}\right)$ exists for each $x \in X$ and the mapping $A : X \to X$ is a fuzzy n-Jordan $*$-derivation satisfying

$$N(f(x)-A(x),t) \geq \frac{3(1-L)t}{3(1-L)t+L\phi\left(x,2x,0\right)} \tag{2.5}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $\mu = 1, y = 2x$, $z = 0$ in (2.3), we have

$$N\left(3f\left(x\right)-f(3x),t\right) \geq \frac{t}{t+\phi\left(x,2x,0\right)} \tag{2.6}$$

and so

$$N\left(3f\left(\frac{x}{3}\right)-f(x),t\right) \geq \frac{t}{t+\phi\left(\frac{x}{3},\frac{2x}{3},0\right)} = \frac{t}{t+\frac{L}{3}\phi\left(x,2x,0\right)}$$

for all $x \in X$. Thus

$$N\left(3f\left(\frac{x}{3}\right)-f(x),\frac{L}{3}t\right) \geq \frac{\frac{L}{3}t}{\frac{L}{3}t+\frac{L}{3}\phi\left(x,2x,0\right)} = \frac{t}{t+\phi\left(x,2x,0\right)} \tag{2.7}$$

for all $x \in X$.

Consider the set

$$G := \{g : X \to X\}$$

and introduce the generalized metric on G:

$$d(g,h) := \inf\{a \in \mathbb{R}^+ : N(g(x)-h(x),at) \geq \frac{t}{t+\phi\left(x,2x,0\right)}\}$$

for all $x \in X$ and all $t > 0$, where $\inf\phi = +\infty$. It is easy to show that (S, d) is complete (see the proof of [32, Lemma 2.1]

Now, we consider the linear mapping $Q : G \to G$ such that

$$Qg(x) := 3g\left(\frac{x}{3}\right)$$

for all $x \in X$.

Let $g, h \in G$ be given such that $d(g,h)=\varepsilon$. Then

$$N(g(x)-h(x),\varepsilon t) \geq \frac{t}{t+\phi(x,2x,0)}$$

for all $x \in X$ and all $t>0$. Hence

$$\begin{aligned} N(Qg(x)-Qh(x),L\varepsilon t) &= N\left(3g\left(\frac{x}{3}\right)-3h\left(\frac{x}{3}\right),L\varepsilon t\right) = N\left(g\left(\frac{x}{3}\right)-h\left(\frac{x}{3}\right),\frac{L}{3}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{3}}{\frac{Lt}{3}+\phi\left(\frac{x}{3},\frac{2x}{3},0\right)} \geq \frac{\frac{Lt}{3}}{\frac{Lt}{3}+\frac{L}{3}\phi(x,2x,0)} \\ &= \frac{t}{t+\phi(x,2x,0)} \end{aligned}$$

for all $x \in X$ and all $t>0$. Thus $d(g,h)=\varepsilon$ implies that $d(Qg,Qh) \leq L\varepsilon$. This means that

$$d(Qg,Qh) \leq Ld(g,h)$$

for all $g,h \in G$.

It follows from (2.7) that $d(f,Qf) \leq \frac{L}{3}$.

By Theorem 1.1, there exists a mapping $A : X \to X$ satisfying the following:

(1) A is a fixed point of Q, *i.e.*,

$$A\left(\frac{x}{3}\right) = \frac{1}{3}A(x) \tag{2.8}$$

for all $x \in X$. The mapping A is a unique fixed point of Q in the set

$$M = \{g \in G : d(f,g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.8) such that there exists an $a \in (0,\infty)$ satisfying

$$N(f(x)-A(x),at) \geq \frac{t}{t+\phi(x,2x,0)}$$

for all $x \in X$.

(2) $d(Q^k f, A) \to 0$ as $k \to \infty$. This implies the equality

$$N-\lim_{k\to\infty} 3^k f\left(\frac{x}{3^k}\right) = A(x)$$

for all $x \in X$;

(3) $d(f,A) \leq \frac{1}{1-L}d(f,Qf)$, which implies the inequality

$$d(f,A) \leq \frac{L}{3(1-L)}.$$

This implies that the inequality (2.5) holds.

Next we show that A is additive. It follows from (2.2) that

$$\begin{aligned}\sum_{k=0}^{\infty} 3^k \phi\left(\frac{x}{3^k}, \frac{y}{3^k}, \frac{z}{3^k}\right) &= \phi(x,y,z) + 3\phi\left(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}\right) + 3^2\phi\left(\frac{x}{3^2}, \frac{y}{3^2}, \frac{z}{3^2}\right) + \cdots \\ &\leq \phi(x,y,z) + L\phi(x,y,z) + L^2\phi(x,y,z) + \cdots \\ &= \frac{1}{1-L}\phi(x,y,z) < \infty\end{aligned}$$

for all $x, y, z \in X$.

By (2.3),

$$\begin{aligned}&N\left(3^k f\left(\mu\frac{y-x}{3^k}\right) + 3^k f\left(\mu\frac{x-z}{3^k}\right) + f\left(\mu\frac{3x-y+z}{3^k}\right) - 3^k\mu f\left(\frac{3}{3^k}x\right), 3^k t\right) \\ &\geq \frac{t}{t + \phi\left(\frac{x}{3^k}, \frac{y}{3^k}, \frac{z}{3^k}\right)}\end{aligned}$$

and so

$$\begin{aligned}&N\left(3^k f\left(\mu\frac{y-x}{3^k}\right) + 3^k f\left(\mu\frac{x-z}{3^k}\right) + 3^k f\left(\mu\frac{3x-y+z}{3^k}\right) - 3^k\mu f\left(\frac{3}{3^k}x\right), t\right) \\ &\geq \frac{\frac{t}{3^k}}{\frac{t}{3^k} + \phi\left(\frac{x}{3^k}, \frac{y}{3^k}, \frac{z}{3^k}\right)} = \frac{t}{t + 3^k\phi\left(\frac{x}{3^k} \cdot \frac{y}{3^k}, \frac{z}{3^k}\right)}\end{aligned}$$

for all $x, y, z \in X$, all $t > 0$ and all $\mu \in \mathbb{T}^1$. Since $\lim_{k\to\infty} \frac{t}{t+3^k\phi\left(\frac{x}{3^k}, \frac{y}{3^k}, \frac{z}{3^k}\right)} = 1$ for all $x, y, z \in X$ and all $t > 0$,

$$N\left(A\left(\mu(y-x)\right) + A\left(\mu(x-z)\right) + A\left(\mu(3x-y+z)\right) - \mu A\left(3x\right), t\right) = 1$$

for all $x, y, z \in X$, all $t > 0$ and all $\mu \in \mathbb{T}^1$. So

$$A\left(\mu(y-x)\right) + A\left(\mu(x-z)\right) + A\left(\mu(3x-y+z)\right) = \mu A\left(3x\right) \tag{2.9}$$

for all $x, y, z \in X$, all $t > 0$ and all $\mu \in \mathbb{T}^1$. Letting $x = y = z = 0$ in (2.9), we have $A(0) = 0$. Let $\mu = 1$, $x = 0$ and replace z by $-z$ in (2.9). By the same reasoning as in the proof of Lemma 2.1, one can easily show that A is additive. Letting $y = 2x$, $z = 0$ in (2.9), we get

$$\mu A(x) = 3A\left(\mu\frac{x}{3}\right) = A(\mu x)$$

for all $x \in X$ and $\mu \in \mathbb{T}^1$. The mapping $A : X \to X$ is $\mathbb{C}$-linear by [35, Theorem 2.1].

By (2.4) and letting $v = 0$ in (2.4), we get

$$\begin{aligned}&N\left(3^{nk} f\left(\frac{w^n}{3^{nk}}\right) - 3^{nk} f\left(\frac{w}{3^k}\right)\left(\frac{w}{3^k}\right)^{n-1} - 3^{nk}\frac{w}{3^k} f\left(\frac{w}{3^k}\right)\left(\frac{w}{3^k}\right)^{n-2} - \cdots\right. \\ &\left.-3^{nk}\left(\frac{w}{3^k}\right)^{n-2} f\left(\frac{w}{3^k}\right) w - 3^{nk}\left(\frac{w}{3^k}\right)^{n-1} f\left(\frac{w}{3^k}\right), 3^{nk} t\right) \geq \frac{t}{t + \phi(\frac{w}{3^k}, 0, 0)}\end{aligned}$$

for all $w \in X$ and all $t > 0$. Thus

$$\begin{aligned} &N\left(3^{nk} f\left(\frac{w^n}{3^{nk}}\right) - 3^{nk} f\left(\frac{w}{3^k}\right)\left(\frac{w}{3^k}\right)^{n-1} - 3^{nk}\frac{w}{3^k} f\left(\frac{w}{3^k}\right)\left(\frac{w}{3^k}\right)^{n-2} - \cdots \right.\\ &\left. -3^{nk}\left(\frac{w}{3^k}\right)^{n-2} f\left(\frac{w}{3^k}\right) w - 3^{nk}\left(\frac{w}{3^k}\right)^{n-1} f\left(\frac{w}{3^k}\right), t\right) \ge \frac{\frac{t}{3^{nk}}}{\frac{t}{3^{nk}} + \phi(\frac{w}{3^k}, 0, 0)} \\ &\ge \frac{t}{t + (3^{n-1}L)^k \phi(w, 0, 0)} \end{aligned}$$

for all $w \in X$ and all $t > 0$. Since $\lim_{k\to\infty} \frac{t}{t+(3^{n-1}L)^k\phi(w,0,0)} = 1$ for all $w \in X$ and all $t > 0$, we get

$$N(D(w^n) - D(w)w^{n-1} - wD(w)w^{n-2} - \cdots - w^{n-2}D(w)w - w^{n-1}D(w), t) = 1$$

for all $x \in X$ and all $t > 0$. So

$$D(w^n) - D(w)w^{n-1} - wD(w)w^{n-2} - \cdots - w^{n-2}D(w)w - w^{n-1}D(w) = 0$$

for all $w \in X$.

Letting $w = 0$ in (2.4), similarly, we get $D(v^*) - D(v)^* = 0$ for all $v \in X$.

Therefore, the mapping $D : X \to X$ is a fuzzy n-Jordan $*$-derivation. □

Corollary 2.3. *Let p be a real number with $p > 1$, $\theta \ge 0$, and X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to X$ be a mapping satisfying*

$$\begin{aligned} &N\left(f\left(\mu(y-x)\right) + f\left(\mu(x-z)\right) + f\left(\mu(3x-y+z)\right) - \mu f\left(3x\right), t\right) \\ &\ge \frac{t}{t + \theta(\|x\|^p + \|y\|^p + \|z\|^p)}, \end{aligned} \tag{2.10}$$

$$\begin{aligned} &N\left(f(w^n) - f(w)w^{n-1} - wf(w)w^{n-2} - \cdots - w^{n-2}f(w)w - w^{n-1}f(w)\right.\\ &\left.+f(v^*) - f(v)^*, t\right) \ge \frac{t}{t + \theta(\|w\|^p + \|v\|^p)} \end{aligned} \tag{2.11}$$

for all $x, y, w, v \in X$, all $t > 0$ and all $\mu \in \mathbb{T}^1$. Then the limit $A(x) = N\!-\!\lim_{n\to\infty} 3^n f\left(\frac{x}{3^n}\right)$ exists for each $x \in X$ and the mapping $A : X \to X$ is a fuzzy n-Jordan $*$-derivation satisfying

$$N(f(x) - A(x), t) \ge \frac{(3^p - 3)t}{(3^p - 3)t + \theta(1 + 2^p)\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.2 by taking

$$\phi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

and $L = 3^{1-p}$. □

Theorem 2.4. *Let $\phi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$3L\phi\left(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}\right) \le \phi(x, y, z) \tag{2.12}$$

for all $x, y, z \in X$. Let $f : X \to X$ be a mapping satisfying (2.3) and (2.4). Then the limit $A(x) = N - \lim_{n\to\infty} \frac{1}{3^n} f(3^n x)$ exists for each $x \in X$ and the mapping $A : X \to X$ is a fuzzy n-Jordan $*$-derivation satisfying

$$N(f(x) - A(x), t) \geq \frac{3(1-L)t}{3(1-L)t + \phi(x, 2x, 0)} \tag{2.13}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (G, d) be generalized metric space defined in the proof of Theorem 2.2. Consider the linear mapping $Q : G \to G$ such that

$$Qg(x) := \frac{1}{3} g(3x)$$

for all $x \in X$.

It follow from (2.6) that

$$N\left(f(x) - \frac{1}{3} f(3x), \frac{1}{3} t\right) \geq \frac{t}{t + \phi(x, 2x, 0)}$$

for all $x \in X$ and all $t > 0$. Thus $d(f, Qf) \leq \frac{1}{3}$. Hence

$$d(f, A) \leq \frac{1}{3(1-L)},$$

which implies that the inequality (2.13) holds.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $\theta \geq 0$ and let p be a positive real number with $p < 1$. Let X be a normed vector space with normed $\|\cdot\|$. Let $f : X \to X$ be a mapping satisfying (2.10) and (2.11). Then $A(x) = N - \lim_{n\to\infty} \frac{1}{3^n} f(3^n x)$ exists for each $x \in X$ and defines a fuzzy n-Jordan $*$-derivation $A : X \to X$ such that*

$$N(f(x) - A(x), t) \geq \frac{(3 - 3^p)t}{(3 - 3^p)t + \theta(1 + 2^p)\|x\|^p}$$

for every $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.4 by taking

$$\phi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

and $L = 3^{p-1}$. □

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Global stability analysis of a delayed viral infection model with antibodies and general nonlinear incidence rate

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Abstract

In this paper, we study the global properties of a viral infection model with antibody immune response. The incidence rate is given by a general function of the populations of the uninfected target cells, infected cells and free viruses. The model contains two types of intracellular discrete time delays to describe the time required for viral contacting an uninfected target cell and viral emission. We have established a set of conditions on the general incidence rate function and determined two threshold parameters R_0 (the basic infection reproduction number) and R_1 (the antibody immune response activation number) which are sufficient to determine the global behavior of the model. The global asymptotic stability of the equilibria of the model has been proven by using direct Lyapunov method and applying LaSalle's invariance principle.

Keywords: Virus dynamics; Intracellular delay; global stability; antibody immune response; Lyapunov functional.

1 Introduction

In recent years, several works have been devoted to study and develop mathematical models of the virus dynamics such as human immunodeficiency virus (HIV) (see e.g. [1]-[14]), hepatitis B virus (HBV) [15]-[18], hepatitis C virus (HCV) [19]-[21] and human T cell leukemia HTLV [22], etc. Mathematical models of viral infection can help for understanding the viral dynamics and developing antiviral drug therapies. In reality, the immune response needs an indispensable components to do its job such as antibodies, cytokines, natural killer cells, and T cells. The antibody immune response is a part of the adaptive system in which the body responds to pathogens by primarily using antibodies that produced from the B cells. While the other part is the Cytotoxic T Lymphocytes (CTL) immune response where the CTL attacks and kills the infected cells [4]. In some infections such as in malaria, the CTL immune response is less effective than the antibody immune response [23]. Mathematical models of viral infection with antibody immune response have been proposed and analyzed in ([24]-[29]). The basic model of viral infection with antibody immune response has introduced by Murase et. al. [24] and Shifi Wang [29] as:

$$\dot{x}(t) = s - dx(t) - \beta v(t)x(t), \tag{1}$$

$$\dot{y}(t) = \beta v(t)x(t) - ay(t), \tag{2}$$

$$\dot{v}(t) = ky(t) - bz(t)v(t) - cv(t), \tag{3}$$

$$\dot{z}(t) = rz(t)v(t) - \mu z(t), \tag{4}$$

where $x(t)$, $y(t)$, $v(t)$ and $z(t)$ denote the populations of uninfected target cells, infected cells, free virus particles and antibody immune cells at time t, respectively. Parameters s, k and r represent, respectively, the rate at which new healthy cells are generated from the source within the body, the generation rate constant of free viruses produced from the infected cells and the proliferation rate constant of antibody immune cells. Parameters d, a, c and μ are the natural death rate constants of the uninfected cells, infected cells, free virus particles and antibody immune cells, respectively. Parameter β is the infection rate constant and b is the removal rate constant of the virus due to the antibodies. All the parameters given in model (1)-(4) are positive.

The intracellular time delay between the time of the virus contacting the target cells and the time of generating new infectious viruses has been neglected in system (1)-(4). In fact, the intracellular delay in the infection process is actually exists (see e.g. [8]-[12]). Note that, the infection rate in model (1)-(4) is presented to be bilinear in x and v, which can not be completely describe the interaction between the uninfected target cells and viruses. Nevertheless, there are many types of improved incidence rates which are more commonly used due to their benefit for helping us gain the unification theory through passing over the unessential details (see e.g. [30] and [31]). Variety of viral infection models with antibody immune response have been considered with different forms of the incidence rate such as saturated incidence rate, $\frac{\beta x v}{1+\alpha v}$ where $\alpha \geq 0$, [27], Beddington-DeAngelis functional response, $\frac{\beta x v}{1+\gamma x+\alpha v}$, $\alpha, \gamma \geq 0$ [26], and general form, $\psi(x, v) v$ [28]. In [28], a discrete time delay has been incorporated within the model. However, the infection rate does not depend on the infected cells y. In some viral infections such as HBV, the infection rate depends on x, y and v [17], [16]. In [32], the infection rate is given by $\psi(x, y, v) v$, however the antibody immune response has been neglected. Our aim in this paper is to investigate the global stability analysis of a viral infection model with general incidence rate function and antibody immune response taking into consideration two types of discrete time delays.

The rest of the paper is designed as follows. In the next section, we introduce the model and discuss the non-negativity and boundedness of the solutions. In Section 3, we define two threshold parameters and discuss the existence of the model's equilibria. In Section 4, we study the global asymptotic stability of the equilibria using suitable Lyapunov functional and applying LaSalle's invariance principle. Finally, conclusion is given in Section 5.

2 The mathematical model

In this section, we consider the following viral infection model with general incidence rate taking into consideration the antibody immune response.

$$\dot{x}(t) = s - dx(t) - \psi(x(t), y(t), v(t))v(t), \tag{5}$$

$$\dot{y}(t) = e^{-\mu_1\tau_1}\psi(x(t-\tau_1), y(t-\tau_1), v(t-\tau_1))v(t-\tau_1) - ay(t), \tag{6}$$

$$\dot{v}(t) = ke^{-\mu_2\tau_2}y(t-\tau_2) - bz(t)v(t) - cv(t), \tag{7}$$

$$\dot{z}(t) = rz(t)v(t) - \mu z(t), \tag{8}$$

where τ_1 and τ_2 are the delay parameters. We assume that, the virus contacts an uninfected target cell at time $t-\tau_1$, the cell becomes infected at time t. The term $e^{-\mu_1\tau_1}$ represents the probability of surviving the contacted cell during the time delay interval, where μ_1 is the death rate constant of the contacted cells. In addition, we assume that a cell infected at time $t-\tau_2$ starts to generate new infectious viruses at time t. The term $e^{-\mu_2\tau_2}$ denotes the probability of surviving the infected cell during the time delay interval, where μ_2 is a constant. The definitions of all variables and parameters are identical to those given in Section 1. The incidence rate of infection is presented by a general function in the form $\psi(x, y, v)v$, where ψ is continuously differentiable and satisfies the following assumptions [28] and [32]:

Assumption A1. $\psi(0, y, v) = 0$ for all $y, v \geq 0$ and $\psi(x, y, v) > 0$ for all $x > 0$, $y \geq 0$, $v \geq 0$.
Assumption A2. $\dfrac{\partial\psi(x, y, v)}{\partial x} > 0$ for all $x > 0$, $y \geq 0$ and $v \geq 0$.
Assumption A3. $\dfrac{\partial\psi(x, y, v)}{\partial y} < 0$, $\dfrac{\partial\psi(x, y, v)}{\partial v} < 0$ for all $x, y, v > 0$.
Assumption A4. $\dfrac{\partial\left(\psi(x, y, v)v\right)}{\partial v} > 0$ for all $x, y, v > 0$.

Let the initial states of system (5)-(8) be given as:

$$\begin{aligned} & x(\eta) = \zeta_1(\eta),\ y(\eta) = \zeta_2(\eta), v(\eta) = \zeta_3(\eta),\ z(\eta) = \zeta_4(\eta), \\ & \zeta_j(\eta) \geq 0,\ \eta \in [-\tau, 0),\quad j = 1, ..., 4, \\ & \zeta_j(0) > 0,\ j = 1, ..., 4, \end{aligned} \tag{9}$$

where $\tau = \max\{\tau_1, \tau_2\}$, $(\zeta_1(\eta), \zeta_2(\eta), \zeta_3(\eta), \zeta_4(\eta)) \in C([-\tau, 0], \mathbb{R}^4_{\geq 0})$. We denote by $C = C([-\tau, 0], \mathbb{R}^4_{\geq 0})$ the

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Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^4_{\geq 0}$; with norm $\|\zeta\| = \sup_{-\tau \leq \eta \leq 0} |\zeta(\eta)|$ for $\zeta \in C$. We note that the system (5)-(8) with initial states (9) has a unique solution [33].

2.1 Non-negativity and boundedness of solutions

In this section, we show that the solutions of model (5)-(8) with initial states (9) are non-negative and ultimately bounded.

Proposition 1. Assume that Assumption A1 is satisfied. Then the solutions of (5)-(8) with the initial states (9) are non-negative and ultimately bounded.

Proof. At the beginning, we show that $x(t)$ is positive for all $t \geq 0$. Let us assume in contrary that $x(t) \leq 0$ on the time interval $[0, \gamma]$ where γ is a constant, and let where $\bar{t} \in [0, \gamma]$ be such that $x(\bar{t}) = 0$, . Then from Eq. (5) we get $\dot{x}(\bar{t}) = s > 0$. Thus, for sufficiently small $\varepsilon > 0$, we have $x(t) > 0$ for some $t \in (\bar{t}, \bar{t} + \varepsilon)$. This contradicts our assumption and then $x(t) > 0$, $\forall\ t \geq 0$. Now from Eqs. (6)-(8) we get

$$y(t) = y(0)e^{-at} + e^{-\mu_1 \tau_1} \int_0^t e^{-a(t-\eta)} \psi(x(\eta - \tau_1), y(\eta - \tau_1), v(\eta - \tau_1)) v(\eta - \tau_1) d\eta,$$

$$v(t) = v(0)e^{-\int_0^t (c + bz(\xi)) d\xi} + ke^{-\mu_2 \tau_2} \int_0^t e^{-\int_\eta^t (c + bz(\xi)) d\xi} y(\eta - \tau_2) d\eta,$$

$$z(t) = z(0)e^{-\int_0^t (\mu - rv(\xi)) d\xi},$$

which yield $y(t)$, $v(t)$, $z(t) \geq 0$ for all $t \in [0, \tau]$. By a recursive argument, we get that $y(t), v(t), z(t) \geq 0$ for all $t \geq 0$.

Next we prove the ultimate bound of the solutions of system (5)-(8). From Eq. (5) we get $\dot{x}(t) \leq s - dx(t)$ and thus $\limsup_{t \to \infty} x(t) \leq \frac{s}{d}$. Let $T_1(t) = e^{-\mu_1 \tau_1} x(t - \tau_1) + y(t)$, then

$$\begin{aligned}
\dot{T}_1(t) &= e^{-\mu_1 \tau_1} \left(s - dx(t - \tau_1) - \psi(x(t - \tau_1), y(t - \tau_1), v(t - \tau_1)) v(t - \tau_1) \right) \\
&\quad + e^{-\mu_1 \tau_1} \psi(x(t - \tau_1), y(t - \tau_1), v(t - \tau_1)) v(t - \tau_1) - ay(t), \\
&= se^{-\mu_1 \tau_1} - de^{-\mu_1 \tau_1} x(t - \tau_1) - ay(t) \leq se^{-\mu_1 \tau_1} - \sigma_1 \left(e^{-\mu_1 \tau_1} x(t - \tau_1) + y(t) \right) \\
&= se^{-\mu_1 \tau_1} - \sigma_1 T_1(t) \leq s - \sigma_1 T_1(t),
\end{aligned}$$

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where $\sigma_1 = \min\{d, a\}$. Hence $\limsup_{t\to\infty} T_1(t) \leq L_1$, where $L_1 = \dfrac{s}{\sigma_1}$. Since $x(t)$ and $y(t)$ are non-negative, then $\limsup_{t\to\infty} y(t) \leq L_1$. Moreover, let $T_2(t) = v(t) + \frac{b}{r}z(t)$, then

$$\begin{aligned}\dot{T}_2(t) &= ke^{-\mu_2\tau_2}y(t-\tau_2) - cv(t) - \frac{b\mu}{r}z(t) \leq ke^{-\mu_2\tau_2}L_1 - \sigma_2(v(t) + \frac{b}{r}z(t))\\ &= ke^{-\mu_2\tau_2}L_1 - \sigma_2 T_2(t) \leq kL_1 - \sigma_2 T_2(t),\end{aligned}$$

where $\sigma_2 = \min\{c, \mu\}$. It follows that, $\limsup_{t\to\infty} T_2(t) \leq L_2$, where $L_2 = \dfrac{kL_1}{\sigma_2}$. Since $v(t)$ and $z(t)$ are non-negative, then $\limsup_{t\to\infty} v(t) \leq L_2$ and $\limsup_{t\to\infty} z(t) \leq L_3$, where $L_3 = \frac{r}{b}L_2$. Therefore, all the state variables of the model are ultimately bounded.

2.2 The equilibria and threshold parameters

At any equilibrium we have

$$s - dx - \psi(x, y, v)v = 0, \tag{10}$$

$$e^{-\mu_1\tau_1}\psi(x, y, v)v - ay = 0, \tag{11}$$

$$ke^{-\mu_2\tau_2}y - bvz - cv = 0, \tag{12}$$

$$(rv - \mu)z = 0. \tag{13}$$

From Eq. (13), either $z = 0$ or $z \neq 0$. If $z = 0$, then from Eqs. (10)-(12) we get

$$y = \frac{s - dx}{ae^{\mu_1\tau_1}} = \frac{c}{ke^{-\mu_2\tau_2}}v, \qquad v = \frac{k(s - dx)}{ace^{\mu_1\tau_1+\mu_2\tau_2}}. \tag{14}$$

Substituting from Eq. (14) into Eq. (11) we get:

$$\left[\psi\left(x, \frac{s - dx}{ae^{\mu_1\tau_1}}, \frac{k(s - dx)}{ace^{\mu_1\tau_1+\mu_2\tau_2}}\right) - \frac{ac}{k}e^{\mu_1\tau_1+\mu_2\tau_2}\right]v = 0. \tag{15}$$

Eq. (15) has two possible solutions $v = 0$ or $v \neq 0$. If $v = 0$, then from Eqs. (10) and (11), we get $x = s/d$ and $y = 0$ which leads to the infection-free equilibrium $E_0(x_0, 0, 0, 0)$ where $x_0 = s/d$. If $v \neq 0$, then we have

$$\psi\left(x, \frac{s - dx}{ae^{\mu_1\tau_1}}, \frac{k(s - dx)}{ace^{\mu_1\tau_1+\mu_2\tau_2}}\right) - \frac{ac}{k}e^{\mu_1\tau_1+\mu_2\tau_2} = 0.$$

Let

$$\Phi_1(x) = \psi\left(x, \frac{s - dx}{ae^{\mu_1\tau_1}}, \frac{k(s - dx)}{ace^{\mu_1\tau_1+\mu_2\tau_2}}\right) - \frac{ac}{k}e^{\mu_1\tau_1+\mu_2\tau_2} = 0,$$

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then, we have

$$\Phi_1'(x) = \frac{\partial\psi}{\partial x} - \frac{d}{ae^{\mu_1\tau_1}}\frac{\partial\psi}{\partial y} - \frac{kd}{ace^{\mu_1\tau_1+\mu_2\tau_2}}\frac{\partial\psi}{\partial v}.$$

Because of Assumptions A2 and A3, we have $\Phi_1'(x) > 0$ which implies that function $\Phi_1(x)$ is strictly increasing w.r.t. x. Moreover,

$$\Phi_1(0) = \psi\left(0, \frac{s}{ae^{\mu_1\tau_1}}, \frac{ks}{ace^{\mu_1\tau_1+\mu_2\tau_2}}\right) - \frac{ac}{k}e^{\mu_1\tau_1+\mu_2\tau_2} = -\frac{ac}{k}e^{\mu_1\tau_1+\mu_2\tau_2} < 0,$$

$$\Phi_1(x_0) = \psi(x_0, 0, 0) - \frac{ac}{k}e^{\mu_1\tau_1+\mu_2\tau_2} = \frac{ac}{k}e^{\mu_1\tau_1+\mu_2\tau_2}\left(\frac{k\psi(x_0,0,0)}{ac}e^{-\mu_1\tau_1-\mu_2\tau_2} - 1\right).$$

Therefore, if $\frac{k\psi(x_0,0,0)}{ac}e^{-\mu_1\tau_1-\mu_2\tau_2} > 1$, then there exist a unique $x_1 \in (0, x_0)$ such that $\Phi_1(x_1) = 0$. It follows from (12) and (14) that $y_1 = \frac{d(x_0 - x_1)}{ae^{\mu_1\tau_1}} > 0$ and $v_1 = \frac{kd(x_0 - x_1)}{ace^{\mu_1\tau_1+\mu_2\tau_2}} > 0$. It means that, a chronic-infection equilibrium without antibody immune response $E_1(x_1, y_1, v_1, 0)$ exists when $\frac{k\psi(x_0,0,0)}{ac}e^{-\mu_1\tau_1-\mu_2\tau_2} > 1$. Let us define the basic infection reproduction number as:

$$R_0 = \frac{k\psi(x_0,0,0)}{ac}e^{-\mu_1\tau_1-\mu_2\tau_2}.$$

The parameter R_0 determines whether a chronic-infection can be established. The other possibility of Eq. (13) is $z \neq 0$ which leads to $v_2 = \frac{\mu}{r}$. From Eq. (10) we let

$$\Phi_2(x) = s - dx - \psi(x, \frac{s - dx}{ae^{\mu_1\tau_1}}, v_2)v_2 = 0.$$

According to Assumptions A2 and A3, we know that Φ_2 is a decreasing function of x. Clearly, $\Phi_2(0) = s > 0$ and $\Phi_2(x_0) = -\psi(x_0, 0, v_2)v_2 < 0$. Thus, there exists a unique $x_2 \in (0, x_0)$ such that $\Phi_2(x_2) = 0$. It follows from Eq. (14) that, $y_2 = \frac{d(x_0 - x_2)}{ae^{\mu_1\tau_1}} > 0$ and $z_2 = \frac{k\psi(x_2,y_2,v_2)}{abe^{\mu_1\tau_1+\mu_2\tau_2}} - \frac{c}{b} = \frac{c}{b}\left(\frac{k\psi(x_2,y_2,v_2)}{ace^{\mu_1\tau_1+\mu_2\tau_2}} - 1\right)$. Then, if $\frac{k\psi(x_2,y_2,v_2)}{ace^{\mu_1\tau_1+\mu_2\tau_2}} > 1$ then $z_2 > 0$. Now we define the antibody immune response activation number as

$$R_1 = \frac{k\psi(x_2,y_2,v_2)}{ace^{\mu_1\tau_1+\mu_2\tau_2}},$$

which determines whether a persistent antibody immune response can be established. Hence, z_2 can be rewritten as $z_2 = \frac{c}{b}(R_1 - 1)$. It follows that, there is a chronic-infection equilibrium with antibody immune response $E_2(x_2, y_2, v_2, z_2)$ when $R_1 > 1$.

Clearly from Assumptions A2 and A3, we have

$$R_1 = \frac{k\psi(x_2,y_2,v_2)}{ace^{\mu_1\tau_1+\mu_2\tau_2}} < \frac{k\psi(x_0,y_2,v_2)}{ace^{\mu_1\tau_1+\mu_2\tau_2}} < \frac{k\psi(x_0,0,0)}{ace^{\mu_1\tau_1+\mu_2\tau_2}} = R_0.$$

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2.3 Global stability analysis

In this section, the global asymptotic stability of the three equilibria of model (5)-(8) will be established by using direct Lyapunov method and applying LaSalle's invariance principle. In the remaining parts of the paper we shall use the following function: $H:(0,\infty)\to[0,\infty)$,

$$H(u)=u-1-\ln u.$$

Theorem 1. Let Assumptions A1-A3 be hold true and $R_0\le 1$, then the infection-free equilibrium E_0 is globally asymptotically stable (GAS).

Proof. We construct a Lyapunov functional as:

$$\begin{aligned}U_0 &= x-x_0-\int_{x_0}^{x}\frac{\psi(x_0,0,0)}{\psi(\eta,0,0)}d\eta+e^{\mu_1\tau_1}y+\frac{a}{k}e^{\mu_1\tau_1+\mu_2\tau_2}v+\frac{ab}{rk}e^{\mu_1\tau_1+\mu_2\tau_2}z\\ &+\int\limits_{t-\tau_1}^{t}\psi(x(\eta),y(\eta),v(\eta))v(\eta)d\eta+ae^{\mu_1\tau_1}\int\limits_{t-\tau_2}^{t}y(\eta)d\eta. \end{aligned} \tag{16}$$

We calculate $\frac{dU_0}{dt}$ along the solutions of model (5)-(8) as:

$$\begin{aligned}\frac{dU_0}{dt} &= \left(1-\frac{\psi(x_0,0,0)}{\psi(x,0,0)}\right)(s-dx-\psi(x,y,v)\,v)+\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)-ae^{\mu_1\tau_1}y\\ &+ae^{\mu_1\tau_1}y(t-\tau_2)-\frac{ac}{k}e^{\mu_1\tau_1+\mu_2\tau_2}v-\frac{ab}{k}e^{\mu_1\tau_1+\mu_2\tau_2}zv+\frac{ab}{k}e^{\mu_1\tau_1+\mu_2\tau_2}zv-\frac{ab\mu}{rk}e^{\mu_1\tau_1+\mu_2\tau_2}z\\ &+\psi(x,y,v)v-\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)+ae^{\mu_1\tau_1}(y-y(t-\tau_2))\\ &= s\left(1-\frac{\psi(x_0,0,0)}{\psi(x,0,0)}\right)\left(1-\frac{x}{x_0}\right)+\left(\psi(x,y,v)\frac{\psi(x_0,0,0)}{\psi(x,0,0)}-\frac{ac}{k}e^{\mu_1\tau_1+\mu_2\tau_2}\right)v-\frac{ab\mu}{rk}e^{\mu_1\tau_1+\mu_2\tau_2}z\\ &= s\left(1-\frac{\psi(x_0,0,0)}{\psi(x,0,0)}\right)\left(1-\frac{x}{x_0}\right)+\frac{ac}{k}e^{\mu_1\tau_1+\mu_2\tau_2}\left(\frac{\psi(x,y,v)}{\psi(x,0,0)}R_0-1\right)v-\frac{ab\mu}{rk}e^{\mu_1\tau_1+\mu_2\tau_2}z. \end{aligned} \tag{17}$$

From Assumptions A2-A3 we know that $\psi(x,y,v)$ is an increasing function of x and decreasing function of y and v. Then, the first term of Eq. (17) is less than or equal zero and

$$\psi(x,y,v)<\psi(x,0,0),\qquad x,y,v>0.$$

It follows that

$$\frac{dU_0}{dt}\le s\left(1-\frac{\psi(x_0,0,0)}{\psi(x,0,0)}\right)\left(1-\frac{x}{x_0}\right)+\frac{ac}{k}e^{\mu_1\tau_1+\mu_2\tau_2}\left(R_0-1\right)v-\frac{ab\mu}{rk}e^{\mu_1\tau_1+\mu_2\tau_2}z. \tag{18}$$

Therefore, if $R_0 \leq 1$, then $\frac{dU_0}{dt} \leq 0$ for all $x, y, v, z > 0$. We note that the solutions of system (5)-(8) converge to Ω, the largest invariant subset of $\left\{\frac{dU_0}{dt} = 0\right\}$[33]. From (18), we have $\frac{dU_0}{dt} = 0$ iff $x = x_0$, $v = 0$ and $z = 0$. The set Ω is invariant and for any element belongs to Ω satisfies $v = 0$ and $z = 0$. We can see from Eq. (7) that

$$\dot{v} = 0 = ke^{-\mu_2\tau_2}y(t-\tau_2).$$

It follows that, $y = 0$. Hence $\frac{dU_0}{dt} = 0$ iff $x = x_0$ and $y = v = z = 0$. Using LaSalle's invariance principle, we derive that E_0 is GAS.

Assumption A5.

$$\left(1 - \frac{\psi(x,y,v)}{\psi(x,y_i,v_i)}\right)\left(\frac{\psi(x,y_i,v_i)}{\psi(x,y,v)} - \frac{v}{v_i}\right) \leq 0, \ i = 1,2 \text{ for all } x, y, v > 0.$$

Theorem 2. Let Assumptions A1-A5 be hold true and $R_1 \leq 1 < R_0$, then the chronic-infection equilibrium without antibody immune response E_1 is GAS.

Proof. Define:

$$\begin{aligned}
U_1 &= x - x_1 - \int_{x_1}^{x} \frac{\psi(x_1,y_1,v_1)}{\psi(\eta,y_1,v_1)} d\eta + e^{\mu_1\tau_1} y_1 H\left(\frac{y}{y_1}\right) \\
&+ \frac{a}{k} e^{\mu_1\tau_1+\mu_2\tau_2} v_1 H\left(\frac{v}{v_1}\right) + \frac{ab}{rk} e^{\mu_1\tau_1+\mu_2\tau_2} z \\
&+ \psi(x_1,y_1,v_1)v_1 \int_{t-\tau_1}^{t} H\left(\frac{\psi(x(\eta),y(\eta),v(\eta))v(\eta)}{\psi(x_1,y_1,v_1)v_1}\right) d\eta + a e^{\mu_1\tau_1} y_1 \int_{t-\tau_2}^{t} H\left(\frac{y(\eta)}{y_1}\right) d\eta. \qquad (19)
\end{aligned}$$

Calculating the time derivative of U_1 along the trajectories of system (5)-(8), we obtain

$$\begin{aligned}
\frac{dU_1}{dt} &= \left(1 - \frac{\psi(x_1, y_1, v_1)}{\psi(x, y_1, v_1)}\right)(s - dx - \psi(x, y, v)\, v) \\
&+ e^{\mu_1\tau_1}\left(1 - \frac{y_1}{y}\right)\left(e^{-\mu_1\tau_1}\psi(x(t-\tau_1), y(t-\tau_1), v(t-\tau_1))v(t-\tau_1) - ay\right) \\
&+ \frac{a}{k}e^{\mu_1\tau_1+\mu_2\tau_2}\left(1 - \frac{v_1}{v}\right)\left(ke^{-\mu_2\tau_2}y(t-\tau_2) - cv - bvz\right) + \frac{ab}{rk}e^{\mu_1\tau_1+\mu_2\tau_2}(rvz - \mu z) \\
&+ \psi(x, y, v)v - \psi(x(t-\tau_1), y(t-\tau_1), v(t-\tau_1))v(t-\tau_1) \\
&+ \psi(x_1, y_1, v_1)v_1 \ln\left(\frac{\psi(x(t-\tau_1), y(t-\tau_1), v(t-\tau_1))v(t-\tau_1)}{\psi(x, y, v)v}\right) \\
&+ ae^{\mu_1\tau_1}\left(y - y(t-\tau_2) + y_1 \ln\left(\frac{y(t-\tau_2)}{y}\right)\right) \\
&= \left(1 - \frac{\psi(x_1, y_1, v_1)}{\psi(x, y_1, v_1)}\right)(s - dx) + \psi(x_1, y_1, v_1)\frac{\psi(x, y, v)v}{\psi(x, y_1, v_1)} \\
&- \frac{y_1}{y}\psi(x(t-\tau_1), y(t-\tau_1), v(t-\tau_1))v(t-\tau_1) + ay_1e^{\mu_1\tau_1} \\
&- \frac{ac}{k}e^{\mu_1\tau_1+\mu_2\tau_2}v - ay(t-\tau_2)\frac{v_1}{v}e^{\mu_1\tau_1} + \frac{ac}{k}e^{\mu_1\tau_1+\mu_2\tau_2}v_1 \\
&+ \frac{ab}{k}e^{\mu_1\tau_1+\mu_2\tau_2}v_1z - \frac{ab\mu}{rk}e^{\mu_1\tau_1+\mu_2\tau_2}z \\
&+ \psi(x_1, y_1, v_1)v_1 \ln\left(\frac{\psi(x(t-\tau_1), y(t-\tau_1), v(t-\tau_1))v(t-\tau_1)}{\psi(x, y, v)v}\right) \\
&+ ae^{\mu_1\tau_1}y_1 \ln\left(\frac{y(t-\tau_2)}{y}\right). \qquad (20)
\end{aligned}$$

Using the equilibrium conditions for E_1:

$$s = dx_1 + ae^{\mu_1\tau_1}y_1, \quad \psi(x_1, y_1, v_1)v_1 = ae^{\mu_1\tau_1}y_1 = \frac{ac}{k}e^{\mu_1\tau_1+\mu_2\tau_2}v_1,$$

we obtain

$$\begin{aligned}\frac{dU_1}{dt} &= dx_1\left(1-\frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)}\right)\left(1-\frac{x}{x_1}\right)+3ae^{\mu_1\tau_1}y_1\\ &- ae^{\mu_1\tau_1}y_1\frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)}+ae^{\mu_1\tau_1}y_1\frac{\psi(x,y,v)v}{\psi(x,y_1,v_1)v_1}\\ &- ae^{\mu_1\tau_1}y_1\frac{y_1\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{y\psi(x_1,y_1,v_1)v_1}\\ &- ae^{\mu_1\tau_1}y_1\frac{v}{v_1}-ae^{\mu_1\tau_1}y_1\frac{v_1y(t-\tau_2)}{vy_1}\\ &+ ae^{\mu_1\tau_1}y_1\ln\left(\frac{\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{\psi(x,y,v)v}\right)\\ &+ ae^{\mu_1\tau_1}y_1\ln\left(\frac{y(t-\tau_2)}{y}\right)+\frac{ab}{k}e^{\mu_1\tau_1+\mu_2\tau_2}\left(v_1-\frac{\mu}{r}\right)z. \end{aligned} \tag{21}$$

Using the following equalities:

$$\begin{aligned}\ln\left(\frac{\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{\psi(x,y,v)v}\right) &= \ln\left(\frac{y_1\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{y\psi(x_1,y_1,v_1)v_1}\right)\\ &+\ln\left(\frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)}\right)+\ln\left(\frac{\psi(x,y_1,v_1)}{\psi(x,y,v)}\right)+\ln\left(\frac{v_1y}{vy_1}\right),\\ \ln\left(\frac{y(t-\tau_2)}{y}\right) &= \ln\left(\frac{vy_1}{v_1y}\right)+\ln\left(\frac{v_1y(t-\tau_2)}{vy_1}\right),\end{aligned}$$

we get

$$\begin{aligned}\frac{dU_1}{dt} &= dx_1\left(1-\frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)}\right)\left(1-\frac{x}{x_1}\right)+ae^{\mu_1\tau_1}y_1\left(\frac{\psi(x,y,v)v}{\psi(x,y_1,v_1)v_1}-\frac{v}{v_1}-1+\frac{\psi(x,y_1,v_1)}{\psi(x,y,v)}\right)\\ &- ae^{\mu_1\tau_1}y_1\left[\left(\frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)}-1-\ln\left(\frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)}\right)\right)\right.\\ &+\left(\frac{y_1\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{y\psi(x_1,y_1,v_1)v_1}-1-\ln\left(\frac{y_1\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{y\psi(x_1,y_1,v_1)v_1}\right)\right)\\ &+\left(\frac{v_1y(t-\tau_2)}{vy_1}-1-\ln\left(\frac{v_1y(t-\tau_2)}{vy_1}\right)\right)\\ &+\left.\left(\frac{\psi(x,y_1,v_1)}{\psi(x,y,v)}-1-\ln\left(\frac{\psi(x,y_1,v_1)}{\psi(x,y,v)}\right)\right)\right]+\frac{ab}{k}e^{\mu_1\tau_1+\mu_2\tau_2}\left(v_1-\frac{\mu}{r}\right)z. \end{aligned} \tag{22}$$

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Eq. (22) can be simplified as:

$$\begin{aligned}
\frac{dU_1}{dt} &= dx_1\left(1-\frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)}\right)\left(1-\frac{x}{x_1}\right)\\
&+ae^{\mu_1\tau_1}y_1\left(1-\frac{\psi(x,y,v)}{\psi(x,y_1,v_1)}\right)\left(\frac{\psi(x,y_1,v_1)}{\psi(x,y,v)}-\frac{v}{v_1}\right)\\
&-ae^{\mu_1\tau_1}y_1\left[H\left(\frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)}\right)+H\left(\frac{v_1y(t-\tau_2)}{vy_1}\right)\right.\\
&\left.+H\left(\frac{y_1\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{y\psi(x_1,y_1,v_1)v_1}\right)+H\left(\frac{\psi(x,y_1,v_1)}{\psi(x,y,v)}\right)\right]\\
&+\frac{ab}{k}e^{\mu_1\tau_1+\mu_2\tau_2}\left(v_1-\frac{\mu}{r}\right)z. \qquad (23)
\end{aligned}$$

From Assumptions A1 and A5, we get that the first and second terms of Eq. (23) are less than or equal zero. Now we show that if $R_1 \le 1$ then $v_1 \le \frac{\mu}{r} = v_2$. Let $R_0 > 1$, then we want to show that

$$sgn(x_2-x_1) = sgn(v_1-v_2) = sgn(y_1-y_2) = sgn(R_1-1).$$

From Assumptions A2-A4, for $x_1, x_2, y_1, y_2, v_1, v_2 > 0$, we have

$$(\psi(x_2,y_2,v_2)-\psi(x_1,y_2,v_2))(x_2-x_1) > 0, \qquad (24)$$

$$(\psi(x_1,y_1,v_1)-\psi(x_1,y_2,v_1))(y_2-y_1) > 0, \qquad (25)$$

$$(\psi(x_1,y_1,v_1)-\psi(x_1,y_1,v_2))(v_2-v_1) > 0, \qquad (26)$$

$$(\psi(x_2,y_2,v_2)v_2-\psi(x_2,y_2,v_1)v_1)(v_2-v_1) > 0. \qquad (27)$$

First, we claim $sgn(x_2-x_1) = sgn(v_1-v_2)$. Suppose this is not true, i.e., $sgn(x_2-x_1) = sgn(v_2-v_1)$. Using the conditions of the equilibria E_1 and E_2 we have

$$\begin{aligned}
(s-dx_2)-(s-dx_1) &= \psi(x_2,y_2,v_2)v_2-\psi(x_1,y_1,v_1)v_1\\
&= ae^{\mu_1\tau_1}(y_2-y_1). \qquad (28)
\end{aligned}$$

Then,

$$sgn(x_2-x_1) = sgn(y_1-y_2) \qquad (29)$$

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Moreover,

$$(s - dx_2) - (s - dx_1) = \psi(x_2, y_2, v_2)v_2 - \psi(x_1, y_1, v_1)v_1$$
$$= (\psi(x_2, y_2, v_2)v_2 - \psi(x_2, y_2, v_1)v_1) + (\psi(x_2, y_2, v_1)v_1 - \psi(x_1, y_2, v_1)v_1)$$
$$+ (\psi(x_1, y_2, v_1)v_1 - \psi(x_1, y_1, v_1)v_1).$$

Therefore, from inequalities (24) and (29) we get:

$$sgn\,(x_1 - x_2) = sgn\,(x_2 - x_1)\,,$$

which leads to contradiction. Thus, $sgn\,(x_2 - x_1) = sgn\,(v_1 - v_2)$. Using the equilibrium conditions for E_1 we have $\frac{k\psi(x_1,y_1,v_1)}{ace^{\mu_1\tau_1+\mu_2\tau_2}} = 1$, then

$$R_1 - 1 = \frac{k\psi(x_2, y_2, v_2)}{ace^{\mu_1\tau_1+\mu_2\tau_2}} - \frac{k\psi(x_1, y_1, v_1)}{ace^{\mu_1\tau_1+\mu_2\tau_2}}$$
$$= \frac{k}{ac}e^{-\mu_1\tau_1-\mu_2\tau_2}[\psi(x_2, y_2, v_2) - \psi(x_2, y_2, v_1) + \psi(x_2, y_2, v_1)$$
$$-\psi(x_1, y_2, v_1) + \psi(x_1, y_2, v_1) - \psi(x_1, y_1, v_1)]\,.$$

We get $sgn(R_1 - 1) = sgn(v_1 - v_2)$. Hence, if $R_0 > 1$, then $x_1, y_1, v_1 > 0$, and if $R_1 \leq 1$, then $v_1 \leq v_2 = \frac{\mu}{r}$. It follows from the above discussion that $\frac{dU_1}{dt} \leq 0$ for all $x, y, v, z > 0$. The solutions of system (5)-(8) converge to Ω, the largest invariant subset of $\left\{(x, y, v, z) : \frac{dU_1}{dt} = 0\right\}$ [33]. We have $\frac{dU_1}{dt} = 0$ iff $x = x_1, v = v_1$, $z = 0$ and $H = 0$ i.e.

$$\frac{y_1\psi(x(t-\tau_1), y(t-\tau_1), v(t-\tau_1))v(t-\tau_1)}{y\psi(x_1, y_1, v_1)v_1} = \frac{v_1 y(t-\tau_2)}{vy_1} = 1 \text{ for almost all } \tau_i \in [0, \tau], i = 1, 2. \tag{30}$$

From Eq. (30), if $v = v_1$ then $y = y_1$ and hence $\frac{dU_1}{dt} = 0$ iff $x = x_1, y = y_1, v = v_1$ and $z = 0$. So Ω contains a unique point, that is E_1. Thus, the global asymptotic stability of the chronic-infection equilibrium without antibody immune response E_1 follows from LaSalle's invariance principle.

Theorem 3. Let Assumptions A1-A5 be hold true and $R_1 > 1$, then the chronic-infection equilibrium with antibody immune response E_2 is GAS.

Proof. We construct a Lyapunov functional as follows:

$$\begin{aligned}U_2 &= x - x_2 - \int_{x_2}^{x} \frac{\psi(x_2,y_2,v_2)}{\psi(\eta,y_2,v_2)} d\eta + e^{\mu_1\tau_1} y_2 H\left(\frac{y}{y_2}\right) \\ &+ \frac{a}{k} e^{\mu_1\tau_1+\mu_2\tau_2} v_2 H\left(\frac{v}{v_2}\right) + \frac{ab}{rk} e^{\mu_1\tau_1+\mu_2\tau_2} z_2 H\left(\frac{z}{z_2}\right) \\ &+ \psi(x_2,y_2,v_2)v_2 \int\limits_{t-\tau_1}^{t} H\left(\frac{\psi(x(\eta),y(\eta),v(\eta))v(\eta)}{\psi(x_2,y_2,v_2)v_2}\right) d\eta + a e^{\mu_1\tau_1} y_2 \int\limits_{t-\tau_2}^{t} H\left(\frac{y(\eta)}{y_2}\right) d\eta. \end{aligned} \tag{31}$$

Function U_2 satisfies:

$$\begin{aligned}\frac{dU_2}{dt} &= \left(1 - \frac{\psi(x_2,y_2,v_2)}{\psi(x,y_2,v_2)}\right)(s - dx - \psi(x,y,v)v) \\ &+ e^{\mu_1\tau_1}\left(1 - \frac{y_2}{y}\right)\left(e^{-\mu_1\tau_1}\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1) - ay\right) \\ &+ \frac{a}{k} e^{\mu_1\tau_1+\mu_2\tau_2}\left(1 - \frac{v_2}{v}\right)\left(ke^{-\mu_2\tau_2} y(t-\tau_2) - cv - bvz\right) + \frac{ab}{rk} e^{\mu_1\tau_1+\mu_2\tau_2}\left(1 - \frac{z_2}{z}\right)(rvz - \mu z) \\ &+ \psi(x,y,v)v - \psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1) \\ &+ \psi(x_2,y_2,v_2)v_2 \ln\left(\frac{\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{\psi(x,y,v)v}\right) \\ &+ a e^{\mu_1\tau_1}\left(y - y(t-\tau_2) + y_2 \ln\left(\frac{y(t-\tau_2)}{y}\right)\right). \end{aligned} \tag{32}$$

Applying $s = dx_2 + ae^{\mu_1\tau_1} y_2$, we get

$$\begin{aligned}\frac{dU_2}{dt} &= d\left(1 - \frac{\psi(x_2,y_2,v_2)}{\psi(x,y_2,v_2)}\right)(x_2 - x) + ae^{\mu_1\tau_1} y_2 - ae^{\mu_1\tau_1} y_2 \frac{\psi(x_2,y_2,v_2)}{\psi(x,y_2,v_2)} \\ &+ \psi(x,y,v)v\frac{\psi(x_2,y_2,v_2)}{\psi(x,y_2,v_2)} - \psi(x_2,y_2,v_2)v_2 \frac{y_2\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{y\psi(x_2,y_2,v_2)v_2} \\ &+ ae^{\mu_1\tau_1} y_2 - \frac{ac}{k} e^{\mu_1\tau_1+\mu_2\tau_2} v - ae^{\mu_1\tau_1} y(t-\tau_2)\frac{v_2}{v} + \frac{ac}{k} e^{\mu_1\tau_1+\mu_2\tau_2} v_2 \\ &+ \frac{ab}{k} e^{\mu_1\tau_1+\mu_2\tau_2} v_2 z - \frac{ab\mu}{rk} e^{\mu_1\tau_1+\mu_2\tau_2} z - \frac{ab}{k} e^{\mu_1\tau_1+\mu_2\tau_2} z_2 v + \frac{ab\mu}{rk} e^{\mu_1\tau_1+\mu_2\tau_2} z_2 \\ &+ \psi(x_2,y_2,v_2)v_2 \ln\left(\frac{\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{\psi(x,y,v)v}\right) + ae^{\mu_1\tau_1} y_2 \ln\left(\frac{y(t-\tau_2)}{y}\right) \end{aligned} \tag{33}$$

By using the equilibrium conditions of E_2

$$\psi(x_2,y_2,v_2)v_2 = ae^{\mu_1\tau_1} y_2, \quad cv_2 = \ ke^{-\mu_2\tau_2} y_2 - bv_2 z_2, \quad \mu = rv_2,$$

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and the following equalities

$$cv = cv_2\frac{v}{v_2} = \left(ke^{-\mu_2\tau_2}y_2 - bv_2z_2\right)\frac{v}{v_2},$$

$$\ln\left(\frac{\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{\psi(x,y,v)v}\right) = \ln\left(\frac{y_2\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{y\psi(x_2,y_2,v_2)v_2}\right) + \ln\left(\frac{\psi(x_2,y_2,v_2)}{\psi(x,y_2,v_2)}\right) + \ln\left(\frac{\psi(x,y_2,v_2)}{\psi(x,y,v)}\right) + \ln\left(\frac{v_2y}{vy_2}\right),$$

$$\ln\left(\frac{y(t-\tau_2)}{y}\right) = \ln\left(\frac{vy_2}{v_2y}\right) + \ln\left(\frac{v_2y(t-\tau_2)}{vy_2}\right),$$

we obtain

$$\begin{aligned}\frac{dU_2}{dt} &= d\left(1-\frac{\psi(x_2,y_2,v_2)}{\psi(x,y_2,v_2)}\right)(x_2-x) + ae^{\mu_1\tau_1}y_2\left(\frac{\psi(x,y,v)v}{\psi(x,y_2,v_2)v_2} - \frac{v}{v_2} - 1 + \frac{\psi(x,y_2,v_2)}{\psi(x,y,v)}\right)\\ &- ae^{\mu_1\tau_1}y_2\left[\left(\frac{\psi(x_2,y_2,v_2)}{\psi(x,y_2,v_2)} - 1 - \ln\left(\frac{\psi(x_2,y_2,v_2)}{\psi(x,y_2,v_2)}\right)\right)\right.\\ &+\left(\frac{y_2\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{y\psi(x_2,y_2,v_2)v_2} - 1 - \ln\left(\frac{y_2\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{y\psi(x_2,y_2,v_2)v_2}\right)\right)\\ &\left.+\left(\frac{v_2y(t-\tau_2)}{vy_2} - 1 - \ln\left(\frac{v_2y(t-\tau_2)}{vy_2}\right)\right) + \left(\frac{\psi(x,y_2,v_2)}{\psi(x,y,v)} - 1 - \ln\left(\frac{\psi(x,y_2,v_2)}{\psi(x,y,v)}\right)\right)\right]. \end{aligned} \tag{34}$$

We can rewrite (34) as

$$\begin{aligned}\frac{dU_2}{dt} &= dx_2\left(1-\frac{\psi(x_2,y_2,v_2)}{\psi(x,y_2,v_2)}\right)\left(1-\frac{x}{x_2}\right) + ae^{\mu_1\tau_1}y_2\left(1-\frac{\psi(x,y,v)}{\psi(x,y_2,v_2)}\right)\left(\frac{\psi(x,y_2,v_2)}{\psi(x,y,v)} - \frac{v}{v_2}\right)\\ &- ae^{\mu_1\tau_1}y_2\left[H\left(\frac{\psi(x_2,y_2,v_2)}{\psi(x,y_2,v_2)}\right) + H\left(\frac{y_2\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{y\psi(x_2,y_2,v_2)v_2}\right)\right.\\ &\left.+H\left(\frac{v_2y(t-\tau_2)}{vy_2}\right) + H\left(\frac{\psi(x,y_2,v_2)}{\psi(x,y,v)}\right)\right]. \end{aligned} \tag{35}$$

We note that from Assumptions A2 and A5, the first and second terms of Eq. (35) are less than or equal zero. Noting that $x, y, v, z > 0$, we have that $\frac{dU_2}{dt} \le 0$. The solutions of model (5)-(8) converge to Ω, the largest invariant subset of $\left\{(x, y, v, z) : \frac{dU_2}{dt} = 0\right\}$[33]. We have $\frac{dU_2}{dt} = 0$ iff $x = x_2, v = v_2$ and $H = 0$ i.e.,

$$\frac{y_2\psi(x(t-\tau_1),y(t-\tau_1),v(t-\tau_1))v(t-\tau_1)}{y\psi(x_2,y_2,v_2)v_2} = \frac{v_2y(t-\tau_2)}{vy_2} = 1 \text{ for almost all } \tau_i \in [0,\tau], i = 1,2. \tag{36}$$

If $v = v_2$, then from Eq. (36) we get $y = y_2$. The set Ω is invariant and for any element belongs to Ω satisfies $v = v_2 = \frac{\mu}{r}$. From Eq. (7) we get $z = z_2$. Therefore, $\frac{dU_2}{dt} = 0$ iff $x = x_2, y = y_2, v = v_2$ and $z = z_2$. The global asymptotic stability of the chronic-infection equilibrium with antibody immune response E_2 follows from LaSalle's invariance principle.

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3 Conclusion

In this paper, we have proposed a delayed viral infection model with general incidence rate function and antibody immune response. The model has been incorporated with two kinds of discrete time delays representing the time needed for infecting an uninfected target cell and viral production. We have derived a set of conditions on the general functional response and have determined two threshold parameters R_0 and R_1 to prove the existence and the global stability of the model's equilibria. The global asymptotic stability of the three equilibria, infection-free, chronic-infection without antibody immune response and chronic-infection with antibody immune response has been proven by using direct Lyapunov method and LaSalle's invariance principle.

4 Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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STABILITY OF GENERALIZED CUBIC SET-VALUED FUNCTIONAL EQUATIONS

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Abstract. We will show the general solution of the functional equation

$$\begin{aligned} & f(ax+by)+f(bx-ay)+(a+b)^2(a-b)f(y) \\ =\ & a^2bf(x+y)+ab^2f(x-y)+(a+b)(a-b)^2f(x) \end{aligned}$$

and investigate the Hyers-Ulam stability of cubic set-valued functional equation when $b=1$.

1. Introduction

The theory of set-valued functions in Banach spaces is connected to the control theory and the mathematical economics. Aumann [4] and Debreu [8] wrote papers that were motivated from the topic. We refer the reader to the papers by [1], [18], [10], [3], [17], [7] and [9].

The stability problem of functional equations originated from a question of Ulam [25] concerning the stability of group homomorphisms. Hyers [11] gave a first affirmative partial answer to the question of Ulam. Afterwards, the result of Hyers was generalized by Aoki [2] for additive mapping and by Rassias [23] for linear mappings by considering a unbounded Cauchy difference. Later, the result of Rassias has provided a lot of influence in the development of what we call Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. For further information about the topic, we also refer the reader to [13], [12], [5] and [6].

Jun and Kim [15] introduced the following cubic functional equation:

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x)$$

and established a general solution. Najati [20] investigated the following generalized cubic functional equation:

$$f(ax+y)+f(ax-y)=af(x+y)+af(x-y)+2(a^3-a)f(x)\,. \tag{1.1}$$

In this paper, we deal with the following functional equation:

$$\begin{aligned} & f(ax+by)+f(bx-ay)+(a+b)^2(a-b)f(y) \\ =\ & a^2bf(x+y)+ab^2f(x-y)+(a+b)(a-b)^2f(x) \end{aligned} \tag{1.2}$$

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for all $x, y \in X$ and integers $a, b\,(a > b \geq 1)$. We will show the general solution of the functional equation (1.2) and investigate the Hyers-Ulam stability of cubic set-valued functional equation when $b = 1$.

2. A generalized cubic functional equation

In this section let X and Y be vector spaces and we investigate the general solution of the functional equation (1.2).

Theorem 2.1. *A function $f : X \to Y$ satisfies the functional equation (1.1) if and only if it satisfies the functional equation*

$$\begin{aligned} f(ax+y) + f(x-ay) - a^2 f(x+y) - af(x-y) \\ = (a-1)(a^2-1)f(x) - (a+1)(a^2-1)f(y) \end{aligned} \tag{2.1}$$

Proof. See [16, Theorem 2.1]. □

Theorem 2.2. *A function $f : X \to Y$ satisfies the functional equation (1.1) if and only if it satisfies the functional equation (1.2).*

Proof. Suppose that f satisfies the equation (1.1). Since f satisfies the equation (1.1), it is easy to show $f(0) = 0$, $f(x) = -f(-x)$ and $f(ax) = a^3 f(x)$ for all $x \in X$ and integer a ($a \neq 0, \pm 1$). Replacing x and y in the equation (1.1), we obtain

$$f(x+ay) - f(x-ay) = a[f(x+y) - f(x-y)] + 2a(a^2-1)f(y) \tag{2.2}$$

for all $x, y \in X$ and an integer a ($a \neq 0, \pm 1$). By letting $x = ax$ in the equation (2.2), we have

$$f(ax+y) - f(ax-y) = a^2[f(x+y) - f(x-y)] + 2(1-a^2)f(y) \tag{2.3}$$

for all $x, y \in X$ and an integer a ($a \neq 0, \pm 1$). By replacing x and y in the equation (2.3), we get

$$f(x+ay) + f(x-ay) = a^2[f(x+y) + f(x-y)] + 2(1-a^2)f(x) \tag{2.4}$$

for all $x, y \in X$ and an integer a ($a \neq 0, \pm 1$). Replacing a by b in the equation (1.1), we have

$$f(bx+y) + f(bx-y) = bf(x+y) + bf(x-y) + 2(b^3-b)f(x) \tag{2.5}$$

Letting $y = by$ in the equation (1.1),

$$f(ax+by) + f(ax-by) = af(x+by) + af(x-by) + 2(a^3-a)f(x) \tag{2.6}$$

Letting $y = ay$ in equation (2.5),

$$f(bx+ay) + f(bx-ay) = bf(x+ay) + bf(x-ay) + 2(b^3-b)f(x) \tag{2.7}$$

Replacing x and y in the equation (2.7),

$$f(ax+by) - f(ax-by) = bf(ax+y) - bf(ax-by) + 2(b^3-b)f(y) \tag{2.8}$$

Replacing x and y in equation (2.6),

$$f(bx+ay) - f(bx-ay) = af(bx+y) - af(bx-y) + 2(a^3-a)f(y) \tag{2.9}$$

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Adding two equations (2.6) and (2.8), we obtain

$$2f(ax+by) = af(x+by) + af(x-by) + 2(a^3-a)f(x) + bf(ax+y) - bf(ax-y) + 2(b^3-b)f(y) \tag{2.10}$$

Subtracting (2.9) from (2.7), we have

$$2f(bx-ay) = bf(x+ay) + bf(x-ay) + 2(b^3-b)f(x) - af(bx+y) + af(bx-y) - 2(a^3-a)f(y) \tag{2.11}$$

Now, adding two equations (2.10) and (2.11), we get

$$\begin{aligned} 2[f(ax+by) + f(bx-ay)] &= a[f(x+by) + f(x-by)] \\ &+ b[f(ax+y) - f(ax-y)] + 2(a^3-a)f(x) + 2(b^3-b)f(y) \\ &+ b[f(x+ay) + f(x-ay)] - a[f(bx+y) - f(bx-y)] \\ &+ 2(b^3-b)f(x) - 2(a^3-a)f(y) \end{aligned} \tag{2.12}$$

The desired result is obtained from the equation (2.12) by using the equations (2.3) and (2.4). Conversely, suppose that f satisfies the equation (1.2). Letting $b=1$ in the equation (1.2), we have the equation (2.1). The remains follow from Theorem 2.1. □

If f satisfies the equation (1.2), we call f a *generalized cubic mapping.*

3. Stability of the generalized cubic set-valued functional equation

In this section, we first introduce some definitions and notations which are needed to prove the main theorems. Let Y be a Banach space. The family of all closed subsets, containing 0, of Y will be denoted by $C_z(Y)$. Let A, B be nonempty subsets of a real vector space X and λ a real number. We define

$$\begin{aligned} A+B &= \{a+b \in X \mid a \in A, b \in B\} \\ \lambda A &= \{\lambda a \in X \mid a \in A\}. \end{aligned}$$

Lemma 3.1 ([21]). *Let λ and μ be real numbers. If A and B are nonempty subset of a real vector space, then*

$$\begin{aligned} \lambda(A+B) &= \lambda A + \lambda B \\ (\lambda+\mu)A &\subseteq \lambda A + \mu A. \end{aligned}$$

Moreover, if A is a convex set and $\lambda\mu \geq 0$, then we have

$$(\lambda+\mu)A = \lambda A + \mu A.$$

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A subset $A \subseteq X$ is said to be a *cone* if $A + A \subseteq A$ and $\lambda A \subseteq A$ for all $\lambda > 0$. If the zero vector in X belongs to A, then we say that A is a *cone with zero*.

Let $C_b(Y)$ be the set of all closed bounded subsets of Y, $C_c(Y)$ the set of all closed convex subsets of Y and $C_{cb}(Y)$ the set of all closed bounded convex subsets of Y. For elements A, B of $C_c(Y)$ and positive real values λ, μ, we denote

$$A \oplus B = \overline{A + B}\,.$$

For a subset A of Y, the distance function $d(\cdot, A)$ and the support function $s(\cdot, A)$ are defined by

$$\begin{aligned} d(x, A) &:= \inf\{||x - y||\,|\, y \in A\} \text{ for all } x \in Y \\ s(x^*, A) &:= \sup\{\langle x^*, x\rangle||\,|\, x \in A\} \text{ for all } x^* \in Y^*. \end{aligned}$$

For $A, A' \in C_b(Y)$, the *Hausdorff distance* $h(A, A')$ between A and A' is defined by

$$h(A, A') := \inf\{\alpha \geq 0 \,|\, A \subseteq A' + \alpha B_Y, A' \subseteq A + \alpha B_Y\},$$

where B_Y is the closed unit ball in Y. Castaing and Valadier [7] proved that $(C_{cb}(Y), \oplus, h)$ is a complete metric semigroup. Rådström [22] showed that $(C_{cb}(Y), \oplus, h)$ is isometrically embedded in a Banach space. The following remark is directly obtained from the notion of the Hausdorff distance.

Remark 3.2. Let $A, A', B, B', C \in C_{cb}(Y)$ and $\alpha > 0$. Then the following properties hold:

(1) $h(A \oplus A', B \oplus B') \leq h(A, B) + h(A', B')$
(2) $h(\alpha A, \alpha B) = \alpha h(A, B)$
(3) $h(A, B) = h(A \oplus C, B \oplus C)$.

First, let X be a real vector space, $A \subset X$ a cone with zero and Y a Banach space.

Theorem 3.3. *If $f : A + (-1)A \to C_z(Y)$ is a set-valued mapping with $f(0) = \{0\}$ satisfying*

$$\begin{aligned} &f(ax + y) + f(x - ay) + (a^2 - 1)(a + 1)f(y) \\ &\subseteq\; a^2 f(x + y) + a f(x - y) + (a^2 - 1)(a - 1)f(x) \end{aligned} \tag{3.1}$$

and

$$\sup\{\text{diam}(f(x)) \,|\, x \in A\} < \infty$$

for all $x, y \in A$ and an integer $a\,(a \geq 2)$, then there exists a unique generalized cubic mapping $C : A + (-1)A \to Y$ such that $C(x) \in f(x)$ for all $x \in A$.

Proof. Letting $y = 0$ in (3.1), we have

$$f(ax) \subseteq a^3 f(x) \tag{3.2}$$

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for all $x \in A$ and an integer $a\,(a \geq 2)$. Replacing x by $a^n x\,, n \in \mathbb{N}$ in (3.2), we get

$$f(a^{n+1}x) \subseteq a^3 f(a^n x)$$

and

$$\frac{1}{a^{3(n+1)}} f(a^{n+1}x) \subseteq \frac{1}{a^{3n}} f(a^n x)$$

for all $x \in A$ and an integer $a\,(a \geq 2)$. Let $f_n(x) = \frac{1}{a^{3n}} f(a^n x)$ for each $x \in A\,, n \in \mathbb{N}$. Then $\{f_n(x)\}_{n \geq 0}$ is a decreasing sequence of closed subsets of the Banach space Y. Also, we obtain

$$\operatorname{diam}(f_n(x)) = \frac{1}{a^{3n}} \operatorname{diam}(f(a^n x)).$$

Since $\sup\{\operatorname{diam}(f(x)) \,|\, x \in A\} < \infty$, we have

$$\lim_{n \to \infty} \operatorname{diam}(f_n(x)) = 0.$$

Using the Cantor theorem for the sequence $\{f_n(x)\}_{n \geq 0}$, we get that $\cap_{n \geq 0} f_n(x)$ is a singleton set and we denote this intersection by $C(x)$ for all $x \in A$. Hence we obtain a map $C : A + (-1)A \to Y$ and

$$C(x) \in f_0(x) = f(x)$$

for all $x \in A$. We claim that C is generalized cubic. We note that

$$\begin{aligned}
& f_n(ax+y) + f_n(x-ay) + (a^2-1)(a+1) f_n(y) \\
&= \frac{f(a^n(ax+y))}{a^{3n}} + \frac{f(a^n(x-ay))}{a^{3n}} + \frac{(a^2-1)(a+1) f(a^n y)}{a^{3n}} \\
&\subseteq \frac{a^2 f(a^n(x+y))}{a^{3n}} + \frac{a f(a^n(x-y))}{a^{3n}} + \frac{(a^2-1)(a-1) f(a^n x)}{a^{3n}} \\
&= a^2 f_n(x+y) + a f_n(x-y) + (a^2-1)(a-1) f_n(x)
\end{aligned}$$

for all $x \in A$ and an integer $a\,(a \geq 2)$. By the definition of C, we obtain

$$\begin{aligned}
& C(ax+y) + C(x-ay) + (a^2-1)(a+1) C(y) \\
&= \cap_{n=0}^{\infty} \Big(f_n(ax+y) + f_n(x-ay) + (a^2-1)(a+1) f_n(y) \Big) \\
&\subseteq \cap_{n=0}^{\infty} \Big(a^2 f_n(x+y) + a f_n(x-y) + (a^2-1)(a-1) f_n(x) \Big)
\end{aligned}$$

for all $x \in A$ and an integer $a\,(a \geq 2)$. Hence we have

$$\begin{aligned}
& ||C(ax+y) + C(x-ay) + (a^2-1)(a+1) C(y) \\
& \qquad - a^2 C(x+y) - a C(x-y) - (a^2-1)(a-1) C(x)|| \\
& \leq a^2 \operatorname{diam}(f_n(x+y)) + a \operatorname{diam}(f_n(x-y)) + (a^2-1)(a-1) \operatorname{diam}(f_n(x)),
\end{aligned}$$

which tends to zero as $n \to \infty$. Thus C satisfies the equality (1.2). Hence C is a generalized cubic, as claimed. Next, let us prove the uniqueness of C. Assume f has two generalized cubic functional equations C_1 and C_2 from $A + (-1)A$ into Y. Then we have

$$(an)^3 C_i(x) = C_i(anx) \in f(anx)$$

for all $x \in X$, $n \in \mathbb{N}$ and $i \in \{1, 2\}$. Then we have

$$\begin{aligned}(an)^3||C_1(x) - C_2(x)|| &= ||(an)^3C_1(x) - (an)^3C_2(x)|| \\ &= ||(C_1(anx) - C_2(anx)|| \\ &\leq \operatorname{diam}(f(anx))\end{aligned}$$

for all $x \in X$, $n \in \mathbb{N}$. Since $\sup\{\operatorname{diam}(f(x)) \,|\, x \in A\} < \infty$, $C_1(x) = C_2(x)$, for all $x \in X$. □

Definition 3.4. Let $f : X \to C_{cb}(Y)$. The generalized cubic set-valued functional equation is defined by

$$\begin{aligned}f(ax+y) \oplus f(x-ay) \oplus (a^2-1)(a+1)f(y) \\ = a^2 f(x+y) \oplus af(x-y) \oplus (a^2-1)(a-1)f(x)\end{aligned} \tag{3.3}$$

for all $x \in A$ and an integer $a\,(a \geq 2)$. Every solution of the generalized cubic set-valued functional equation is called a *generalized cubic set-valued mapping.*

Theorem 3.5. *Let $\phi : X \times X \to [0, \infty)$ be a function such that*

$$\widetilde{\phi}(x, y) := \sum_{j=0}^{\infty} \frac{1}{a^{3j}} \phi(a^j x,\, a^j y) < \infty \tag{3.4}$$

for all $x, y \in X$ and an integer $a\,(a \geq 2)$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping with $f(0) = \{0\}$ satisfying

$$\begin{aligned}h\Big(f(ax+y) \oplus f(x-ay) \oplus (a^2-1)(a+1)f(y), \\ a^2 f(x+y) \oplus af(x-y) \oplus (a^2-1)(a-1)f(x)\Big) \leq \phi(x,\, y)\end{aligned} \tag{3.5}$$

for all $x, y \in X$ and an integer $a\,(a \geq 2)$. Then there exists a unique generalized cubic set-valued mapping $C : X \to (C_{cb}(Y),\, h)$ such that

$$h(f(x),\, C(x)) \leq \frac{1}{a^3} \widetilde{\phi}(x, 0) \tag{3.6}$$

for all $x, y \in X$ and an integer $a\,(a \geq 2)$.

Proof. Let $y = 0$ in the inequality (3.5). Since $f(x)$ is convex, we have

$$h\Big(f(ax) \oplus f(x),\, a^2 f(x) \oplus af(x) \oplus (a^2-1)(a-1)f(x)\Big) \leq \phi(x,\, 0)\,,$$

that is,

$$h\Big(f(x),\, \frac{1}{a^3} f(ax)\Big) \leq \frac{1}{a^3} \phi(x, 0) \tag{3.7}$$

for all $x \in X$. Replacing x by $a^k x$, $k \in \mathbb{N}$, we get

$$h\Big(f(a^k x), \frac{1}{a^3} f(a^{k+1} x)\Big) \leq \frac{1}{a^3} \phi(a^k x, 0)$$

and

$$h\Big(\frac{1}{a^{3k}} f(a^k x), \frac{1}{a^{3(k+1)}} f(a^{k+1} x)\Big) \leq \frac{1}{a^{3(k+1)}} \phi(a^k x, 0)$$

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for all $x \in X$. Using the induction on k , we obtain

$$h\Big(f(x), \frac{1}{a^{3n}} f(a^n x)\Big) \le \frac{1}{a^3} \sum_{k=0}^{n-1} \frac{1}{a^{3k}} \phi(a^k x, 0) \tag{3.8}$$

for all $x \in X$ and $n \in \mathbb{N}$. Dividing the inequality (3.8) by a^{3m} and replacing x by $a^m x$, we have

$$h\Big(\frac{1}{a^{3m}} f(a^m x), \frac{1}{a^{3(n+m)}} f(a^{n+m} x)\Big) \le \frac{1}{a^3} \frac{1}{a^m} \sum_{k=0}^{n-1} \frac{1}{a^{3k}} \phi(a^{m+k} x, 0) \tag{3.9}$$

for all $x \in X$ and $n, m \in \mathbb{N}$. Since the right-hand side of the inequality (3.9) tends to zero as $m \to \infty$, the sequence $\{\frac{1}{a^{3n}} f(a^n x)\}$ is a Cauchy sequence in $(C_{cb}(Y), h)$. By the completeness of $C_{cb}(Y)$, we can define

$$C(x) := \lim_{n\to\infty} \frac{1}{a^{3n}} f(a^n x)$$

for all $x \in X$ and an integer $a\,(a \ge 2)$. We note that

$$\begin{aligned} & h\Big(\frac{f(a^n(ax+y))}{a^{3n}} \oplus \frac{f(a^n(x-ay))}{a^{3n}} \oplus \frac{(a^2-1)(a+1)f(a^n y)}{a^{3n}}, \\ & \frac{a^2 f(a^n(x+y))}{a^{3n}} \oplus \frac{af(a^n(x-y))}{a^{3n}} \oplus \frac{(a^2-1)(a-1)f(a^n x)}{a^{3n}}\Big) \\ & \le \frac{1}{a^{3n}} \phi(a^n x, a^n y) \end{aligned}$$

for all $x, y \in X$ and an integer $a\,(a \ge 2)$. By the definition of C , we have

$$\begin{aligned} & h\Big(C(ax+y) \oplus C(x-ay) \oplus (a^2-1)(a+1)C(y), \\ & \qquad a^2 C(x+y) \oplus aC(x-y) \oplus (a^2-1)(a-1)C(x)\Big) \\ & \le \lim_{n\to\infty} \frac{1}{a^{3n}} \phi(a^n x, a^n y) = 0 . \end{aligned}$$

Hence C is a generalized cubic set-valued mapping. Now, by taking $n \to \infty$ in the inequality (3.8), we have the inequality (3.6). It remains to show the uniqueness of C . Assume $C' : X \to (C_{cb}(Y), h)$ is another generalized cubic set-valued mapping satisfying the inequality (3.6). Then

$$\begin{aligned} h\Big(C(x), C'(x)\Big) &= \frac{1}{a^{3n}} h\Big(C(a^n x), C'(a^n x)\Big) \\ &\le \frac{1}{a^{3n}} h\Big(C(a^n x), f(a^n x)\Big) + \frac{1}{a^{3n}} h\Big(f(a^n x), C'(a^n x)\Big) \\ &\le \frac{2}{a^{3(n+1)}} \widetilde{\phi}(a^n x, 0) \end{aligned}$$

for all $x \in X$. Since $\frac{2}{a^{3(n+1)}} \widetilde{\phi}(a^n x, 0) \to 0$ as $n \to \infty$, we may conclude that the generalized cubic set-valued mapping C is unique. □

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Corollary 3.6. *Let $0 < p < 3\,, \theta \geq 0$ be real numbers and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y),\, h)$ is a mapping with $f(0) = \{0\}$ satisfying*

$$h\Big(f(ax+y) \oplus f(x-ay) \oplus (a^2-1)(a+1)f(y),$$
$$a^2 f(x+y) \oplus af(x-y) \oplus (a^2-1)(a-1)f(x)\Big) \leq \theta(||x||^p + ||y||^p)$$

for all $x,\, y \in X$ and an integer $a\,(a \geq 2)$. Then there exists a unique generalized cubic set-valued mapping $C : X \to (C_{cb}(Y),\, h)$ satisfying

$$h(f(x),\, C(c)) \leq \frac{\theta}{a^3 - a^p}||x||^p$$

for all $x,\, y \in X$ and an integer $a\,(a \geq 2)$.

Proof. It follows from Theorem 3.5 by letting $\phi(x,\, y) = \theta(||x||^p + ||y||^p)$ for all $x,\, y \in X$. □

4. Stability of set-valued functional equation by the fixed point method

Now, we will investigate the stability of the given functional equation (3.3) using the alternative fixed point method. Before proceeding the proof, we will state the theorem, the alternative of fixed point; see [19] and [24].

Definition 4.1. Let X be a set. A function $d : X \times X \to [0,\, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x,\, y) = 0$ if and only if $x = y$;
(2) $d(x,\, y) = d(y,\, x)$ for all $x,\, y \in X$;
(3) $d(x,\, z) \leq d(x,\, y) + d(y,\, z)$ for all $x,\, y,\, z \in X$.

Theorem 4.2. *[The alternative of fixed point* [19], [24] *] Suppose that we are given a complete generalized metric space (Ω, d) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then for each given $x \in \Omega$, either*

$$d(T^n x,\, T^{n+1}x) = \infty \text{ for all } n \geq 0\,,$$

or there exists a natural number n_0 such that

(1) *$d(T^n x,\, T^{n+1}x) < \infty$ for all $n \geq n_0$;*
(2) *The sequence $(T^n x)$ is convergent to a fixed point y^* of T;*
(3) *y^* is the unique fixed point of T in the set*

$$\triangle = \{y \in \Omega | d(T^{n_0}x, y) < \infty\}\,;$$

(4) *$d(y, y^*) \leq \frac{1}{1-L}\, d(y, Ty)$ for all $y \in \triangle$.*

Theorem 4.3. *Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping with $f(0) = \{0\}$ satisfying*

$$h\Big(f(ax+y) \oplus f(x-ay) \oplus (a^2-1)(a+1)f(y), \tag{4.1}$$

$$a^2f(x+y)\oplus af(x-y)\oplus(a^2-1)(a-1)f(x)\Big)\le\phi(x,\,y)$$

for all $x,\,y\in X$ *and an integer* $a\,(a\ge2)$ *and there exists a constant* L *with* $0<L<1$ *for which the function* $\phi: X^2\to\mathbb{R}^+$ *satisfies*

$$\phi(ax,0)\le a^3L\phi(x,0) \tag{4.2}$$

for all $x\in X$. *Then there exists a unique generalized cubic set-valued mapping* $C: X\to(C_{cb}(Y),\,h)$ *given by* $C(x)=\lim_{n\to\infty}\frac{f(a^nx)}{a^{3n}}$ *such that*

$$h(f(x),\,C(x))\le\frac{1}{a^3(1-L)}\widetilde{\phi}(x,0) \tag{4.3}$$

for all $x,\,y\in X$ *and an integer* $a\,(a\ge2)$.

Proof. Consider the set

$$\Omega=\{g\,|\,g: X\to C_{cb}(Y)\,,g(0)=\{0\}\}$$

and introduce the generalized metric on Ω defined by

$$d(g_1,\,g_2)=\inf\{\mu\in(0,\infty)\,|\,h(g_1(x)\,,g_2(x))\le\mu\phi(x,0)\,,\text{for all }x\in X\}\,.$$

We note that $\inf\emptyset:=\infty$. It is easy to show that (Ω,d) is complete; see [14]. Now we define a function $T:\Omega\to\Omega$ by

$$T(g)(x)=\frac{1}{a^3}g(ax) \tag{4.4}$$

for all $x\in X$. Note that for all $g_1,\,g_2\in\Omega$, let $\mu\in(0,\,\infty)$ be an arbitrary constant with $d(g_1,\,g_2)=\mu$. Then

$$h(\frac{1}{a^3}g_1(ax)\,,\frac{1}{a^3}g_2(ax))\le\frac{\mu}{a^3}\phi(ax,0) \tag{4.5}$$

for all $x\in X$. By using (4.2), we have

$$h(\frac{1}{a^3}g_1(ax)\,,\frac{1}{a^3}g_2(ax))\le\mu\,L\phi(x,0) \tag{4.6}$$

for all $x\in X$. Hence we obtain

$$d(Tg_1,\,Tg_2)\le Ld(g_1,\,g_2)$$

for all $g_1\,,g_2\in\Omega$, that is, T is a strictly self-mapping of Ω with the Lipschitz constant L. Letting $y=0$ in the inequality (4.1), we get

$$h(\frac{1}{a^3}f(ax)\,,f(x))\le\frac{1}{a^3}\phi(x,0)$$

for all $x\in X$. This means that

$$d(Tf,f)\le\frac{1}{a^3}\,.$$

By Theroem 4.2, there exists a fixed point $C: X\to(C_{cb}(Y),\,h)$ of T in $\{g\in\Omega\,|\,d(g_1\,,g_2)<\infty\}$ such that $\{T^kf\}\to0$ ad $k\to\infty$. Hence we have

$$C(x)=\lim_{n\to\infty}\frac{f(a^nx)}{a^{3n}}\,, \tag{4.7}$$

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for all $x \in X$. Also, we have

$$d(f, C) \leq \frac{1}{1-L} d(Tf, f) \leq \frac{1}{a^3} \frac{1}{1-L} .$$

This implies that the inequality (4.3) holds for all $x \in X$. By the inequalities (4.1) and (4.2), we have

$$\begin{aligned} & h\Big(C(ax+y) \oplus C(x-ay) \oplus (a^2-1)(a+1)C(y), \\ & \qquad a^2 C(x+y) \oplus aC(x-y) \oplus (a^2-1)(a-1)C(x)\Big) \\ & \leq \lim_{n \to \infty} L^n \phi(a^n x,\, a^n y) = 0 \end{aligned}$$

for all $x, y \in X$ and an integer $a\,(a \geq 2)$. Thus C is a unique generalized cubic set-valued mapping. □

Corollary 4.4. *Let $0 < p < 3$ and $\theta \geq 0$ be real numbers and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping with $f(0) = \{0\}$ satisfying*

$$\begin{aligned} & h\Big(f(ax+y) \oplus f(x-ay) \oplus (a^2-1)(a+1)f(y), \\ & \quad a^2 f(x+y) \oplus af(x-y) \oplus (a^2-1)(a-1)f(x)\Big) \leq \theta(||x||^p + ||y||^p) \end{aligned} \tag{4.8}$$

for all $x, y \in X$ and an integer $a\,(a \geq 2)$. Then there exists a unique generalized cubic set-valued mapping $C : X \to (C_{cb}(Y),\, h)$ such that

$$h(f(x),\, C(x)) \leq \frac{\theta}{a^3 - a^p} ||x||^p \tag{4.9}$$

for all $x \in X$ and an integer $a\,(a \geq 2)$.

Proof. It follows from Theorem 4.3 by letting $\phi(x,\, y) = \theta(||x||^p + ||y||^p)$ for all $x, y \in X$. Then we can choose $L = a^{p-3}$ and hence we have the desired result. □

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A new regularity (p-regularity) of stratified L-generalized convergence spaces

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Abstract: In the classical theory of convergence spaces, both regularity (p-regularity) and topologicallness (p-topologicallness) are important notions. It is well known that topologicallness (p-topologicallness) can be described by a sophisticated Fischer-type diagonal condition, and regularity (p-regularity) can be described by dualizing that diagonal condition. Additionally, regularity (p-regularity) can also be characterized by the notion of closures of filters. In this paper, for stratified L-generalized convergence spaces, a new regularity (p-regularity) is defined by duzlizing a Fischer-type diagonal condition, which is used to describe the L-topologicallness of stratified L-convergence spaces (a subcategory of stratified L-generalized convergence spaces). Additionally, a characterization on this new regularity (p-regularity) by a notion of closures of stratified L-filters, is also presented.

Keywords: Topology; Lattice-valued topology; Lattice-valued convergence space; regularity; Diagonal condition

1 Introduction

p-topologicallness [17] and p-regularity [11] are dual notions in the classical theory of convergence spaces [16]. For a set X, let $\mathbb{F}(X)$ denote the set of all filters on X. Let q and p be convergence structures on a set X. Then the space (X, q) is called p-topological if it satisfies either of the two conditions below.

(1) $\mathbb{U}_p(\mathbb{F}) \xrightarrow{q} x$ whenever $\mathbb{F} \xrightarrow{q} x$, where $\mathbb{U}_p(\mathbb{F})$ is the neighborhood of $\mathbb{F}$ w.r.t p.

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(2) (Fischer-type diagonal condition) Let J be any set, $\psi : J \longrightarrow X$, and let $\sigma : J \longrightarrow \mathbb{F}(X)$ have the condition that $\sigma(j) \xrightarrow{p} \psi(j)$, for all $j \in J$. If $\mathbb{F} \in \mathbb{F}(J)$ is such that $\psi(\mathcal{F}) \xrightarrow{q} x$, then $k\sigma\mathbb{F} \xrightarrow{q} x$. Here, $k\sigma\mathbb{F} = \bigcup\limits_{F\in\mathbb{F}} \bigcap\limits_{j\in F} \sigma(j) \in \mathbb{F}(X)$ is called the compression of $\mathbb{F}$ relative to σ.

The space (X, q) is called p-regular if it satisfies either of the two conditions below.

(1) $\overline{\mathbb{F}}_p \xrightarrow{q} x$ whenever $\mathbb{F} \xrightarrow{q} x$, where $\overline{\mathbb{F}}_p$ is the closure of $\mathbb{F}$ w.r.t p.

(2) (Dual Fischer-type diagonal condition) Let J be any set, $\psi : J \longrightarrow X$, and let $\sigma : J \longrightarrow \mathbb{F}(X)$ have the condition that $\sigma(j) \xrightarrow{p} \psi(j)$, for all $j \in J$. If $\mathbb{F} \in \mathbb{F}(J)$ is such that $k\sigma\mathbb{F} \xrightarrow{q} x$, then $\psi(\mathcal{F}) \xrightarrow{q} x$.

When $p = q$, p-topologicallness and p-regularity are refereed to topologicallness and regularity [1, 3, 12], respectively.

Stratified L-generalized convergence spaces defined by Jäger [7] are lattice- valued extensions of convergence spaces. In [9], Jäger studied a regularity of stratified L-generalized convergence spaces both by a dual Fischer-type diagonal condition and a notion of α-lever closures of stratified L-filters. Later, Li and Jin [14] generalized Jäger's regularity to p-regularity. Quite recently, by modifying Jäger's Fischer-type diagonal condition, the first author and his co-author [15] introduced a new Fischer-type diagonal condition, and proved that this condition happens to characterize the topologicallness of stratified L-convergence spaces [4, 13] (a subcategory of stratified L-generalized convergence spaces). In this paper, by dualizing that diagonal condition, a new regularity (p-regularity) of stratified L-generalized convergence spaces is defined, and a characterization on this new regularity (p-regularity) by the notion of closures of stratified L-filters, is also presented.

The contents are arranged as follows. Section 2 fixes some notions and notations used in this note. Section 3 recalls the Fischer-type diagonal notion such that stratified L-convergence spaces are L-topological. Section 4 presents the main results. That is, by dualizing a Fischer-type diagonal condition in Section 3, we define a new regularity (p-regularity) of stratified L-generalized convergence spaces and then present a characterization on that regularity (p-regularity) by a notion of closures of stratified L-filters.

In this paper, if not otherwise specified, $L = (L, \leq)$ is always a complete lattice with a top element 1 and a bottom element 0, which satisfies the distributive law $\alpha \wedge (\bigvee_{i\in I} \beta_i) = \bigvee_{i\in I}(\alpha \wedge \beta_i)$. A lattice with these conditions is called a complete Heyting algebra or a frame. The operation $\rightarrow: L \times L \longrightarrow L$ given by $\alpha \rightarrow \beta = \bigvee\{\gamma \in L : \alpha \wedge \gamma \leq \beta\}$, is called the residuation with respect to $\wedge$. For the properties of $\wedge$ and $\rightarrow$, please refer to the literatures [6, 7, 14].

For a set X, the set L^X of functions from X to L with the pointwise order becomes a complete lattice. Each element of L^X is called an L-set (or a fuzzy subset) of X. And

we make no difference between a constant function and its value since no confusion will arise. Let $f : X \longrightarrow Y$ be a function. We define$f^{\leftarrow} : L^Y \longrightarrow L^X$ [6] by $f^{\leftarrow}(\mu) = \mu \circ f$ for $\mu \in L^Y$.

Let X be a set. A fuzzy partial order (or, an L-partial order) on X [2] is a reflexive, transitive and antisymmetric fuzzy relation on X. The pair (X, R) is called an L-partially ordered set. Let $[L^X] : L^X \times L^X \longrightarrow L$ be a function defined by $[L^X](\lambda, \mu) = \bigwedge_{x\in X}(\lambda(x) \to \mu(x))$. Then $[L^X]$ is an L-partial order on L^X [2, 19]. The value $[L^X](\lambda, \mu) \in L$ is interpreted as the degree that λ is contained in μ. In the sequel, we use the symbol $[\lambda, \mu]$ to denote $[L^X](\lambda, \mu)$ for simplicity. The following lemma is useful to the subsequent section.

Lemma 1.1. *[14] Let $f : X \longrightarrow Y$ be an function. For any $\lambda, \mu, \nu \in L^X$ and any $\{\lambda_i\}_{i\in I}, \{\mu_i\}_{i\in I} \subseteq L^X$, we have (1) $\lambda \leq \mu$ implies $[\lambda, \nu] \geq [\mu, \nu]$; (2) $[\lambda, \wedge_{i\in I}\mu_i] = \wedge_{i\in I}[\lambda, \mu_i]$; (3) $\lambda \wedge [\lambda, \mu] \leq \mu$; (4) $[\vee_{i\in I}\lambda_i, \mu] = \wedge_{i\in I}[\lambda_i, \mu]$; (5) $[\lambda, \mu] \leq [f^{\rightarrow}(\lambda), f^{\rightarrow}(\mu)]$.*

A stratified L-filter [6] on a set X is a function $\mathcal{F} : L^X \longrightarrow L$ such that: (F0) $\mathcal{F}(0) = 0$, (F1) $\mathcal{F}(1) = 1$, (F2) $\forall \lambda, \mu \in L^X$, $\mathcal{F}(\lambda) \wedge \mathcal{F}(\mu) = \mathcal{F}(\lambda \wedge \mu)$, (Fs) $\forall \alpha \in L$, $\mathcal{F}(\alpha) \geq \alpha$. The set $\mathcal{F}^s_L(X)$ of all stratified L-filters on X is ordered by $\mathcal{F} \leq \mathcal{G} \Leftrightarrow \forall \lambda \in L^X, \mathcal{F}(\lambda) \leq \mathcal{G}(\lambda)$. There is a natural fuzzy partial order on $\mathcal{F}^s_L(X)$ inherited from $L^{(L^X)}$. Precisely, for all $\mathcal{F}, \mathcal{G} \in \mathcal{F}^s_L(X)$, if we let $[\mathcal{F}^s_L(X)](\mathcal{F}, \mathcal{G}) = [L^{L^X}](\mathcal{F}, \mathcal{G}) = \bigwedge_{\lambda\in L^X}(\mathcal{F}(\lambda) \to \mathcal{G}(\lambda))$, then $[\mathcal{F}^s_L(X)]$ is an L-partially order.

Example 1.2. (1) For each point x in a set X, the function $[x] : L^X \longrightarrow L, [x](\lambda) = \lambda(x)$ is a stratified L-filter on X. (2) If $\{\mathcal{F}_j | j \in J\} \subseteq \mathcal{F}^s_L(X)$, then $\bigwedge_{j\in J} \mathcal{F}_j \in \mathcal{F}^s_L(X)$.

(3) Let $f : X \longrightarrow Y$ be a function. If $\mathcal{F} \in \mathcal{F}^s_L(X)$, then the function $f^{\Rightarrow}(\mathcal{F}) \in \mathcal{F}^s_L(Y)$, where $f^{\Rightarrow}(\mathcal{F}) : L^Y \longrightarrow L$ is defined by $\lambda \mapsto \mathcal{F}(\lambda \circ f) = \mathcal{F}(f^{\leftarrow}(\lambda))$.

2 Fischer-type diagonal condition of stratified L-convergence spaces

In this section, we shall recall the Fischer-type diagonal condition such that a stratified L-convergence space is L-topological.

Definition 2.1. A stratified L-generalized convergence structure [7, 18] on a set X is a function $\lim^q : \mathcal{F}^s_L(X) \longrightarrow L^X$ satisfying **(LC1)** $\forall x \in X, \lim^q[x](x) = 1$; and **(LC2)** $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}^s_L(X)$, $\mathcal{F} \leq \mathcal{G} \Longrightarrow \lim^q\mathcal{F} \leq \lim^q\mathcal{G}$. The pair $(X, \lim^q)$ is called a stratified L-generalized convergence space. The pair $(X, \lim^q)$ is called a stratified L-convergence space [13] (or, a stratified L-ordered convergence space in [4]) if $\lim : \mathcal{F}^s_L(X) \longrightarrow L^X$ is a function satisfying **(LC1)** and **(LC2′)** $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}^s_L(X), [\mathcal{F}^s_L(X)](\mathcal{F}, \mathcal{G}) \leq [L^X](\lim^q\mathcal{F}, \lim^q\mathcal{G})$. Because **(LC2)′⇒(LC2)**, a stratified L-convergence space is a

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stratified L-generalized convergence space. A function $f : X \longrightarrow X'$ between two stratified L-generalized convergence spaces $(X, \lim^q)$, $(X', \lim^{q'})$ is called continuous if for all $\mathcal{F} \in \mathcal{F}_L^s(X)$ and all $x \in X$ we have $\lim^q \mathcal{F}(x) \le \lim^{q'} f^{\Rightarrow}(\mathcal{F})(f(x))$.

For a given source $(X \xrightarrow{f_i} (X_i, \lim^{q_i}))_{i\in I}$, the *initial structure*, $\lim^q$ on X is defined by $\forall \mathcal{F} \in \mathcal{F}_L^s(X), \forall x \in X$, $\lim^q\mathcal{F}(x) = \bigwedge\limits_{i\in I} \lim^{q_i} f_i^{\Rightarrow}(\mathcal{F})(f_i(x))$.

For a given sink $((X_i, \lim^{q_i}) \xrightarrow{f_i} X)_{i\in I}$, the *final structure*, $\lim^q$ on X is defined by

$$\lim^q\mathcal{F}(x) = \begin{cases} 1, & \mathcal{F} \ge [x]; \\ \bigvee_{i\in I, x_i\in X_i, \mathcal{G}_i\in\mathcal{F}_L^s(X_i), f_i(x_i)=x, f_i^{\Rightarrow}(\mathcal{G}_i)\le\mathcal{F}} \lim^{q_i}\mathcal{G}_i(x_i), & \mathcal{F} \not\ge [x]. \end{cases}$$

When $X = \cup_{i\in I} f_i(X_i)$, the final structure $\lim^q$ can be simplified as [14]

$$\lim^q\mathcal{F}(x) = \bigvee_{i\in I, x_i\in X_i, \mathcal{G}_i\in\mathcal{F}_L^s(X_i), f_i(x_i)=x, f_i^{\Rightarrow}(\mathcal{G}_i)\le\mathcal{F}} \lim^{q_i}\mathcal{G}_i(x_i).$$

In the theory of convergence spaces, Fischer-type diagonal condition is formulated by the aid of the notion of compression. The situation with lattice-valued convergence is similar. In [8], Jäger introduced an lattice-valued version of compression, which first appeared in [5] with a slightly different formalization.

Let $\sigma : J \longrightarrow \mathcal{F}_L^s(X)$ be a function and $\mathcal{F} \in \mathcal{F}_L^s(X)$. Then the function $k_L\sigma\mathcal{F} : L^X \longrightarrow L$ defined by

$$\forall \lambda \in L^X, k_L\sigma\mathcal{F}(\lambda) := \mathcal{F}(\widehat{\sigma}(\lambda)), \text{ where } \widehat{\sigma}(\lambda) = \sigma(-)(\lambda) \in L^J$$

forms a stratified L-filter on X; and it is called the compression of $\mathcal{F}$ w.r.t σ.

In [15], the first author and his co-author modified Jäger's compression and introduced a Fischer-type diagonal condition. It was proved that a stratified L-convergence space with this diagonal condition is L-topological.

Note that when a function $\sigma : J \longrightarrow \mathcal{F}_L^s(X)$ being given, that means an L-filter $\sigma(j)$ is selected for each $j \in J$. In this sense, we call $\sigma : J \longrightarrow \mathcal{F}_L^s(X)$ an L-filter select function. The definition below generalizes that notion.

Definition 2.2. [15] A function $\sigma = (\sigma_1, \sigma_2) : J \longrightarrow \mathcal{F}_L^s(X) \times L_0$, where $L_0 = L - \{0\}$, is said to be an L-filter select degree function. For any $j \in J$, the value $\sigma_2(j) \in L$ is interpreted as the degree to which the stratified L-filter $\sigma_1(j)$ is selected. Obviously, an L-filter select function can be regarded as an L-filter select degree function with $\sigma_2 \equiv 1$.

Definition 2.3. [15] Let $\sigma : J \longrightarrow \mathcal{F}_L^s(X) \times L_0$ be an L-filter select degree function and $\mathcal{F} \in \mathcal{F}_L^s(J)$. If the function $k_L\sigma\mathcal{F} : L^X \longrightarrow L$ defined by

$$\forall \lambda \in L^X, k_L\sigma\mathcal{F}(\lambda) := \mathcal{F}(\widehat{\sigma}(\lambda)), \text{ where } \widehat{\sigma}(\lambda) = \sigma_2(-) \to \sigma_1(-)(\lambda) \in L^J$$

forms a stratified L-filter on X, then we call such $\mathcal{F}$ compressible w.r.t σ and call $k_L\sigma\mathcal{F}$ as the compression of $\mathcal{F}$ w.r.t σ. It is easily seen that $k_L\sigma\mathcal{F}$ satisfies (F1), (F2) and (Fs) for any $\mathcal{F} \in \mathcal{F}_L^s(J)$.

If $\sigma : J \longrightarrow \mathcal{F}_L^s(X) \times L_0$ is an L-filter select function, then $k_L\sigma\mathcal{F} \in \mathcal{F}_L^s(X)$ for any $\mathcal{F} \in \mathcal{F}_L^s(J)$. In this case, $k_L\sigma\mathcal{F}$ coincides with Jäger's compression. Thus, Definition 2.3 generalizes Jäger's compression.

Theorem 2.4. *[15] Let $(X, \lim^q)$ be a stratified L-convergence spaces. Then $(X, \lim^q)$ is L-topological if and only if it satisfies the following condition* **(Lf)**.

(Lf) *Let J be any set, $\psi : J \longrightarrow X$, and let $\sigma : J \longrightarrow \mathcal{F}_L^s(X) \times L_0$. If $\mathcal{F} \in \mathcal{F}_L^s(J)$ is compressible w.r.t σ, then for each $x \in X$,*

$$\lim{}^q \psi^{\Rightarrow}(\mathcal{F})(x) * \bigwedge_{j\in J} \lim{}^q \sigma(j)(\psi(j)) \leq \lim{}^q k_L\sigma\mathcal{F}(x),$$

where $\lim^q \sigma(j)(\psi(j)) := \sigma_2(j) \rightarrow \lim^q \sigma_1(j)(\psi(j))$.

Obviously, the condition **(Lf)** implies the following condition **(Lfw)**.

(Lfw): Let J be any set, $\psi : J \longrightarrow X$, and let $\sigma : J \longrightarrow \mathcal{F}_L^s(X) \times L_0$ have the condition $\forall j \in J$, $\sigma_2(j) = \lim^q \sigma_1(j)(\psi(j))$ (which means that $\lim^q \sigma(j)(\psi(j)) \equiv 1$). If $\mathcal{F} \in \mathcal{F}_L^s(J)$ is compressible w.r.t σ, then $\lim^q \psi^{\Rightarrow}(\mathcal{F})(x) \leq \lim^q k_L\sigma\mathcal{F}(x)$ for each $x \in X$.

Note that in the proof of the sufficiency of Theorem 2.4, the selected σ, ψ satisfies the condition $\sigma_2(j) = \lim^q \sigma_1(j)(\psi(j))$ (see Theorem 4.9 in [15]). It follows immediately that **(Lf)**$\Leftrightarrow$**(Lfw)**. In addition, the characterization on L-topologicallness of stratified L-convergence spaces by the notion of neighborhoods of stratified L-filters, was presented in [10].

3 regularity and p-regularity of stratified L-generalized convergence spaces

In this section, by dualizing the condition **(Lfw)** we define a new regularity (p-regularity) of stratified L-generalized convergence spaces. Then we also present a characterization on that regularity (p-regularity) by a notion of closures of stratified L-filters.

Let $(X, \lim^p, \lim^q)$ be a pair of stratified L-generalized convergence spaces.

p-**(DLfw)**: Let J be any set, $\psi : J \longrightarrow X$, and let $\sigma : J \longrightarrow \mathcal{F}_L^s(X) \times L_0$ have the condition $\forall j \in J$, $\sigma_2(j) = \lim^p \sigma_1(j)(\psi(j))$. If $\mathcal{F} \in \mathcal{F}_L^s(J)$ is compressible w.r.t σ, then $\lim^q k_L\sigma\mathcal{F}(x) \leq \lim^q \psi^{\Rightarrow}(\mathcal{F})(x)$ for each $x \in X$.

When $\lim^p = \lim^q$, the condition p-**(DLfw)** is denoted as **(DLfw)**. Obviously, the condition **(DLfw)** is obtained by dualizing the condition **(Lfw)**.

It is easily seen that when $L = \{0, 1\}$, the condition p-**(DLfw)** is equivalent to the crisp dual Fischer-type diagonal condition.

Definition 3.1. Let $(X, \lim^p, \lim^q)$ be a pair of stratified L-generalized convergence spaces. Then $(X, \lim^q)$ is called p-regular if it satisfies the dual Fischer-type diagonal condition p-**(DLfw)**. When $\lim^p = \lim^q$, then $(X, \lim^q)$ is called regular if it is p-regular.

In the following, we shall give a characterization on regularity (p-regularity) by the notion of closures of stratified L-filters.

Definition 3.2. Let $(X, \lim^p)$ be a stratified L-generalized convergence space, and let $\lambda \in L^X$. Then the L-set $\overline{\lambda}_p \in L^X$ defined by

$$\forall x \in X, \overline{\lambda}_p(x) = \bigvee_{\mathcal{F} \in \mathcal{F}_L^s(X):\ \lim^p \mathcal{F}(x) \neq 0} (\lim^p \mathcal{F}(x) \to \mathcal{F}(\lambda))$$

is called the closure of λ w.r.t $(X, \lim^p)$.

Remark 3.3. When $L = \{0, 1\}$, a stratified L-generalized convergence space reduces to a convergence space. Then it is easily seen that $x \in \overline{\lambda}_p \Leftrightarrow \exists \mathbb{F} \xrightarrow{p} x$ s.t. $\lambda \in \mathbb{F}$. This shows that closure is precisely the crisp closure in [16] when $L = \{0, 1\}$.

Lemma 3.4. *Let $(X, \lim^p)$ be a stratified L-generalized convergence space. Then for all $\lambda, \mu \in L^X$ and all $\alpha \in L$ we get* (1) $\lambda \leq \overline{\lambda}_p$; (2) $\lambda \leq \mu$ *implies* $\overline{\lambda}_p \leq \overline{\mu}_p$; (3) $\overline{\alpha}_p \geq \alpha$.

Proof. (1) For each $x \in X$, by $\lim^p[x](x) = 1$ we get $\overline{\lambda}_p(x) \geq [x](\lambda) = \lambda(x)$. So, $\lambda \leq \overline{\lambda}_p$. Take $\lambda = 1$ in (1), we obtain $\overline{1}_p = 1$.

(2) It follows from the property (F2) of stratified L-filters.

(3) For each $x \in X$ we have

$$\overline{\alpha}_p(x) = \bigvee_{\lim \mathcal{F}(x) \neq 0} (\lim^p \mathcal{F}(x) \to \mathcal{F}(\alpha)) \overset{\text{(Fs)}}{\geq} \bigvee_{\lim \mathcal{F}(x) \neq 0} (\lim^p \mathcal{F}(x) \to \alpha) \geq \alpha. \quad \square$$

Theorem 3.5. *Let $(X, \lim^p)$ be a stratified L-generalized convergence space. For each $\mathcal{F} \in \mathcal{F}_L^s(X)$, the function $\overline{\mathcal{F}}_p : L^X \longrightarrow L$, defined by $\forall \lambda \in L^X, \overline{\mathcal{F}}_p(\lambda) = \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge [\overline{\mu}_p, \lambda])$, is a stratified L-filter, called the closure of $\mathcal{F}$ w.r.t $(X, \lim^p)$.*

Proof. (F1) That $\overline{\mathcal{F}}_p(1) = 1$ is obvious. That $\overline{\mathcal{F}}_p(0) = 0$ follows by

$$\overline{\mathcal{F}}_p(\lambda) = \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge [\overline{\mu}_p, \lambda]) \leq \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge [\mu, \lambda]) \leq \mathcal{F}(\lambda).$$

(F2) Obviously, $\overline{\mathcal{F}}_p(\lambda \wedge \mu) \leq \overline{\mathcal{F}}_p(\lambda) \wedge \overline{\mathcal{F}}_p(\mu)$. Conversely,

$$\begin{aligned}
\overline{\mathcal{F}}_p(\lambda) \wedge \overline{\mathcal{F}}_p(\mu) &= \bigvee_{a \in L^X} (\mathcal{F}(a) \wedge [\overline{a}_p, \lambda]) \wedge \bigvee_{b \in L^X} (\mathcal{F}(b) \wedge [\overline{b}_p, \mu]) \\
&= \bigvee_{a,b \in L^X} (\mathcal{F}(a) \wedge \mathcal{F}(b) \wedge [\overline{a}_p, \lambda] \wedge [\overline{b}_p, \mu]) \\
&\leq \bigvee_{a,b \in L^X} (\mathcal{F}(a \wedge b) \wedge [\overline{(a \wedge b)}_p, \lambda \wedge \mu]) \\
&\leq \bigvee_{c \in L^X} (\mathcal{F}(c) \wedge [\overline{c}_p, \lambda \wedge \mu]) = \overline{\mathcal{F}}_p(\lambda \wedge \mu).
\end{aligned}$$

(Fs) By $\overline{1}_p = 1$, it follows that $\overline{\mathcal{F}}_p(\alpha) = \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge [\overline{\mu}_p, \alpha]) \geq \mathcal{F}(1) \wedge \alpha = \alpha$. □

Remark 3.6. When $L = \{0, 1\}$, a stratified L-generalized convergence space reduces to a convergence space. It is easily seen that $\overline{\mathcal{F}}_p$ is precisely the filter generated by $\{\overline{A} : A \in \mathbb{F}\}$ as a filterbasis [16].

Lemma 3.7. *Let J, X, σ, ψ satisfy the condition in* p-**(DLfw)**. *Then for any $\lambda, \mu \in L^X$ we have $[\overline{\mu}_p, \lambda] \leq [\hat{\phi}(\mu), \psi^{\leftarrow}(\lambda)]$.*

Proof. Note that $\forall j \in J$, $\sigma_2(j) = \lim^p \sigma_1(j)(\psi(j)) \neq 0$. Then

$$\begin{aligned}
[\overline{\mu}_p, \lambda] &= \bigwedge_{x \in X} \Big(\bigvee_{\mathcal{G} \in \mathcal{F}_L^s(X) : \lim^p \mathcal{G}(x) \neq 0} (\lim^p \mathcal{G}(x) \to \mathcal{G}(\mu)) \to \lambda(x)\Big) \\
&= \bigwedge_{x \in X} \bigwedge_{\mathcal{G} \in \mathcal{F}_L^s(X) : \lim^p \mathcal{G}(x) \neq 0} ((\lim^p \mathcal{G}(x) \to \mathcal{G}(\mu)) \to \lambda(x)) \\
&\leq \bigwedge_{j \in J} (\lim^p \sigma_1(j)(\psi(j)) \to \sigma_1(j)(\mu)) \to \lambda(\psi(j))) \\
&\leq \bigwedge_{j \in J} (\sigma_2(j) \to \sigma_1(j)(\mu)) \to \psi^{\leftarrow}(\lambda)(j)) \\
&= \bigwedge_{j \in J} (\hat{\sigma}(\mu)(j) \to \psi^{\leftarrow}(\lambda)(j)) = [\hat{\sigma}(\mu), \psi^{\leftarrow}(\lambda)]. \quad \square
\end{aligned}$$

Lemma 3.8. *Let J, X, σ, ψ satisfy the condition in* p-**(DLfw)**, *and let $\mathcal{F} \in \mathcal{F}_L^s(X)$. Then the function $\mathcal{F}^\sigma : L^J \longrightarrow L$, defined by $\mathcal{F}^\sigma(\lambda) = \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge [\hat{\sigma}(\mu), \lambda])$, satisfies (F1), (F2), (Fs) and $k_L \sigma \mathcal{F}^\sigma \geq \mathcal{F}$.*

Proof. (F1): It is obvious. (F2): Obviously, $\mathcal{F}^\sigma(\lambda \wedge \mu) \leq \mathcal{F}^\sigma(\lambda) \wedge \mathcal{F}^\sigma(\mu)$. Conversely,

$$\begin{aligned}
\mathcal{F}^\sigma(\lambda) \wedge \mathcal{F}^\sigma(\mu) &= \bigvee_{a \in L^X} (\mathcal{F}(a) \wedge [\widehat{\sigma}(a), \lambda]) \wedge \bigvee_{b \in L^X} (\mathcal{F}(b) \wedge [\widehat{\sigma}(b), \mu]) \\
&= \bigvee_{a,b \in L^X} (\mathcal{F}(a) \wedge \mathcal{F}(b) \wedge [\widehat{\sigma}(a), \lambda] \wedge [\widehat{\sigma}(b), \mu]) \\
&\leq \bigvee_{a,b \in L^X} (\mathcal{F}(a \wedge b) \wedge [\widehat{\sigma}(a \wedge b), \lambda \wedge \mu]) \\
&\leq \bigvee_{c \in L^X} (\mathcal{F}(c) \wedge [\widehat{\sigma}(c), \lambda \wedge \mu]) = \mathcal{F}^\sigma(\lambda \wedge \mu).
\end{aligned}$$

(Fs): For any $\beta \in L$, we have

$$\mathcal{F}^\sigma(\beta) = \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge [\widehat{\sigma}(\mu), \beta]) \geq \mathcal{F}(1) \wedge [\widehat{\sigma}(1), \beta] = 1 \wedge \beta = \beta.$$

It follows by the following inequality that $k_L \sigma \mathcal{F}^\sigma \geq \mathcal{F}$. For any $\lambda \in L^X$,

$$k_L \sigma \mathcal{F}^\sigma(\lambda) = \mathcal{F}^\sigma(\hat{\sigma}(\lambda)) = \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge [\hat{\sigma}(\mu), \hat{\sigma}(\lambda)]) \geq \mathcal{F}(\lambda). \quad \square$$

Theorem 3.9. *Let* $(X, \lim^p, \lim^q)$ *be a pair of stratified* L*-generalized convergence spaces. Then* $(X, \lim^q)$ *is* p*-regular if and only if* $\lim^q \mathcal{F} \leq \lim^q \overline{\mathcal{F}}_p$ *for any* $\mathcal{F} \in \mathcal{F}_L^s(X)$.

Proof. Necessity: Let

$$J = \{(\mathcal{G}, y) \in \mathcal{F}_L^s(X) \times X | \lim^p \mathcal{G}(y) \neq 0\},\ \psi : J \longrightarrow X, (\mathcal{G}, y) \mapsto y,$$

$$\sigma : J \longrightarrow \mathcal{F}_L^s(X) \times L_0, (\mathcal{G}, y) \mapsto (\mathcal{G}, \lim^p \mathcal{G}(y)).$$

Then (1) $\sigma_2(j) = \lim^p \sigma_1(j)(\psi(j)) \neq 0$. (2) For any $\mathcal{F} \in \mathcal{F}_L^s(X)$ we have $\mathcal{F}^\sigma \in \mathcal{F}_L^s(J)$.

Indeed, by Lemma 3.8, we need only to check that $\mathcal{F}^\sigma(0) = 0$.

$$\begin{aligned}
\mathcal{F}^\sigma(0) &= \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge [\hat{\sigma}(\mu), 0]) = \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge \bigwedge_{j \in J} (\hat{\sigma}(\mu)(j) \to 0)) \\
&\leq \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge (\bigwedge_{y \in X} (\hat{\sigma}(\mu)([y], y) \to 0)) \\
&= \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge (\bigwedge_{y \in X} ((\lim^p [y](y) \to [y](\mu)) \to 0)) \\
&= \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge (\bigwedge_{y \in X} (\mu(y) \to 0)) = \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge [\mu, 0]) \\
&\leq \bigvee_{\mu \in L^X} \mathcal{F}(\mu \wedge [\mu, 0]) \leq \mathcal{F}(0) = 0.
\end{aligned}$$

(3) $\psi^\Rightarrow(\mathcal{F}^\sigma) = \overline{\mathcal{F}}_p$. For any $\lambda, \mu \in L^X$,

$$\begin{aligned}
[\overline{\mu}_p, \lambda] &= \bigwedge_{x \in X} (\bigvee_{\mathcal{G} \in \mathcal{F}_L^s(X) : \lim^p \mathcal{G}(x) \neq 0} (\lim^p \mathcal{G}(x) \to \mathcal{G}(\mu)) \to \lambda(x)) \\
&= \bigwedge_{x \in X} \bigwedge_{\mathcal{G} \in \mathcal{F}_L^s(X) : \lim^p \mathcal{G}(x) \neq 0} ((\lim^p \mathcal{G}(x) \to \mathcal{G}(\mu)) \to \lambda(x)) \\
&= \bigwedge_{j \in J} (\lim^p \sigma_1(j)(\psi(j)) \to \sigma_1(j)(\mu)) \to \lambda(\psi(j))) \\
&= \bigwedge_{j \in J} (\sigma_2(j) \to \sigma_1(j)(\mu)) \to \psi^\leftarrow(\lambda)(j)) \\
&= \bigwedge_{j \in J} (\hat{\sigma}(\mu)(j) \to \psi^\leftarrow(\lambda)(j)) = [\hat{\sigma}(\mu), \psi^\leftarrow(\lambda)].
\end{aligned}$$

It follows that

$$\psi^{\Rightarrow}(\mathcal{F}^{\sigma})(\lambda)=\mathcal{F}^{\sigma}(\psi^{\leftarrow}(\lambda))=\bigvee_{\mu\in L^X}(\mathcal{F}(\mu)\wedge[\hat{\sigma}(\mu),\psi^{\leftarrow}(\lambda)])=\bigvee_{\mu\in L^X}(\mathcal{F}(\mu)\wedge[\overline{\mu}_p,\lambda])=\overline{\mathcal{F}}_p(\lambda).$$

(4) $\mathcal{F}^{\sigma}$ is compressible w.r.t. σ. For any $\lambda,\mu\in L^X$,

$$\begin{aligned}[\widehat{\sigma}(\lambda),\widehat{\sigma}(\mu)] &= \bigwedge_{j\in J}(\widehat{\sigma}(\lambda)(j)\to\widehat{\sigma}(\mu)(j))\\ &= \bigwedge_{(\mathcal{G},y):\lim^p\mathcal{G}(y)\neq 0}((\sigma_2(j)\to\sigma_1(j)(\lambda))\to(\sigma_2(j)\to\sigma_1(j)(\mu))\\ &\le \bigwedge_{([y],y):y\in X}((\lim^p[y](y)\to[y](\lambda))\to(\lim^p[y](y)\to[y](\mu)))\\ &= \bigwedge_{y\in X}(\lambda(y)\to\mu(y))=[\lambda,\mu].\end{aligned}$$

Therefore, for any $\lambda\in L^X$,

$$k_L\sigma\mathcal{F}^{\sigma}(\lambda)=\mathcal{F}^{\sigma}(\hat{\sigma}(\lambda))=\bigvee_{\mu\in L^X}(\mathcal{F}(\mu)\wedge[\hat{\sigma}(\mu),\hat{\sigma}(\lambda)])\le\bigvee_{\mu\in L^X}(\mathcal{F}(\mu)\wedge[\mu,\lambda])\le\mathcal{F}(\lambda).$$

By Lemma 3.8, we have $k_L\sigma\mathcal{F}^{\sigma}=\mathcal{F}\in\mathcal{F}_L^s(X)$. Thus $k_L\sigma\mathcal{F}^{\sigma}$ is compressible w.r.t. σ.

Applying (1)-(4) in p-**(DLfw)** we have $\lim^q\mathcal{F}\le\lim^q\overline{\mathcal{F}}_p$.

Sufficiency: Let J,X,σ,ψ satisfy the condition in **(DLfw)**. Then for any $\mathcal{F}\in\mathcal{F}_L^s(J)$, by $(X,\lim^q)$ is p-regular we have that $\lim^q k_L\sigma\mathcal{F}\le\lim^q\overline{k_L\sigma\mathcal{F}}_p(x)$. For any $\lambda\in L^X$, by Lemma 3.7 we have

$$\begin{aligned}\overline{k_L\sigma\mathcal{F}}_p(\lambda) &= \bigvee_{\mu\in L^X}(k_L\sigma\mathcal{F}(\mu)\wedge[\overline{\mu}_p,\lambda])=\bigvee_{\mu\in L^X}(\mathcal{F}(\hat{\sigma}(\mu))\wedge[\overline{\mu}_p,\lambda])\\ &\le \bigvee_{\mu\in L^X}(\mathcal{F}(\hat{\sigma}(\mu))\wedge[\hat{\sigma}(\mu),\psi^{\leftarrow}(\lambda)])\le\mathcal{F}(\psi^{\leftarrow}(\lambda))=\psi^{\Rightarrow}(\mathcal{F})(\lambda).\end{aligned}$$

So, $\overline{k_L\sigma\mathcal{F}}_p\le\psi^{\Rightarrow}(\mathcal{F})$, and hence $\lim^q\psi^{\Rightarrow}(\mathcal{F})\ge\lim^q\overline{k_L\sigma\mathcal{F}}_p\ge\lim^q k_L\sigma\mathcal{F}$, i.e., the condition p-**(DLfw)** holds. □

The next two theorems show that p-regularity behave reasonably well relative to initial and final structures.

Definition 3.10. Let $f:(X,\lim^q)\longrightarrow(Y,\lim^p)$ be a function between stratified L-generalized convergence spaces. Then f is said to be a closure function if $f^{\rightarrow}(\overline{\lambda}_q)\ge\overline{f^{\rightarrow}(\lambda)}_p$ for all $\lambda\in L^X$.

Lemma 3.11. *Let $f:(X,\lim^q)\longrightarrow(Y,\lim^p)$ be a function between stratified L-generalized convergence spaces, and let $\mathcal{F}\in\mathcal{F}_L^s(X)$. (1) If f is continuous, then $f^{\Rightarrow}(\overline{\mathcal{F}}_q)\ge\overline{f^{\Rightarrow}(\mathcal{F})}_p$. (2) If f is a closure function, then $f^{\Rightarrow}(\overline{\mathcal{F}}_q)\le\overline{f^{\Rightarrow}(\mathcal{F})}_p$.*

Proof. (1) Let f be a continuous function. Then for each $\lambda \in L^Y$ we check below that $\overline{(f^{\leftarrow}(\lambda))}_q \leq f^{\leftarrow}(\overline{\lambda}_p)$. Indeed, for each $x \in X$,

$$\begin{array}{rcl} \overline{(f^{\leftarrow}(\lambda))}_q(x) & = & \bigvee\limits_{\mathcal{G}\in\mathcal{F}_L^s(X):\lim^q\mathcal{G}(x)\neq 0} \big(\lim^q\mathcal{G}(x) \to \mathcal{G}(f^{\leftarrow}(\lambda))\big) \\ & \leq & \bigvee\limits_{\mathcal{G}\in\mathcal{F}_L^s(X):\lim^q\mathcal{G}(x)\neq 0} \big(\lim^p f^{\Rightarrow}(\mathcal{G})(f(x)) \to f^{\Rightarrow}(\mathcal{G})(\lambda)\big) \\ & \leq & \bigvee\limits_{\mathcal{H}\in\mathcal{F}_L^s(Y):\lim^p\mathcal{H}(x)\neq 0} \big(\lim^p\mathcal{H}(f(x)) \to \mathcal{H}(\lambda)\big) = f^{\leftarrow}(\overline{\lambda}_p)(x), \end{array}$$

where the first inequality holds for the continuity of f. Then for each $\mathcal{F} \in \mathcal{F}_L^s(X)$ and each $\lambda \in L^Y$

$$\begin{array}{rcl} f^{\Rightarrow}(\overline{\mathcal{F}}_q)(\lambda) & = & \overline{\mathcal{F}}_q(f^{\leftarrow}(\lambda)) = \bigvee\limits_{\mu\in L^X}([\overline{\mu}_q, f^{\leftarrow}(\lambda)] \wedge \mathcal{F}(\mu)) \\ & \geq & \bigvee\limits_{\nu\in L^Y}([\overline{(f^{\leftarrow}(\nu))}_q, f^{\leftarrow}(\lambda)] \wedge \mathcal{F}(f^{\leftarrow}(\nu))) \geq \bigvee\limits_{\nu\in L^Y}([f^{\leftarrow}(\overline{\nu}_p), f^{\leftarrow}(\lambda)] \wedge \mathcal{F}(f^{\leftarrow}(\nu))) \\ & \geq & \bigvee\limits_{\nu\in L^Y}([\overline{\nu}_p, \lambda] \wedge f^{\Rightarrow}(\mathcal{F})(\nu)) = \overline{f^{\Rightarrow}(\mathcal{F})}_p(\lambda). \end{array}$$

Thus $f^{\Rightarrow}(\overline{\mathcal{F}}_q) \geq \overline{f^{\Rightarrow}(\mathcal{F})}_p$. (2) Let f be a closure function. Then for each $\lambda \in L^Y$,

$$\begin{array}{rcl} \overline{f^{\Rightarrow}(\mathcal{F})}_p(\lambda) & = & \bigvee\limits_{\mu\in L^Y}(f^{\Rightarrow}(\mathcal{F})(\mu) \wedge [\overline{\mu}_p, \lambda]) = \bigvee\limits_{\mu\in L^Y}(\mathcal{F}(f^{\leftarrow}(\mu)) \wedge [\overline{\mu}_p, \lambda]) \\ & \geq & \bigvee\limits_{\nu\in L^X}(\mathcal{F}(f^{\leftarrow}f^{\rightarrow}(\nu)) \wedge [\overline{f^{\rightarrow}(\nu)}_p, \lambda]) \geq \bigvee\limits_{\nu\in L^X}(\mathcal{F}(\nu) \wedge [\overline{f^{\rightarrow}(\nu)}_p, \lambda]) \\ & \geq & \bigvee\limits_{\nu\in L^X}(\mathcal{F}(\nu) \wedge [f^{\rightarrow}(\overline{\nu}_q), \lambda]) = \bigvee\limits_{\nu\in L^X}(\mathcal{F}(\nu) \wedge [\overline{\nu}_q, f^{\leftarrow}(\lambda)]) \\ & = & \overline{\mathcal{F}}_q(f^{\leftarrow}(\lambda)) = f^{\Rightarrow}(\overline{\mathcal{F}}_q)(\lambda), \end{array}$$

where the third inequality holds for f being a closure function, and the third equality follows from Lemma 1.1 (7). By the arbitrariness of λ, we get $f^{\Rightarrow}(\overline{\mathcal{F}}_q) \leq \overline{f^{\Rightarrow}(\mathcal{F})}_p$. □

Theorem 3.12. *Let $\{(X_i, \lim^{q_i}, \lim^{p_i})\}_{i\in I}$ be pairs of stratified L-generalized convergence spaces and let $\lim^q$ (resp., $\lim^p$) be the initial structure on X relative to the source $(X \xrightarrow{f_i} (X_i, \lim^{q_i}))_{i\in I}$ (resp., $(X \xrightarrow{f_i} (X_i, \lim^{p_i}))_{i\in I}$). If each $\lim^{q_i}$ is p_i-regular, then $(X, \lim^q)$ is p-regular.*

Proof. Let $\mathcal{F} \in \mathcal{F}_L^s(X)$ and $x \in X$. Then by Lemma 3.11 (1) we have $f_i^{\Rightarrow}(\overline{\mathcal{F}}_p) \geq \overline{f_i^{\Rightarrow}(\mathcal{F})}_{p_i}$ for all $i \in I$. It follows by each $(X_i, \lim^{q_i})$ being p_i-regular that

$$\begin{array}{rcl} \lim^q\overline{\mathcal{F}}_p(x) & = & \bigwedge\limits_{i\in I}\lim^{q_i} f_i^{\Rightarrow}(\overline{\mathcal{F}}_p)(f_i(x)) \geq \bigwedge\limits_{i\in I}\lim^{q_i}\overline{f_i^{\Rightarrow}(\mathcal{F})}_{p_i}(f_i(x)) \\ & \geq & \bigwedge\limits_{i\in I}\lim^{q_i} f_i^{\Rightarrow}(\mathcal{F})(f_i(x)) = \lim^q\mathcal{F}(x). \end{array}$$

Thus $(X, \lim^q)$ is p-regular. □

Theorem 3.13. *Let $\{(X_i, \lim^{q_i}, \lim^{p_i})\}_{i\in I}$ be pairs of stratified L-generalized convergence spaces, and let $\lim^q$ be the final structure on X w.r.t. the sink $((X_i, \lim^{q_i}) \xrightarrow{f_i} X)_{i\in I}$ with $X = \cup_{i\in I} f_i(X_i)$. If each $\lim^{q_i}$ is p_i-regular and $\lim^p$ is a stratified L-generalized convergence structure on X such that each $f_i : (X_i, \lim^{p_i}) \longrightarrow (X, \lim^p)$ is a closure function, then $(X, \lim^q)$ is p-regular.*

Proof. Let $\mathcal{F} \in \mathcal{F}_L^s(X)$ and $x \in X$. Then

$$\begin{aligned}
\lim^q \mathcal{F}(x) &= \bigvee_{i\in I, x_i\in X_i, \mathcal{G}_i\in\mathcal{F}_L^s(X_i), f_i(x_i)=x, f_i^{\Rightarrow}(\mathcal{G}_i)\le\mathcal{F}} \lim^{q_i}\mathcal{G}_i(x_i) \\
&\le \bigvee_{i\in I, x_i\in X_i, \mathcal{G}_i\in\mathcal{F}_L^s(X_i), f_i(x_i)=x, f_i^{\Rightarrow}(\mathcal{G}_i)\le\mathcal{F}} \lim^{q_i}\overline{\mathcal{G}}_{ip_i}(x_i) \\
&\le \bigvee_{i\in I, x_i\in X_i, \mathcal{G}_i\in\mathcal{F}_L^s(X_i), f_i(x_i)=x, \overline{f_i^{\Rightarrow}(\mathcal{G}_i)}_p\le\overline{\mathcal{F}}_p} \lim^{q_i}\overline{\mathcal{G}}_{ip_i}(x_i) \\
&= \bigvee_{i\in I, x_i\in X_i, \mathcal{G}_i\in\mathcal{F}_L^s(X_i), f_i(x_i)=x, f_i^{\Rightarrow}(\overline{\mathcal{G}}_{ip_i})\le\overline{f_i^{\Rightarrow}(\mathcal{G}_i)}_p\le\overline{\mathcal{F}}_p} \lim^{q_i}\overline{\mathcal{G}}_{ip_i}(x_i) \\
&\le \bigvee_{i\in I, x_i\in X_i, \mathcal{H}_i\in\mathcal{F}_L^s(X_i), f_i(x_i)=x, f_i^{\Rightarrow}(\mathcal{H}_i)\le\overline{\mathcal{F}}_p} \lim^{q_i}\mathcal{H}_i(x_i) = \lim^q\overline{\mathcal{F}}_p(x),
\end{aligned}$$

where the first inequality holds for p_i-regularity of $\lim^{q_i}$, the second equality follows from Lemma 3.11 (2). Then $\lim^q$ is p-regular by $\lim^q \mathcal{F} \le \lim^q \overline{\mathcal{F}}_p$. □

The regularity has similar characterization and properties, we omit them here.

4 Conclusion

In this paper, by dualizing the Fischer-type diagonal condition **(Lfw)**, which is used to describe the L-topologicallness of stratified L-convergence spaces, we define a new regularity (p-regularity) of stratified L-generalized convergence spaces. Then we also present a characterization on that regularity (p-regularity) by the notion of closures of stratified L-filters. The regularity (p-regularity) is proved to behave reasonably well relative to initial and final structures.

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Uni-soft filters and uni-soft G-filters in residuated lattices

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Abstract

The notions of uni-soft filters and uni-soft G-filters in residuated lattices are introduced, and their relations, properties and characterizations are investigated. Conditions for a uni-soft filter to be a uni-soft G-filter are provided.

Keywords: Residuated lattice, Uni-soft filter, Uni-soft G-filter.

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1 Introduction

Non-classical logic has become a formal and useful tool in dealing with fuzzy and uncertain informations. Various logical algebras have been proposed as the semantical systems of non-classical logic systems. Residuated lattices are important algebraic structures which are basic of BL-algebras, MV-algebras, MTL-algebras, Gödel algebras, R_0-algebras, lattice implication algebras, and so forth. The (fuzzy) filter theory in the logical algebras has an important role in studying these algebras and completeness of the corresponding non-classical logics, and it is studied in the papers [1], [2], [3], [9], [12], [13] and [14]. Filter theory, which is an important notion, in residuated lattices is studied by Shen and Zhang [11] and Zhu and Xu [16]. Uncertainty is an attribute of information. As a new mathematical tool for dealing with uncertainties, Molodtsov [10] introduced the concept of soft sets. Since then several authors studied (fuzzy) algebraic structures based on soft

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set theory in several algebraic structures. In order to lay a foundation for providing a soft algebraic tool in considering many problems that contain uncertainties, Jun [7] discussed the union soft sets with applications in BCK/BCI-algebras. Also, Jun et al. [8] discussed uni-soft sets applied to commutative BCI-ideals.

In this paper, we introduce uni-soft filters and uni-soft G-filters in residuated lattices, and investigate their properties. We consider characterizations of uni-soft filters and uni-soft G-filters. We provide conditions for a uni-soft filter to be a uni-soft G-filter.

2 Preliminaries

Definition 2.1 ([1, 5, 6]). A residuated lattice is an algebra $\mathcal{L} := (L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that

(1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice.

(2) $(L, \odot, 1)$ is a commutative monoid.

(3) $\odot$ and $\rightarrow$ form an adjoint pair, that is,

$$(\forall x, y, z \in L)\,(x \leq y \rightarrow z \;\Leftrightarrow\; x \odot y \leq z)\,.$$

In a residuated lattice $\mathcal{L}$, the ordering $\leq$ is defined as follows:

$$(\forall x, y \in L)\,(x \leq y \;\Leftrightarrow\; x \wedge y = x \;\Leftrightarrow\; x \vee y = y \;\Leftrightarrow\; x \rightarrow y = 1)$$

and x' will be reserved for $x \rightarrow 0$, and $x'' = (x')'$, etc. for all $x \in L$.

Proposition 2.2 ([1, 5, 6, 12, 13]). *In a residuated lattice L, the following properties are valid.*

$$1 \rightarrow x = x,\; x \rightarrow 1 = 1,\; x \rightarrow x = 1,\; 0 \rightarrow x = 1,\; x \rightarrow (y \rightarrow x) = 1. \tag{2.1}$$

$$x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z = y \rightarrow (x \rightarrow z). \tag{2.2}$$

$$x \leq y \;\Rightarrow\; z \rightarrow x \leq z \rightarrow y,\; y \rightarrow z \leq x \rightarrow z. \tag{2.3}$$

$$z \rightarrow y \leq (x \rightarrow z) \rightarrow (x \rightarrow y),\; z \rightarrow y \leq (y \rightarrow x) \rightarrow (z \rightarrow x). \tag{2.4}$$

Definition 2.3 ([11]). A nonempty subset F of a residuated lattice $\mathcal{L}$ is called a filter of $\mathcal{L}$ if it satisfies the conditions:

$$(\forall x, y \in L)\,(x, y \in F \;\Rightarrow\; x \odot y \in F)\,. \tag{2.5}$$

$$(\forall x, y \in L)\,(x \in F,\; x \leq y \;\Rightarrow\; y \in F)\,. \tag{2.6}$$

Proposition 2.4 ([11])**.** *A nonempty subset F of a residuated lattice $\mathcal{L}$ is a filter of $\mathcal{L}$ if and only if it satisfies:*

$$1 \in F. \tag{2.7}$$

$$(\forall x \in F)\,(\forall y \in L)\,(x \to y \in F \;\Rightarrow\; y \in F)\,. \tag{2.8}$$

Definition 2.5 ([15])**.** A nonempty subset F of $\mathcal{L}$ is called a G-filter of $\mathcal{L}$ if it is a filter of $\mathcal{L}$ that satisfies the following condition:

$$(\forall x, y \in L)\,((x \odot x) \to y \in F \;\Rightarrow\; x \to y \in F)\,. \tag{2.9}$$

A soft set theory is introduced by Molodtsov [10], and Çağman et al. [4] provided new definitions and various results on soft set theory.

In what follows, let U be an initial universe set and E be a set of parameters. Let $\mathscr{P}(U)$ denotes the power set of U and $A, B, C, \cdots \subseteq E$.

Definition 2.6 ([4, 10])**.** A soft set $(\tilde{f}, A)$ over U is defined to be the set of ordered pairs

$$(\tilde{f}, A) := \left\{(x, \tilde{f}_A(x)) : x \in E,\; \tilde{f}_A(x) \in \mathscr{P}(U)\right\},$$

where $\tilde{f}_A : E \to \mathscr{P}(U)$ such that $\tilde{f}(x) = \emptyset$ if $x \notin A$. The soft set $(\tilde{f}, A)$ is simply denoted by $\tilde{f}_A$.

For a soft set $\tilde{f}_A$ over U and a subset τ of U, the τ-exclusive set of $\tilde{f}_A$, denoted by $e(\tilde{f}_A; \tau)$, is defined to be the set

$$e(\tilde{f}_A; \tau) := \left\{x \in A \mid \tilde{f}_A(x) \subseteq \tau\right\}.$$

3 Uni-soft filters

In what follows, we take a residuated lattice $\mathcal{L}$ as a set of parameters.

Definition 3.1. A soft set $\tilde{f}_{\mathcal{L}}$ over U is called a uni-soft filter of $\mathcal{L}$ if it satisfies:

$$(\forall x, y \in L)\left(x \leq y \;\Rightarrow\; \tilde{f}_{\mathcal{L}}(x) \supseteq \tilde{f}_{\mathcal{L}}(y)\right), \tag{3.1}$$

$$(\forall x, y \in L)\left(\tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(y) \supseteq \tilde{f}_{\mathcal{L}}(x \odot y)\right). \tag{3.2}$$

Proposition 3.2. *Every uni-soft filter $\tilde{f}_{\mathcal{L}}$ of $\mathcal{L}$ satisfies:*

$$(\forall x \in L)\left(\tilde{f}_{\mathcal{L}}(x) \supseteq \tilde{f}_{\mathcal{L}}(1)\right). \tag{3.3}$$

$$(\forall x, y \in L)\left(\tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x \to y) \supseteq \tilde{f}_{\mathcal{L}}(y)\right). \tag{3.4}$$

Proof. Let $x, y \in L$. Since $x \leq 1$, we have $\tilde{f}_{\mathcal{L}}(x) \supseteq \tilde{f}_{\mathcal{L}}(1)$ by (3.1). Since $x \odot (x \to y) \leq y$, it follows from (3.2) and (3.1) that

$$\tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x \to y) \supseteq \tilde{f}_{\mathcal{L}}(x \odot (x \to y)) \supseteq \tilde{f}_{\mathcal{L}}(y).$$

This completes the proof. □

Lemma 3.3. *If a soft set $\tilde{f}_{\mathcal{L}}$ over U satisfies two conditions (3.3) and (3.4), then*

$$(\forall x, y, z \in L)\left(x \leq y \to z \;\Rightarrow\; \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(y) \supseteq \tilde{f}_{\mathcal{L}}(z)\right), \tag{3.5}$$

$$(\forall x, y, z \in L)\left(x \odot y \leq z \;\Rightarrow\; \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(y) \supseteq \tilde{f}_{\mathcal{L}}(z)\right). \tag{3.6}$$

Proof. Assume that $x \leq y \to z$ for all $x, y, z \in L$. Then $x \to (y \to z) = 1$, and so

$$\begin{aligned}
\tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(y) &= \left(\tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(1)\right) \cup \tilde{f}_{\mathcal{L}}(y) \\
&= \left(\tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x \to (y \to z))\right) \cup \tilde{f}_{\mathcal{L}}(y) \\
&\supseteq \tilde{f}_{\mathcal{L}}(y) \cup \tilde{f}_{\mathcal{L}}(y \to z) \supseteq \tilde{f}_{\mathcal{L}}(z).
\end{aligned}$$

Since $x \leq y \to z \;\Leftrightarrow\; x \odot y \leq z$, we know that (3.5) induces (3.6). □

We consider characterizations of uni-soft filters.

Theorem 3.4. *A soft set $\tilde{f}_{\mathcal{L}}$ over U is a uni-soft filter of $\mathcal{L}$ if and only if it satisfies two conditions (3.3) and (3.4).*

Proof. The necessity is from Proposition 3.2.

Conversely, let $\tilde{f}_{\mathcal{L}}$ be a soft set over U that satisfies (3.3) and (3.4). Let $x, y \in L$ be such that $x \leq y$. Then $x \to y = 1$ and so

$$\tilde{f}_{\mathcal{L}}(x) = \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(1) = \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x \to y) \supseteq \tilde{f}_{\mathcal{L}}(y).$$

Since $x \odot y \leq x \odot y$ for all $x, y \in L$, it follows from (3.6) that $\tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(y) \supseteq \tilde{f}_{\mathcal{L}}(x \odot y)$ for all $x, y \in L$. Therefore $\tilde{f}_{\mathcal{L}}$ is a uni-soft filter of $\mathcal{L}$. □

Theorem 3.5. *A soft set $\tilde{f}_{\mathcal{L}}$ over U is a uni-soft filter of $\mathcal{L}$ if and only if it satisfies the condition* (3.5).

Proof. The necessity is from Lemma 3.3 and Theorem 3.4.

Conversely let $\tilde{f}_{\mathcal{L}}$ be a soft set over U satisfying (3.5). Since

$$x \leq x \to 1 \text{ and } x \to y \leq x \to y$$

for all $x, y \in L$, it follows from (3.5) that

$$\tilde{f}_{\mathcal{L}}(x) = \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x) \supseteq \tilde{f}_{\mathcal{L}}(1) \text{ and } \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x \to y) \supseteq \tilde{f}_{\mathcal{L}}(y)$$

for all $x, y \in L$. Hence $\tilde{f}_{\mathcal{L}}$ is a uni-soft filter of $\mathcal{L}$ by Theorem 3.4. □

Proposition 3.6. *Every uni-soft filter $\tilde{f}_{\mathcal{L}}$ of $\mathcal{L}$ satisfies the following condition:*

$$(\forall x, y, z \in L)\left(\tilde{f}_{\mathcal{L}}(x \to (y \to z)) \cup \tilde{f}_{\mathcal{L}}(x \to y) \supseteq \tilde{f}_{\mathcal{L}}(x \to (x \to z))\right). \tag{3.7}$$

Proof. Let $x, y, z \in L$. Using (2.2) and (2.4), we have

$$x \to (y \to z) = y \to (x \to z) \leq (x \to y) \to (x \to (x \to z)).$$

It follows from Theorem 3.5 that

$$\tilde{f}_{\mathcal{L}}(x \to (y \to z)) \cup \tilde{f}_{\mathcal{L}}(x \to y) \supseteq \tilde{f}_{\mathcal{L}}(x \to (x \to z)).$$

This completes the proof. □

Theorem 3.7. *A soft set $\tilde{f}_{\mathcal{L}}$ over U is a uni-soft filter of $\mathcal{L}$ if and only if $\tilde{f}_{\mathcal{L}}$ satisfies the condition* (3.3) *and*

$$(\forall x, y, z \in L)\left(\tilde{f}_{\mathcal{L}}(x \to (y \to z)) \cup \tilde{f}_{\mathcal{L}}(y) \supseteq \tilde{f}_{\mathcal{L}}(x \to z)\right). \tag{3.8}$$

Proof. Assume that $\tilde{f}_{\mathcal{L}}$ is a uni-soft filter of $\mathcal{L}$. Then the condition (3.3) is valid. Using (3.4) and (2.2), we have

$$\begin{aligned}\tilde{f}_{\mathcal{L}}(x \to z) &\subseteq \tilde{f}_{\mathcal{L}}(y) \cup \tilde{f}_{\mathcal{L}}(y \to (x \to z)) \\ &= \tilde{f}_{\mathcal{L}}(y) \cup \tilde{f}_{\mathcal{L}}(x \to (y \to z))\end{aligned}$$

for all $x, y, z \in L$.

Conversely, let $\tilde{f}_{\mathcal{L}}$ be a soft set over U satisfying (3.3) and (3.8). Taking $x := 1$ in (3.8) and using (2.1), we have

$$\begin{aligned}\tilde{f}_{\mathcal{L}}(z) = \tilde{f}_{\mathcal{L}}(1 \to z) &\subseteq \tilde{f}_{\mathcal{L}}(1 \to (y \to z)) \cup \tilde{f}_{\mathcal{L}}(y) \\ &= \tilde{f}_{\mathcal{L}}(y \to z) \cup \tilde{f}_{\mathcal{L}}(y)\end{aligned}$$

for all $y, z \in L$. Thus $\tilde{f}_{\mathcal{L}}$ is a uni-soft filter of $\mathcal{L}$ by Theorem 3.4. □

Proposition 3.8. *Every uni-soft filter $\tilde{f}_{\mathcal{L}}$ of $\mathcal{L}$ satisfies the following condition:*

$$(\forall a, x \in L)\left(\tilde{f}_{\mathcal{L}}(a) \supseteq \tilde{f}_{\mathcal{L}}((a \to x) \to x)\right). \tag{3.9}$$

Proof. If we take $y := (a \to x) \to x$ and $x := a$ in (3.4), then

$$\begin{aligned}
\tilde{f}_{\mathcal{L}}((a \to x) \to x) &\subseteq \tilde{f}_{\mathcal{L}}(a) \cup \tilde{f}_{\mathcal{L}}(a \to ((a \to x) \to x)) \\
&= \tilde{f}_{\mathcal{L}}(a) \cup \tilde{f}_{\mathcal{L}}((a \to x) \to (a \to x)) \\
&= \tilde{f}_{\mathcal{L}}(a) \cup \tilde{f}_{\mathcal{L}}(1) = \tilde{f}_{\mathcal{L}}(a).
\end{aligned}$$

This completes the proof. □

Theorem 3.9. *A soft set $\tilde{f}_{\mathcal{L}}$ over U is a uni-soft filter of $\mathcal{L}$ if and only if it satisfies the following conditions:*

$$(\forall x, y \in L)\left(\tilde{f}_{\mathcal{L}}(x) \supseteq \tilde{f}_{\mathcal{L}}(y \to x)\right), \tag{3.10}$$

$$(\forall x, a, b \in L)\left(\tilde{f}_{\mathcal{L}}(a) \cup \tilde{f}_{\mathcal{L}}(b) \supseteq \tilde{f}_{\mathcal{L}}((a \to (b \to x)) \to x)\right). \tag{3.11}$$

Proof. Assume that $\tilde{f}_{\mathcal{L}}$ is a uni-soft filter of $\mathcal{L}$. Using (3.4), (2.1) and (3.3), we have

$$\tilde{f}_{\mathcal{L}}(y \to x) \subseteq \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x \to (y \to x)) = \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(1) = \tilde{f}_{\mathcal{L}}(x)$$

for all $x, y \in L$. Using (3.8) and (3.9), we get

$$\tilde{f}_{\mathcal{L}}((a \to (b \to x)) \to x) \subseteq \tilde{f}_{\mathcal{L}}((a \to (b \to x)) \to (b \to x)) \cup \tilde{f}_{\mathcal{L}}(b) \subseteq \tilde{f}_{\mathcal{L}}(a) \cup \tilde{f}_{\mathcal{L}}(b)$$

for all $a, b, x \in L$.

Conversely, let $\tilde{f}_{\mathcal{L}}$ be a soft set over U satisfying two conditions (3.10) and (3.11). If we take $y := x$ in (3.10), then $\tilde{f}_{\mathcal{L}}(x) \supseteq \tilde{f}_{\mathcal{L}}(x \to x) = \tilde{f}_{\mathcal{L}}(1)$ for all $x \in L$. Using (3.11) induces

$$\tilde{f}_{\mathcal{L}}(y) = \tilde{f}_{\mathcal{L}}(1 \to y) = \tilde{f}_{\mathcal{L}}(((x \to y) \to (x \to y)) \to y) \subseteq \tilde{f}_{\mathcal{L}}(x \to y) \cup \tilde{f}_{\mathcal{L}}(x)$$

for all $x, y \in L$. Therefore $\tilde{f}_{\mathcal{L}}$ is a uni-soft filter of $\mathcal{L}$ by Theorem 3.4. □

Theorem 3.10. *A soft set $\tilde{f}_{\mathcal{L}}$ over U is a uni-soft filter of $\mathcal{L}$ if and only if the nonempty τ-exclusive set of $\tilde{f}_{\mathcal{L}}$ is a filter of $\mathcal{L}$ for all $\tau \in \mathscr{P}(U)$.*

Proof. Assume that $\tilde{f}_{\mathcal{L}}$ is a uni-soft filter of $\mathcal{L}$ and let $\tau \in \mathscr{P}(U)$ be such that $e(\tilde{f}_{\mathcal{L}};\tau) \neq \emptyset$. Let $x, y \in L$ be such that $x \in e(\tilde{f}_{\mathcal{L}};\tau)$ and $x \to y \in e(\tilde{f}_{\mathcal{L}};\tau)$. Then $\tau \supseteq \tilde{f}_{\mathcal{L}}(x)$ and $\tau \supseteq \tilde{f}_{\mathcal{L}}(x \to y)$. It follows from (3.3) and (3.4) that $\tilde{f}_{\mathcal{L}}(1) \subseteq \tilde{f}_{\mathcal{L}}(x) \subseteq \tau$ and $\tilde{f}_{\mathcal{L}}(y) \subseteq \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x \to y) \subseteq \tau$. Hence $1 \in e(\tilde{f}_{\mathcal{L}};\tau)$ and $y \in e(\tilde{f}_{\mathcal{L}};\tau)$, and therefore $e(\tilde{f}_{\mathcal{L}};\tau)$ is a filter of $\mathcal{L}$ by Proposition 2.4.

Conversely, suppose that $e(\tilde{f}_{\mathcal{L}};\tau)$ is a filter of $\mathcal{L}$ for all $\tau \in \mathscr{P}(U)$ with $e(\tilde{f}_{\mathcal{L}};\tau) \neq \emptyset$. For any $x \in L$, let $\tilde{f}_{\mathcal{L}}(x) = \delta$. Then $x \in e(\tilde{f}_{\mathcal{L}};\delta)$ and $e(\tilde{f}_{\mathcal{L}};\delta)$ is a filter of $\mathcal{L}$. Hence $1 \in e(\tilde{f}_{\mathcal{L}};\delta)$ and so $\tilde{f}_{\mathcal{L}}(x) = \delta \supseteq \tilde{f}_{\mathcal{L}}(1)$. For any $x, y \in L$, let $\tilde{f}_{\mathcal{L}}(x) = \delta_x$ and $\tilde{f}_{\mathcal{L}}(x \to y) = \delta_{x\to y}$. If we take $\delta = \delta_x \cup \delta_{x\to y}$, then $x \in e(\tilde{f}_{\mathcal{L}};\delta)$ and $x \to y \in e(\tilde{f}_{\mathcal{L}};\delta)$ which imply that $y \in e(\tilde{f}_{\mathcal{L}};\delta)$. Thus

$$\tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x \to y) = \delta_x \cup \delta_{x\to y} = \delta \supseteq \tilde{f}_{\mathcal{L}}(y).$$

Therefore $\tilde{f}_{\mathcal{L}}$ is a uni-soft filter of $\mathcal{L}$ by Theorem 3.4. □

Theorem 3.11. *For a soft set $\tilde{f}_{\mathcal{L}}$ over U, let $\tilde{f}^*_{\mathcal{L}}$ be a soft set over U which is given as follows:*

$$\tilde{f}^*_{\mathcal{L}} : L \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tilde{f}_{\mathcal{L}}(x) & \text{if } x \in e(\tilde{f}_{\mathcal{L}};\tau), \\ U & \text{otherwise,} \end{cases}$$

where $\tau \in \mathscr{P}(U)$ with $\tau \neq U$. If $\tilde{f}_{\mathcal{L}}$ is a uni-soft filter of $\mathcal{L}$, then so is $\tilde{f}^*_{\mathcal{L}}$.

Proof. Suppose that $\tilde{f}_{\mathcal{L}}$ is a uni-soft filter of $\mathcal{L}$. Then $e(\tilde{f}_{\mathcal{L}};\tau)$ is a filter of $\mathcal{L}$ for all $\tau \in \mathscr{P}(U)$ with $e(\tilde{f}_{\mathcal{L}};\tau) \neq \emptyset$ by Theorem 3.10. Thus $1 \in e(\tilde{f}_{\mathcal{L}};\tau)$, and so $\tilde{f}^*_{\mathcal{L}}(1) = \tilde{f}_{\mathcal{L}}(1) \subseteq \tilde{f}_{\mathcal{L}}(x) \subseteq \tilde{f}^*_{\mathcal{L}}(x)$ for all $x \in L$. Let $x, y \in L$. If $x \in e(\tilde{f}_{\mathcal{L}};\tau)$ and $x \to y \in e(\tilde{f}_{\mathcal{L}};\tau)$, then $y \in e(\tilde{f}_{\mathcal{L}};\tau)$. Hence

$$\tilde{f}^*_{\mathcal{L}}(x) \cup \tilde{f}^*_{\mathcal{L}}(x \to y) = \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x \to y) \supseteq \tilde{f}_{\mathcal{L}}(y) = \tilde{f}^*_{\mathcal{L}}(y).$$

If $x \notin e(\tilde{f}_{\mathcal{L}};\tau)$ or $x \to y \notin e(\tilde{f}_{\mathcal{L}};\tau)$, then $\tilde{f}^*_{\mathcal{L}}(x) = U$ or $\tilde{f}^*_{\mathcal{L}}(x \to y) = U$. Thus

$$\tilde{f}^*_{\mathcal{L}}(x) \cup \tilde{f}^*_{\mathcal{L}}(x \to y) = U \supseteq \tilde{f}^*_{\mathcal{L}}(y).$$

Therefore $\tilde{f}^*_{\mathcal{L}}$ is a uni-soft filter of $\mathcal{L}$. □

Theorem 3.12. *If $\tilde{f}_{\mathcal{L}}$ is a uni-soft filter of L, then the set*

$$\mathcal{L}_a := \{x \in L \mid \tilde{f}_{\mathcal{L}}(a) \supseteq \tilde{f}_{\mathcal{L}}(x)\}$$

is a filter of $\mathcal{L}$ for every $a \in L$.

Proof. Since $\tilde{f}_{\mathcal{L}}(1) \subseteq \tilde{f}_{\mathcal{L}}(a)$ for all $a \in L$, we have $1 \in \mathcal{L}_a$. Let $x, y \in L$ be such that $x \in \mathcal{L}_a$ and $x \to y \in \mathcal{L}_a$. Then $\tilde{f}_{\mathcal{L}}(x) \subseteq \tilde{f}_{\mathcal{L}}(a)$ and $\tilde{f}_{\mathcal{L}}(x \to y) \subseteq \tilde{f}_{\mathcal{L}}(a)$. Since $\tilde{f}_{\mathcal{L}}$ is a uni-soft filter of L, it follows from (3.4) that

$$\tilde{f}_{\mathcal{L}}(a) \supseteq \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x \to y) \supseteq \tilde{f}_{\mathcal{L}}(y)$$

so that $y \in \mathcal{L}_a$. Hence $\mathcal{L}_a$ is a filter of $\mathcal{L}$ by Proposition 2.4. □

Theorem 3.13. *Let $a \in L$ and let $\tilde{f}_{\mathcal{L}}$ be a soft set over U. Then*

(1) *If $\mathcal{L}_a$ is a filter of L, then $\tilde{f}_{\mathcal{L}}$ satisfies the following condition:*

$$(\forall x, y \in L)\,(\tilde{f}_{\mathcal{L}}(a) \supseteq \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x \to y) \ \Rightarrow\ \tilde{f}_{\mathcal{L}}(a) \supseteq \tilde{f}_{\mathcal{L}}(y)). \tag{3.12}$$

(2) *If $\tilde{f}_{\mathcal{L}}$ satisfies (3.3) and (3.12), then $\mathcal{L}_a$ is a filter of L.*

Proof. (1) Assume that $\mathcal{L}_a$ is a filter of L. Let $x, y \in L$ be such that

$$\tilde{f}_{\mathcal{L}}(a) \supseteq \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x \to y).$$

Then $x \to y \in \mathcal{L}_a$ and $x \in \mathcal{L}_a$. Using (2.8), we have $y \in \mathcal{L}_a$ and so $\tilde{f}_{\mathcal{L}}(a) \supseteq \tilde{f}_{\mathcal{L}}(y)$.

(2) Suppose that $\tilde{f}_{\mathcal{L}}$ satisfies (3.3) and (3.12). Then $1 \in \mathcal{L}_a$ by (3.3). Let $x, y \in L$ be such that $x \in \mathcal{L}_a$ and $x \to y \in \mathcal{L}_a$. Then $\tilde{f}_{\mathcal{L}}(a) \supseteq \tilde{f}_{\mathcal{L}}(x)$ and $\tilde{f}_{\mathcal{L}}(a) \supseteq \tilde{f}_{\mathcal{L}}(x \to y)$, which imply that $\tilde{f}_{\mathcal{L}}(a) \supseteq \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x \to y)$. Thus $\tilde{f}_{\mathcal{L}}(a) \supseteq \tilde{f}_{\mathcal{L}}(y)$ by (3.12), and so $y \in \mathcal{L}_a$. Therefore $\mathcal{L}_a$ is a filter of $\mathcal{L}$ by Proposition 2.4. □

4 Uni-soft G-filters

Definition 4.1. A soft set $\tilde{f}_{\mathcal{L}}$ over U is called a uni-soft G-filter of $\mathcal{L}$ if it is a uni-soft filter of $\mathcal{L}$ that satisfies:

$$(\forall x, y \in L)\left(\tilde{f}_{\mathcal{L}}((x \odot x) \to y) \supseteq \tilde{f}_{\mathcal{L}}(x \to y)\right). \tag{4.1}$$

Note that the condition (4.1) is equivalent to the following condition:

$$(\forall x, y \in L)\left(\tilde{f}_{\mathcal{L}}(x \to (x \to y)) \supseteq \tilde{f}_{\mathcal{L}}(x \to y)\right). \tag{4.2}$$

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Example 4.2. Let $L := [0, 1]$ (unit interval). For any $a, b \in L$, define

$$a \vee b = \max\{a, b\}, \;\; a \wedge b = \min\{a, b\},$$
$$a \to b = \begin{cases} 1 & \text{if } \; a \leq b, \\ b & \text{otherwise,} \end{cases} \quad \text{and} \quad a \odot b = \min\{a, b\}.$$

Then $\mathcal{L} := (L, \vee, \wedge, \odot, \to, 0, 1)$ is a residuated lattice. Let $\tilde{f}_{\mathcal{L}}$ be a soft set over U defined by

$$\tilde{f}_{\mathcal{L}} : L \to \mathscr{P}(U), \;\; x \mapsto \begin{cases} \tau & \text{if } \; x \in [\frac{1}{2}, 1], \\ U & \text{otherwise,} \end{cases}$$

where $\tau \in \mathscr{P}(U)$ with $\tau \neq U$. Then $\tilde{f}_{\mathcal{L}}$ is a uni-soft G-filter of $\mathcal{L}$.

Theorem 4.3. *Let $\tilde{f}_{\mathcal{L}}$ be a soft set over U. Then $\tilde{f}_{\mathcal{L}}$ is a uni-soft G-filter of $\mathcal{L}$ if and only if it is a uni-soft filter of $\mathcal{L}$ that satisfies the following condition:*

$$(\forall x, y, z \in L) \left(\tilde{f}_{\mathcal{L}}(x \to (y \to z)) \cup \tilde{f}_{\mathcal{L}}(x \to y) \supseteq \tilde{f}_{\mathcal{L}}(x \to z) \right). \tag{4.3}$$

Proof. Assume that $\tilde{f}_{\mathcal{L}}$ is a uni-soft G-filter of $\mathcal{L}$. Then $\tilde{f}_{\mathcal{L}}$ is a uni-soft filter of $\mathcal{L}$. Note that $x \leq 1 = (x \to y) \to (x \to y)$, and thus $x \to y \leq x \to (x \to y)$ for all $x, y \in L$. It follows from (3.1) that $\tilde{f}_{\mathcal{L}}(x \to y) \supseteq \tilde{f}_{\mathcal{L}}(x \to (x \to y))$. Combining this and (4.2), we have

$$\tilde{f}_{\mathcal{L}}(x \to y) = \tilde{f}_{\mathcal{L}}(x \to (x \to y)) \tag{4.4}$$

for all $x, y \in L$. Using (3.7) and (4.4), we have

$$\tilde{f}_{\mathcal{L}}(x \to (y \to z)) \cup \tilde{f}_{\mathcal{L}}(x \to y) \supseteq \tilde{f}_{\mathcal{L}}(x \to z)$$

for all $x, y, z \in L$.

Conversely, let $\tilde{f}_{\mathcal{L}}$ be a uni-soft filter of $\mathcal{L}$ that satisfies the condition (4.3). If we put $y = x$ and $z = y$ in (4.3) and use (2.1) and (3.3), then

$$\begin{aligned} \tilde{f}_{\mathcal{L}}(x \to y) &\subseteq \tilde{f}_{\mathcal{L}}(x \to (x \to y)) \cup \tilde{f}_{\mathcal{L}}(x \to x) \\ &= \tilde{f}_{\mathcal{L}}(x \to (x \to y)) \cup \tilde{f}_{\mathcal{L}}(1) \\ &= \tilde{f}_{\mathcal{L}}(x \to (x \to y)) \end{aligned}$$

for all $x, y \in L$. Therefore $\tilde{f}_{\mathcal{L}}$ is a uni-soft G-filter of $\mathcal{L}$. □

Theorem 4.4. *Let $\tilde{f}_{\mathcal{L}}$ be a soft set over U that satisfies the condition (3.3) and*

$$(\forall x, y, z \in L)\left(\tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}((y \to z) \to (x \to y)) \supseteq \tilde{f}_{\mathcal{L}}(y)\right). \tag{4.5}$$

Then $\tilde{f}_{\mathcal{L}}$ is a uni-soft G-filter of $\mathcal{L}$.

Proof. If we take $z := 1$ in (4.5) and use (2.1), then

$$\begin{aligned}\tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x \to y) &= \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(1 \to (x \to y)) \\ &= \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}((y \to 1) \to (x \to y)) \\ &\supseteq \tilde{f}_{\mathcal{L}}(y).\end{aligned}$$

Hence $\tilde{f}_{\mathcal{L}}$ is a uni-soft filter of $\mathcal{L}$ by Theorem 3.4. Let $x, y, z \in L$. Since

$$x \to (y \to z) \leq (x \to y) \to (x \to (x \to z))$$

by (2.2) and (2.4), we have $\tilde{f}_{\mathcal{L}}(x \to (y \to z)) \supseteq \tilde{f}_{\mathcal{L}}((x \to y) \to (x \to (x \to z)))$ by (3.1). It follows from (3.1), (3.3), (3.4), (2.4) and (4.5) that

$$\begin{aligned}\tilde{f}_{\mathcal{L}}(x \to y) \cup \tilde{f}_{\mathcal{L}}(x \to (y \to z)) &\supseteq \tilde{f}_{\mathcal{L}}(x \to y) \cup \tilde{f}_{\mathcal{L}}((x \to y) \to (x \to (x \to z))) \\ &\supseteq \tilde{f}_{\mathcal{L}}(x \to (x \to z)) \\ &\supseteq \tilde{f}_{\mathcal{L}}(((x \to z) \to z) \to (x \to z)) \\ &= \tilde{f}_{\mathcal{L}}(((x \to z) \to z) \to (1 \to (x \to z))) \\ &\supseteq \tilde{f}_{\mathcal{L}}(x \to z).\end{aligned}$$

Therefore $\tilde{f}_{\mathcal{L}}$ is a uni-soft G-filter of $\mathcal{L}$ by Theorem 4.3. □

The following example shows that any uni-soft G-filter may not satisfy the condition (4.5).

Example 4.5. The uni-soft G-filter $\tilde{f}_{\mathcal{L}}$ of $\mathcal{L}$ in Example 4.2 does not satisfy the condition (4.5) since

$$\tilde{f}_{\mathcal{L}}(\tfrac{2}{3}) \cup \tilde{f}_{\mathcal{L}}((\tfrac{1}{3} \to \tfrac{1}{4}) \to (\tfrac{2}{3} \to \tfrac{1}{3})) = \tilde{f}_{\mathcal{L}}(\tfrac{2}{3}) \cup \tilde{f}_{\mathcal{L}}(1) = \tau \not\supseteq U = \tilde{f}_{\mathcal{L}}(\tfrac{1}{3}).$$

Proposition 4.6. *For a uni-soft filter $\tilde{f}_{\mathcal{L}}$ of $\mathcal{L}$, the condition (4.5) is equivalent to the following condition.*

$$(\forall x, y \in L)\left(\tilde{f}_{\mathcal{L}}((x \to y) \to x) \supseteq \tilde{f}_{\mathcal{L}}(x)\right). \tag{4.6}$$

Proof. Assume that the condition (4.5) is valid. It follows from (3.3) and (2.1) that

$$\begin{aligned}\tilde{f}_{\mathcal{L}}((x \to y) \to x) &= \tilde{f}_{\mathcal{L}}(1) \cup \tilde{f}_{\mathcal{L}}((x \to y) \to x)\\ &= \tilde{f}_{\mathcal{L}}(1) \cup \tilde{f}_{\mathcal{L}}((x \to y) \to (1 \to x))\\ &\supseteq \tilde{f}_{\mathcal{L}}(x)\end{aligned}$$

for all $x, y \in L$.

Conversely, suppose that the condition (4.6) is valid. It follows from (2.2) and (3.4) that

$$\begin{aligned}\tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}((y \to z) \to (x \to y)) &= \tilde{f}_{\mathcal{L}}(x) \cup \tilde{f}_{\mathcal{L}}(x \to ((y \to z) \to y))\\ &\supseteq \tilde{f}_{\mathcal{L}}((y \to z) \to y) \supseteq \tilde{f}_{\mathcal{L}}(y)\end{aligned}$$

for all $x, y \in L$. □

Combining Theorem 4.4 and Proposition 4.6, we have the following result.

Theorem 4.7. *Every uni-soft filter satisfying the condition* (4.6) *is a uni-soft G-filter.*

Proposition 4.8. *Every uni-soft filter $\tilde{f}_{\mathcal{L}}$ of $\mathcal{L}$ with the condition* (4.5) *satisfies the following condition.*

$$(\forall x, y \in L)\left(\tilde{f}_{\mathcal{L}}((x \to y) \to y) \supseteq \tilde{f}_{\mathcal{L}}((y \to x) \to x)\right). \tag{4.7}$$

Proof. Let $\tilde{f}_{\mathcal{L}}$ be a uni-soft filter of $\mathcal{L}$ that satisfies the condition (4.5) and let $x, y \in L$. Since $x \to ((y \to x) \to x) = (y \to x) \to (x \to x) = (y \to x) \to 1 = 1$, that is, $x \leq (y \to x) \to x$, we have $((y \to x) \to x) \to y \leq x \to y$ by (2.3). It follows from (2.4), (2.2) and (2.3) that

$$\begin{aligned}(x \to y) \to y &\leq (y \to x) \to ((x \to y) \to x)\\ &= (x \to y) \to ((y \to x) \to x)\\ &\leq (((y \to x) \to x) \to y) \to ((y \to x) \to x).\end{aligned}$$

Using (3.1), (3.3), (2.1), (2.2) and (4.5), we have

$$\begin{aligned}\tilde{f}_{\mathcal{L}}((x \to y) \to y) &\supseteq \tilde{f}_{\mathcal{L}}((((y \to x) \to x) \to y) \to ((y \to x) \to x))\\ &= \tilde{f}_{\mathcal{L}}(1) \cup \tilde{f}_{\mathcal{L}}(1 \to ((((y \to x) \to x) \to y) \to ((y \to x) \to x)))\\ &= \tilde{f}_{\mathcal{L}}(1) \cup \tilde{f}_{\mathcal{L}}((((y \to x) \to x) \to y) \to (1 \to ((y \to x) \to x)))\\ &\supseteq \tilde{f}_{\mathcal{L}}((y \to x) \to x).\end{aligned}$$

Hence the condition (4.7) is valid. □

Corollary 4.9. *Every uni-soft filter $\tilde{f}_{\mathcal{L}}$ of $\mathcal{L}$ with the condition (4.6) satisfies the condition (4.7).*

Proposition 4.10. *Every uni-soft G-filter $\tilde{f}_{\mathcal{L}}$ of $\mathcal{L}$ with the condition (4.7) satisfies the condition (4.5).*

Proof. Let $\tilde{f}_{\mathcal{L}}$ be a uni-soft G-filter of $\mathcal{L}$ that satisfies the condition (4.7). For any $x, y, z \in L$, we have

$$\begin{aligned}\tilde{f}_{\mathcal{L}}(z) \cup \tilde{f}_{\mathcal{L}}((x \to y) \to (z \to x)) &= \tilde{f}_{\mathcal{L}}(z) \cup \tilde{f}_{\mathcal{L}}(z \to ((x \to y) \to x))\\ &\supseteq \tilde{f}_{\mathcal{L}}((x \to y) \to x)\\ &\supseteq \tilde{f}_{\mathcal{L}}((x \to y) \to ((x \to y) \to y))\\ &\supseteq \tilde{f}_{\mathcal{L}}((x \to y) \to y)\\ &\supseteq \tilde{f}_{\mathcal{L}}((y \to x) \to x)\end{aligned}$$

by (2.2), (3.4), (3.1), (2.4), (4.2) and (4.7). Since $(x \to y) \to x \leq y \to x \leq z \to (y \to x)$, it follows from (3.1) that $\tilde{f}_{\mathcal{L}}((x \to y) \to x) \supseteq \tilde{f}_{\mathcal{L}}(z \to (y \to x))$ and so from (3.4) that

$$\begin{aligned}\tilde{f}_{\mathcal{L}}(z) \cup \tilde{f}_{\mathcal{L}}((x \to y) \to (z \to x)) &\supseteq \tilde{f}_{\mathcal{L}}(z) \cup \tilde{f}_{\mathcal{L}}((x \to y) \to x)\\ &\supseteq \tilde{f}_{\mathcal{L}}(z) \cup \tilde{f}_{\mathcal{L}}(z \to (y \to x))\\ &\supseteq \tilde{f}_{\mathcal{L}}(y \to x).\end{aligned}$$

Therefore

$$\tilde{f}_{\mathcal{L}}(z) \cup \tilde{f}_{\mathcal{L}}((x \to y) \to (z \to x)) \supseteq \tilde{f}_{\mathcal{L}}(y \to x) \cup \tilde{f}_{\mathcal{L}}((y \to x) \to x) \supseteq \tilde{f}_{\mathcal{L}}(x).$$

Hence the condition (4.5) is valid. □

Theorem 4.11. *Let $\tilde{f}_{\mathcal{L}}$ be a uni-soft filter of $\mathcal{L}$. Then $\tilde{f}_{\mathcal{L}}$ is a uni-soft G-filter of $\mathcal{L}$ if and only if the following condition holds:*

$$(\forall x \in L)\left(\tilde{f}_{\mathcal{L}}(x \to (x \odot x)) = \tilde{f}_{\mathcal{L}}(1)\right). \tag{4.8}$$

Proof. Suppose that $\tilde{f}_{\mathcal{L}}$ is a uni-soft G-filter of L. Since $x \to (x \to (x \odot x)) = 1$ for all $x \in L$, we have $\tilde{f}_{\mathcal{L}}(x \to (x \to (x \odot x))) = \tilde{f}_{\mathcal{L}}(1)$. It follows from (4.3) and (2.1) that

$$\tilde{f}_{\mathcal{L}}(x \to (x \odot x)) \subseteq \tilde{f}_{\mathcal{L}}(x \to (x \to (x \odot x))) \cup \tilde{f}_{\mathcal{L}}(x \to x) = \tilde{f}_{\mathcal{L}}(1)$$

and so from (3.3) that $\tilde{f}_{\mathcal{L}}(x \to (x \odot x)) = \tilde{f}_{\mathcal{L}}(1)$ for all $x \in L$.

Conversely, let $\tilde{f}_{\mathcal{L}}$ be a uni-soft filter of $\mathcal{L}$ which satisfies the condition (4.8) and let $x, y \in L$. Since

$$x \to (x \to y) = (x \odot x) \to y \leq (x \to (x \odot x)) \to (x \to y)$$

by (2.2) and (2.4), it follows from (3.1) that

$$\tilde{f}_{\mathcal{L}}(x \to (x \to y)) \supseteq \tilde{f}_{\mathcal{L}}((x \to (x \odot x)) \to (x \to y)).$$

Hence, we have

$$\begin{aligned}\tilde{f}_{\mathcal{L}}(x \to y) &\subseteq \tilde{f}_{\mathcal{L}}((x \to (x \odot x)) \to (x \to y)) \cup \tilde{f}_{\mathcal{L}}(x \to (x \odot x)) \\ &\subseteq \tilde{f}_{\mathcal{L}}(x \to (x \to y)) \cup \tilde{f}_{\mathcal{L}}(x \to (x \odot x)) \\ &= \tilde{f}_{\mathcal{L}}(x \to (x \to y)) \cup \tilde{f}_{\mathcal{L}}(1) \\ &= \tilde{f}_{\mathcal{L}}(x \to (x \to y))\end{aligned}$$

by using (3.4), (4.8) and (3.3). Hence $\tilde{f}_{\mathcal{L}}$ is a uni-soft G-filter of $\mathcal{L}$. $\square$

Theorem 4.12. *A soft set $\tilde{f}_{\mathcal{L}}$ over U is a uni-soft G-filter of $\mathcal{L}$ if and only if it is a uni-soft filter of $\mathcal{L}$ with an additional condition:*

$$(\forall x, y \in L)\left(\tilde{f}_{\mathcal{L}}(x \to y) = \tilde{f}_{\mathcal{L}}(x \to (x \to y))\right). \tag{4.9}$$

Proof. Suppose that $\tilde{f}_{\mathcal{L}}$ is a uni-soft G-filter of $\mathcal{L}$. Then $\tilde{f}_{\mathcal{L}}$ is a uni-soft filter of $\mathcal{L}$. Let $x, y \in L$. Since $x \to y \leq x \to (x \to y)$, we have $\tilde{f}_{\mathcal{L}}(x \to y) \supseteq \tilde{f}_{\mathcal{L}}(x \to (x \to y))$ by (3.1). Hence $\tilde{f}_{\mathcal{L}}(x \to y) = \tilde{f}_{\mathcal{L}}(x \to (x \to y))$ by using (4.2).

Conversely, let $\tilde{f}_{\mathcal{L}}$ be a uni-soft filter of $\mathcal{L}$ with the condition (4.9). It follows from Proposition 3.6 that

$$\tilde{f}_{\mathcal{L}}(x \to (y \to z)) \cup \tilde{f}_{\mathcal{L}}(x \to y) \supseteq \tilde{f}_{\mathcal{L}}(x \to (x \to z)) = \tilde{f}_{\mathcal{L}}(x \to z)$$

for all $x, y, z \in L$. Therefore $\tilde{f}_{\mathcal{L}}$ is a uni-soft G-filter of $\mathcal{L}$ by Theorem 4.3. $\square$

Proposition 4.13. *Every uni-soft G-filter $\tilde{f}_{\mathcal{L}}$ of $\mathcal{L}$ satisfies the following conditions:*

$$(\forall x, y, z \in L)\left(\tilde{f}_{\mathcal{L}}(x \to (y \to z)) \supseteq \tilde{f}_{\mathcal{L}}((x \to y) \to (x \to z))\right). \tag{4.10}$$

$$(\forall x, y, z \in L)\left(\tilde{f}_{\mathcal{L}}(x \to (y \to z)) = \tilde{f}_{\mathcal{L}}((x \to y) \to (x \to z))\right). \tag{4.11}$$

Proof. Let $\tilde{f}_{\mathcal{L}}$ be a uni-soft G-filter of $\mathcal{L}$. Using (2.2), (4.3), (2.4) and (3.3), we have

$$\begin{aligned}\tilde{f}_{\mathcal{L}}((x \to y) \to (x \to z)) &= \tilde{f}_{\mathcal{L}}(x \to ((x \to y) \to z)) \\ &\subseteq \tilde{f}_{\mathcal{L}}(x \to (y \to z)) \cup \tilde{f}_{\mathcal{L}}(x \to ((y \to z) \to ((x \to y) \to z))) \\ &= \tilde{f}_{\mathcal{L}}(x \to (y \to z)) \cup \tilde{f}_{\mathcal{L}}((y \to z) \to ((x \to y) \to (x \to z))) \\ &= \tilde{f}_{\mathcal{L}}(x \to (y \to z)) \cup \tilde{f}_{\mathcal{L}}(1) \\ &= \tilde{f}_{\mathcal{L}}(x \to (y \to z))\end{aligned}$$

for all $x, y, z \in L$. Thus (4.10) holds. Since $(x \to y) \to (x \to z) \leq x \to (y \to z)$ for all $x, y, z \in L$, it follows from (3.1) that $\tilde{f}_{\mathcal{L}}((x \to y) \to (x \to z)) \supseteq \tilde{f}_{\mathcal{L}}(x \to (y \to z))$ and so that

$$\tilde{f}_{\mathcal{L}}(x \to (y \to z)) = \tilde{f}_{\mathcal{L}}((x \to y) \to (x \to z))$$

for all $x, y, z \in L$ by using (4.10). □

Proposition 4.14. *Assume that $\mathcal{L}$ satisfies the divisibility, that is, $x \wedge y = x \odot (x \to y)$ for all $x, y \in L$. If $\tilde{f}_{\mathcal{L}}$ is a uni-soft G-filter of $\mathcal{L}$ satisfying (4.11), then the following equality is true.*

$$(\forall x, y, z \in L) \left(\tilde{f}_{\mathcal{L}}((x \odot y) \to z) = \tilde{f}_{\mathcal{L}}((x \wedge y) \to z) \right). \tag{4.12}$$

Proof. Using the divisibility and (2.2), we have

$$(x \wedge y) \to z = (x \odot (x \to y)) \to z = (x \to y) \to (x \to z)$$

for all $x, y, z \in L$. It follows from (2.2) and (4.11) that

$$\begin{aligned}\tilde{f}_{\mathcal{L}}((x \odot y) \to z) &= \tilde{f}_{\mathcal{L}}(x \to (y \to z)) \\ &= \tilde{f}_{\mathcal{L}}((x \to y) \to (x \to z)) \\ &= \tilde{f}_{\mathcal{L}}((x \wedge y) \to z)\end{aligned}$$

for all $x, y, z \in L$. □

Theorem 4.15. *Let $\mathcal{L}$ satisfy the divisibility, that is, $x \wedge y = x \odot (x \to y)$ for all $x, y \in L$. Then every uni-soft filter $\tilde{f}_{\mathcal{L}}$ of $\mathcal{L}$ satisfying the condition (4.12) is a uni-soft G-filter of $\mathcal{L}$.*

Proof. Using Proposition 3.6, (2.2) and (4.12), we have

$$\begin{aligned}&\tilde{f}_{\mathcal{L}}(x \to (y \to z)) \cup \tilde{f}_{\mathcal{L}}(x \to y) \supseteq \tilde{f}_{\mathcal{L}}(x \to (x \to z)) \\ &= \tilde{f}_{\mathcal{L}}((x \odot x) \to z) = \tilde{f}_{\mathcal{L}}((x \wedge x) \to z) = \tilde{f}_{\mathcal{L}}(x \to z)\end{aligned}$$

for all $x, y, z \in L$. Therefore $\tilde{f}_{\mathcal{L}}$ is a uni-soft G-filter of $\mathcal{L}$ by Theorem 4.3. □

Theorem 4.16. *Let $\tilde{f}_{\mathcal{L}}$ and $\tilde{g}_{\mathcal{L}}$ be uni-soft filters of $\mathcal{L}$ such that $\tilde{f}_{\mathcal{L}}(1) = \tilde{g}(1)$ and $\tilde{f}_{\mathcal{L}} \supseteq \tilde{g}_{\mathcal{L}}$, i.e., $\tilde{f}_{\mathcal{L}}(x) \supseteq \tilde{g}_{\mathcal{L}}(x)$ for all $x \in L$. If $\tilde{f}_{\mathcal{L}}$ is a uni-soft G-filter of $\mathcal{L}$, then so is $\tilde{g}_{\mathcal{L}}$.*

Proof. Assume that $\tilde{f}_{\mathcal{L}}$ is a uni-soft G-filter of $\mathcal{L}$. Using (2.2) and (2.1), we have

$$x \to (x \to ((x \to (x \to y)) \to y)) = (x \to (x \to y)) \to (x \to (x \to y)) = 1$$

for all $x, y \in L$. Thus

$$\begin{aligned}\tilde{g}(x \to ((x \to (x \to y)) \to y)) &\subseteq \tilde{f}_{\mathcal{L}}(x \to ((x \to (x \to y)) \to y)) \\ &= \tilde{f}_{\mathcal{L}}(x \to (x \to ((x \to (x \to y)) \to y))) \\ &= \tilde{f}_{\mathcal{L}}(1) = \tilde{g}(1)\end{aligned}$$

by hypotheses and (4.4), and so

$$\tilde{g}(x \to ((x \to (x \to y)) \to y)) = \tilde{g}(1)$$

for all $x, y \in L$ by (3.3). Since $\tilde{g}_{\mathcal{L}}$ is a uni-soft filter of $\mathcal{L}$, it follows from (3.4), (2.2) and (3.3) that

$$\begin{aligned}\tilde{g}(x \to y) &\subseteq \tilde{g}(x \to (x \to y)) \cup \tilde{g}((x \to (x \to y)) \to (x \to y)) \\ &= \tilde{g}(x \to (x \to y)) \cup \tilde{g}(x \to ((x \to (x \to y)) \to y)) \\ &= \tilde{g}(x \to (x \to y)) \cup \tilde{g}(1) \\ &= \tilde{g}(x \to (x \to y))\end{aligned}$$

for all $x, y \in L$. Therefore $\tilde{g}_{\mathcal{L}}$ is a uni-soft G-filter of $\mathcal{L}$. $\square$

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Mathematical analysis of a general viral infection model with immune response

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Abstract

In this paper, we study the global dynamics of a viral infection model with antibody immune response. The incidence rate is given by a general function of the population of the uninfected target cells, infected cells and free viruses. We have established a set of conditions on the general incidence rate function and determined two threshold parameters R_0 (the basic infection reproduction number) and R_1 (the antibody immune response activation number) which are sufficient to determine the global behavior of the model. The global asymptotic stability of the equilibria of the model has been proven by using direct Lyapunov method and applying LaSalle's invariance principle.

Keywords: Virus dynamics; global stability; antibody immune response; Lyapunov functional.

Mathematics Subject Classification: 34D20; 34D23; 37N25; 92D30

1 Introduction

Several works have been devoted to propose mathematical models of viral infectious dynamics such as human immunodeficiency virus (HIV) (see, for example, [1]-[22]), hepatitis B virus (HBV) [23]-[26], hepatitis C virus (HCV) [27]-[29] and human T cell leukemia HTLV [30], etc. Mathematical models of viral infection can help for understanding the viral dynamics and developing antiviral drug therapies. In reality, the immune response needs an indispensable components to do its job such as antibodies, cytokines, natural killer cells, and T cells. The antibody immune response is a part of the adaptive system in which the body responds to pathogens by primarily using antibodies that produced from the B cells. While the other part is the Cytotoxic T Lymphocytes (CTL) immune response where the CTL attacks and kills the infected cells [7]. In some infections such as malaria, the CTL immune response is less effective than the antibody immune response [31]. Mathematical models of viral infection with antibody immune response have been proposed and analyzed in ([32]-[39]). The basic model of viral infection with antibody immune response has introduced by Murase et. al. [32] and Shifi Wang [39] as:

$$\dot{x} = s - dx - \beta vx, \tag{1}$$

$$\dot{y} = \beta vx - ay, \tag{2}$$

$$\dot{v} = ky - bzv - cv, \tag{3}$$

$$\dot{z} = rzv - \mu z, \tag{4}$$

where x, y, v and z denote the populations of uninfected target cells, infected cells, free virus particles and antibody immune cells at time t, respectively. Parameters s, k and r represent, respectively, the rate at which new healthy cells are generated from the source within the body, the generation rate constant of free viruses produced from the infected cells and the proliferation

rate constant of antibody immune cells. Parameters d, a, c and μ are the natural death rate constants of the uninfected target cells, infected cells, free virus particles and antibody immune cells, respectively. Parameter β is the infection rate constant and b is the removal rate constant of the viruses due to the antibodies. All the parameters given in model (1)-(4) are positive.

Note that, the infection rate in model (1)-(4) is presented to be bilinear in x and v, which can not be completely describe the interaction between the uninfected target cells and viruses. Nevertheless, there are many types of an improved incidence rate which are more commonly used due to their benefit for helping us gain the unification theory through passing over the unessential details (see e.g. [40] and [41]). Variety of viral infection models with antibody immune response have been considered different forms of the incidence rate such as saturated incidence rate, $\frac{\beta xv}{1+\alpha v}$ where $\alpha \geq 0$ [42], [37], [35], Beddington-DeAngelis functional response, $\frac{\beta xv}{1+\gamma x+\alpha v}$, $\alpha, \gamma \geq 0$ [36], and general form, $\psi(x, v)v$ [38].

However the infection rate does not depend on the infected cells y. In some viral infections such as HBV, the infection rate depends on x, y and v [25], [24]. In [43], the infection rate is given by $\psi(x, y, v)v$, however the antibody immune response has been neglected. Our aim in this paper is to investigate the global stability analysis of the viral infection model with general incidence rate function and antibody immune response.

The rest of the paper is designed as follows. In the next section, we introduce the model and discuss the non-negativity and boundedness of the solutions. In Section 3, we define two threshold parameters and discuss the existence of the model's equilibria. In Section 4, we study the global asymptotic stability of the equilibria using suitable Lyapunov functional and applying LaSalle's invariance principle. Finally, conclusion is given in Section 5.

2 The mathematical model

In this section, we consider the following viral infection model with general incidence rate taking into consideration the antibody immune response.

$$\dot{x} = s - dx - \psi(x, y, v)v, \tag{5}$$

$$\dot{y} = \psi(x, y, v)v - ay, \tag{6}$$

$$\dot{v} = ky - bzv - cv, \tag{7}$$

$$\dot{z} = rzv - \mu z. \tag{8}$$

The definitions of all variables and parameters are identical to those given in Section 1. The incidence rate of infection is presented by a general function in the form $\psi(x, y, v)v$, where ψ is continuously differentiable and satisfies the following assumptions (see [38] and [43]):

Assumption A1. $\psi(x, y, v) > 0$ for all $x > 0$, $y \geq 0$, $v \geq 0$, and $\psi(0, y, v) = 0$ for all $y \geq 0$, $v \geq 0$.

Assumption A2. $\dfrac{\partial \psi(x, y, v)}{\partial x} > 0$ for all $x > 0$, $y \geq 0$ and $v \geq 0$.

Assumption A3. $\dfrac{\partial \psi(x, y, v)}{\partial y} < 0$, $\dfrac{\partial \psi(x, y, v)}{\partial v} < 0$ for all $x > 0$, $y > 0$ and $v > 0$.

Assumption A4. $\dfrac{\partial\left(\psi(x, y, v)v\right)}{\partial v} > 0$ for all $x > 0$, $y > 0$ and $v > 0$.

2.1 Positive invariance

In the following proposition, we show that the non-negative orthant $\mathbb{R}^4_{\geq 0}$ is the positively invariant and there exists a compact set which is positively invariant for model (5)-(8).

Proposition 1. Assume that Assumption A1 is satisfied. Then there exist positive numbers L_i, $i = 1, 2, 3$, such that the compact set

$$\Gamma = (x, y, v, z) \in \mathbb{R}^4_{\geq 0} : 0 \leq x, y \leq L_1, 0 \leq v \leq L_2, 0 \leq z \leq L_3$$

is positively invariant.

Proof. First, we prove that the orthant $\mathbb{R}^4_{\geq 0}$ is positively invariance for system (5)-(8). We have

$$\dot{x}\mid_{x=0}= s > 0,$$

$$\dot{y}\mid_{y=0}= \psi(x,0,v)v \geq 0 \text{ for all } x > 0, v \geq 0,$$

$$\dot{v}\mid_{v=0}= ky \geq 0 \text{ for all } y \geq 0,$$

$$\dot{z}\mid_{z=0}= 0.$$

Hence, all the solutions are nonnegative.

Next we show that the solutions of system are bounded. Let $T_1(t) = x(t) + y(t)$, then

$$\dot{T}_1(t) = (s - dx - \psi(x,y,v)v) + \psi(x,y,v)v - ay,$$

$$= s - dx - ay \leq s - \sigma_1(x+y) = s - \sigma_1 T_1(t),$$

where $\sigma_1 = \min\{d, a\}$. Hence $0 \leq T_1(t) \leq \frac{s}{\sigma_1}$ for all $t \geq 0$ if $T_1(0) \leq \frac{s}{\sigma_1}$. It follows that, $0 \leq x(t), y(t) \leq L_1$ for all $t \geq 0$ if $x(0) + y(0) \leq L_1$, where $L_1 = \frac{s}{\sigma_1}$. Moreover, let $T_2(t) = v(t) + \frac{b}{r}z(t)$, then

$$\dot{T}_2(t) = ky - cv - \frac{b\mu}{r}z \leq kL_1 - \sigma_2(v + \frac{b}{r}z) = kL_1 - \sigma_2 T_2(t),$$

where $\sigma_2 = \min\{c, \mu\}$. Hence $0 \leq T_2(t) \leq L_2$ for all $t \geq 0$ when $T_2(0) \leq L_2$. It follows that $0 \leq v(t) \leq L_2$ and $0 \leq z(t) \leq L_3$ for all $t \geq 0$ if $v(0) + \frac{b}{r}z(0) \leq L_2$, where $L_2 = \dfrac{kL_1}{\sigma_2}$ and $L_3 = \frac{r}{b}L_2$. Therefore, $x(t), y(t), v(t)$ and $z(t)$ are all bounded.

2.2 The equilibria and threshold parameters

At any equilibrium we have

$$s - dx - \psi(x, y, v)v = 0, \tag{9}$$

$$\psi(x, y, v)v - ay = 0, \tag{10}$$

$$ky - bzv - cv = 0, \tag{11}$$

$$rzv - \mu z = 0. \tag{12}$$

From Eq. (12), either $z = 0$ or $z \neq 0$. If $z = 0$, then from Eqs. (9)-(11) we get

$$y = \frac{s - dx}{a} = \frac{c}{k}v, \qquad v = \frac{k(s - dx)}{ac}. \tag{13}$$

Substituting from Eq. (13) into Eq. (10) we get:

$$\left[\psi\left(x, \frac{s - dx}{a}, \frac{k(s - dx)}{ac}\right) - \frac{ac}{k}\right] v = 0. \tag{14}$$

Eq. (14) has two possible solutions $v = 0$ or $v \neq 0$. If $v = 0$, then from Eqs. (9) and (10), we get $x = s/d$ and $y = 0$ which leads to the infection-free equilibrium $E_0(x_0, 0, 0, 0)$ where $x_0 = s/d$. If $v \neq 0$, then we have

$$\psi\left(x, \frac{s - dx}{a}, \frac{k(s - dx)}{ac}\right) - \frac{ac}{k} = 0.$$

Let

$$\Phi_1(x) = \psi\left(x, \frac{s - dx}{a}, \frac{k(s - dx)}{ac}\right) - \frac{ac}{k} = 0.$$

Then, we have

$$\Phi_1'(x) = \frac{\partial \psi}{\partial x} - \frac{d}{a}\frac{\partial \psi}{\partial y} - \frac{kd}{ac}\frac{\partial \psi}{\partial v}.$$

Because of Assumptions A2 and A3, we have $\Phi_1'(x) > 0$ which implies that function $\Phi_1(x)$ is strictly increasing w.r.t. x. Moreover,

$$\Phi_1(0) = \psi\left(0, \frac{s}{a}, \frac{ks}{ac}\right) - \frac{ac}{k} = -\frac{ac}{k} < 0,$$
$$\Phi_1(x_0) = \psi(x_0, 0, 0) - \frac{ac}{k} = \frac{ac}{k}\left(\frac{k\psi(x_0, 0, 0)}{ac} - 1\right).$$

Therefore, if $\frac{k\psi(x_0,0,0)}{ac} > 1$, then there exists a unique $x_1 \in (0, x_0)$ such that $\Phi_1(x_1) = 0$. Therefore from Eq. (13) we obtain $y_1 = \frac{d(x_0 - x_1)}{a} > 0$ and $v_1 = \frac{kd(x_0 - x_1)}{ac} > 0$. It follows that, if $\frac{k\psi(x_0,0,0)}{ac} > 1$, then there exists a chronic-infection equilibrium without antibody immune response $E_1(x_1, y_1, v_1, 0)$.

Let us define the basic reproduction number as:

$$R_0 = \frac{k\psi(x_0, 0, 0)}{ac}.$$

The parameter R_0 determines whether a chronic-infection can be established. The other possibility of Eq. (12) is $z \neq 0$ which leads to $v_2 = \frac{\mu}{r}$. From Eq. (9) we let

$$\Phi_2(x) = s - dx - \psi\left(x, \frac{s - dx}{a}, v_2\right) v_2 = 0.$$

Assumptions A2 and A3 provide that Φ_2 is a decreasing function of x. Clearly, $\Phi_2(0) = s > 0$ and $\Phi_2(x_0) = -\psi(x_0, 0, v_2)v_2 < 0$. Thus, there exists a unique $x_2 \in (0, x_0)$ such that $\Phi_2(x_2) = 0$. It follows from Eqs. (11) and (13) that, $y_2 = \frac{d(x_0 - x_2)}{a} > 0$ and $z_2 = \frac{k\psi(x_2, y_2, v_2)}{ab} - \frac{c}{b} = \frac{c}{b}\left(\frac{k\psi(x_2, y_2, v_2)}{ac} - 1\right)$. Then if $\frac{k\psi(x_2, y_2, v_2)}{ac} > 1$ then $z_2 > 0$. Now we Define the antibody immune response activation number as:

$$R_1 = \frac{k\psi(x_2, y_2, v_2)}{ac},$$

which determines whether a persistent antibody immune response can be established. Hence, z_2 can be rewritten as $z_2 = \frac{c}{b}(R_1 - 1)$. It follows that, there is a chronic-infection equilibrium with

antibody immune response $E_2(x_2, y_2, v_2, z_2)$ iff $R_1 > 1$.

Clearly from Assumptions A2 and A3, we have

$$R_1 = \frac{k\psi(x_2, y_2, v_2)}{ac} < \frac{k\psi(x_0, y_2, v_2)}{ac} < \frac{k\psi(x_0, 0, 0)}{ac} = R_0.$$

2.3 Global stability analysis

In this section, the global asymptotic stability of the three equilibria of model (5)-(8) will be established by using direct Lyapunov method and applying LaSalle's invariance principle. Let us define the function $H : (0, \infty) \to [0, \infty)$ as

$$H(w) = w - 1 - \ln w.$$

Theorem 1. Let Assumptions A1-A3 be hold true and $R_0 \leq 1$, then the infection-free equilibrium E_0 is globally asymptotically stable (GAS).

Proof. We construct a Lyapunov functional as:

$$U_0 = x - x_0 - \int_{x_0}^{x} \frac{\psi(x_0, 0, 0)}{\psi(\eta, 0, 0)} d\eta + y + \frac{a}{k} v + \frac{ab}{rk} z. \tag{15}$$

We calculate $\frac{dU_0}{dt}$ along the solutions of model (5)-(8) as:

$$\begin{aligned} \frac{dU_0}{dt} &= d\left(1 - \frac{\psi(x_0, 0, 0)}{\psi(x, 0, 0)}\right)(x_0 - x) + \left(\psi(x, y, v)\frac{\psi(x_0, 0, 0)}{\psi(x, 0, 0)} - \frac{ac}{k}\right)v - \frac{ab\mu}{rk} z \\ &= s\left(1 - \frac{\psi(x_0, 0, 0)}{\psi(x, 0, 0)}\right)\left(1 - \frac{x}{x_0}\right) + \frac{ac}{k}\left(\frac{\psi(x, y, v)}{\psi(x, 0, 0)} R_0 - 1\right)v - \frac{ab\mu}{rk} z. \end{aligned} \tag{16}$$

From Assumptions A2 and A3 we know that $\psi(x, y, v)$ is an increasing function of x and decreasing function of y and v. Then the first term of Eq. (16) is less than or equal zero and

$$\psi(x, y, v) < \psi(x, 0, 0), \quad x, y, v > 0.$$

It follows that

$$\frac{dU_0}{dt} \leq s\left(1-\frac{\psi(x_0,0,0)}{\psi(x,0,0)}\right)\left(1-\frac{x}{x_0}\right)+\frac{ac}{k}\left(R_0-1\right)v-\frac{ab\mu}{rk}z. \tag{17}$$

Therefore, if $R_0 \leq 1$, then $\frac{dU_0}{dt} \leq 0$ for all $x, y, v, z > 0$. We note that the solutions of system (5)-(8) converge to Ω, the largest invariant subset of $\left\{\frac{dU_0}{dt}=0\right\}$ [44]. From (17), we have $\frac{dU_0}{dt}=0$ iff $x=x_0$, $v=0$ and $z=0$. The set Ω is invariant and for any element belong to Ω satisfies $v=0$ and $z=0$. We can see from Eq. (7) that

$$\dot{v}=0=ky.$$

It follows that, $y=0$. Hence $\frac{dU_0}{dt}=0$ iff $x=x_0$ and $y=v=z=0$. Using LaSalle's invariance principle, we derive that E_0 is GAS.

Assumption A5

$$\left(1-\frac{\psi(x,y,v)}{\psi(x,y_i,v_i)}\right)\left(\frac{\psi(x,y_i,v_i)}{\psi(x,y,v)}-\frac{v}{v_i}\right) \leq 0,\ i=1,2 \text{ for all } x,y,v>0.$$

Theorem 2. Assume that Assumptions A1-A5 are satisfied and $R_1 \leq 1 < R_0$, then the chronic-infection equilibrium without antibody immune response E_1 is GAS.

Proof. Define a Lyapunov functional as:

$$U_1 = x - x_1 - \int_{x_1}^{x}\frac{\psi(x_1,y_1,v_1)}{\psi(\eta,y_1,v_1)}d\eta + y_1 H\left(\frac{y}{y_1}\right)+\frac{a}{k}v_1 H\left(\frac{v}{v_1}\right)+\frac{ab}{rk}z.$$

Calculating the time derivative of U_1 along the trajectories of system (5)-(8), we obtain

$$\begin{aligned}
\frac{dU_1}{dt} &= \left(1-\frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)}\right)\left(s-dx-\psi(x,y,v)\,v\right)+\left(1-\frac{y_1}{y}\right)\left(\psi(x,y,v)v-ay\right)\\
&+\frac{a}{k}\left(1-\frac{v_1}{v}\right)\left(ky-bzv-cv\right)+\frac{ab}{rk}(rzv-\mu z)\\
&=\left(1-\frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)}\right)(s-dx)+\psi(x_1,y_1,v_1)\frac{\psi(x,y,v)v}{\psi(x,y_1,v_1)}\\
&-\frac{y_1}{y}\psi(x,y,v)v+ay_1-\frac{ac}{k}v-ay\frac{v_1}{v}+\frac{ac}{k}v_1+\frac{ab}{k}v_1z-\frac{ab\mu}{rk}z.
\end{aligned} \tag{18}$$

Using the equilibrium conditions for E_1:

$$s = dx_1 + ay_1, \quad \psi(x_1, y_1, v_1)v_1 = ay_1 = \frac{ac}{k}v_1,$$

we obtain

$$\begin{aligned}\frac{dU_1}{dt} &= d\left(1 - \frac{\psi(x_1, y_1, v_1)}{\psi(x, y_1, v_1)}\right)(x_1 - x) + 3ay_1 - ay_1\frac{\psi(x_1, y_1, v_1)}{\psi(x, y_1, v_1)} + ay_1\frac{\psi(x, y, v)v}{\psi(x, y_1, v_1)v_1} \\ &- ay_1\frac{y_1\psi(x, y, v)v}{y\psi(x_1, y_1, v_1)v_1} - ay_1\frac{v}{v_1} - ay_1\frac{v_1 y}{vy_1} + \frac{ab}{k}\left(v_1 - \frac{\mu}{r}\right)z. \end{aligned} \tag{19}$$

Collecting terms of Eq. (19) we get

$$\begin{aligned}\frac{dU_1}{dt} &= dx_1\left(1 - \frac{\psi(x_1, y_1, v_1)}{\psi(x, y_1, v_1)}\right)\left(1 - \frac{x}{x_1}\right) \\ &+ ay_1\left(\frac{\psi(x, y, v)v}{\psi(x, y_1, v_1)v_1} - \frac{v}{v_1} - 1 + \frac{\psi(x, y_1, v_1)}{\psi(x, y, v)}\right) \\ &+ ay_1\left[4 - \frac{\psi(x_1, y_1, v_1)}{\psi(x, y_1, v_1)} - \frac{y_1\psi(x, y, v)v}{y\psi(x_1, y_1, v_1)v_1} - \frac{v_1 y}{vy_1} - \frac{\psi(x, y_1, v_1)}{\psi(x, y, v)}\right] \\ &+ \frac{ab}{k}\left(v_1 - \frac{\mu}{r}\right)z. \end{aligned} \tag{20}$$

Eq. (20) can be simplified as:

$$\begin{aligned}\frac{dU_1}{dt} &= dx_1\left(1 - \frac{\psi(x_1, y_1, v_1)}{\psi(x, y_1, v_1)}\right)\left(1 - \frac{x}{x_1}\right) \\ &+ ay_1\left(1 - \frac{\psi(x, y, v)}{\psi(x, y_1, v_1)}\right)\left(\frac{\psi(x, y_1, v_1)}{\psi(x, y, v)} - \frac{v}{v_1}\right) \\ &+ ay_1\left[4 - \frac{\psi(x_1, y_1, v_1)}{\psi(x, y_1, v_1)} - \frac{y_1\psi(x, y, v)v}{y\psi(x_1, y_1, v_1)v_1} - \frac{v_1 y}{vy_1} - \frac{\psi(x, y_1, v_1)}{\psi(x, y, v)}\right] \\ &+ \frac{ab}{k}\left(v_1 - \frac{\mu}{r}\right)z. \end{aligned} \tag{21}$$

From Assumptions A1 and A5, we get that the first and second terms of Eq. (21) is less than or equal zero. Since the geometrical mean is less than or equal to the arithmetical mean, then the third term of Eq. (21) is also less than or equal zero.

Now we show that if $R_1 \le 1$ then $v_1 \le \frac{\mu}{r} = v_2$. Let $R_0 > 1$, then we want to show that

$$sgn(x_2 - x_1) = sgn(v_1 - v_2) = sgn(y_1 - y_2) = sgn(R_1 - 1).$$

From Assumptions A2-A4, for $x_1, x_2, y_1, y_2, v_1, v_2 > 0$, we have

$$(\psi(x_2, y_2, v_2) - \psi(x_1, y_2, v_2))(x_2 - x_1) > 0, \tag{22}$$

$$(\psi(x_1, y_1, v_1) - \psi(x_1, y_2, v_1))(y_2 - y_1) > 0 \tag{23}$$

$$(\psi(x_1, y_1, v_1) - \psi(x_1, y_1, v_2))(v_2 - v_1) > 0, \tag{24}$$

$$(\psi(x_2, y_2, v_2)v_2 - \psi(x_2, y_2, v_1)v_1)(v_2 - v_1) > 0. \tag{25}$$

First, we claim $sgn(x_2-x_1) = sgn(v_1-v_2)$. Suppose this is not true, i.e., $sgn(x_2-x_1) = sgn(v_2-v_1)$. Using the conditions of the equilibria E_1 and E_2 we have

$$\begin{aligned}(s - dx_2) - (s - dx_1) &= \psi(x_2, y_2, v_2)v_2 - \psi(x_1, y_1, v_1)v_1 \\ &= a(y_2 - y_1),\end{aligned} \tag{26}$$

then $sgn(x_1 - x_2) = sgn(y_2 - y_1)$. Moreover

$$\begin{aligned}(s - dx_2) - (s - dx_1) &= \psi(x_2, y_2, v_2)v_2 - \psi(x_1, y_1, v_1)v_1 \\ &= (\psi(x_2, y_2, v_2)v_2 - \psi(x_2, y_2, v_1)v_1) + (\psi(x_2, y_2, v_1)v_1 - \psi(x_1, y_2, v_1)v_1) \\ &+ (\psi(x_1, y_2, v_1)v_1 - \psi(x_1, y_1, v_1)v_1).\end{aligned}$$

Therefore, from inequalities (22)-(26) we get:

$$sgn\,(x_1 - x_2) = sgn\,(x_2 - x_1)\,,$$

which leads to contradiction. Thus, $sgn\,(x_2 - x_1) = sgn\,(v_1 - v_2)$. Using the equilibrium conditions for E_1 we have $\frac{k\psi(x_1,y_1,v_1)}{ac} = 1$, then

$$\begin{aligned}R_1 - 1 &= \frac{k\psi(x_2, y_2, v_2)}{ac} - \frac{k\psi(x_1, y_1, v_1)}{ac} \\ &= \frac{k}{ac}(\psi(x_2, y_2, v_2) - \psi(x_2, y_2, v_1) + \psi(x_2, y_2, v_1) \\ &- \psi(x_1, y_2, v_1) + \psi(x_1, y_2, v_1) - \psi(x_1, y_1, v_1)).\end{aligned}$$

We get $sgn(R_1 - 1) = sgn(v_1 - v_2)$. Hence, if $R_0 > 1$, then $x_1, y_1, v_1 > 0$, and if $R_1 \leq 1$, then $v_1 \leq v_2 = \frac{\mu}{r}$. It follows from the above discussion that $\frac{dU_1}{dt} \leq 0$ for all $x, y, v, z > 0$ and $\frac{dU_1}{dt} = 0$ iff $x = x_1, y = y_1, v = v_1$ and $z = 0$. So Ω contains a unique point, the equilibrium E_1. Thus, we prove the global asymptotic stability of the chronic-infection equilibrium without antibody immune response E_1 by using LaSalle's invariance principle.

Theorem 3. Let Assumptions A1-A5 be hold true and $R_1 > 1$, then the chronic-infection equilibrium with antibody immune response E_2 is GAS.

Proof. We construct a Lyapunov functional as follows:

$$U_2 = x - x_2 - \int_{x_2}^{x} \frac{\psi(x_2, y_2, v_2)}{\psi(\eta, y_2, v_2)} d\eta + y_2 H\left(\frac{y}{y_2}\right) + \frac{a}{k} v_2 H\left(\frac{v}{v_2}\right) + \frac{ab}{rk} z_2 H\left(\frac{z}{z_2}\right). \tag{27}$$

Function U_2 satisfies:

$$\begin{aligned}\frac{dU_2}{dt} &= \left(1 - \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)}\right)(s - dx - \psi(x, y, v)v) + \left(1 - \frac{y_2}{y}\right)(\psi(x, y, v)v - ay) \\ &+ \frac{a}{k}\left(1 - \frac{v_2}{v}\right)(ky - bzv - cv) + \frac{ab}{rk}\left(1 - \frac{z_2}{z}\right)(rzv - \mu z).\end{aligned} \tag{28}$$

Applying $s = dx_2 + ay_2$, we get

$$\begin{aligned}\frac{dU_2}{dt} &= d\left(1 - \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)}\right)(x_2 - x) + ay_2 \\ &- ay_2 \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)} + \psi(x, y, v)v \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)} \\ &- \psi(x_2, y_2, v_2)v_2 \frac{y_2 \psi(x, y, v)v}{y\psi(x_2, y_2, v_2)v_2} + ay_2 - \frac{ac}{k}v - ay\frac{v_2}{v} \\ &+ \frac{ac}{k}v_2 + \frac{ab}{k}v_2 z - \frac{ab\mu}{rk}z - \frac{ab}{k}z_2 v + \frac{ab\mu}{rk}z_2.\end{aligned} \tag{29}$$

By using the equilibrium conditions of E_2

$$\psi(x_2, y_2, v_2)v_2 = ay_2, \quad cv_2 = ky_2 - bv_2 z_2, \quad \mu = rv_2,$$

and the following equality

$$cv = cv_2 \frac{v}{v_2} = \frac{v}{v_2}(ky_2 - bv_2 z_2),$$

we obtain

$$\frac{dU_2}{dt} = d\left(1 - \frac{\psi(x_2,y_2,v_2)}{\psi(x,y_2,v_2)}\right)(x_2 - x) + ay_2\left(\frac{\psi(x,y,v)v}{\psi(x,y_2,v_2)v_2} - \frac{v}{v_2} - 1 + \frac{\psi(x,y_2,v_2)}{\psi(x,y,v)}\right)$$
$$+ ay_2\left[4 - \frac{\psi(x_2,y_2,v_2)}{\psi(x,y_2,v_2)} - \frac{y_2\psi(x,y,v)v}{y\psi(x_2,y_2,v_2)v_2} - \frac{v_2 y}{v y_2} - \frac{\psi(x,y_2,v_2)}{\psi(x,y,v)}\right]. \tag{30}$$

We can simplify (30) as:

$$\frac{dU_2}{dt} = dx_2\left(1 - \frac{\psi(x_2,y_2,v_2)}{\psi(x,y_2,v_2)}\right)\left(1 - \frac{x}{x_2}\right) + ay_2\left(1 - \frac{\psi(x,y,v)}{\psi(x,y_2,v_2)}\right)\left(\frac{\psi(x,y_2,v_2)}{\psi(x,y,v)} - \frac{v}{v_2}\right)$$
$$+ ay_2\left[4 - \frac{\psi(x_2,y_2,v_2)}{\psi(x,y_2,v_2)} - \frac{y_2\psi(x,y,v)v}{y\psi(x_2,y_2,v_2)v_2} - \frac{v_2 y}{v y_2} - \frac{\psi(x,y_2,v_2)}{\psi(x,y,v)}\right]. \tag{31}$$

We note that from assumptions A2, A5 and the relationship between the arithmetical and geometrical means, we have $\frac{dU_2}{dt} \leq 0$. One can easily see that $\frac{dU_2}{dt} = 0$ at E_2. The global asymptotic stability of the chronic-infection equilibrium with antibody immune response E_2 follows from LaSalle's invariance principle.

3 Conclusion

In this paper, we have proposed a viral infection model with general incidence rate function and antibody immune response. We have derived a set of conditions on the general functional response and have determined two thresholds parameters R_0 and R_1 to prove the existence and global stability of the model's equilibria. The global asymptotic stability of the three equilibria, infection-free, chronic-infection without antibody immune response and chronic-infection with antibody immune response has been proven by using direct Lyapunov method and LaSalle's invariance principle.

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NEWTON'S METHOD FOR COMPUTING THE FIFTH ROOTS OF p-ADIC NUMBERS

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Abstract We consider Newton's method to compute the fifth root of a p-adic number in $\mathbb{Q}_p$. We have the sufficient conditions for the convergence of Newton's method and the speed of its convergence. We also calculate the number of iterations to obtain a number of corrected digits in the approximation.

1. Introduction

Let p be a prime and $\mathbb{Q}_p$ be the field of p-adic numbers. The theory of the field of p-adic numbers introduced by Hensel has been related to several areas of mathematics including number theory, analysis and other modern mathematics, and recently to physics. The study of this field has been an important area of research in mathematics([9]).

The application of classical methods in numerical analysis to p-adic numbers and polynomials and the analysis of their convergence in $\mathbb{Q}_p$ have been a recent development([2-3], [5], [7], [10-11]). Newton's method is the most often used method to find zeros of polynomials. In [7], the authors applied Newton's method to compute the cubic root of a p-adic number. In [2-3], the authors also used Newton-Raphson method to compute square and cube roots of p-adic numbers in $\mathbb{Q}_p$. Computing the q-th root of a p-adic number is useful in the field of computer science and cryptography, specially when q is a prime. In [6], Kim-Choi give the conditions for the existence of the q-th roots of p-adic numbers in $\mathbb{Q}_p$ when $(p,q)=1$, and also have the condition for the existence the fifth roots including $p=q$.

In this paper, we use Newton's method to compute the fifth root of a p-adic number in $\mathbb{Q}_p$. We have the sufficient conditions for the convergence of Newton's method and the speed of its convergence. We also calculate the number of iterations to obtain a number of corrected digits in the approximation.

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2. Preliminaries

The following definitions and results are needed for our discussion. See [4] and [8] for details.

Definition 1. *Let $p \in \mathbb{N}$ be a prime number and $x \in \mathbb{Q}$ $(x \neq 0)$. The p-adic order of x, $\mathrm{ord}_p x$, is defined by*

$$\mathrm{ord}_p x = \begin{cases} \text{the highest power of } p \text{ which divides } x, & \text{if } x \in \mathbb{Z}, \\ \mathrm{ord}_p a - \mathrm{ord}_p b, & \text{if } x = \frac{a}{b},\ a, b \in \mathbb{Z},\ b \neq 0. \end{cases}$$

Consider a map $|\cdot|_p : \mathbb{Q} \to \mathbb{R}^+$ as follows.

Definition 2. *Let $p \in \mathbb{N}$ be a prime number and $x \in \mathbb{Q}$. The p-adic norm $|\cdot|_p$ of x is defined by*

$$|x|_p = \begin{cases} p^{-\mathrm{ord}_p x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

The field of p-adic numbers $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$ of Definition 2. The elements of $\mathbb{Q}_p$ are equivalence classes of Cauchy sequences in $\mathbb{Q}$ with respect to the extension of the p-adic norm defined by

$$|a|_p = \lim_{n \to \infty} |a_n|_p,$$

where $\{a_n\}$ is a Cauchy sequence in $\mathbb{Q}$ representing $a \in \mathbb{Q}_p$.

Theorem 1. *Every equivalence class a in $\mathbb{Q}_p$ satisfying $|a|_p \leq 1$ has exactly one representative Cauchy sequence $\{a_i\}$ such that*
(1) $a_i \in \mathbb{Z}$, $0 \leq a_i < p^i$ for $i = 1, 2, \ldots$,
(2) $a_i \equiv a_{i+1}$ (mod p^i) for $i = 1, 2, \ldots$.

From this, every p-adic number $a \in \mathbb{Q}_p$ has a unique representation

$$a = \sum_{n=-m}^{\infty} a_n p^n,$$

where $a_{-m} \neq 0$ and $a_n \in \{0, 1, 2, \ldots, p-1\}$ for $n \geq -m$. We represent the given p-adic number a as a fraction in the base p as follows:

$$a = \ldots a_n \ldots a_2 a_1 a_0 . a_{-1} \ldots a_{-m}.$$

This representation is called the canonical p-adic expansion of a.

Definition 3. *Let $\mathbb{Z}_p = \{a \in \mathbb{Q}_p |\ a = \sum_{i=0}^{\infty} a_i p^i\}$ be the set of p-adic integers and $\mathbb{Z}_p^{\times} = \{a \in \mathbb{Q}_p |\ a = \sum_{i=0}^{\infty} a_i p^i,\ a_0 \neq 0\}$ be the set of p-adic units.*

From Definition 3, it is easy to see that $\mathbb{Z}_p = \{a \in \mathbb{Q}_p | \ |a|_p \le 1\}$ and $\mathbb{Z}_p^\times = \{a \in \mathbb{Q}_p | \ |a|_p = 1\}$. Hence the following theorem follows.

Theorem 2. *Let a be a p-adic number of norm p^{-n}. Then $a = p^n u$ for some $u \in \mathbb{Z}_p^\times$.*

From now, we discuss the conditions for the existence of p-adic roots.

Definition 4. *A p-adic number $x \in \mathbb{Q}_p$ is said to be a q-th root of $a \in \mathbb{Q}_p$ of order $k \in \mathbb{N}$ if and only if $x^q \equiv a$ (mod p^k).*
When $q = 5$, the q-th root of $a \in \mathbb{Q}_p$ is called the fifth root of a.

The following lemmata are essential for our discussions([4]).

Lemma 3. *Let $a, b \in \mathbb{Q}_p$. Then a and b are congruent modulo p^k and write $a \equiv b$ (mod p^k) if and only if $|a - b|_p \le 1/p^k$.*

Lemma 4. *Let $a, b \in \mathbb{Q}_p$. If $|a - b|_p < |b|_p$, then $|a|_p = |b|_p$.*

The next theorem is the basis for the existence of p-adic roots([8]).

Theorem 5. *(Hensel's lemma) Let $F(x) = c_0 + c_1 x + \cdots + c_n x^n$ be a polynomial whose coefficients are p-adic integers. Let $F'(x) = c_1 + c_2 x + 3c_3 x^2 + \cdots + nc_n x^n$ be the derivative of $F(x)$. Let a_0 be a p-adic integer such that $F(a_0) \equiv 0$ (mod p) and $F'(a_0) \not\equiv 0$ (mod p). Then there exists a unique p-adic integer a such that*

$$F(a) = 0 \quad \textit{and} \quad a \equiv a_0 \ (\textit{mod}\ p).$$

The following theorem follows from Theorem 5, and provides the condition between p-adic numbers and congruence([4]).

Theorem 6. *A polynomial with integer coefficients has a root in $\mathbb{Z}_p$ if and only if it has an integer root modulo p^k for any $k \ge 1$.*

Some results of the existence of square roots of p-adic numbers are obtained from Theorem 6([4]). In [6], we have the conditions for the existence of the fifth roots of p-adic numbers in $\mathbb{Q}_p$ as followings.

Theorem 7. *A rational integer a not divisible by p has a fifth root in $\mathbb{Z}_p$ ($p \ne 5$) if and only if a is a fifth residue modulo p.*

From Theorem 7, we have the following theorem([6]).

Theorem 8. *Let p be a prime number. Then we have:*
(1) If $p \ne 5$, then $a = p^{\mathrm{ord}_p a} u \in \mathbb{Q}_p$ for some $u \in \mathbb{Z}_p^\times$ has a fifth root in $\mathbb{Q}_p$ if and only if $\mathrm{ord}_p a = 5m$ for $m \in \mathbb{Z}$ and $u = v^5$ for some unit $v \in \mathbb{Z}_p^\times$.
(2) If $p = 5$, then $a = 5^{\mathrm{ord}_5 a} u \in \mathbb{Q}_5$ for some $u \in \mathbb{Z}_5^\times$ has a fifth root in $\mathbb{Q}_5$ if and only if $\mathrm{ord}_5 a = 5m$ for $m \in \mathbb{Z}$ and $u \equiv 1$ (mod 25) or $u \equiv k$ (mod 5) for some k ($2 \le k \le 4$).

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3. Newton's Method

Newton's method is a well known numerical method to find zeros of a polynomial $f(x)$ in $\mathbb{R}$([1]). The iterative formula for this method is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots. \tag{3.1}$$

To seek the fifth root of a is to find the zero of $f(x) = x^5 - a$. The iteration (3.1) for Newton's method becomes the recurrence relation

$$x_{n+1} = \frac{4x_n^5 + a}{5x_n^4}, \quad n = 0, 1, 2, \ldots. \tag{3.2}$$

Like for real numbers, we can show that Newton's method also converges quadratically for convergence.

Let $a(\neq 0) \in \mathbb{Q}_p$ be a p-adic number such that

$$|a|_p = p^{-\mathrm{ord}_p a} = p^{-5m}, \quad m \in \mathbb{Z}.$$

The following theorem is the result when $p \neq 5$.

Theorem 9. *Let $p \neq 5$ and $\{x_n\}$ be the sequence of p-adic numbers obtained from the Newton's iteration (3.2). If x_0 is a fifth root of a of order r with $|x_0|_p = p^{-m}$ and $r > 5m$, then*

(1) $|x_n|_p = p^{-m}$, $n = 1, 2, \ldots$,
(2) $x_n^5 \equiv a \ (mod\ p^{2^n r - 5m(2^n - 1)})$,
(3) $\{x_n\}$ converges to the fifth root of a.

Proof. We will prove (1) and (2) by induction.
(i) First, we prove it when $p > 5$. Let $n = 1$. By assumption, we have

$$x_0^5 = a + bp^r \quad (0 < b < p). \tag{3.3}$$

From (3.2), (3.3) and Lemma 4, we have

$$|x_1|_p = \frac{|4x_0^5 + a|_p}{|5x_0^4|_p} = \frac{|5a + 4bp^r|_p}{|5x_0^4|_p} = \frac{\max\{|5a|_p, |4bp^r|_p\}}{|5x_0^4|_p} = p^{-m}. \tag{3.4}$$

Also by (3.2), we have

$$x_1^5 - a = \frac{(x_0^5 - a)^2}{3125x_0^{20}}(1024x_0^{15} + 203ax_0^{10} + 22a^2x_0^5 + a^3). \tag{3.5}$$

To calculate the p-adic norm of $x_1^5 - a$, we let

$$h(x) = 1024x^{15} + 203ax^{10} + 22a^2x^5 + a^3. \tag{3.6}$$

From (3.3), we have

$$h(x_0) = 1250a^3 + 3500a^2bp^r + 3275ab^2p^{2r} + 1024b^3p^{3r}. \tag{3.7}$$

Using the strong triangle inequality, we have from (3.7) that

$$\begin{aligned}
&|h(x_0)|_p \\
&\le \max\left\{|2\cdot 5^4a^3|_p, |2^25^37a^2bp^r|_p, |5^2131ab^2p^{2r}|_p, |2^{10}b^3p^{3r}|_p\right\} \\
&= \max\left\{p^{-15m}, p^{-10m-r}, p^{-5m-2r}, p^{-3r}\right\} \\
&= p^{-15m}.
\end{aligned} \tag{3.8}$$

Also the p-adic norm of the denominator of the right hand of (3.5) is

$$|3125x_0^{20}|_p = |5^5x_0^{20}|_p = p^{-20m}. \tag{3.9}$$

Since x_0 is a fifth root of a of order r, we have

$$|(x_0^5-a)^2|_p = p^{-2r}. \tag{3.10}$$

By (3.5), (3.8), (3.9) and (3.10), we have

$$|x_1^5-a|_p \le p^{5m-2r}.$$

By Lemma 3, $x_1^5-a\equiv 0 \pmod{p^{2r-5m}}$. Hence (1) and (2) is true when $n=1$.

Now assume that

$$|x_{n-1}|_p = p^{-m}, \tag{3.11}$$

$$x_{n-1}^5 = a \pmod{p^{2^{n-1}r-5m(2^{n-1}-1)}}, \tag{3.12}$$

and so

$$x_{n-1}^5 = a + bp^{2^{n-1}r-5m(2^{n-1}-1)} \quad (0<b<p). \tag{3.13}$$

From (3.2), (3.11) and (3.13), we have

$$\begin{aligned}
|x_n|_p &= \frac{|4x_{n-1}^5+a|_p}{|5x_{n-1}^4|_p} = \frac{|5a+4bp^{2^{n-1}r-5m(2^{n-1}-1)}|_p}{|5x_{n-1}^4|_p} \\
&= \frac{\max\{|5a|_p, |4bp^{2^{n-1}r-5m(2^{n-1}-1)}|_p\}}{|5x_{n-1}^4|_p} = p^{-m}.
\end{aligned} \tag{3.14}$$

Thus (1) is proved by (3.4), (3.11) and (3.14). Also from (3.2), it follows that

$$x_n^5 - a = \frac{(x_{n-1}^5-a)^2}{5^5x_{n-1}^{20}}h(x_{n-1}). \tag{3.15}$$

Let $Q = p^{2^{n-1}r-5m(2^{n-1}-1)}$ for simplicity. From (3.13),

$$h(x_{n-1}) = 2\cdot 5^4a^3 + 2^2\cdot 5^3\cdot 7a^2bQ + 5^2\cdot 131ab^2Q^2 + 2^{10}b^3Q^3. \tag{3.16}$$

Since $r>5m$, the p-adic norm of $h(x_{n-1})$ in (3.16) is

$$\begin{aligned}
|h(x_{n-1})|_p &\le \max\{p^{-15m}, p^{-15m-2^{n-1}(r-5m)}, \\
&\qquad p^{-15m-2^n(r-5m)}, p^{-15m-3\cdot 2^{n-1}(r-5m)}\} \\
&= p^{-15m}.
\end{aligned} \tag{3.17}$$

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Since x_{n-1} is a fifth root of a of order $2^{n-1}r - 5m(2^{n-1}-1)$, we have from (3.12), (3.15) and (3.17) that

$$|x_n^5 - a|_p \le p^{-2^n r + 5m(2^n - 1)}.$$

By Lemma 3, we have $x_n^5 - a \equiv 0 \pmod{p^{2^n r - 5m(2^n-1)}}$. Thus (2) is true for all $n \in \mathbb{N}$.

(ii) When $p < 5$, there are two cases, $p = 3$ and $p = 2$.

The proof is the same with (i) when the first case $p = 3$, because 3 is no factor of any coefficients of terms of $h(x_0)$ in (3.7). It means that $|h(x_0)|_p \le p^{-15m}$, and so $x_1^5 \equiv a \pmod{p^{2r-5m}}$. By assuming $x_{n-1}^5 \equiv a$ $\pmod{p^{2^{n-1}r-5m(2^{n-1}-1)}}$, we have $x_n^5 \equiv a \pmod{p^{2^n r - 5m(2^n-1)}}$ using the same process of (i). Moreover we can check easily $|x_n|_3 = 3^{-m}$ by induction.

The other case is $p = 2$. Let $n = 1$, $|x_1|_p = p^{-m}$ is obtained easily from (3.4). And we have

$$x_1^5 - a = \frac{(x_0^5 - a)^2}{3125 x_0^{20}} h(x_0), \tag{3.18}$$

where $h(x)$ is the polynomial in (3.6). Since $r > 5m$, we have

$$|h(x_0)|_p \le \max\{p^{-15m-1},\, p^{-10m-r-2},\, p^{-5m-2r},\, p^{-3r-10}\} \le p^{-15m}. \tag{3.19}$$

In (3.18), we have

$$|3125 x_0^{20}|_p = p^{-20m}, \tag{3.20}$$

and, by assumption,

$$|(x_0^5 - a)^2|_p = p^{-2r}. \tag{3.21}$$

Also (3.19), (3.20) and (3.21) imply $|x_1^5 - a|_p \le p^{-2r+5m}$, and so $x_1^5 \equiv a \pmod{p^{2r-5m}}$. Thus (1) and (2) are true when $n = 1$ if $p = 2$.

Assume that $|x_{n-1}|_p = p^{-m}$ and $x_{n-1}^5 \equiv a \pmod{p^{2^{n-1}r-5m(2^{n-1}-1)}}$. That is,

$$x_{n-1}^5 = a + b p^{2^{n-1}r - 5m(2^{n-1}-1)} \ (0 < b < p). \tag{3.22}$$

It follows (3.15) and (3.16), and so we have

$$\begin{aligned} |h(x_{n-1})|_p \le \max\{&p^{-15m-1},\, p^{-15m-2-2^{n-1}(r-5m)},\\ &p^{-15m-2^n(r-5m)},\, p^{-15m-10-3\cdot 2^{n-1}(r-5m)}\}\\ \le p^{-15m}&. \end{aligned} \tag{3.23}$$

By (3.15), (3.17), (3.20) and (3.23), we have

$$|x_n^5 - a|_p \le p^{-2^n r + 5m(2^n - 1)}.$$

Hence we have that for all $n \in \mathbb{N}$, $x_n^5 \equiv a \pmod{p^{2^n r - 5m(2^n-1)}}$. We note that $|x_n|_2 = 2^{-m}$ is obtained easily from (3.14). So we complete the proof of (1) and (2).

From (2), we have

$$|x_n^5 - a|_p \leq p^{-2^n r + 5m(2^n-1)} \tag{3.24}$$

for each prime $p (\neq 5)$. (3) follows immediately from the inequality (3.24) as $n \to \infty$. ☐

When $p = 5$, we have the following theorem.

Theorem 10. *Let $p = 5$ and $\{x_n\}$ be the sequence of p-adic numbers obtained from the Newton's iteration (3.2). If x_0 is a fifth root of a of order r with $|x_0|_p = p^{-m}$ and $r > 5m+1$, then*

(1) $|x_n|_p = p^{-m}$, $n = 1, 2, \ldots,$
(2) $x_n^5 \equiv a \pmod{p^{2^n r - (5m+1)(2^n-1)}}$,
(3) $\{x_n\}$ converges to the fifth root of a.

Proof. (1) and (2) will be proved by induction. Let $n = 1$. By assumption $x_0^5 \equiv a \pmod{p^r}$, and from (3.2) and Lemma 4, we have

$$|x_1|_p = \frac{|5a + 4bp^r|_p}{|5x_0^4|_p} = \frac{\max\{|5a|_p, |4bp^r|_p\}}{|5x_0^4|_p} = \frac{p^{-5m-1}}{p^{-4m-1}} = p^{-m}.$$

By calculating the p-adic norms of $h(x_0)$ in (3.7), we have

$$|h(x_0)|_p \leq \max\{p^{-15m-4}, p^{-10m-r-3}, p^{-5m-2r-2}, p^{-3r}\} = p^{-15m-4},$$

since $r > 5m+1$. Also we have $|3125x_0^{20}|_p = p^{-20m-5}$. Thus

$$|x_1^5 - a|_p \leq p^{-2r+5m+1},$$

and so $x_1^5 \equiv a \pmod{p^{2r-(5m+1)}}$ by Lemma 3. Hence it is true when $n = 1$. Now we assume that

$$|x_{n-1}|_p = p^{-m}$$

and

$$x_{n-1}^5 \equiv a \pmod{p^{2^{n-1}r - (5m+1)(2^{n-1}-1)}}.$$

In the similar manner as (3.14), (3.16) and (3.17), we have

$$\begin{aligned} |x_n|_p = \frac{|4x_{n-1}^5 + a|_p}{|5x_{n-1}^4|_p} &= \frac{|5a + 4bp^{2^{n-1}r-(5m+1)(2^{n-1}-1)}|_p}{|5x_{n-1}^4|_p} \\ &= \frac{\max\{|5a|_p, |4bp^{2^{n-1}r-(5m+1)(2^{n-1}-1)}|_p\}}{|5x_{n-1}^4|_p} = \frac{p^{-5m-1}}{p^{-4m-1}} = p^{-m} \end{aligned}$$

and

$$\begin{aligned}|h(x_{n-1})|_p &\leq \max\{p^{-15m-4},\ p^{-15m-4-2^{n-1}[r-(5m+1)]}\\ &\qquad p^{-15m-4-2^n[r-(5m+1)]},\ p^{-15m-3-3\cdot 2^{n-1}[r-(5m+1)]}\}\\ &= p^{-15m-4}.\end{aligned}$$

And so we have

$$|x_n^5 - a|_p \leq p^{-2^n r+(5m+1)(2^n-1)}. \tag{3.25}$$

It follows that (1) and (2) are true for all $n \in \mathbb{N}$.
(3) follows from the inequality (3.25) as $n \to \infty$. □

To determine the rate of convergence of the sequence $\{x_n\}$ given by (3.2), we consider the sequence $\{e_n\}$ defined by

$$e_n = x_{n+1} - x_n, \quad \forall n \in \mathbb{N}. \tag{3.26}$$

From Theorem 9 and Theorem 10, we obtain the following theorem.

Theorem 11. *If x_0 is the fifth root of a of order r, then the sequence $\{e_n\}$ in (3.26) is $e_n \equiv 0$ (mod p^{α_n}), where*

$$\alpha_n = \begin{cases} 2^n r - 5m\cdot 2^n + m, & \text{if } p \neq 5,\\ 2^n r - (5m+1)\cdot 2^n + m, & \text{if } p = 5.\end{cases}$$

Proof. (i) First, let $p \neq 5$. Then, from the Newton's iteration formula (3.2), we have

$$e_n = x_{n+1} - x_n = \frac{1}{5x_n^4}(a - x_n^5), \quad \forall n \in \mathbb{N}. \tag{3.27}$$

By computing the p-adic norms of each side of the equation (3.27), we have from Theorem 8 that

$$|e_n|_p = |x_{n+1} - x_n|_p = \left|\frac{1}{5x_n^4}\right|_p \cdot |a - x_n|_p \leq p^{-2^n r+5m\cdot 2^n - m}.$$

Hence $e_n \equiv 0$ (mod p^{α_n}) by Lemma 3.
(ii) Let $p = 5$. By a similar way as (i), we have from Theorem 9 that

$$|e_n|_p = \left|\frac{1}{5x_n^4}\right|_p \cdot |a - x_n|_p \leq p^{-2^n r+(5m+1)\cdot 2^n - m}.$$

Hence $e_n \equiv 0$ (mod p^{α_n}) by Lemma 3. This completes the proof. □

From Theorem 11, we have that the rate of convergence of the sequence $\{x_n\}$ is of order α_n. Thus the number of correct digits in the approximation increases by α_n for every iteration.

We can compute the number of iterations to obtain certain finite digits. From Theorem 9 and Theorem 10, we have the following corollary.

Corollary 12. *(1) For $p \neq 5$, let $\{x_n\}$ be the sequence of approximation in Theorem 9. Then the number of iterations to obtain at least M correct digits is*

$$n = \left\lceil \frac{\ln\left(\frac{M-4m}{r-5m}\right)}{\ln 2} \right\rceil. \tag{3.28}$$

(2) Let $p = 5$ and $\{x_n\}$ be the sequence of approximation in Theorem 10. Then the number of iterations to obtain at least M correct digits is

$$n = \left\lceil \frac{\ln\left(\frac{M-(4m+1)}{r-(5m+1)}\right)}{\ln 2} \right\rceil. \tag{3.29}$$

Proof. (1) Since we need M correct digits in the approximation, we must set the order to $M+m$ to find the number of iterations with M correct digits. That is,

$$2^n r - 5m(2^n - 1) = M + m. \tag{3.30}$$

From (3.30), we have

$$2^n = \frac{M-4m}{r-5m}.$$

Since $\{x_n\}$ converges to the fifth root of a by Theorem 8 (3) and $r > 5m$, we have the equation (3.28).

(2) As in the proof of (1), we set

$$2^n r - (5m+1)(2^n - 1) = M + m. \tag{3.31}$$

From (3.31), we have

$$2^n = \frac{M-4m-1}{r-(5m+1)}. \tag{3.32}$$

Since $r > 5m+1$, the result follows from (3.32). $\square$

The numbers in (3.28) and (3.29) are sufficient numbers of iterations to provide at least M correct digits in the approximation.

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Solution of the Ulam stability problem for Euler-Lagrange $(\alpha, \beta; k)$-quadratic mappings

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Abstract. In 1940 S. M. Ulam proposed at the University of Wisconsin the problem: "Give conditions in order for a linear mapping near an approximately linear mapping to exist". In 1982-2013, the second author solved the above Ulam problem for a variety of quadratic mappings. Interesting stability results have been achieved by S. A. Mohiuddine et al., since 2009. In this paper, we solve the Ulam stability problem for Euler-Lagrange $(\alpha, \beta; k)$ quadratic mapping. The other authors of this research area have established important results also on functional inequalities.

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1. Introduction

In 1940 S. M. Ulam [36] proposed the famous "Ulam stability problem", which was solved by D. H. Hyers [4], in 1941, for additive mappings. In 1950 T. Aoki [3] solved this Ulam problem for weaker additive mappings. In 1978 Th. M. Rassias [33] generalized the theorem of Hyers for linear mappings. In 1982-1999, J. M. Rassias ([23–30]) generalized this problem. For more detail of Ulam stability problem, we refer to [5, 6, 8–11, 19, 20, 32, 34] and references therein.

In 1992, the second author [23, 24] introduced the term "Euler-Lagrange functional equation" and "Euler-Lagrange quadratic mappings", of satisfying

$$Q(x+y) + Q(x-y) = 2\big[Q(x) + Q(y)\big] \tag{1.1}$$

and then solved the Ulam stability problem of the Euler-Lagrange quadratic functional equation (1.1). In 1996, J. M. Rassias [30] established the Ulam stablity of the general Euler-Lagrange quadratic functional equation

$$Q(\alpha x + \beta y) + Q(\beta x - \alpha y) = (\alpha^2 + \beta^2)\big[Q(x) + Q(y)\big]. \tag{1.2}$$

In 2009-2014, S. A. Mohiuddine et al. ([1,2,12–18]) solved this problem in several normed spaces. In 2008-2012 J. M. Rassias et al. ([21, 22, 31, 37]) solved the generalized Ulam problem via various methods. In 2010, M. E. Gordji et al [7] established Ulam stabilities on Banach algebras. Also J. Rätz [35] results are interesting on orthogonal mappings.

In this paper, we solve the Ulam stability problem for the Euler-Lagrange $(\alpha, \beta; k)$ quadratic mapping satisfying

$$kQ(\alpha x + \beta y) + Q(k\beta x - \alpha y) = (\alpha^2 + k\beta^2)\big[kQ(x) + Q(y)\big]. \tag{1.3}$$

Let us note that $Q(x) = |x|^2$ satisfies equation (1.3) because the following Euler-Lagrange quadratic identity

$$k|\alpha x + \beta y|^2 + |k\beta x - \alpha y|^2 = (\alpha^2 + k\beta^2)\big[k|x|^2 + |y|^2\big] \tag{1.4}$$

holds with any fixed reals α, β and k.

Definition 1.1. Let X be a normed linear space and let Y be a real complete normed linear space. Then a non-linear mapping $Q : X \to Y$ is called Euler-Lagrange quadratic if equation (1.3) holds for all 2-dimensional vectors $(x, y) \in X^2$, and any fixed reals α, β and k. We note that Q may be called quadratic because the above Euler-Lagrange identity (1.4) holds and because the functional equation

$$Q(m^n x) = (m^n)^2 Q(x) \tag{1.5}$$

holds for all $x \in X$, all $n \in \mathbb{N}$:

$$m = \alpha^2 + k\beta^2. \tag{1.6}$$

Assume $m \in \mathbb{R} - \{0, 1\}$ and $k \in \mathbb{R} - \{-1, 0\}$.

In fact, substitution of $x = y = 0$ in equation (1.3) yields

$$(k + 1)(1 - m)Q(0) = 0,$$

or

$$Q(0) = 0, \quad m \neq 1 \quad (\text{and } \ k \neq -1). \tag{1.7}$$

Substituting $x = x, y = 0$ in (1.3), one gets that

$$kQ(\alpha x) + Q(k\beta x) = kmQ(x) + mQ(0), \tag{1.8}$$

or

$$Q(\alpha x) + \frac{1}{k}Q(k\beta x) = mQ(x) + \frac{m}{k}Q(0), \tag{1.9}$$

holds for all $x \in X$, and any fixed real $k \neq 0$. Employing (1.7), we obtain from (1.8) that

$$Q(\alpha x) + Q(k\beta x) = kmQ(x). \tag{1.10}$$

Moreover, substitution $x \to \alpha x$, $y = k\beta x$ in (1.3), we find that

$$kQ(mx) + Q(0) = m\big[kQ(\alpha x) + Q(k\beta x)\big],$$

or

$$kQ(\alpha x) + Q(k\beta x) = km^{-1}Q(mx) + \frac{1}{m}Q(0), \tag{1.11}$$

or

$$Q(\alpha x) + \frac{1}{k}Q(k\beta x) = m^{-1}Q(mx) + \frac{1}{km}Q(0) \tag{1.12}$$

holds for all $x \in X$, and any fixed reals $k \neq 0$, $m \neq 0$. Functional Equations (1.8) and (1.11), or (1.9) and (1.12) yield

$$km^{-1}Q(mx) + \frac{1}{m}Q(0) = kmQ(x) + mQ(0),$$

or

$$km\big[Q(x) - m^{-2}Q(mx)\big] = \left(\frac{1}{m} - m\right)Q(0),$$

or

$$km\big[Q(x) - m^2 Q(mx)\big] = \left(\frac{1-m^2}{m}\right)Q(0),$$

or

$$Q(x) - m^{-2}Q(mx) = \left(\frac{1}{k}\ \frac{1-m^2}{m^2}\right)Q(0). \tag{1.13}$$

Employing (1.7), one gets

$$Q(x) = m^{-2}Q(mx), \tag{1.14}$$

or

$$Q(mx) = m^2 Q(x) \tag{1.15}$$

Replaying $x \to mx$ in (1.15), we find

$$Q(m^2x) = m^2 Q(mx),$$

or

$$Q(m^2x) = m^4 Q(x) \tag{1.16}$$

Then by induction on $n \in \mathbb{N}$ with $x \to m^{n-1}x$ yields equation (1.5).

Definition 1.2. Let X be a normed linear space and let Y be a real complete normed linear space. Then we call the non-linear mapping $\bar{Q} : X \to Y$, a 2-dimensional quadratic weighted mean if

$$\bar{Q}(x) = \frac{kQ(\alpha x) + Q(k\beta x)}{km} \tag{1.17}$$

holds for all $x \in X$ and any fixed reals $k, m \neq 0$.

Let us note that from (1.8) and (1.17), one get

$$\bar{Q}(x) = \frac{kmQ(x) + mQ(o)}{km},$$

or

$$\bar{Q}(x) = Q(x) + \frac{1}{k}Q(o), \tag{1.18}$$

for all $x \in X$, and any fixed real $k \neq 0$. From (1.7) and (1.18), we obtain

$$\bar{Q}(x) = Q(x), \tag{1.19}$$

for all $x \in X$.

2. Stability for Euler-Lagrange quadratic mappings

Let us introduce the Euler-Lagrange $(\alpha, \beta; k)$ quadratic functional inequality

$$\big\|kf(\alpha x + \beta y) + f(k\beta x - \alpha y) - (\alpha^2 + k\beta^2)\big[kf(x) + f(y)\big]\big\| \le c, \tag{2.1}$$

for all 2-dimensional vectors $(x, y) \in X^2$ and any fixed reals α, β and k as well as $m = \alpha^2 + k\beta^2$, with $m \in \mathbb{R} - \{0, 1\}$ $\quad (k \in \mathbb{R} - \{-1, 0\})$, and $c(:=$ constant inde of $x, y) \ge 0$.

Then we prove the following theorem.

Theorem 2.1. Let X be a normed linear space and let Y be a real complete normed linear space. Let us denote,

$$\bar{f}(x) = \frac{kf(\alpha x) + f(k\beta x)}{km} \tag{2.2}$$

holds for all $x \in X$ and any fixed reals $k, m \neq 0$. Also let us assume $m : |m| > 1$. Then the limit

$$Q(x) = \lim_{n\to\infty} m^{-2n} f(m^n x), \tag{2.3}$$

exists for all $x \in X$, all $n \in \mathbb{N}$, and any fixed real $m : |m| > 1$ and $Q : X \to Y$ is the unique quadratic mapping satisfying functional equation (1.3) such that

$$\|f(x) - Q(x)\| \leq c_3 = \frac{c_2}{m^2 - 1}, \ |m| > 1, \tag{2.4}$$

where

$$c_2 = m^2 c_1 = \frac{|k+1|\big(1+|m|\big) + |1+m|}{|k|\ |k+1|} c,$$
$$c_3 = \frac{c_2}{m^2-1}.$$

Moreover , identity

$$Q(x) = m^{-2n} Q(m^n x) \tag{2.5}$$

holds for all $x \in X$ all $n \in \mathbb{N}$, and any fixed reals: α, β; k, $m : |m| > 1$ with $m \in \mathbb{R} - \{0, 1\}$, $(k \in \mathbb{R} - \{-1, 0\})$.

Proof of Existence in Theorem 2.1.

In fact, substitution of $x = y = 0$ in equality (2.1) yields

$$|k+1|\ |1-m|\ \|f(0)\| \leq c,$$

or

$$\|f(0)\| \leq \frac{c}{|k+1|\ |1-m|}, k \neq -1, m \neq 1. \tag{2.6}$$

Substituting $x = x$, $y = 0$ in (2.1), one gets that

$$\|kmf(x) - [kf(\alpha x) + f(k\beta x)] + mf(0)\| \leq c,$$

or

$$\|f(x) - \bar{f}(x) + \frac{1}{k} f(0)\| \leq \frac{c}{|k||m|}, k \neq 0, m \neq 0, |m| > 1 \tag{2.7}$$

from (2.2). Moreover substitution $x \to \alpha x, y = k\beta x$ in (2.1), we find that

$$\|kf(mx) + f(0) - m[kf(\alpha x) + f(k\beta x)]\| \leq c,$$

or

$$\|kf(\alpha x) + f(k\beta x) - km^{-1} f(mx) - \frac{1}{m} f(0)\| \leq \frac{c}{|m|},$$

or

$$\|\bar{f}(x) - m^{-2} f(mx) - \frac{1}{km^2} f(0)\| \leq \frac{c}{|k|\ m^2}, \tag{2.8}$$

Functional inequalities (2.6),(2.7),(2.8) and triangle inequality yields

$$\begin{aligned}
\|f(x) - m^{-2} f(mx)\| &\leq \|f(x) - \bar{f}(x) + \frac{1}{k} f(0)\| + \|\bar{f}(x) - m^{-2} f(mx) - \frac{1}{km^2} f(0)\| \\
&\quad + \Big\| \frac{1}{km^2} f(0) - \frac{1}{k} f(0) \Big\| \\
&\leq \frac{c}{|k|\ |m|} + \frac{c}{|k|\ m^2} + \frac{|1-m^2|}{|k| m^2} \|f(0)\| \\
&= \frac{1+|m|}{|k|\ m^2} c + \frac{|1-m^2|}{|k| m^2} \|f(0)\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\frac{1+|m|}{|k|\,m^2}+\frac{|1-m^2|}{|k|m^2}\;\frac{1}{|k+1||1-m|}\right)c\\
&= \left(\frac{1+|m|}{|k|\,m^2}+\frac{|1+m|}{|k||k+1|m^2}\right)c\\
&= c_1=\frac{|k+1|\big(1+|m|\big)+|1+m|}{|k||k+1|m^2}c,
\end{aligned}
$$

or

$$\big\|f(x)-m^{-2}f(mx)\big\|\leq c_1=\frac{c_2}{m^2}, \tag{2.9}$$

where

$$c_2=m^2c_1=\frac{|k+1|\big(1+|m|\big)+|1+m|}{|k||k+1|}c, \tag{2.10}$$

holds for fixed k, $m\neq 0$, $m\neq 1$, $m>1$. Replacing $x\to mx$ in (2.9) and then multiplying by m^{-2}, we find

$$\big\|m^{-2}f(mx)-m^{-4}f(m^2x)\big\|\leq m^{-2}c_1, m\neq 0 \tag{2.11}$$

From (2.9) and (2.11), one gets

$$\big\|f(x)-m^{-4}f(m^2x)\big\|\leq\big\|f(x)-m^{-2}f(mx)\big\|+\big\|m^{-2}f(mx)-m^{-4}f(m^2x)\big\|\leq 1+m^{-2}c_1,$$

or

$$\big\|f(x)-m^{-4}f(m^2x)\big\|\leq(1+m^{-2})c_1, m\neq 0. \tag{2.12}$$

Employing (2.9) and (2.12) without induction, we obtain

$$
\begin{aligned}
\big\|f(x)-m^{-2n}f(m^nx)\big\| &\leq \big\|f(x)-m^{-2}f(mx)\big\|+\big\|m^{-2}f(mx)-m^{-4}f(m^2x)\big\|+\cdots\\
&\quad+\big\|m^{-2(n-1)}f(m^{n-1}x)-m^{-2n}f(m^nx)\big\|\\
&\leq \big(1+m^{-2}+\cdots+m^{-2(n-1)}\big)c_1,
\end{aligned}
$$

or

$$\big\|f(x)-m^{-2n}f(m^nx)\big\|\leq\frac{1-m^{-2n}}{1-m^{-2}}c_1=\frac{m^2}{m^2-1}\big(1-m^{-2n}\big)c_1, \tag{2.13}$$

or the general inequality:

$$\big\|f(x)-m^{-2n}f(m^nx)\big\|\leq\frac{1}{m^2-1}\big(1-m^{-2n}\big)c_2, \tag{2.14}$$

where $|m|>1$, $c_2=m^2c_1$.

Claim now that the sequence

$$\big\{f_n(x)\big\}, f_n(x)=\big\{m^{-2n}f(m^nx)\big\} \tag{2.15}$$

converges. Note that from the general inequality (2.14) and the completeness of Y, one proves that the above sequence (2.15) is a Cauchy sequence. In fact, if $i>j>0$, then

$$
\begin{aligned}
\big\|f_i(x)-f_j(x)\big\| &= \big\|m^{-2i}f(m^ix)-m^{-2j}f(m^jx)\big\|\\
&= m^{-2j}\big\|m^{-2(i-j)}f(m^ix)-f(m^jx)\big\|\\
&= m^{-2j}\big\|f(m^jx)-m^{-2(i-j)}f(m^{i-j}\cdot m^jx)\big\|\\
&\leq m^{-2j}\cdot\frac{1}{m^2-1}(1-m^{-2(i-j)})c_2,
\end{aligned}
$$

or

$$\big\|f_i(x)-f_j(x)\big\|\leq\frac{1}{m^2-1}(m^{-2j}-m^{-2i})c_2,\ \ |m|>1, \tag{2.16}$$

or

$$0 \leq \lim_{i>j\to\infty} \big\|f_i(x) - f_j(x)\big\| \leq 0,$$

or

$$\lim_{i>j\to\infty} \big\|f_i(x) - f_j(x)\big\| = 0, \tag{2.17}$$

completing the proof that the sequence $\{f_n(x)\}$ converges. Hence $Q = Q(x)$ is well-defined via the formula (2.3). This means that the limit (2.3) exists for all $x \in X$.

In addition claim that mapping Q satisfies the functional equation (1.3) for all vectors $(x, y) \in X^2$.

In fact, it is clear from functional inequality (2.1) and the limit (2.3) that inequality

$$\begin{aligned} \Big\|k \lim_{n\to\infty} m^{-2n} f\big[m^n(\alpha x + \beta y)\big] \quad &+ \quad \lim_{n\to\infty} m^{-2n} f\big[m^n(k\beta x - \alpha y)\big] \\ &-(\alpha^2 + k\beta^2)\big[k \lim_{n\to\infty} m^{-2n} f(m^n x) + \lim_{n\to\infty} m^{-2n} f(m^n y)\big]\Big\| \\ &\leq \quad c(\lim_{n\to 0} m^{-2n}) = 0, \qquad |m| > 1, \qquad (2.18) \end{aligned}$$

or

$$\big\|kQ(\alpha x + \beta y) + Q(k\beta x - \alpha y) - (\alpha^2 + k\beta^2)[kQ(x) + Q(y)\big\| = 0,$$

or mapping Q satisfies the functional equation (1.3) for all $x, y \in X$, and $|m| > 1$. Thus Q is a 2-dimensional quadratic mapping. It is now clear from general inequality (2.14), $n \to \infty$, and the formula (2.3) that inequality (2.4) holds in X, completing the existence proof of this Theorem 2.1.

Proof of Uniqueness in Theorem 2.1.

Let $Q' : X \to Y$ be another 2-dimensional quadratic mapping satisfying equation (1.3), such that

$$\big\|f(x) - Q'(x)\big\| \leq c_3 \Big(= \frac{c_2}{m^2 - 1}\Big), \tag*{(2.4)$'$}$$

for all $x \in X$, and any fixed real $m : |m| > 1$.

To prove the above-mentioned uniqueness employ (2.5) for Q and Q', as well, so that

$$Q'(x) = m^{-2n} Q'(m^n x) \tag*{(2.5)$'$}$$

holds for all $x \in X$, all $n \in \mathbb{N}$, and any fixed real $m : |m| > 1$.

Moreover, the triangle inequality and functional inequalities (2.4)-(2.4)$'$ yield

$$\big\|Q(m^n x) - Q'(m^n x)\big\| \leq \big\|Q(m^n x) - f(m^n x)\big\| + \big\|f(m^n x) - Q'(m^n x)\big\|,$$

or

$$\big\|Q(m^n x) - Q'(m^n x)\big\| \leq 2c_3, \tag{2.19}$$

for all $x \in X$, all $n \in \mathbb{N}$, and any fixed real $m : |m| > 1$. Then from (2.5)-(2.5)$'$, and (2.19), one proves that

$$\big\|Q(x) - Q'(x)\big\| = \big\|m^{-2n} Q(m^n x) - m^{-2n} Q'(m^n x)\big\|,$$

or

$$\big\|Q(x) - Q'(x)\big\| \leq 2m^{-2n} c_3, \tag{2.20}$$

holds for all $x \in X$, all $n \in \mathbb{N}$, and any fixed real $m : |m| > 1$. Therefore from (2.20), and $n \to \infty$, one establishes

$$0 \leq \lim_{n\to\infty} \big\|Q(x) - Q'(x)\big\| \leq 2\Big(\lim_{n\to\infty} m^{-2n}\Big) c_3 = 0, \quad |m| > 1,$$

or

$$\big\|Q(x) - Q'(x)\big\| = 0,$$

or

$$Q(x) = Q'(x), \quad |m| > 1, \tag{2.21}$$

for all $x \in X$, completing the proof of uniqueness and thus the stability of Theorem 2.1.

Theorem 2.2. Let X be a normed linear space and let Y be a real complete normed linear space. Let us denote

$$\bar{\bar{f}}(x) = m^2\bar{f}(m^{-1}x) = \frac{m}{k}\left[kf\left(\frac{1}{m}\alpha x\right)+f\left(\frac{k}{m}\beta x\right)\right] \tag{2.2$'$}$$

holds for all $x \in X$ and any fixed reals $k, m \neq 0$. Also let us assume $|m| < 1$. Then the limit

$$Q(x) = \lim_{n\to\infty} m^{2n} f(m^{-n}x), \tag{2.3$'$}$$

exists for all $x \in X$, all $n \in \mathbb{N}$, and any fixed real $m : |m| < 1$, and $Q : X \to Y$ is the unique quadratic mapping satisfying functional equation $(2.3)'$, such that

$$\big\|f(x) - Q(x)\big\| \leq c_4 = \frac{c_1}{1-m^2}.$$

Moreover, identity

$$Q(x) = m^{2n}Q(m^{-n}x) \tag{2.5$'$}$$

holds for all $x \in X$, $n \in \mathbb{N}$ and $|m| < 1, m \neq 0$. From (2.7) with $x \to m^{-1}x(m \neq 0, |m| < 1)$ and multiplying by m^2, one find

$$\left\|m^2f(m^{-1}x) - \bar{\bar{f}}(x) + \frac{m^2}{k}f(0)\right\|\leq \frac{|m|}{|k|}c, \tag{2.22}$$

where

$$\bar{\bar{f}}(x) = m^2\bar{f}(m^{-1}x) = \frac{m}{k}\left[kf\big(m^{-1}\alpha x\big) + f\left(\frac{k}{m}\beta x\right)\right], \quad m \neq 0, |m| < 1. \tag{2.23}$$

From (2.8) with $x \to m^{-1}x \quad (m \neq 0, |m| < 1)$, one obtains

$$\left\|\bar{f}(m^{-1}x) - m^{-2}f(x) - \frac{1}{km^2}f(0)\right\|\leq \frac{c}{|k|\,m^2}.$$

Multiplying by m^2, we get

$$\left\|\bar{\bar{f}}(x) - f(x) - \frac{1}{k}f(0)\right\|\leq \frac{c}{|k|}. \tag{2.24}$$

Functional inequalities (2.6),(2.23),(2.24) and triangle inequality yield

$$\begin{aligned}
\big\|f(x) - m^2f(m^{-1}x)\big\| &\leq \left\|f(x) - \bar{\bar{f}}(x) + \frac{1}{k}f(0)\right\| + \left\|\bar{\bar{f}}(x) - m^2f(m^{-1}x) - \frac{m^2}{k}f(0)\right\| \\
&\quad + \left\|\frac{m^2}{k}f(0) - \frac{1}{k}f(0)\right\| \\
&\leq \frac{c}{|k|} + \frac{|m|}{|k|}c + \frac{|m^2-1|}{|k|}\big\|f(0)\big\| \\
&= \frac{1+|m|}{|k|}c + \frac{|1-m^2|}{|k|}\|f(0)\| \\
&\leq \left(\frac{1+|m|}{|k|} + \frac{|1+m|}{|k||k+1|}\right)c \\
&= \frac{|k+1|\big(1+|m|\big) + |1+m|}{|k||k+1|}c = c_2,
\end{aligned}$$

or

$$\big\|f(x)-m^2f(m^{-1}x)\big\|\le c_2, \tag{2.25}$$

where

$$c_2=\frac{|k+1|\big(1+|m|\big)+|1+m|}{|k||k+1|}c, \quad |m|<1, k\ne 0, k\ne -1, m\ne 0.$$

Replacing $x\to m^{-1}x$ in (2.25) and multiplying by m^2, we get

$$m^2f(m^{-1}x)-m^4f(m^{-2}x)\|\le m^2c_2, \tag{2.26}$$

From (2.25)-(2.26), one finds

$$\big\|f(x)-m^4f(m^{-2}x)\big\|\le\big\|f(x)-m^2f(m^{-1}x)\big\|+\big\|m^2f(m^{-1}x)-m^4f(m^{-2}x)\big\|\le(1+m^2)c_2,$$

or

$$\big\|f(x)-m^4f(m^{-2}x)\big\|\le\big(1+m^2\big)c_2,\ m\ne 0. \tag{2.27}$$

Employing (2.25) and (2.27), without induction, we get

$$\begin{aligned}\big\|f(x)-m^{2n}f(m^{-n}x)\big\| &\le \big\|f(x)-m^2f(m^{-1}x)\big\|+\big\|m^2f(m^{-1}x)-m^4f(m^{-2}x)\big\|+\cdots\\ &\quad+\big\|m^{2(n-1)}f(m^{-(n-1)}x)-m^{2n}f(m^{-n}x)\big\|\\ &\le \big(1+m^2+\cdots+m^{2(n-1)}\big)c_2\end{aligned}$$

or

$$\big\|f(x)-m^{2n}f(m^{-n}x)\big\|\le\frac{1-m^{2(n-1)}}{1-m^2}c_2=\frac{c_2}{1-m^2}(1-m^{2(n-1)}), \tag{2.28}$$

or the general inequality:

$$\big\|f(x)-m^{2n}f(m^{-n}x)\big\|\le\frac{c_2}{1-m^2}, \tag{2.29}$$

where $|m|<1, m\ne 0$.

Rest of the proof is similar to the proof of Theorem 2.1.

Assume the following condition on f:

$$f(0)=0. \tag{2.30}$$

From (2.30) and (2.7)-(2.8), we get

$$\big\|f(x)-\bar f(x)\big\|\le\frac{c}{|k|\ |m|}, \tag{2.31}$$

and

$$\big\|\bar f(x)-m^{-2}f(mx)\big\|\le\frac{c}{|k|\ m^2},\quad k\ne 0, m\ne 0, |m|>1. \tag{2.32}$$

From (2.31)-(2.32), one obtains

$$\big\|f(x)-m^{-2}f(mx)\big\|\le\big\|f(x)-\bar f(x)\big\|+\big\|\bar f(x)-m^{-2}f(mx)\big\|,$$

or

$$\big\|f(x)-m^{-2}f(mx)\big\|\le c_1'=\frac{|m|+1}{|k|\ m^2}c,\quad k\ne 0, m\ne 0, |m|>1. \tag{2.33}$$

Thus

$$\begin{aligned}\big\|f(x)-m^{-2n}f(m^nx)\big\| &\le \big\|f(x)-m^{-2}f(mx)\big\|+\big\|m^{-2}f(mx)-m^{-4}f(m^2x)\big\|\\ &\quad+\cdots+\big\|m^{-2(n-1)}f(m^{n-1}x)-m^{-2n}f(m^nx)\big\|\\ &\le \big(1+m^{-2}+\cdots+m^{-2(n-1)}\big)c_1',\end{aligned}$$

or

$$\left\|f(x)-m^{-2n}f(m^n x)\right\| \le \frac{1-m^{-2n}}{1-m^{-2}}c_1' = \frac{m^2}{m^2-1}\left(1-m^{-2n}\right)c_1',$$

or

$$\left\|f(x)-m^{-2n}f(m^n x)\right\| \le \frac{1}{m^2-1}\left(1-m^{-2n}\right)c_2', \tag{2.34}$$

where

$$|m|>1, \text{ with } c_2' = m^2 c_1' = \frac{|m|+1}{|k|}c.$$

Therefore the following Theorem 2.1a holds.

Theorem 2.1a. Let X be a normed linear space and let Y be a real complete normed linear space. Then the limit (2.3) exists for all $x \in X$, all $n \in \mathbb{N}$, $|m|>1$ and $Q: X \to Y$ is the unique quadratic mapping satisfying equation (1.3), such that

$$\left\|f(x)-Q(x)\right\| \le \frac{c_2'}{m^2-1} = \frac{|m|+1}{m^2-1}\ \frac{1}{|k|}c, \quad k \neq 0, |m|>1. \tag{2.35}$$

The proof of this Theorem 2.1a is similar to the proof of the previous Theorem 2.1.

Alternatively: $|m|<1,\ f(0)=0$:

From (2.30) and (2.22) , (2.24), we get

$$\left\|f(x)-\bar{\bar{f}}(x)\right\| \le \frac{c}{|k|}, \tag{2.36}$$

and

$$\left\|\bar{\bar{f}}-m^2 f(m^{-1}x)\right\| \le \frac{|m|}{|k|}c, \tag{2.37}$$

$k \neq 0, m \neq 0, |m|<1$. From (2.36)-(2.37), one obtains

$$\left\|f(x)-m^2 f(m^{-1}x)\right\| \le \left\|f(x)-\bar{\bar{f}}(x)\right\| + \left\|\bar{\bar{f}}(x)-m^2 f(m^{-1}x)\right\|$$

or

$$\left\|f(x)-m^2 f(m^{-1}x)\right\| \le c_2' = \frac{|m|+1}{|k|}c \tag{2.38}$$

$k \neq 0, m \neq 0, |m|<1$. Thus

$$\begin{aligned}\left\|f(x)-m^{2n}f(m^{-n}x)\right\| &\le \left\|f(x)-m^{-2}f(mx)\right\| \\ &\quad + \cdots + \left\|m^{2(n-1)}f(m^{-(n-1)}x)-m^{2n}f(m^{-n}x)\right\| \\ &\le \left(1+m^2+\cdots+m^{2(n-1)}\right)c_2',\end{aligned}$$

or

$$\left\|f(x)-m^{2n}f(m^{-n}x)\right\| \le \frac{1}{1-m^2}\left(1-m^{2n}\right)c_2', \tag{2.39}$$

where $|m|<1, m\neq 0$.

Therefore the following Theorem 2.2a (analogous to Theorem 2.1a) holds for $|m|<1$, $m \neq 0$.

Theorem 2.2a. Let X be a normed linear space, and Y a real complete normed linear space. Then the limit $(2.3)'$ exists for all $x \in X$, $n \in \mathbb{N}$, $|m|<1$; $m \neq 0$, and $Q: X \to Y$ is the unique quadratic mapping satisfying equation (1.3), such that

$$\left\|f(x)-Q(x)\right\| \le \frac{c_2'}{1-m^2} = \frac{1+|m|}{1-m^2}\ \frac{1}{|k|}c, \tag{2.40}$$

$k \neq 0$, $|m| < 1$; $m \neq 0$.

Special case: Replacing $\alpha = \beta = 1$ in equation (1.3) and (2.1), one gets

$$kf(x+y) + f(kx-y) = (k+1)\big[kf(x) + f(y)\big], \quad k \in \mathbb{R} - \{-1, 0\}. \tag{2.41}$$

Thus

$$m = k + 1 \in \mathbb{R} - \{0, 1\}.$$

Also

$$\big\|kf(x+y) + f(kx-y) - (k+1)\big[kf(x) + f(y)\big]\big\| \leq c, \quad k \in \mathbb{R} - \{-1, 0\}. \tag{2.42}$$

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Some integral inequalities via $(h-(\alpha,m))-$logarithmically convexity

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Abstract. In this paper, we introduce the concept of $(h-(\alpha,m))-$logarithmically convex functions and establish some new integral inequalities of these classes of functions.

Keywords: Hermite's inequalities; $m-$logarithmically convex; $(\alpha,m)-$logarithmically convex; $(h-(\alpha,m))-$logarithmically convex;

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1 Introduction and preliminaries

The mathematical inequalities play an important role in the mathematical branches and their enormous application can not be underestimated. Afterwards, many researchers[1-13] studied the properties of convexity and achieve some different integral inequalities. The purpose of this paper is to introduce the definition of $(h-(\alpha,m))-$logarithmically convex functions and establish some new integral inequalities of these classes of functions. Before stating our results, we need recall some notions.

Throughout this paper, by $\Re$, we denote the set of all real numbers.

Definition 1.1 *Let $f : I \subset \Re \to \Re$ be a function define on interval I of real numbers. Then f is called convex (see[4]) if*

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$$

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for all $x, y \in I$ and $t \in [0,1]$.

In [2] , Toader gave the definition of $m-$convexity as follows.

Definition 1.2 *The function $f : [a,b] \to \Re$, $0 \le a < b$ is said to be $m-$convex , where $m \in [0,1]$, if*

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y)$$

holds for all $x, y \in [0,1]$ and $t \in [0,1]$. We say that f is $m-$concave if $-f$ is $m-$convex.

In [3] , Mihesan gave the definition of $(\alpha, m)-$convexity as follows.

Definition 1.3 *The function $f : [a,b] \to \Re$, $0 \le a < b$ is said to be $(\alpha, m)-$convex , where $(\alpha, m) \in [0,1]^2$, if*

$$f(tx + m(1-t)y) \le t^{\alpha} f(x) + m(1-t^{\alpha})f(y)$$

holds for all $x, y \in [0,1]$ and $t \in [0,1]$.

In [1], Özedemir et al. gave the definition of $(h-(\alpha, m))-$convexity as follows.

Definition 1.4 *Let $h : K \subset \Re \to \Re$ be a nonnegative function, $h \neq 0$. The function $f : L \subset \Re \to \Re$ is said to be $(h-(\alpha, m))-$convex function if f is non-negative and for all $x, y \in [0,1]$ and $t \in (0,1)$ for $(\alpha, m) \in [0,1]^2$, we have*

$$f(tx + m(1-t)y) \le h^{\alpha}(t)f(x) + m(1-h^{\alpha}(t))f(y).$$

In [5], Bai gave the definition of $m-$ and $(\alpha, m)-$logarithmically convex functions as follows.

Definition 1.5 *The function $f : [a,b] \to (0,\infty)$,$0 \le a < b$ is said to be $m-$logarithmically convex, where $m \in (0,1]$, if*

$$f(tx + m(1-t)y) \le [f(x)]^t [f(y)]^{m(1-t)}$$

holds for all $x, y \in [0,1]$ and $t \in [0,1]$.

Definition 1.6 *The function $f : [a,b] \to (0,\infty)$,$0 \le a < b$ is said to be $(\alpha, m)-$logarithmically convex, where $(\alpha, m) \in (0,1]^2$, if*

$$f(tx + m(1-t)x) \le [f(x)]^{t^{\alpha}} [f(x)]^{m(1-t^{\alpha})}$$

holds for all $x, y \in [0,1]$ and $t \in [0,1]$.

2 Main results

In this section, we will introduce the concept of $(h-(\alpha,m))-$logarithmically convex functions. We give some new integral inequalities of these classes of functions. First, we present the definition of $(h-(\alpha,m))-$logarithmically convex functions as follow.

Definition 2.1 *Let $h : K \subset \Re \to \Re$ be a nonnegative function, $h \neq 0$. The function $f : L \subset \Re \to \Re$ is said to be $(h-(\alpha,m))-$logarithmically convex function if f is nonnegative and for all $x, y \in L$ and $t \in (0,1)$ for $(\alpha,m) \in (0,1]^2$, we have*

$$f(tx+m(1-t)y) \leq [f(x)]^{h^\alpha(t)}[f(y)]^{m\left(1-h^\alpha(t)\right)}.$$

Obviously, if $h(t)=t$, then $(h-(\alpha,m))-$logarithmically convex function is a $(\alpha,m)-$logarithmically convex function; if $h(t)=t$,$\alpha=1$, then $(h-(\alpha,m))-$logarithmically convex function is a $m-$logarithmically convex function.

Before giving our results, we need the following lemma which is proved by Özdemir et al. [13].

Lemma 2.1 *Let $f : [a,b] \to \Re$, $0 \leq a < b$ be continuous on $[a,b]$ such that $f \in L([a,b])$. Then the equality*

$$\int_a^b (x-a)^p(x-b)^q f(x)dx = (b-a)^{p+q+1}\int_0^1 (1-t)^p t^q f(tx+(1-t)y)dt$$

holds for some fixed $p, q > 0$.

Theorem 2.1 *Let $f : [a,b] \to \Re$, $0 \leq a < b$ be continuous on $[a,b]$ such that $f \in L([a,b])$. If the mapping f is $(h-(\alpha,m))-$logarithmically convex on $[a,b]$ for all $t \in (0,1)$ and $(\alpha,m) \in (0,1]^2$, then*

$$\begin{aligned}\int_a^b (x-a)^p(x-b)^q f(x)dx \leq (b-a)^{p+q+1}[\beta(\frac{q}{1-m}+1,\frac{p}{1-m}+1)]^{1-m} \\ \times\left\{\int_0^1 [f(a)]^{\frac{h^\alpha(t)}{m}} f(\frac{b}{m})^{1-h^\alpha(t)}dt\right\}^m\end{aligned} \tag{2.1}$$

where $\beta(x,y)=\int_0^1 (t)^{x-1}(1-t)^{y-1}dt$.

Proof. Using Lemma 2.1 , we have

$$\int_a^b (x-a)^p(x-b)^q f(x)dx = (b-a)^{p+q+1}\int_0^1 (1-t)^p t^q f(ta+(1-t)b)dt. \tag{2.2}$$

Since f is $(h-(\alpha,m))-$logarithmically convex on $[a,b]$, we know that for every $t \in (0,1)$

$$f(ta+(1-t)b) = f(ta+m(1-t)(\frac{b}{m})) \leq [f(a)]^{h^\alpha(t)}[f(\frac{b}{m})]^{m\left(1-h^\alpha(t)\right)}. \tag{2.3}$$

From (2.1), (2.2), (2.3) and Hölder inequality, we can conclude that

$$\begin{aligned}
&\int_a^b (x-a)^p (x-b)^q f(x)dx \\
&= (b-a)^{p+q+1} \int_0^1 (1-t)^p t^q f(ta+(1-t)b)dt \\
&= (b-a)^{p+q+1} \int_0^1 (1-t)^p t^q f(ta+m(1-t)\frac{b}{m})dt \\
&\le (b-a)^{p+q+1} \int_0^1 (1-t)^p t^q [f(a)]^{h^\alpha(t)} [f(\frac{b}{m})]^{m\left(1-h^\alpha(t)\right)} dt \\
&\le (b-a)^{p+q+1} \left\{ \int_0^1 [(1-t)^p t^q]^{\frac{1}{1-m}} dt \right\}^{1-m} \left\{ \int_0^1 \{[f(a)]^{h^\alpha(t)} [f(\frac{b}{m})^{m(1-h^\alpha(t))}]\}^{\frac{1}{m}} dt \right\}^m \\
&\le (b-a)^{p+q+1} \left\{ \int_0^1 [(1-t)^{\frac{p}{1-m}} t^{\frac{q}{1-m}} dt \right\}^{1-m} \left\{ \int_0^1 \{[f(a)]^{\frac{h^\alpha(t)}{m}} [f(\frac{b}{m})^{1-h^\alpha(t)}]\} dt \right\}^m \\
&\le (b-a)^{p+q+1} [\beta(\frac{q}{1-m}+1, \frac{p}{1-m}+1)]^{1-m} \left\{ \int_0^1 \{[f(a)]^{\frac{h^\alpha(t)}{m}} [f(\frac{b}{m})^{1-h^\alpha(t)}]\} dt \right\}^m .
\end{aligned}$$

Hence, the proof of theorem 2.1 is completed.

Remark 2.1 *If $\alpha = 1$, then we can conclude the following inequality:*

$$\begin{aligned}
\int_a^b (x-a)^p (x-b)^q f(x)dx \le\ & (b-a)^{p+q+1} [\beta(\frac{q}{1-m}+1, \frac{p}{1-m}+1)]^{1-m} \\
& \times \left\{ \int_0^1 \{[f(a)]^{\frac{h(t)}{m}} [f(\frac{b}{m})^{1-h(t)}]\} dt \right\}^m .
\end{aligned}$$

Theorem 2.2 *Let $f : [a,b] \to \Re$, $0 \le a < b$ be continuous on $[a,b]$ such that $f \in L([a,b])$. If the mapping $|f|^{\frac{k}{k-1}}$ $(k > 1)$ is $(h-(\alpha,m))-$logarithmically convex on $[a,b]$ for all $t \in (0,1)$ and $(\alpha,m) \in (0,1]^2$, then*

$$\begin{aligned}
\int_a^b (x-a)^p (x-b)^q f(x)dx \le\ & (b-a)^{p+q+1} [\beta(kq+1, kp+1)]^{\frac{1}{k}} \left[\int_0^1 |f(a)|^{\frac{k^2 h^\alpha(t)}{k-1}} dt \right]^{\frac{k-1}{k^2}} \\
& \times \left[\int_0^1 |f(\frac{b}{m})|^{\frac{k^2 m}{(k-1)^2}(1-h^\alpha(t))} \right]^{\frac{(k-1)^2}{k^2}}
\end{aligned} \tag{2.4}$$

where $\beta(x,y) = \int_0^1 (t)^{x-1}(1-t)^{y-1}dt$.

Proof. Using Lemma 2.1 , we have

$$\int_a^b (x-a)^p (x-b)^q f(x)dx = (b-a)^{p+q+1} \int_0^1 (1-t)^p t^q f(ta+(1-t)b)dt. \tag{2.5}$$

Taking into account that $|f|^{\frac{k}{k-1}}$ is $(h-(\alpha,m))-$logarithmically convex on $[a,b]$, we deduce that

$$|f(ta+(1-t)b)|^{\frac{k}{k-1}} = |f(ta+m(1-t)(\frac{b}{m}))|^{\frac{k}{k-1}} \le |f(a)|^{\frac{k}{k-1}h^\alpha(t)} |f(\frac{b}{m})|^{\frac{k}{k-1}m(1-h^\alpha(t))}. \tag{2.6}$$

Hence, from (2.4), (2.5), (2.6) and Hölder inequality, we can achieve the following inequality:

$$\begin{aligned}
&\int_a^b (x-a)^p(x-b)^q f(x)dx \\
&= (b-a)^{p+q+1}\int_0^1 (1-t)^p t^q f(ta+(1-t)b)dt \\
&\le (b-a)^{p+q+1}\left[\int_0^1 (1-t)^{kp}t^{kq}dt\right]^{\frac{1}{k}}\left\{\int_0^1 \left|f(ta+m(1-t)\frac{b}{m})\right|^{\frac{k}{k-1}}dt\right\}^{\frac{k-1}{k}} \\
&= (b-a)^{p+q+1}[\beta(kq+1,kp+1)]^{\frac{1}{k}}\left\{\int_0^1 \left|f(ta+m(1-t)\frac{b}{m})\right|^{\frac{k}{k-1}}dt\right\}^{\frac{k-1}{k}} \\
&\le (b-a)^{p+q+1}[\beta(kq+1,kp+1)]^{\frac{1}{k}}\left[\int_0^1 |f(a)|^{\frac{k}{k-1}h^\alpha(t)}|f(\frac{b}{m})|^{\frac{k}{k-1}m(1-h^\alpha(t))}dt\right]^{\frac{k-1}{k}}.
\end{aligned} \tag{2.7}$$

Using Hölder inequality again, we have

$$\begin{aligned}
&\left[\int_0^1 |f(a)|^{\frac{k}{k-1}h^\alpha(t)}|f(\frac{b}{m})|^{\frac{k}{k-1}m(1-h^\alpha(t))}dt\right]^{\frac{k-1}{k}} \\
&\le \left\{\left[\int_0^1 |f(a)|^{\frac{k^2}{k-1}h^\alpha(t)}dt\right]^{\frac{1}{k}}\left[\int_0^1 \left[|f(\frac{b}{m})|^{\frac{k}{k-1}m(1-h^\alpha(t))}\right]^{\frac{k}{k-1}}dt\right]^{\frac{k-1}{k}}\right\}^{\frac{k-1}{k}} \\
&\le \left\{\left[\int_0^1 |f(a)|^{\frac{k^2}{k-1}h^\alpha(t)}dt\right]^{\frac{1}{k}}\left[\int_0^1 \left[|f(\frac{b}{m})|^{(\frac{k}{k-1})^2 m(1-h^\alpha(t))}\right]dt\right]^{\frac{k-1}{k}}\right\}^{\frac{k-1}{k}}
\end{aligned} \tag{2.8}$$

Combining with (2.7) and (2.8), we can conclude that (2.4) holds. Hence, the proof of theorem 2.2 is completed.

Remark 2.2 *If $\alpha = 1$, then we can conclude the following inequality:*

$$\begin{aligned}
\int_a^b (x-a)^p(x-b)^q f(x)dx \le (b-a)^{p+q+1}[\beta(kq+1,kp+1)]^{\frac{1}{k}}\left[\int_0^1 |f(a)|^{\frac{k^2 h(t)}{k-1}}dt\right]^{\frac{k-1}{k^2}} \\
\times\left[\int_0^1 |f(\frac{b}{m})|^{\frac{k^2 m}{(k-1)^2}(1-h(t))}\right]^{\frac{(k-1)^2}{k^2}}.
\end{aligned}$$

Theorem 2.3 *Let $f : [a,b] \to \Re$, $0 \le a < b$ be continuous on $[a,b]$ such that $f \in L([a,b])$. If the mapping $|f|^l$ $(l \ge 1)$ is $(h-(\alpha,m))-$logarithmically convex on $[a,b]$ for all $t \in (0,1)$ and $(\alpha,m) \in (0,1]^2$, then*

$$\begin{aligned}
\int_a^b (x-a)^p(x-b)^q f(x)dx \le (b-a)^{p+q+1}[\beta(q+1,p+1)]^{\frac{l-1}{l}}\left[\beta(ql+1,pl+1)\right]^{\frac{1}{l^2}}\left[\int_0^1 |f(a)|^{\frac{l^3 h^\alpha(t)}{l-1}}dt\right]^{\frac{l-1}{l^3}} \\
\times\left[\int_0^1 |f(\frac{b}{m})|^{\frac{l^2 m(1-h^\alpha(t))}{(l-1)^2}}dt\right]^{\frac{(l-1)^2}{l^3}}
\end{aligned} \tag{2.9}$$

where $\beta(x,y) = \int_0^1 (t)^{x-1}(1-t)^{y-1}dt$.

Proof. Using Lemma 2.1 , we have

$$\int_a^b (x-a)^p(x-b)^q f(x)dx = (b-a)^{p+q+1}\int_0^1 (1-t)^p t^q f(ta+(1-t)b)dt. \tag{2.10}$$

Since, $|f|^l$ is $(h-(\alpha,m))-$logarithmically convex on $[a,b]$, we have

$$|f(ta+(1-t)b)|^l = |f(ta+m(1-t)(\frac{b}{m}))|^l \leq |f(a)|^{lh^\alpha(t)}|f(\frac{b}{m})|^{lm(1-h^\alpha(t))}. \tag{2.11}$$

From (2.9), (2.10), (2.11) and Hölder inequality, we can achieve the following inequality:

$$\begin{aligned}
&\int_a^b (x-a)^p(x-b)^q f(x)dx \\
&= (b-a)^{p+q+1}\int_0^1 (1-t)^p t^q f(ta+m(1-t)(\frac{b}{m}))dt \\
&\leq (b-a)^{p+q+1}\int_0^1 [(1-t)^p(t)^q]^{\frac{l-1}{l}}[(1-t)^p(t)^q]^{\frac{1}{l}} f(ta+m(1-t)(\frac{b}{m}))dt \\
&\leq (b-a)^{p+q+1}\left[\int_0^1 (1-t)^p(t)^q dt\right]^{\frac{l-1}{l}}\left\{\int_0^1 [(1-t)^p(t)^q]|f(ta+m(1-t)(\frac{b}{m}))|^l dt\right\}^{\frac{1}{l}} \\
&= (b-a)^{p+q+1}[\beta(q+1,p+1)]^{\frac{l-1}{l}}\left\{\int_0^1 [(1-t)^p(t)^q]|f(ta+m(1-t)(\frac{b}{m}))|^l dt\right\}^{\frac{1}{l}}.
\end{aligned} \tag{2.12}$$

Using Hölder inequality again, we have

$$\begin{aligned}
&\left\{\int_0^1 [(1-t)^p(t)^q]|f(ta+m(1-t)(\frac{b}{m}))|^l dt\right\}^{\frac{1}{l}} \\
&\leq \left\{\int_0^1 [(1-t)^p(t)^q]|f(a)|^{lh^\alpha(t)}|f(\frac{b}{m})|^{lm(1-h^\alpha(t))}dt\right\}^{\frac{1}{l}} \\
&\leq \left\{\{\int_0^1 [(1-t)^p(t)^q]^l dt\}^{\frac{1}{l}}\{\int_0^1 \left[|f(a)|^{lh^\alpha(t)}|f(\frac{b}{m})|^{lm(1-h^\alpha(t))}\right]^{\frac{l}{l-1}} dt\}^{\frac{l-1}{l}}\right\}^{\frac{1}{l}} \\
&\leq \left[\beta(ql+1,pl+1)\right]^{\frac{1}{l^2}}\left[\int_0^1 |f(a)|^{\frac{l^2h^\alpha(t)}{l-1}}|f(\frac{b}{m})|^{\frac{l^2m(1-h^\alpha(t))}{l-1}}dt\right]^{\frac{l-1}{l^2}} \\
&\leq \left[\beta(ql+1,pl+1)\right]^{\frac{1}{l^2}}\left[\int_0^1 |f(a)|^{\frac{l^3h^\alpha(t)}{l-1}}dt\right]^{\frac{l-1}{l^3}}\left[\int_0^1 |f(\frac{b}{m})|^{\frac{l^3m(1-h^\alpha(t))}{(l-1)^2}}dt\right]^{\frac{(l-1)^2}{l^3}}.
\end{aligned} \tag{2.13}$$

By (2.12) and (2.13), we can achieve that (2.9) holds. Hence, the proof of theorem 2.3 is completed.

Remark 2.3 *If $\alpha=1$, then we can conclude the following inequality:*

$$\begin{aligned}
\int_a^b (x-a)^p(x-b)^q f(x)dx \leq (b-a)^{p+q+1}[\beta(q+1,p+1)]^{\frac{l-1}{l}}\left[\beta(ql+1,pl+1)\right]^{\frac{1}{l^2}}\left[\int_0^1 |f(a)|^{\frac{l^3h(t)}{l-1}}dt\right]^{\frac{l-1}{l^3}} \\
\times\left[\int_0^1 |f(\frac{b}{m})|^{\frac{l^2m(1-h(t))}{(l-1)^2}}dt\right]^{\frac{(l-1)^2}{l^3}}.
\end{aligned}$$

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On Gosper's q-Trigonometric Function

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Abstract. In this paper, we study about periodicity of q-trigonometric function which was introduced by Gosper and also we rewrite the q-analogue of Legendres duplication formula with the same bases. Furthermore, we modify some identities involving q-shifted factorial.

Keywords. Gosper's q-trigonometric function, q-Gamma function, Legendres duplication formula.

Mathematics Subject Classification. 11B65, 33D05.

1 Introduction

The q-shifted factorial [1, 3] is defined by

$$(a;q)_n = \begin{cases} 1 & n = 0, \\ \prod_{m=0}^{n-1}(1-aq^m) & n = 1, 2, ... \end{cases} \tag{1}$$

and it is assumed that $a \neq q^{-m},\ m = 0, 1, ...$ The q-shifted factorial [1, 3] is also defined for any complex number α,

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}, \tag{2}$$

where $(a;q)_\infty := \lim_{n\to\infty} \prod_{m=0}^{n}(1-aq^m)$ and the principal value of q^α is taken and it is assumed that $0 < q < 1$.

The q-Gamma function was introduced by Thomae [6] and Jackson [5], (see [3], page 20)

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad 0 < q < 1. \tag{3}$$

A q-analogue of Legendre's duplication formula [5, 7] has the form

$$\Gamma_q(2x)\Gamma_{q^2}(\frac{1}{2}) = (1+q)^{2x-1}\Gamma_{q^2}(x)\Gamma_{q^2}(x+\frac{1}{2}). \tag{4}$$

Gosper [4] defined q-trigonometric functions as follows:

$$sin_q(\pi z) := q^{(z-1/2)^2}\frac{(q^{2z};q^2)_\infty(q^{2-2z};q^2)_\infty}{(q;q^2)_\infty^2}, \quad 0 < q < 1, \tag{5}$$

$$cos_q(\pi z) := q^{z^2}\frac{(q^{1+2z};q^2)_\infty(q^{1-2z};q^2)_\infty}{(q;q^2)_\infty^2}, \quad 0 < q < 1. \tag{6}$$

1

It can be seen [4] that

$$cos_q(z) = sin_q(\frac{\pi}{2} - z). \tag{7}$$

By using (5), (6) and (7), one can see that, for the cases $x = 0$ and $x = \frac{\pi}{2}$, $sin_q(x)$ and $cos_q(x)$ are;

$$\begin{array}{ll} sin_q(0) = 0, & sin_q(\frac{\pi}{2}) = 1, \\ cos_q(0) = 1, & cos_q(\frac{\pi}{2}) = 0. \end{array} \tag{8}$$

There are many identities involving q-shifted factorial [1, 3], but in this paper we are using the following identities;

For all $a \in \mathbb{C}$ and $n \in \mathbb{N}$, following equations hold

$$(q^{2a}; q^2)_n = (q^a; q)_n(-q^a; q)_n, \tag{9}$$

$$(a; q)_{2n} = (a; q^2)_n(aq; q^2)_n, \tag{10}$$

$$(q^{1-a-n}; q)_n = (q^a; q)_n(-1)^n \, q^{-\binom{n}{2}-an}. \tag{11}$$

2 Main result

In the next lemma we show that the equations (9) and (10) are also valid for any complex number α,

Lemma 1. *For all a, $\alpha \in \mathbb{C}$, the following equations hold*

$$(q^{2a}; q^2)_\alpha = (q^a; q)_\alpha(-q^a; q)_\alpha, \tag{12}$$

$$(a; q)_{2\alpha} = (a; q^2)_\alpha(aq; q^2)_\alpha. \tag{13}$$

Proof. To prove (12) we use (2), then we have

$$(q^{2a}; q^2)_\alpha = \frac{(q^{2a}; q^2)_\infty}{(q^{2a+2\alpha}; q^2)_\infty}.$$

By using the definition of q-shifted factorial (1), we obtain

$$\begin{aligned} (q^{2a}; q^2)_\alpha &= \frac{(q^{2a}; q^2)_\infty}{(q^{2a+2\alpha}; q^2)_\infty} \\ &= \frac{\prod_{i=0}^{\infty}(1 - q^{2a+2i})}{\prod_{i=0}^{\infty}(1 - q^{2a+2\alpha+2i})} \\ &= \frac{\prod_{i=0}^{\infty}(1 - q^{a+i})(1 + q^{a+i})}{\prod_{i=0}^{\infty}(1 - q^{a+\alpha+i})(1 + q^{a+\alpha+i})} \\ &= \frac{\prod_{i=0}^{\infty}(1 - q^{a+i})}{\prod_{i=0}^{\infty}(1 - q^{a+\alpha+i})} \frac{\prod_{i=0}^{\infty}(1 + q^{a+i})}{\prod_{i=0}^{\infty}(1 + q^{a+\alpha+i})} \\ &= \frac{(q^a; q)_\infty}{(q^{a+\alpha}; q)_\infty} \frac{(-q^a; q)_\infty}{(-q^{a+\alpha}; q)_\infty} \\ &= (q^a; q)_\alpha(-q^a; q)_\alpha. \end{aligned}$$

2

The proof of (12) is complete. To Prove the next equation, we use (1) and (2), then we have

$$\begin{aligned}\frac{(a;q^2)_\alpha(aq;q^2)_\alpha}{(a;q)_{2\alpha}} &= \frac{(a;q^2)_\infty(aq;q^2)_\infty(aq^{2\alpha};q)_\infty}{(aq^{2\alpha};q^2)_\infty(aq^{2\alpha+1};q^2)_\infty(a;q)_\infty}\\ &= \frac{(a;q^2)_\infty(aq;q^2)_\infty}{(a;q)_\infty}\frac{(aq^{2\alpha};q)_\infty}{(aq^{2\alpha};q^2)_\infty(aq^{2\alpha+1};q^2)_\infty},\end{aligned}$$

each fraction in the last line is equal 1, since $(c;q)_\infty(cq^{\frac{1}{2}};q)_\infty = (c;q^{\frac{1}{2}})_\infty$ (see [8] , page 13). The proof of (13) is complete.

□

In the next lemma, we want to modify the equation (11).

Lemma 2. *For all α and $\beta \in \mathbb{C}$, the following equation holds*

$$(q^{1-\alpha-\beta};q)_\alpha = (q^\beta;q)_\alpha \frac{sin_{\sqrt{q}}\pi(\alpha+\beta)}{sin_{\sqrt{q}}\pi(\beta)}\, q^{-\binom{\alpha}{2}-\alpha\beta}, \tag{14}$$

where sin_q is defined as in (5).

Proof. After applying the equation (2) for both numerator and denominator of the left hand side of the following equation, we obtain that

$$\frac{(q^{1-\alpha-\beta};q)_\alpha}{(q^\beta;q)_\alpha} = \frac{(q^{1-\alpha-\beta};q)_\infty(q^{\alpha+\beta};q)_\infty}{(q^{1-\beta};q)_\infty(q^\beta;q)_\infty}$$

and by using the definition of sin_q which is written in (5), we have

$$\frac{(q^{1-\alpha-\beta};q)_\infty(q^{\alpha+\beta};q)_\infty}{(q^{1-\beta};q)_\infty(q^\beta;q)_\infty} = \frac{sin_{\sqrt{q}}\pi(\alpha+\beta)}{sin_{\sqrt{q}}\pi(\beta)}\, q^{-\binom{\alpha}{2}-\alpha\beta}.$$

Therefore proof is complete. □

Theorem 1. *For all $n \in \mathbb{N}$ and $x \in \mathbb{C}$, the following equations hold*

$$sin_q(x+n\pi) = (-1)^n sin_q(x), \tag{15}$$

$$cos_q(x+n\pi) = (-1)^n cos_q(x), \tag{16}$$

$$tan_q(x+n\pi) = tan_q(x), \tag{17}$$

$$cot_q(x+n\pi) = cot_q(x). \tag{18}$$

Proof. We use lemma 2 for prove the equation (15). Taking any arbitrary n $\in \mathbb{N}$ and a $\in \mathbb{C}$, then we have

$$(q^{1-n-a};q)_n = (q^a;q)_n \frac{sin_{\sqrt{q}}\pi(a+n)}{sin_{\sqrt{q}}\pi(a)}\, q^{-\binom{n}{2}-na}. \tag{19}$$

By comparing the equations (11) and (19), we obtain

$$\frac{sin_{\sqrt{q}}\pi(a+n)}{sin_{\sqrt{q}}\pi(a)} = (-1)^n.$$

Substituting q with $\sqrt{q}$ and x with $a\pi$ completes the proof of equation (15).

By using (7) and (15), one can shows that (16) is valid for all n $\in \mathbb{N}$ and $x \in \mathbb{C}$, and the last two equations (17) and (18) comes from $\frac{sin_q(x)}{cos_q(x)}$ and $\frac{cos_q(x)}{sin_q(x)}$, respectively.

□

Remark 1. *The $cos_q(x)$ is an even function, its come from the definition (6) directly. And the $sin_q(x)$ is an odd function, since by using (7), we can write $sin_q(x) = cos_q(\frac{\pi}{2}-x)$ and also we know that $cos_q(x)$ is an even function then we have $sin_q(x) = cos_q(x-\frac{\pi}{2})$, again apply (7), we obtain $cos_q(x-\frac{\pi}{2}) = sin_q(\pi-x)$. Now by using the Theorem 1, we obtain $sin_q(\pi - x) = -sin_q(-x)$. Therefore $sin_q(x) = -sin_q(-x)$.*

Lemma 3. *For all $k \in \mathbb{Z}$, zeroes of q-sine and q-cosine functions are $k\pi$ and $\frac{(2k+1)\pi}{2}$, respectively.*

Proof. Since sin_q is an odd function, therefore its enough to prove the lemma for positive value of k. We prove the lemma for positive value of k by induction. For $k = 1$ and using (15), we have

$$sin_q(\pi) = sin_q(0+\pi) = sin_q(0) = 0,$$

since $sin_q(0) = 0$ comes from definition of sin_q. Then lemma is valid for $k = 1$. Assume that $sin_q(n\pi) = 0$ is true. We need to show that $sin_q((n+1)\pi) = 0$ is also true. By using (15), we have

$$sin_q((n+1)\pi) = sin_q(n\pi+\pi) = (-1)sin_q(n\pi) = 0.$$

Therefore zeroes of $sin_q(x)$ are $k\pi$, for all $k \in \mathbb{Z}$. About the zeroes of $cos_q(x)$, take any arbitrary $k \in \mathbb{Z}$, and by using the (7), we have

$$cos_q(\frac{(2k+1)\pi}{2}) = cos_q(k\pi + \frac{\pi}{2}) = sin_q(-k\pi) = 0.$$

Therefore zeroes of $cos_q(x)$ are $\frac{(2k+1)\pi}{2}$, for all $k \in \mathbb{Z}$.

□

Lemma 4. *For all $z \in \mathbb{C}$, the following equation holds*

$$(q^{z+1};q)_z = (-q^{\frac{1}{2}};q^{\frac{1}{2}})_{2z}(q^{\frac{1}{2}};q)_z.$$

4

Proof. Taking $a = 1$, $\alpha = 2z$ and substituting q with $q^{\frac{1}{2}}$ in (12), and applying to $(-q^{\frac{1}{2}}; q^{\frac{1}{2}})_{2z}$, we obtain

$$(-q^{\frac{1}{2}}; q^{\frac{1}{2}})_{2z} \frac{(q^{\frac{1}{2}}; q)_z}{(q^{z+1}; q)_z} = \frac{(q; q)_{2z}}{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_{2z}} \frac{(q^{\frac{1}{2}}; q)_z}{(q^{z+1}; q)_z}. \tag{20}$$

By using (2), the right hand side of (20) can be written as

$$\begin{aligned} \frac{(q; q)_{2z}}{(q^{z+1}; q)_z} \frac{(q^{\frac{1}{2}}; q)_z}{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_{2z}} &= \frac{\frac{(q;q)_\infty}{(q^{2z+1};q)_\infty}}{\frac{(q^{z+1};q)_\infty}{(q^{2z+1};q)_\infty}} \frac{(q^{\frac{1}{2}}; q)_z}{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_{2z}}, \\ &= \frac{(q; q)_\infty}{(q^{z+1}; q)_\infty} \frac{(q^{\frac{1}{2}}; q)_z}{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_{2z}}, \\ &= (q; q)_z \frac{(q^{\frac{1}{2}}; q)_z}{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_{2z}}. \end{aligned} \tag{21}$$

By substituting q with $q^{\frac{1}{2}}$ and then taking $a = q^{\frac{1}{2}}$ in equation (13), one can see that the right hand side of (21) is equal 1 and this completes the proof.

□

Lemma 5. *For all $z \in \mathbb{C}$, the following equation holds*

$$(q^{\frac{1}{2}-z}; q)_z = (q^{z+1}; q)_z \frac{q^{-\frac{z^2}{2}}}{(-q^{\frac{1}{2}}; q^{\frac{1}{2}})_{2z}} cos_{\sqrt{q}}(\pi z). \tag{22}$$

Proof. By using the Lemma 4 the equation (22) can be written as

$$(q^{\frac{1}{2}-z}; q)_z = (q^{\frac{1}{2}}; q)_z q^{-\frac{z^2}{2}} cos_{\sqrt{q}}(\pi z). \tag{23}$$

The equation (23) is a special case of lemma 2 when $\beta = \frac{1}{2}$, since $cos_q(z) = sin_q(\frac{\pi}{2} - z)$ and also cos_q is an even function.

□

Corollary 1. *For the positive integers value of n, Lemma 5 deduce to*

$$(q^{\frac{1}{2}-n}; q)_n = (-1)^n (q^{n+1}; q)_n \frac{q^{-\frac{n^2}{2}}}{(-q^{\frac{1}{2}}; q^{\frac{1}{2}})_{2n}}.$$

Proof. The result is obtained by using Theorem 1.

□

Euler (see [2], page 271 or [3], page 222) found the following formula in connection with partitions,

$$(-q; q)_\infty (q; q^2)_\infty = 1.$$

In the next lemma, we want to generalize this Euler's formula.

Theorem 2. *For all $z \in \mathbb{C}$, the following equation holds*

$$(q^{z+1};q)_z = (-q;q)_z(q;q^2)_z.$$

Proof. By substituting q with $q^{\frac{1}{2}}$ and then taking $a = -q^{\frac{1}{2}}$ in equation (13), we have

$$(-q^{\frac{1}{2}};q^{\frac{1}{2}})_{2z} = (-q^{\frac{1}{2}};q)_z(-q;q)_z,$$

now, we apply the result to Lemma 5 and obtain

$$(q^{z+1};q)_z = (-q^{\frac{1}{2}};q)_z(-q;q)_z(q^{\frac{1}{2}};q)_z. \tag{24}$$

Taking $a = \frac{1}{2}$ in the equation (12) and then applying to the right hand side of the equation (24) completes the proof.

□

Theorem 3. *For all $x \in \mathbb{C}$, the following equation holds,*

$$\Gamma_q(2x)\Gamma_q(\frac{1}{2}) = \Gamma_q(x)\Gamma_q(x+\frac{1}{2})(-q^{\frac{1}{2}};q^{\frac{1}{2}})_{2x-1}.$$

Proof. By using the definition of q-Gamma function (3) and then applying the equation (2), we can write

$$\frac{\Gamma_q(2x)\Gamma_q(\frac{1}{2})}{\Gamma_q(x)\Gamma_q(x+\frac{1}{2})} = \frac{(q^x;q)_\infty(q^{x+\frac{1}{2}};q)_\infty}{(q^{2x};q)_\infty(q^{\frac{1}{2}};q)_\infty} = \frac{(q^x;q)_x}{(q^{\frac{1}{2}};q)_x}, \tag{25}$$

the last equation holds since $(q^x;q)_x = \frac{(q^x;q)_\infty}{(q^{2x};q)_\infty}$ and $(q^{\frac{1}{2}};q)_x = \frac{(q^{\frac{1}{2}};q)_\infty}{(q^{\frac{1}{2}+x};q)_\infty}$. Taking $\beta = \frac{1}{2}$ in Lemma 2 and applying for the denominator of the last fraction in (25), we get

$$\begin{aligned}\frac{(q^x;q)_x}{(q^{\frac{1}{2}};q)_x} &= \frac{(q^x;q)_x}{(q^{\frac{1}{2}-x};q)_x}\frac{sin_{\sqrt{q}}\pi(\frac{1}{2}+x)}{sin_{\sqrt{q}}\pi(\frac{1}{2})}\,q^{-\frac{x^2}{2}},\\ &= \frac{(q^x;q)_x}{(q^{\frac{1}{2}-x};q)_x}cos_{\sqrt{q}}(\pi x)\,q^{-\frac{x^2}{2}}.\end{aligned}$$

Now, by using Lemma 4, we have

$$\frac{(q^x;q)_x}{(q^{\frac{1}{2}-x};q)_x}cos_{\sqrt{q}}(\pi x)\,q^{-\frac{x^2}{2}} = \frac{(q^x;q)_x}{(q^{x+1};q)_x}(-q^{\frac{1}{2}};q^{\frac{1}{2}})_{2x}.$$

Making use of (2), we have

$$\frac{(q^x;q)_x}{(q^{x+1};q)_x}(-q^{\frac{1}{2}};q^{\frac{1}{2}})_{2x} = \frac{(q^x;q)_\infty}{(q^{2x};q)_\infty}\frac{(q^{2x+1};q)_\infty}{(q^{x+1};q)_\infty}(-q^{\frac{1}{2}};q^{\frac{1}{2}})_{2x}.$$

After expanding the first and second fractions and then a simplification, yields

$$\begin{aligned}\frac{(q^x;q)_\infty}{(q^{2x};q)_\infty}\frac{(q^{2x+1};q)_\infty}{(q^{x+1};q)_\infty}(-q^{\frac{1}{2}};q^{\frac{1}{2}})_{2x} &= \frac{1-q^x}{1-q^{2x}}(-q^{\frac{1}{2}};q^{\frac{1}{2}})_{2x},\\ &= \frac{1}{1+q^x}(-q^{\frac{1}{2}};q^{\frac{1}{2}})_{2x},\\ &= (-q^{\frac{1}{2}};q^{\frac{1}{2}})_{2x-1}.\end{aligned}$$

□

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APPROXIMATE QUADRATIC FORMS ON RESTRICTED DOMAINS

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ABSTRACT. Let r, s be nonzero real numbers with $r + s = 1$. In [9], Najati and Jung investigated a quadratic functional equation $g(rx + sy) + rs\,g(x - y) = rg(x) + sg(y)$. We introduce a functional equation $f(rx + sy, rz + sw) + rsf(x - y, z - w) = rf(x, z) + sf(y, w)$ and investigate the relation between the above two functional equations. And we find out the general solution and the Hyers-Ulam stability of the latter on restricted domains.

1. Introduction

In 1940 and in 1968, Ulam [12] proposed the general Ulam stability problem:

"When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?"

In 1941, Hyers [7] solved this problem for linear mappings. In 1950, Aoki [2] provided a generalization of the Hyers' theorem for additive mappings. This stability concept is also applied to the case of other functional equations. For more results on the stability of functional equations (see [5, 6, 11]). In 1998, S.-M. Jung [8] investigated the Hyers-Ulam stability for additive and quadratic mappings on restricted domains.

Let X and Y be real vector spaces. For a mapping $g : X \to Y$, consider the quadratic functional equation:

$$g(x + y) + g(x - y) = 2g(x) + 2g(y). \tag{1.1}$$

In 1989, J. Aczel [1] solved the solution of the equation (1.1). Later, many different quadratic functional equations were solved by numerous authors [3, 8, 10]. In recent, A. Najati and S.-M. Jung [9] introduced a generalized quadratic functional equation

$$g(rx + sy) + rs\,g(x - y) = rg(x) + sg(y), \tag{1.2}$$

where r, s are nonzero real numbers with $r + s = 1$. In 2007, the authors [4] solved the solution of the 2-variable quadratic functional equation

$$f(x + y, z + w) + f(x - y, z - w) = 2f(x, z) + 2f(y, w). \tag{1.3}$$

Consider a generalized 2-variable quadratic functional equation

$$f(rx + sy, rz + sw) + rsf(x - y, z - w) = rf(x, z) + sf(y, w), \tag{1.4}$$

where r, s are nonzero real numbers with $r + s = 1$.

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In this paper, we investigate the relation between (1.2) and (1.4) by the same method as the proofs of Theorem 1 and Theorem 2 in [4]. And we find out the general solution and the Hyers-Ulam stability of (1.4) in the spirit of Najati and Jung [9].

2. Relation between (1.2) and (1.4)

The functional equation (1.4) induces the quadratic functional equation (1.2) as follows.

THEOREM 2.1. *Let $f : X \times X \to Y$ be a mapping satisfying* (1.4) *and let $g : X \to Y$ be the mapping given by*

$$g(x) := f(x, x) \tag{2.1}$$

for all $x \in X$, then g satisfies (1.2).

Proof. By (1.4) and (2.1), we obtain

$$\begin{aligned} g(rx+sy)+rsg(x-y) &= f(rx+sy, rx+sy)+rsf(x-y, x-y) \\ &= rf(x,x)+sf(y,y) \\ &= rg(x)+sg(y) \end{aligned}$$

for all $x, y \in X$. □

EXAMPLE 1. *Let X be a real algebra and $D : X \to X$ a derivation on X. Define a mapping $f : X \times X \to X$ by*

$$f(x, y) := D(xy) = xD(y) + D(x)y$$

for all $x, y \in X$. Then we see that

$$\begin{aligned} &f(rx+sy, rz+sw)+rsf(x-y, z-w) = D[(rx+sy)(rz+sw)]+rsD[(x-y)(z-w)] \\ &= (rx+sy)D(rz+sw)+D(rx+sy)(rz+sw)+rs[(x-y)D(z-w)+D(x-y)(z-w)] \\ &= (rx+sy)[rD(z)+sD(w)]+[rD(x)+sD(y)](rz+sw) \\ &\quad + rs\big((x-y)[D(z)-D(w)]+[D(x)-D(y)](z-w)\big) \\ &= r^2xD(z)+s^2yD(w)+r^2D(x)z+s^2D(y)w+rsxD(z)+rsyD(w)+rsD(x)z+rsD(y)w \\ &= r[xD(z)+D(x)z]+s[yD(w)+D(y)w] = rD(xz)+sD(yw) = rf(x,z)+sf(y,w) \end{aligned}$$

for all $x, y, z, w \in X$. Thus f satisfies (1.4). *Define a mapping $g : X \to X$ by*

$$g(x) := D(x^2) = xD(x) + D(x)x$$

for all $x \in X$. Then g satisfies (2.1). *By Theorem* 2.1, *g satisfies* (1.2).

The quadratic functional equation (1.2) induces the functional equation (1.4) with an additional condition.

THEOREM 2.2. *Let $a, b, c \in \mathbb{R}$ and $g : X \to Y$ be a mapping satisfying* (1.2). *If $f : X \times X \to Y$ is the mapping given by*

$$f(x, y) := ag(x) + \frac{b}{4}[g(x+y) - g(x-y)] + cg(y) \tag{2.2}$$

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for all $x, y \in X$, then f satisfies (1.4). *Furthermore,* (2.1) *holds if r is a rational number and $a+b+c=1$.*

Proof. By (1.2) and (2.2), we see that

$$\begin{aligned}
&f(rx+sy, rz+sw)+rsf(x-y, z-w)\\
&= ag(rx+sy)+\frac{b}{4}\big[g\big(r(x+z)+s(y+w)\big)-g\big(r(x-z)+s(y-w)\big)\big]+cg(rz+sw)\\
&\quad +rs\Big(ag(x-y)+\frac{b}{4}\big[g(x-y+z-w)-g(x-y-z+w)\big]+cg(z-w)\Big)\\
&= ag(rx+sy)+rsag(x-y)+\frac{b}{4}\big[g\big(r(x+z)+s(y+w)\big)+rsg\big((x+z)-(y+w)\big)\big]\\
&\quad -\frac{b}{4}\big[g\big(r(x-z)+s(y-w)\big)+rsg\big((x-z)-(y-w)\big)\big]+cg(rz+sw)+rscg(z-w)\\
&= a\big[g(rx+sy)+rsg(x-y)\big]+\frac{b}{4}\big[rg(x+z)+sg(y+w)\big]\\
&\quad -\frac{b}{4}\big[rg(x-z)+sg(y-w)\big]+c\big[g(rz+sw)+rsg(z-w)\big]\\
&= a[rg(x)+sg(y)]+\frac{b}{4}\big(r\big[g(x+z)-g(x-z)\big]+s\big[g(y+w)-g(y-w)\big]\big)+c[rg(z)+sg(w)]\\
&= rf(x,z)+sf(y,w)
\end{aligned}$$

for all $x, y, z, w \in X$.

Let r be a rational number. Since g satisfies (1.2), it also satisfies (1.1) (see Theorem 2.3. in [9]). Letting $x=y=0$ and $y=x$ in (1.1), respectively,

$$g(0)=0 \quad\text{and}\quad g(2x)=4g(x)$$

for all $x \in X$. By (2.2) and the above two equalities,

$$\begin{aligned}
f(x,x) &= ag(x)+\frac{b}{4}[g(2x)-g(0)]+cg(x)\\
&= (a+b+c)g(x)\\
&= g(x)
\end{aligned}$$

for all $x \in X$. □

Example 2. *Consider the function $g : \mathbb{R}^2 \to \mathbb{R}$ given by $g(\mathbf{x}) := \mathbf{x}^T A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$, where A is a 2×2 real matrix. Then we see that*

$$\begin{aligned}
g(r\mathbf{x}+s\mathbf{y})+rs\,g(\mathbf{x}-\mathbf{y}) &= (r\mathbf{x}+s\mathbf{y})^T A(r\mathbf{x}+s\mathbf{y})+rs(\mathbf{x}-\mathbf{y})^T A(\mathbf{x}-\mathbf{y})\\
&= (r\mathbf{x}^T+s\mathbf{y}^T)A(r\mathbf{x}+s\mathbf{y})+rs(\mathbf{x}^T-\mathbf{y}^T)A(\mathbf{x}-\mathbf{y})\\
&= r^2\mathbf{x}^T A\mathbf{x}+rs(\mathbf{x}^T A\mathbf{y}+\mathbf{y}^T A\mathbf{x})+s^2\mathbf{y}^T A\mathbf{y}+rs(\mathbf{x}^T A\mathbf{x}-\mathbf{x}^T A\mathbf{y}-\mathbf{y}^T A\mathbf{x}+\mathbf{y}^T A\mathbf{y})\\
&= r(r+s)\mathbf{x}^T A\mathbf{x}+s(r+s)\mathbf{y}^T A\mathbf{y} = rg(\mathbf{x})+sg(\mathbf{y})
\end{aligned}$$

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for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$*, where* r, s *are nonzero real numbers with* $r + s = 1$*. Thus* g *satisfies* (1.2)*. Let* $a, b, c \in \mathbb{R}$ *and define* $f(\mathbf{x}, \mathbf{y}) := ag(\mathbf{x}) + \frac{b}{4}[g(\mathbf{x}+\mathbf{y}) - g(\mathbf{x}-\mathbf{y})] + cg(\mathbf{y})$ *for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$*. By Theorem* 2.2*, the function* f *satisfies* (1.4)*. In fact,*

$$f(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}^T \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \begin{pmatrix} A\mathbf{x} \\ A\mathbf{y} \end{pmatrix}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$*.*

Example 3. *Let* M_n *be the algebra of* $n \times n$ *real matrices. Consider the mapping* $g : M_n \to M_n$ *given by* $g(A) := A^2$ *for all* $A \in M_n$*. Then we see that*

$$\begin{aligned} g(rA + sB) + rs\, g(A - B) &= (rA + sB)^2 + rs(A - B)^2 \\ &= r^2A^2 + rs(AB + BA) + s^2B^2 + rs(A^2 - AB - BA + B^2) \\ &= r^2A^2 + rsAB + rsBA + s^2B^2 + rsA^2 - rsAB - rsBA + rsB^2 \\ &= r(r + s)A^2 + s(r + s)B^2 = rg(A) + sg(B) \end{aligned}$$

for all $A, B \in \mathbb{R}^2$*, where* r, s *are nonzero real numbers with* $r + s = 1$*. Thus* g *satisfies* (1.2)*. Let* $a, b, c \in \mathbb{R}$ *and define*

$$f(A, B) := aA^2 + bA \circ B + cB^2,$$

where $A \circ B$ *the Jordan product* $\frac{1}{2}(AB + BA)$ *of* A *and* B *for all* $A, B \in M_n$*. Then the mapping* $f : M_n \times M_n \to M_n$ *satisfies* (2.2)*. By Theorem* 2.2*, the mapping* f *satisfies* (1.4)*.*

3. **Solution of the equation (1.4)**

We recall that r, s are nonzero real numbers with $r + s = 1$. In the following theorem, we find out the general solution of the functional equation (1.4).

Theorem 3.1. *Let* $f : X \times X \to Y$ *be a mapping such that* $f(x, y) = f(-x, -y)$ *for all* $x, y \in X$*. Then* f *satisfies* (1.3) *if it satisfies* (1.4)*. If* r *and* s *are rational numbers and* f *satisfies* (1.3)*, then it also satisfies* (1.4)*.*

Proof. Letting $x = y = z = w = 0$ in (1.4), we gain $f(0, 0) = 0$. Putting $y = w = 0$ in (1.4), we get $f(rx, rz) = r^2f(x, z)$ for all $x, z \in X$. Replacing x by $x + y$ and z by $z + w$ in (1.4), we have

$$f(rx + y, rz + w) = rf(x + y, z + w) + sf(y, w) - rsf(x, z) \tag{3.1}$$

for all $x, y, z, w \in X$. Replacing y by $-y$ and w by $-w$ in (3.1), we obtain

$$f(rx - y, rz - w) = rf(x - y, z - w) + sf(y, w) - rsf(x, z)$$

for all $x, y, z, w \in X$. Adding (3.1) to the above equation, we see that

$$f(rx+y, rz+w)+f(rx-y, rz-w) = r[f(x+y, z+w)+f(x-y, z-w)]+2sf(y, w)-2rsf(x, z) \tag{3.2}$$

for all $x, y, z, w \in X$. Replacing y by $x + ry$ and w by $z + rw$ in (3.1), we obtain

$$f(r(x + y) + x, r(z + w) + z) = rf(2x + ry, 2z + rw) + sf(x + ry, z + rw) - rsf(x, z) \tag{3.3}$$

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for all $x, y, z, w \in X$. Replacing x, y, z, w by $2x, ry, 2z, rw$ in (3.1), respectively, we obtain

$$rf(2x+ry, 2z+rw) = r^2 f(2x+y, 2z+w) - r^2 sf(y,w) + rsf(2x, 2z) \tag{3.4}$$

for all $x, y, z, w \in X$. Replacing y by ry and w by rw in (3.1), we obtain

$$sf(x+ry, z+rw) = rsf(x+y, z+w) - rs^2 f(y,w) + s^2 f(x,z) \tag{3.5}$$

for all $x, y, z, w \in X$. Replacing x, y, z, w by $x+y, x, z+w, z$ in (3.1), respectively, we obtain

$$f(r(x+y)+x, r(z+w)+z) = rf(2x+y, 2z+w) + sf(x,z) - rsf(x+y, z+w) \tag{3.6}$$

for all $x, y, z, w \in X$. By (3.3), (3.4), (3.5) and (3.6), we see that

$$f(2x+y, 2z+w) + 2f(x,z) + f(y,w) = 2f(x+y, z+w) + f(2x, 2z) \tag{3.7}$$

for all $x, y, z, w \in X$. Putting $y = -x$ and $w = -z$ in (3.7), we get $f(2x, 2z) = 4f(x,z)$ for all $x, z \in X$. Therefore, it follows from (3.7) that

$$f(2x+y, 2z+w) + f(y,w) = 2f(x+y, z+w) + 2f(x,z)$$

for all $x, y, z, w \in X$. Replacing y by $y-x$ and w by $w-z$ in the above equation, we have

$$f(x+y, z+w) + f(y-x, w-z) = 2f(x,z) + 2f(y,w)$$

for all $x, y, z, w \in X$. Hence f satisfies (1.3).

Conversely, let r and s be rational numbers and let f satisfy (1.3). Then there exist two symmetric bi-additive mappings $S_1, S_2 : X \times X \to Y$ and a bi-additive mapping $B : X \times X \to Y$ such that $f(x,y) = S_1(x,x) + B(x,y) + S_2(y,y)$ for all $x, y \in X$ (see [4]). Since r and s are rational numbers,

$$\begin{aligned}
&rf(x,z) + sf(y,w) - rsf(x-y, z-w) \\
&\quad = r^2 S_1(x,x) + 2rsS_1(x,y) + s^2 S_1(y,y) + r^2 B(x,z) + rsB(x,w) + rsB(y,z) + s^2 B(y,w) \\
&\qquad + r^2 S_2(z,z) + 2rsS_2(z,w) + s^2 S_2(w,w) \\
&\quad = S_1(rx,rx) + 2S_1(rx,sy) + S_1(sy,sy) + B(rx,rz) + B(rx,sw) + B(sy,rz) + B(sy,sw) \\
&\qquad + S_2(rz,rz) + 2S_2(rz,sw) + S_2(sw,sw) \\
&\quad = S_1(rx+sy, rx+sy) + B(rx+sy, rz+sw) + S_2(rz+sw, rz+sw) \\
&\quad = f(rx+sy, rz+sw)
\end{aligned}$$

for all $x, y, z, w \in X$. Therefore f satisfies (1.4). □

4. **Stability of the equation (1.4)**

From now on, let X be a real normed space and Y a Banach space.

The authors proved a generalized Hyers-Ulam stability theorem on a functional equation (1.3). The following theorem is a particular case of Theorem 4 in [4].

THEOREM 4.1 *Let $\delta \geq 0$ be fixed. If a mapping $f : X \times X \to Y$ satisfies the inequality*

$$\|f(x+y, z+w) + f(x-y, z-w) - 2f(x,z) - 2f(y,w)\| \leq \delta \tag{4.1}$$

for all $x, y, z, w \in X$, then there exists a unique 2-variable quadratic mapping $F : X \times X \to Y$ such that $\|f(x,y) - F(x,y)\| \leq \frac{1}{3}\delta$ for all $x, y \in X$.

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Using a similar method used in the paper [8], we obtain the following theorem.

THEOREM 4.2 *Let $d > 0$ and $\delta \geq 0$ be fixed and let $X \neq \{0\}$. If a mapping $f : X \times X \to Y$ satisfies the inequality (4.1) for all $x, y, z, w \in X$ with $\|x+z\|+\|y+w\| \geq d$, then there exists a unique 2-variable quadratic mapping $F : X \times X \to Y$ such that*

$$\|f(x,y) - F(x,y)\| \leq \frac{5}{3}\delta \tag{4.2}$$

for all $x, y \in X$.

Proof. Assume that $\|x+z\| + \|y+w\| < d$. Let

$$\begin{aligned} t &= \frac{1}{2}\Big(1 + \frac{d}{\|x+z\|}\Big)(x+z) \qquad \text{if} \qquad \|x+z\| \geq \|y+w\|; \\ t &= \frac{1}{2}\Big(1 + \frac{d}{\|y+w\|}\Big)(y+w) \qquad \text{if} \qquad \|x+z\| < \|y+w\|. \end{aligned}$$

If $x + z = y + w = 0$, then one can choose a $t \in X$ with $\|t\| = \frac{d}{2}$. Note that

$$\begin{aligned} 2\|t\| &= \|x+z\| + d \geq d \qquad \text{if} \qquad \|x+z\| \geq \|y+w\|; \\ 2\|t\| &= \|y+w\| + d > d \qquad \text{if} \qquad \|x+z\| < \|y+w\|. \end{aligned}$$

Clearly, we see that

$$\begin{aligned} &\|x+z-2t\| + \|y+w+2t\| \geq 4\|t\| - (\|x+z\| + \|y+w\|) \geq 2d - (\|x+z\| + \|y+w\|) \\ &\qquad\qquad\qquad\qquad\qquad \geq 2d > d, \\ &\|x+z-y-w\| + 4\|t\| \geq \|x+z-y-w\| + 2d \geq 2d > d, \\ &\|x+z+2t\| + \|-y-w+2t\| \geq \max\{\|x+z+2t\|, \|-y-w+2t\|\} \geq d, \\ &\|x+z\| + 2\|t\| \geq 2\|t\| \geq d, \qquad 2\|t\| + \|y+w\| \geq 2\|t\| \geq d, \qquad 4\|t\| \geq 2d > d. \end{aligned} \tag{4.3}$$

These inequalities (4.3) come from the corresponding substitutions attached between the right-hand sided parentheses of the following functional identity.

Besides from (4.1) with $x = y = z = w = 0$ we get $\|f(0,0)\| \leq \frac{\delta}{2}$. Therefore from (4.1), (4.3) and the new functional identity

$$\begin{aligned} &2[f(x+y,z+w) + f(x-y,z-w) - 2f(x,z) - 2f(y,w) - f(0,0)] \\ &\quad = [f(x+y,z+w) + f(x-y-2t,z-w-2t) - 2f(x-t,z-t) - 2f(y+t,w+t)] \\ &\qquad - [f(x-y-2t,z-w-2t) + f(x-y+2t,z-w+2t) - 2f(x-y,z-w) - 2f(2t,2t)] \\ &\qquad + [f(x-y+2t,z-w+2t) + f(x+y,z+w) - 2f(x+t,z+t) - 2f(-y+t,-w+t)] \\ &\qquad + 2[f(x+t,z+t) + f(x-t,z-t) - 2f(x,z) - 2f(t,t)] \\ &\qquad + 2[f(t+y,t+w) + f(t-y,t-w) - 2f(t,t) - 2f(y,w)] \\ &\qquad - 2[f(2t,2t) + f(0,0) - 4f(t,t)], \end{aligned}$$

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we get

$$\begin{aligned}2\|f(x+y,z+w)+f(x-y,z-w)-2f(x,z)-2f(y,w)-f(0,0)\|\\ \leq \delta+\delta+\delta+2\delta+2\delta+2\delta=9\delta,\end{aligned}$$

or

$$\|f(x+y,z+w)+f(x-y,z-w)-2f(x,z)-2f(y,w)\| \leq \frac{9}{2}\delta+\|f(0,0)\| \leq 5\delta. \tag{4.4}$$

Applying now Theorem 4.1 and the above inequality, there exists a unique 2-variable quadratic mapping $F : X \times X \to Y$ satisfying (4.2) such that $F(x,y) = \lim_{n\to\infty} 2^{-2n} f(2^n x, 2^n y)$, completing the proof. □

We recall that r, s are nonzero real numbers with $r + s = 1$.

THEOREM 4.3. *Let $d > 0$ and $\delta \geq 0$ be given. Assume that a mapping $f : X \times X \to Y$ such that $f(x,y) = f(-x,-y)$ and*

$$\|f(rx+sy,rz+sw)+rsf(x-y,z-w)-rf(x,z)-sf(y,w)\| \leq \delta \tag{4.5}$$

for all $x,y,z,w \in X$ with $\|x+z\|+\|y+w\| \geq d$. Then there exists $K > 0$ such that f satisfies

$$\|f(x+y,z+w)+f(x-y,z-w)-2f(x,z)-2f(y,w)\| \leq \frac{4(2+|r|+|s|)}{|rs|}\delta \tag{4.6}$$

for all $x,y,z,w \in X$ with $\|x+z\|+\|y+w\| \geq K$.

Proof. Let $x,y,z,w \in X$ with $\|x+z\|+\|y+w\| \geq 2d$. Since $2\|y+w\| = \|x+y+z+w+y+w-x-z\| \leq \|x+y+z+w\|+\|y+w\|+\|x+z\|$, we get

$$2\|y+w\|-\|x+z\| \leq \|x+y+z+w\|+\|y+w\|.$$

Since $\|x+z\| = \|x+y+z+w-y-w\| \leq \|x+y+z+w\|+\|y+w\|$, we have

$$\max\{\|x+z\|, 2\|y+w\|-\|x+z\|\} \leq \|x+y+z+w\|+\|y+w\|. \tag{4.7}$$

If $\|x+z\| < d$, then, since $\|x+z\|+\|y+w\| \geq 2d$, we get $2\|y+w\| > 2d = d+d > d+\|x+z\|$ and $2\|y+w\|-\|x+z\| > d$. So we have

$$\max\{\|x+z\|, 2\|y+w\|-\|x+z\|\} \geq d. \tag{4.8}$$

By (4.7) and (4.8), we have $\|x+y+z+w\|+\|y+w\| \geq d$. So it follows from (4.5) that

$$\|f(rx+y,rz+w)+rsf(x,z)-rf(x+y,z+w)-sf(y,w)\| \leq \delta \tag{4.9}$$

for all $x,y,z,w \in X$ with $\|x+z\|+\|y+w\| \geq 2d$. So

$$\|f(ry+x,rw+z)+rsf(y,w)-rf(x+y,z+w)-sf(x,z)\| \leq \delta \tag{4.10}$$

for all $x,y,z,w \in X$ with $\|x+z\|+\|y+w\| \geq 2d$.

Let $x,y,z,w \in X$ with $\|x+z\|+\|y+w\| \geq 4d\big(1/|r|+\big|1-1/|r|\big|\big)$. If $\|y+w\| > 2d/|r|$, then

$$\|x+z\|+\|x+ry+z+rw\| \geq |r|(\|y+w\|) \geq 2d. \tag{4.11}$$

If $\|y+w\| \le 2d/|r|$, then $\|x+z\| \ge 2d(1/|r| + 2|1-1/|r||)$ and

$$\|x+z\| + \|x+ry+z+rw\| \ge 2\|x+z\| - |r|\cdot\|y+w\| \ge \left(\frac{2}{|r|} + 4\left|1-\frac{1}{|r|}\right| - 1\right) \ge 2d. \tag{4.12}$$

Therefore we get that $\|x+z\| + \|x+ry+z+rw\| \ge 2d$ from (4.11) and (4.12). Hence by (4.9) we have

$$\|f(r(x+y)+x, r(z+w)+z) + rsf(x,z) - rf(2x+ry, 2z+rw) - sf(x+ry, z+rw)\| \le \delta \tag{4.13}$$

for all $x,y,z,w \in X$ with $\|x+z\|+\|y+w\| \ge 4d\big(1/|r| + \big|1-1/|r|\big|\big)$. Set $M := 4d\big(1/|r| + \big|1-1/|r|\big|\big)$. Then

$$\|x+y+z+w\| + \|x+z\| \ge \frac{M}{2} \ge 2d, \quad \|2x+2z\| + \|y+w\| \ge M \ge 4d \tag{4.14}$$

for all $x,y,z,w \in X$ with $\|x+z\| + \|y+w\| \ge M$. From (4.9) and (4.10), we get the following inequalities:

$$\begin{aligned}
&\|f(r(x+y)+x, r(z+w)+z) + rsf(x+y, z+w) - rf(2x+y, 2z+w) - sf(x,z)\| \le \delta,\\
&\|rf(ry+2x, rw+2z) + r^2sf(y,w) - r^2f(2x+y, 2z+w) - rsf(2x,2z)\| \le \delta|r|,\\
&\|sf(ry+x, rw+z) + rs^2f(y,w) - rsf(x+y, z+w) - s^2f(x,z)\| \le \delta|s|.
\end{aligned}$$

Using (4.13) and the above three inequalities, we get

$$\|f(2x+y, 2z+w) + 2f(x,z) + f(y,w) - 2f(x+y, z+w) - f(2x,2z)\| \le \frac{2+|r|+|s|}{|rs|}\delta \tag{4.15}$$

for all $x,y,z,w \in X$ with $\|x+z\| + \|y+w\| \ge M$. If $x,y,z,w \in X$ with $\|x+z\| + \|y+w\| \ge 2M$, then $\|x+z\| + \|y+w-x-z\| \ge M$. So it follows from (4.15) that

$$\|f(x+y, z+w) + 2f(x,z) + f(y-x, w-z) - 2f(y,w) - f(2x,2z)\| \le \frac{2+|r|+|s|}{|rs|}\delta \tag{4.16}$$

for all $x,y,z,w \in X$ with $\|x+z\| + \|y+w\| \ge 2M$.

Letting $y=0$ and $w=0$ in (4.16), we get

$$\|4f(x,z) - f(2x,2z) - 2f(0,0)\| \le \frac{2+|r|+|s|}{|rs|}\delta \tag{4.17}$$

for all $x,z \in X$ with $\|x+z\| \ge 2M$. Letting $x=0$ and $z=0$ (and $y,w \in X$ with $\|y\| \ge 2M$, $\|w\| \ge 2M$) in (4.16), we get

$$\|f(0,0)\| \le \big((2+|r|+|s|)/|rs|\big)\delta. \tag{4.18}$$

Therefore it follows from (4.16), (4.17) and (4.18) that

$$\begin{aligned}
\|f(x+y,&z+w) + f(y-x, w-z) - 2f(x,z) - 2f(y,w)\|\\
&\le \|f(x+y, z+w) + 2f(x,z) + f(y-x, w-z) - 2f(y,w) - f(2x,2z)\|\\
&\qquad + \|4f(x,z) - f(2x,2z) - 2f(0,0)\| + 2\|f(0,0)\|\\
&\le \frac{4(2+|r|+|s|)}{|rs|}\delta
\end{aligned}$$

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for all $x, y, z, w \in X$ with $\|x + z\| \geq 2M$. Since $f(x, y) = f(-x, -y)$ for all $x, y \in X$, the above inequality holds for all $x, y, z, w \in X$ with $\|y + w\| \geq 2M$. Therefore

$$\|f(x + y, z + w) + f(y - x, w - z) - 2f(x, z) - 2f(y, w)\| \leq \frac{4(2 + |r| + |s|)}{|rs|}\delta$$

for all $x, y, z, w \in X$ with $\|x + z\| + \|y + w\| \geq 4M$. This completes the proof by letting $K := 4M$. □

Theorem 4.4 *Let $d > 0$ and $\delta \geq 0$ be given. Assume that a mapping $f : X \times X \to Y$ such that $f(x, y) = f(-x, -y)$ and (4.5) for all $x, y, z, w \in X$ with $\|x + z\| + \|y + w\| \geq d$. Then there exists $K > 0$ such that f satisfies*

$$\|f(x + y, z + w) + f(x - y, z - w) - 2f(x, z) - 2f(y, w)\| \leq \frac{19(2 + |r| + |s|)}{|rs|}\delta$$

for all $x, y, z, w \in X$.

Proof. By Theorem 4.3, there exists $K > 0$ such that f satisfies (4.6) for all $x, y, z, w \in X$ with $\|x + z\| + \|y + w\| > K$. By (4.4) and (4.18), we get that

$$\begin{aligned}\|f(x + y, z + w) + f(x - y, z - w) - 2f(x, z) - 2f(y, w)\| &\leq \frac{18(2 + |r| + |s|)}{|rs|}\delta + \|f(0, 0)\| \\ &\leq \frac{19(2 + |r| + |s|)}{|rs|}\delta\end{aligned}$$

for all $x, y, z, w \in X$. □

Theorem 4.5 *Let $d > 0$ and $\delta \geq 0$ be given. Assume that a mapping $f : X \times X \to Y$ such that (4.5) and $f(x, y) = f(-x, -y)$ for all $x, y, z, w \in X$ with $\|x + z\| + \|y + w\| \geq d$. Then there exists a unique quadratic mapping $F : X \times X \to Y$ such that $F(x, y) = \lim_{n\to\infty} 4^{-n} f(2^n x, 2^n y)$ and*

$$\|f(x, y) - Q(x, y)\| \leq \frac{19(2 + |r| + |s|)}{3|rs|}\delta$$

for all $x, y \in X$.

Proof. The result follows from Theorem 4.1 and Theorem 4.4. □

Corollary 4.6. *Let r and s be rational numbers and a mapping $f : X \times X \to Y$ satisfy $f(x, y) = f(-x, -y)$ for all $x, y \in X$. Then f is quadratic if and only if the asymptotic condition*

(4.19)

$$\|f(rx + sy, rz + sw) + rsf(x - y, z - w) - rf(x, z) - sf(y, w)\| \to 0 \quad \text{as} \quad \|x + z\| + \|y + w\| \to \infty$$

holds true.

Proof. The asymptotic condition (4.19) is equivalent to the condition that there exists a sequence $\{\delta_n\}$ monotonically decreasing to 0 such that

$$\|f(rx + sy, rz + sw) + rsf(x - y, z - w) - rf(x, z) - sf(y, w)\| \leq \delta_n \tag{4.20}$$

for all $x, y, z, w \in X$ with $\|x + z\| + \|y + w\| \geq n$.

It follows from (4.20) and Theorem 4.4 that there exists a unique quadratic mapping $Q_n : X \times X \to Y$ such that

$$\|f(x,y) - Q_n(x,y)\| \le \frac{19(2+|r|+|s|)}{|rs|}\delta_n \tag{4.21}$$

for all $x, y \in X$. Since $\{\delta_n\}$ is monotonically decreasing, the quadratic mapping Q_m satisfies (4.21) for all $m \ge n$. The uniqueness of Q_n implies $Q_m = Q_n$ for all $m \ge n$. By letting $n \to \infty$ in (4.21), we conclude that f is quadratic.

The converse is trivial. □

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2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

Contribution to a Book

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FIXED POINTS IN TOPOLOGICAL VECTOR SPACE(tvs)VALUED CONE METRIC SPACES

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Abstract: We use the notion of topological vector space valued cone metric space and generalized a common fixed point theorem of a pair of mappings satisfying a generalized contractive type condition. Our results extend some well-known recent results in the literature.

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1 Introduction and Preliminaries

Many authors [1, 3, 4, 6, 17, 10, 11, 12, 13, 14, 15, 16, 17, 18, 21] studied fixed points results of mappings satisfying contractive type condition in Banach space valued cone metric spaces. The class of tvs-cone metric spaces is bigger than the class of cone metric spaces studied in [2, 7, 8, 19, 20]. Recently Azam et al. [5] obtain common fixed points of mappings satisfying a generalized contractive type condition in tvs-cone metric spaces. In this paper we continue these investigations to generalize the results in [1, 10].

Let (E, τ) be always a topological vector space (tvs) and P a subset of E. Then, P is called a cone whenever
(i) P is closed, non-empty and $P \neq \{0\}$,
(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b,
(iii) $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. $x < y$ will stand for $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of P.

Definition 1 *Let X be a non-empty set. Suppose the mapping $d : X \times X \to E$ satisfies*
(d$_1$) $0 \preceq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$,
(d$_2$) $d(x, y) = d(y, x)$ *for all* $x, y \in X$,
(d$_3$) $d(x, y) \preceq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.
Then d is called a topological vector space-valued cone metric on X and (X, d) is called a topological vector space-valued cone metric space.

If E is a real Banach space then (X, d) is called (Banach space valued) cone metric space [1, 6, 17, 10, 21]

Definition 2 *[7] Let (X, d) be a tvs-cone metric space, $x \in X$ and $\{x_n\}_{n \geq 1}$ a sequence in X. Then*
(i) *$\{x_n\}_{n \geq 1}$ converges to x whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$.*
(ii) *$\{x_n\}_{n \geq 1}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.*
(iii) *(X, d) is a complete cone metric space if every Cauchy sequence is convergent.*

Lemma 3 *[7] Let (X, d) be a tvs-cone metric space, P be a cone. Let $\{x_n\}$ be a sequence in X and $\{a_n\}$ be a sequence in P converging to $\mathbf{0}$. If $d(x_n, x_m) \preceq a_n$ for every $n \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence.*

The fixed point theorems and other results, in the case of cone metric spaces with non-normal solid cones, cannot be proved by reducing to metric spaces. Further, the vector valued function cone metric is not continuous in the general case.

Remark 4 *[7] Let A, B, C, D, E be non negative real numbers with $A + B + C + D + E < 1$, $B = C$ or $D = E$. If $\lambda = (A + B + D)(1 - C - D)^{-1}$ and $\mu = (A + C + E)(1 - B - E)^{-1}$, then $\lambda\mu < 1$.*

2 Common Fixed Points

The following theorem improves/generalizes the results in [1, 7].

Theorem 5 *Let (X, d) be a complete topological vector space-valued cone metric space, P be a cone and m, n be positive integers. If mappings $F, G : X \to X$ satisfies:*

$$d(Fx, Gy) \preceq A\, d(x, y) + B\, d(x, Fx) + C d(y, Gy) + D\, d(x, Gy) + E\, d(y, Fx) \quad (2.1)$$

for all $x, y \in X$, where A, B, C, D, E are non negative real numbers with $A + B + C + D + E < 1$, $B = C$ or $D = E$. Then F and G have a unique common fixed point.

Proof. For $x_0 \in X$ and $k \geq 0$, define

$$\begin{aligned} x_{2k+1} &= Fx_{2k} \\ x_{2k+2} &= Gx_{2k+1}. \end{aligned}$$

Then,

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) &= d(Fx_{2k}, Gx_{2k+1}) \\ &\preceq Ad(x_{2k}, x_{2k+1}) + Bd(x_{2k}, Fx_{2k}) + Cd(x_{2k+1}, Gx_{2k+1}) \\ &\quad +Dd(x_{2k}, Gx_{2k+1}) + Ed(x_{2k+1}, Fx_{2k}) \\ &\preceq [A+B]\ d(x_{2k}, x_{2k+1}) + Cd(x_{2k+1}, x_{2k+2}) + D\ d(x_{2k}, x_{2k+2}) \\ &\preceq [A+B+D]\ d(x_{2k}, x_{2k+1}) + [C+D]\ d(x_{2k+1}, x_{2k+2}). \end{aligned}$$

It implies that

$$[1-C-D]d(x_{2k+1}, x_{2k+2}) \preceq [A+B+D]\ d(x_{2k}, x_{2k+1}).$$

That is,

$$d(x_{2k+1}, x_{2k+2}) \preceq \lambda\, d(x_{2k},\ x_{2k+1}),$$

where $\lambda = \dfrac{A+B+D}{1-C-D}$. Similarly,

$$\begin{aligned} d(x_{2k+2}, x_{2k+3}) &= d(Fx_{2k+2}, Gx_{2k+1}) \\ &\preceq Ad(x_{2k+2}, x_{2k+1}) + B\ d(x_{2k+2}, Fx_{2k+2}) + Cd(x_{2k+1}, Gx_{2k+1}) \\ &\quad +Dd(x_{2k+2}, Gx_{2k+1}) + E\ d(x_{2k+1}, Fx_{2k+2}) \\ &\preceq A\ d(x_{2k+2}, x_{2k+1}) + B\ d(x_{2k+2},\ x_{2k+3}) + Cd(x_{2k+1},\ x_{2k+2}) \\ &\quad +D\ d(x_{2k+2},\ x_{2k+2}) + E\ d(x_{2k+1},\ x_{2k+3}) \\ &\preceq [A+C+E]\ d(x_{2k+1},\ x_{2k+2}) + [B+E]\ d(x_{2k+2}, x_{2k+3}), \end{aligned}$$

which implies

$$d(x_{2k+2}, x_{2k+3}) \preceq \mu\, d(x_{2k+1},\ x_{2k+2})$$

with $\mu = \dfrac{A+C+E}{1-B-E}$. Now by induction, we obtain for each $k = 0, 1, 2, \ldots$

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) &\preceq \lambda\, d(x_{2k}, x_{2k+1}) \\ &\preceq (\mu)\, d(x_{2k-1}, x_{2k}) \\ &\preceq \lambda(\lambda\mu)\, d(x_{2k-2},\ x_{2k-1}) \\ &\preceq \cdots \preceq \lambda(\lambda\mu)^k d(x_0,\ x_1) \end{aligned}$$

and

$$\begin{aligned} d(x_{2k+2}, x_{2k+3}) &\preceq \mu\, d(x_{2k+1},\ x_{2k+2}) \\ &\preceq \cdots \preceq (\lambda\mu)^{k+1}\, d(x_0,\ x_1). \end{aligned}$$

For $p < q$ and by Remark 1.4, we have

$$\begin{aligned} d(x_{2p+1},\ x_{2q+1}) &\preceq d(x_{2p+1},\ x_{2p+2}) + d(x_{2p+2},\ x_{2p+3}) + d(x_{2p+3},\ x_{2p+4}) \\ &\quad + \cdots + d(x_{2q},\ x_{2q+1}) \\ &\preceq \left[\lambda \sum_{i=p}^{q-1} (\lambda\mu)^i + \sum_{i=p+1}^{q} (\lambda\mu)^i\right] d(x_0, x_1) \\ &\preceq \left[\frac{\lambda(\lambda\mu)^p}{1-\lambda\mu} + \frac{(\lambda\mu)^{p+1}}{1-\lambda\mu}\right] d(x_0, x_1) \\ &\preceq (1+\lambda)\left[\frac{(\lambda\mu)^p}{1-\lambda\mu}\right] d(x_0, x_1). \end{aligned}$$

In analogous way, we deduce

$$d(x_{2p},\ x_{2q+1}) \preceq (1+\lambda)\left[\frac{(\lambda\mu)^p}{1-\lambda\mu}\right] d(x_0, x_1),$$

$$d(x_{2p},\ x_{2q}) \preceq (1+\lambda)\left[\frac{(\lambda\mu)^p}{1-\lambda\mu}\right] d(x_0, x_1)$$

and

$$d(x_{2p+1},\ x_{2q}) \preceq (1+\lambda)\left[\frac{(\lambda\mu)^p}{1-\lambda\mu}\right] d(x_0, x_1).$$

Hence, for $0 < n < m$

$$d(x_n, x_m) \preceq a_n$$

where $a_n = (1+\lambda)\left[\frac{(\lambda\mu)^p}{1-\lambda\mu}\right] d(x_0, x_1)$ with p the integer part of $n/2$. Fix $\mathbf{0} \ll c$ and choose a symmetric neighborhood V of $\mathbf{0}$ such that $c + V \subseteq \mathrm{int}P$. Since $a_n \to \mathbf{0}$ as $n \to \infty$, by Lemma 1.3, we deduce that $\{x_n\}$ is a Cauchy sequence. Since X is a complete, there exist $u \in X$ such that $x_n \to u$. Fix $\mathbf{0} \ll c$ and choose $n_0 \in \mathbb{N}$ be such that

$$d(u, x_{2n}) \ll \frac{c}{3K}, \quad d(x_{2n-1}, x_{2n}) \ll \frac{c}{3K}, \quad d(u, x_{2n-1}) \ll \frac{c}{3K}$$

for all $n \geq n_0$, where

$$K = \max\left\{\frac{1+D}{1-B-E}, \frac{A+E}{1-B-E}, \frac{C}{1-B-E}\right\}.$$

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Now,

$$\begin{aligned} d(u,Fu) &\preceq d(u,x_{2n})+d(x_{2n},Fu) \\ &\preceq d(u,x_{2n})+d(Gx_{2n-1},Fu) \\ &\preceq d(u,x_{2n})+A\ d(u,x_{2n-1})+B\ d(u,Fu)+Cd(x_{2n-1},Gx_{2n-1}) \\ &\quad +D\ d(u,Gx_{2n-1})+E\ d(x_{2n-1},Fu) \\ &\preceq d(u,x_{2n})+A\ d(u,x_{2n-1})+B\ d(u,Fu)+Cd(x_{2n-1},x_{2n}) \\ &\quad +D\ d(u,x_{2n})+E\ d(x_{2n-1},u)+E\ d(u,Fu)] \\ &\preceq (1+D)\ d(u,x_{2n})+(A+E)\ d(u,x_{2n-1})+Cd(x_{2n-1},x_{2n}) \\ &\quad +(B+E)\ d(u,Fu). \end{aligned}$$

So,

$$\begin{aligned} d(u,Fu) &\preceq K\,d(u,x_{2n})+K\,d(u,x_{2n-1})+K\,d(x_{2n-1},x_{2n}) \\ &\ll \frac{c}{3}+\frac{c}{3}+\frac{c}{3}=c \end{aligned}$$

Hence

$$d(u,Fu)\ll \frac{c}{p}$$

for every $p\in\mathbb{N}$. From

$$\frac{c}{p}-d(u,Fu)\in \mathrm{int}P,$$

being P closed, as $p\to\infty$, we deduce $-d(u,Fu)\in P$ and so $d(u,Fu)=\mathbf{0}$. This implies that $u=Fu$. Similarly, by using the inequality,

$$d(u,Gu)\preceq d(u,x_{2n+1})+d(x_{2n+1},Gu),$$

we can show that $u=Gu$, which in turn implies that u is a common fixed point of F,G and, that is

$$u=Fu=Gu.$$

For uniqueness, assume that there exists another point u^* in X such that

$$u^*=Tu^*=Gu^*$$

for some u^* in X. From

$$\begin{aligned} d(u,u^*) &= d(Fu,Gu^*) \\ &\preceq Ad(u,u^*)+Bd(u,Fu)+Cd(u^*,Gu^*) \\ &\quad +Dd(u,Gu^*)+Ed(u^*,Fu) \\ &\preceq Ad(u,u^*)+Bd(u,u)+Cd(u^*,u^*) \\ &\quad +D\ d(u,u^*)+Ed(u,u^*) \\ &\preceq (A+D+E)d(u,u^*), \end{aligned}$$

we obtain that $u^*=u$. ■

By substituting $D=E=0$ in the Theorem 2.1, we obtain the following result.

Corollary 6 *Let (X,d) be a complete topological vector space-valued cone metric space, P be a cone and m,n be positive integers. If mappings $F,G : X \to X$ satisfies:*

$$d(Fx,Gy) \preceq A\ d(x,y) + B\ d(x,Fx) + Cd(y,Gy) \tag{2.2}$$

for all $x,y \in X$, where A,B,C are non negative real numbers with $A+B+C < 1$. Then F and G have a unique common fixed point.

By substituting $B = C = 0$ in the Theorem 2.1, we obtain the following result.

Corollary 7 *Let (X,d) be a complete topological vector space-valued cone metric space, P be a cone and m,n be positive integers. If mappings $F,G : X \to X$ satisfies:*

$$d(Fx,Gy) \preceq A\ d(x,y) + D\ d(x,Gy) + E\ d(y,Fx) \tag{2.3}$$

for all $x,y \in X$, where A,D,E are non negative real numbers with $A+D+E < 1$. Then F and G have a unique common fixed point.

By substituting $F = T^m$, $G = T^n$ in the Theorem 2.1, we obtain the following result.

Corollary 8 *[7] Let (X,d) be a complete topological vector space-valued cone metric space, P be a cone and m,n be positive integers. If a mapping $T : X \to X$ satisfies:*

$$d(T^m x,T^n y) \preceq A\ d(x,y)+B\ d(x,T^m x)+Cd(y,T^n y)+D\ d(x,T^n y)+E\ d(y,T^m x) \tag{2.4}$$

for all $x,y \in X$, where A,B,C,D,E are non negative real numbers with $A + B + C + D + E < 1$, $B = C$ or $D = E$. Then T has a unique fixed point.

Corollary 9 *[1] Let (X,d) be a complete Banach space-valued cone metric space, P be a cone. If a mapping $F,G : X \to X$ satisfies:*

$$d(Fx,Gy) \preceq pd(x,y) + \ q\,[d(x,Fx) + d(y,Gy)] + \ r\,[d(x,Gy) + E\ d(y,Fx)] \tag{2.5}$$

for all $x,y \in X$, where p,q,r are non negative real numbers with $p+2q+2r < 1$. Then F and G have a unique common fixed point.

3 Multivalued Fixed point results in tvs-valued cone metric spaces

In the sequel, let $\mathbb{E}$ be a locally convex Hausdorff tvs with its zero vector θ, P be a proper, closed and convex pointed cone in $\mathbb{E}$ with $int\, P \neq \emptyset$ and $\preccurlyeq$ denotes the induced partial ordering with respect to P.

According to [5] let (X, d) be a tvs-valued cone metric space with a solid cone P and $CB(X)$ be a collection of nonempty closed and bounded subsets of X. Let $T : X \to CB(X)$ be a multi-valued mapping. For any $x \in X$, $A \in CB(X)$, define a set $W_x(A)$ as follows:

$$W_x(A) = \{d(x, a) : a \in A\}.$$

Thus, for any $x, y \in X$, we have

$$W_x(Ty) = \{d(x, u) : u \in Ty\}.$$

Definition 10 *[9] Let (X, d) be a cone metric space with the solid cone P. A multi-valued mapping $S : X \to 2^{\mathbb{E}}$ is said to be bounded from below if, for any $x \in X$, there exists $z(x) \in \mathbb{E}$ such that*

$$Sx - z(x) \subset P.$$

Definition 11 *[9] Let (X, d) be a cone metric space with the solid cone P. A cone P is said to be complete if, for any bounded from above and nonempty subset A of $\mathbb{E}$, $\sup A$ exists in $\mathbb{E}$. Equivalently, a cone P is complete if, for any bounded from below and nonempty subset A of $\mathbb{E}$, $\inf A$ exists in $\mathbb{E}$.*

Definition 12 *[5] Let (X, d) be a tvs-valued cone metric space with the solid cone P. A multi-valued mapping $T : X \to CB(X)$ is said to have the lower bound property (l.b. property) on X if, for any $x \in X$, the multi-valued mapping $S_x : X \to 2^{\mathbb{E}}$ defined by*

$$S_x(y) = W_x(Ty)$$

is bounded from below, that is, for any $x, y \in X$, there exists an element $\ell_x (Ty) \in \mathbb{E}$ such that

$$W_x(Ty) - \ell_x (Ty) \subset P.$$

$\ell_x (Ty)$ is called the lower bound of T associated with (x, y).

Definition 13 *[5] Let (X, d) be a tvs-valued cone metric space with the solid cone P. A multi-valued mapping $T : X \to CB(X)$ is said to have the greatest lower bound property (for short, g.l.b. property) on X if the greatest lower bound of $W_x(Ty)$ exists in $\mathbb{E}$ for all $x, y \in X$. We denote $d(x, Ty)$ by the greatest lower bound of $W_x(Ty)$, that is,*

$$d(x, Ty) = \inf\{d(x, u) : u \in Ty\}.$$

According to [20], we denote

$$s\,(p) = \{q \in \mathbb{E} : p \preccurlyeq q\}$$

for all $q \in \mathbb{E}$ and

$$s\,(a, B) = \underset{b \in B}{\cup} s\,(d\,(a, b)) = \underset{b \in B}{\cup} \{x \in \mathbb{E} : \; d\,(a, b) \preccurlyeq x\}$$

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for all $a \in X$ *and* $B \in CB(X)$. *For any* $A, B \in CB(X)$, *we denote*

$$s(A,B) = \Big(\bigcap_{a \in A} s(a,B) \Big) \cap \Big(\bigcap_{b \in B} s(b,A) \Big).$$

Remark 14 *[20] Let* (X,d) *be a tvs-valued cone metric space. If* $\mathbb{E} = R$ *and* $P = [0,+\infty)$, *then* (X,d) *is a metric space. Moreover, for any* $A, B \in CB(X)$, $H(A,B) = \inf\ s(A,B)$ *is the Hausdorff distance induced by* d.

Now we present the following theorem regarding the common fixed point of multivalued mapping with g.l.b property.

Theorem 15 *Let* (X,d) *be a complete tvs-valued cone metric space with the solid (normal or non-normal) cone* P *and let* $S, T : X \longrightarrow CB(X)$ *be multivalued mappings with g.l.b property such that*

$$A\ d(x,y) + B\ d(x,Sx) + Cd(y,Ty) + Dd(x,Ty) + Ed(y,Sx)) \in s(Sx,Ty) \quad (2.6)$$

ffor all $x, y \in X$, where A, B, C, D, E are non negative real numbers with $A + B + C + D + E < 1$. Then S and T have common fixed point.

Proof. Let x_0 be an arbitrary point in X and $x_1 \in Sx_0$. From (2.6), we have

$$Ad(x_0,x_1) + B(x_0,Sx_0) + Cd(x_1,Tx_1) + Dd(x_0,Tx_1) + Ed(x_1,Sx_0) \in s(Sx_0,Tx_1).$$

This implies that

$$Ad(x_0,x_1) + B(x_0,Sx_0) + Cd(x_1,Tx_1) + Dd(x_0,Tx_1) + Ed(x_1,Sx_0) \in \Big(\bigcap_{x \in Sx_0} s(x,Tx_1) \Big)$$

and

$$Ad(x_0,x_1) + B(x_0,Sx_0) + Cd(x_1,Tx_1) + Dd(x_0,Tx_1) + Ed(x_1,Sx_0) \in\ s(x,Tx_1)\ \ for\ all\ x \in Sx_0.$$

Since $x_1 \in Sx_0$, so we have

$$Ad(x_0,x_1) + B(x_0,Sx_0) + Cd(x_1,Tx_1) + Dd(x_0,Tx_1) + Ed(x_1,Sx_0) \in\ s(x_1,Tx_1)$$

and

$$Ad(x_0,x_1) + B(x_0,Sx_0) + Cd(x_1,Tx_1) + Dd(x_0,Tx_1) + Ed(x_1,Sx_0) \in\ s(x_1,Tx_1) = \bigcup_{x \in Tx_1} s(d(x_1,x)).$$

So there exists some $x_2 \in Tx_1$, such that

$$Ad(x_0,x_1) + B(x_0,Sx_0) + Cd(x_1,Tx_1) + Dd(x_0,Tx_1) + Ed(x_1,Sx_0) \in s(d(x_1,x_2)).$$

That is

$$d(x_1,x_2) \preceq Ad(x_0,x_1) + B(x_0,Sx_0) + Cd(x_1,Tx_1) + Dd(x_0,Tx_1) + Ed(x_1,Sx_0).$$

By using the greatest lower bound property (g.l.b property) of S and T, we get

$$d(x_1,x_2) \preceq Ad(x_0,x_1) + B(x_0,x_1) + Cd(x_1,x_2) + Dd(x_0,x_2) + Ed(x_1,x_1),$$

which implies that

$$d(x_1,x_2) \preceq (A+B+D)d(x_0,x_1) + (C+D)d(x_1,x_2)$$

which further implies that

$$d(x_1,x_2) \preceq \frac{A+B+D}{1-C-D}d(x_0,x_1).$$

Similarly from (2.6), we get

$$Ad(x_1,x_2)+B(x_2,Sx_2)+Cd(x_1,Tx_1)+Dd(x_2,Tx_1)+Ed(x_1,Sx_2) \in s(Tx_1,Sx_2).$$

This implies that

$$Ad(x_1,x_2)+B(x_2,Sx_2)+Cd(x_1,Tx_1)+Dd(x_2,Tx_1)+Ed(x_1,Sx_2) \in \left(\bigcap_{x\in Tx_1} s(x,Sx_2)\right)$$

and

$$Ad(x_1,x_2)+B(x_2,Sx_2)+Cd(x_1,Tx_1)+Dd(x_2,Tx_1)+Ed(x_1,Sx_2) \in s(x,Sx_2) \text{ for all } x \in Tx_1.$$

Since $x_2 \in Tx_1$, so we have

$$Ad(x_1,x_2)+B(x_2,Sx_2)+Cd(x_1,Tx_1)+Dd(x_2,Tx_1)+Ed(x_1,Sx_2) \in s(x_2,Sx_2)$$

and

$$Ad(x_1,x_2)+B(x_2,Sx_2)+Cd(x_1,Tx_1)+Dd(x_2,Tx_1)+Ed(x_1,Sx_2) \in s(x_2,Sx_2) = \bigcup_{x\in Sx_2} s(d(x_2,x)).$$

So there exists some $x_3 \in Sx_2$, such that

$$Ad(x_1,x_2)+B(x_2,Sx_2)+Cd(x_1,Tx_1)+Dd(x_2,Tx_1)+Ed(x_1,Sx_2) \in s(d(x_2,x_3)).$$

That is

$$d(x_2,x_3) \preceq Ad(x_1,x_2)+B(x_2,Sx_2)+Cd(x_1,Tx_1)+Dd(x_2,Tx_1)+Ed(x_1,Sx_2).$$

By using the greatest lower bound property (g.l.b property) of S and T, we get

$$d(x_2,x_3) \preceq Ad(x_1,x_2) + B(x_2,x_3) + Cd(x_1,x_2) + Dd(x_2,x_2) + Ed(x_1,x_3).$$

which implies that

$$d(x_2,x_3) \preceq (A+C+E)d(x_1,x_2) + (B+E)(x_2,x_3).$$

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This further implies

$$d(x_2, x_3) \preceq \frac{A+C+E}{1-B-E} d(x_1, x_2).$$

Let $\delta = \max\{\frac{A+B+D}{1-C-D}, \frac{A+C+E}{1-B-E}\}$. Then $\delta < 1$. Thus inductively, one can easily construct a sequence $\{x_n\}$ in X such that

$$x_{2n+1} \in Sx_{2n}, \quad x_{2n+2} \in Tx_{2n+1}$$

and

$$d(x_{2n}, x_{2n+1}) \preccurlyeq \delta d(x_{2n-1}, x_{2n}).$$

for each $n \geq 0$. We assume that $x_n \neq x_{n+1}$ for each $n \geq 0$. Otherwise, there exists n such that $x_{2n} = x_{2n+1}$. Then $x_{2n} \in Sx_{2n}$ and x_{2n} is a fixed point of S and hence a fixed point of T. Similarly, if $x_{2n+1} = x_{2n+2}$ for some n, then x_{2n+1} is a common fixed point of T and S. Similarly, one can show that

$$d(x_{2n+1}, x_{2n+2}) \preccurlyeq \delta d(x_{2n}, x_{2n+1}).$$

Thus we have

$$d(x_n, x_{n+1}) \preccurlyeq \delta d(x_{n-1}, x_n) \preccurlyeq \delta^2 d(x_{n-2}, x_{n-1}) \preccurlyeq \cdots \preccurlyeq \delta^n d(x_0, x_1)$$

for each $n \geq 0$. Now, for any $m > n$, consider

$$\begin{aligned} d(x_m,\ x_n) &\preccurlyeq d(x_n,\ x_{n+1}) + d(x_{n+1},\ x_{n+2}) + \cdots + d(x_{m-1},\ x_m) \\ &\preccurlyeq \left[\delta^n + \delta^{n+1} + \cdots + \delta^{m-1}\right] d(x_0, x_1) \\ &\preccurlyeq \left[\frac{\delta^n}{1-\delta}\right] d(x_0, x_1). \end{aligned}$$

Let $\theta \ll c$ be given and choose a symmetric neighborhood V of θ such that $c + V \subseteq \text{int}P$. Also, choose a natural number k_1 such that $\left[\frac{\delta^n}{1-\delta}\right] d(x_0, x_1) \in V$ for all $n \geq k_1$. Then $\frac{\delta^n}{1-\delta} d(x_1, x_0) \ll c$ for all $n \geq k_1$. Thus we have

$$d(x_m,\ x_n) \preccurlyeq \left[\frac{\delta^n}{1-\delta}\right] d(x_0, x_1) \ll c$$

for all $m > n$. Therefore, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $\nu \in X$ such that $x_n \to \nu$. Choose a natural number k_2 such that

$$\frac{1+E}{1-C} d(\nu, x_{2n+1}) \ll \frac{c}{3}, \quad \frac{A}{1-C} d(x_{2n}, v) \ll \frac{c}{3} \text{ and } \frac{B}{1-C} d(x_{2n}, x_{2n}) \ll \frac{c}{3} \tag{2.7}$$

for all $n \geq k_2$. Then, for all $n \geq k_2$, we have

$$Ad(x_{2n}, v) + Bd(x_{2n}, Sx_{2n}) + Cd(v, Tv) + Dd(x_{2n}, T\nu) + Ed(\nu, Sx_{2n}) \in s\left(Sx_{2n}, T\nu\right).$$

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This implies that

$$Ad(x_{2n},v)+Bd(x_{2n},Sx_{2n})+Cd(v,Tv)+Dd(x_{2n},T\nu)+Ed(\nu,Sx_{2n}) \in \left(\bigcap_{x\in Sx_{2n}} s\left(x,Tv\right)\right)$$

and we have

$$Ad(x_{2n},v)+Bd(x_{2n},Sx_{2n})+Cd(v,Tv)+Dd(x_{2n},T\nu)+Ed(\nu,Sx_{2n}) \in s\left(x,Tv\right) \ for\ all\ x \in Sx_{2n}.$$

Since $x_{2n+1} \in Sx_{2n}$, so we have

$$Ad(x_{2n},v)+Bd(x_{2n},Sx_{2n})+Cd(v,Tv)+Dd(x_{2n},T\nu)+Ed(\nu,Sx_{2n}) \in s\left(x_{2n+1},Tv\right).$$

By definition, we obtain

$$Ad(x_{2n},v)+Bd(x_{2n},Sx_{2n})+Cd(v,Tv)+Dd(x_{2n},T\nu)+Ed(\nu,Sx_{2n}) \in s\left(x_{2n+1},Tv\right) = \bigcup_{u'\in Tu} s\left(d\left(x_{2n+1},u'\right)\right).$$

There exists some $\nu_n \in Tv$ such that

$$Ad(x_{2n},v)+Bd(x_{2n},Sx_{2n})+Cd(v,Tv)+Dd(x_{2n},T\nu)+Ed(\nu,Sx_{2n}) \in s\left(x_{2n+1},Tv\right) \in s\left(d(x_{2n+1},\nu_n)\right),$$

that is

$$d(x_{2n+1},\nu_n) \preceq Ad(x_{2n},v)+Bd(x_{2n},Sx_{2n})+Cd(v,Tv)+Dd(x_{2n},T\nu)+Ed(\nu,Sx_{2n}).$$

By using the greatest lower bound property (g.l.b property) of S and T, we have

$$d(x_{2n+1},v_n) \preceq Ad(x_{2n},v)+Bd(x_{2n},x_{2n})+Cd(v,\nu_n)+Dd(x_{2n},\nu_n)+Ed(\nu,x_{2n+1}).$$

Now by using the triangular inequality, we get

$$d\left(x_{2n+1},\nu_n\right) \preceq Ad(x_{2n},v)+Bd(x_{2n},x_{2n+1})+Cd(v,x_{2n+1})+Dd(x_{2n},\nu_n)+Ed(\nu,x_{2n+1})$$

and it follows that

$$d\left(x_{2n+1},\nu_n\right) \preceq \frac{A}{1-C}d(x_{2n},v)+\frac{B}{1-C}d(x_{2n},x_{2n}))+\frac{C+E}{1-C}d(\nu,x_{2n+1}).$$

By using again triangular inequality, we get

$$\begin{aligned} d(\nu,\nu_n) &\preceq d(\nu,x_{2n+1})+d(x_{2n+1},\nu_n) \\ &\preceq d(\nu,x_{2n+1})+\frac{A}{1-C}d(x_{2n},v)+\frac{B}{1-C}d(x_{2n},x_{2n}))+\frac{C+E}{1-C}d(\nu,x_{2n+1}) \\ &\preceq \frac{1+E}{1-C}d(\nu,x_{2n+1})+\frac{A}{1-C}d(x_{2n},v)+\frac{B}{1-C}d(x_{2n},x_{2n}) \\ &\ll \frac{c}{3}+\frac{c}{3}+\frac{c}{3}=c \end{aligned}$$

Thus, we get

$$d(v,v_n) \ll \frac{c}{m}$$

for all $m \geq 1$ and so $\frac{c}{m} - d(v,v_n) \in P$ for all $m \geq 1$. Since $\frac{c}{m} \to \theta$ as $m \to \infty$ and P is closed, it follows that $-d(v,v_n) \in P$. But $d(v,v_n) \in P$. Therefore, $d(v,v_n) = \theta$ and $v_n \to v \in Tv$, since Tv is closed. This implies that v is a common point of S and T. This completes the proof. ■

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Corollary 16 *[5] Let (X, d) be a complete tvs-valued cone metric space with the solid (normal or non-normal) cone P and let $S, T : X \longrightarrow CB(X)$ be multivalued mappings with g.l.b property such that*

$$B\ d(x, Sx) + Cd(y, Ty) \in s\,(Sx, Ty)$$

ffor all $x, y \in X$, where B, C are non negative real numbers with $B+C<1$. Then S and T have common fixed point.

Theorem 17 *[5] Let (X, d) be a complete tvs-valued cone metric space with the solid (normal or non-normal) cone P and let $S, T : X \longrightarrow CB(X)$ be multivalued mappings with g.l.b property such that*

$$Dd(x, Ty) + Ed(y, Sx)) \in s\,(Sx, Ty)$$

ffor all $x, y \in X$, where D, E are non negative real numbers with $D+E<1$. Then S and T have common fixed point.

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ON THE TWISTED q-CHANGHEE POLYNOMIALS OF HIGHER ORDER

JIN-WOO PARK

Abstract. The q-Changhee polynomials and numbers are introduced by T. Kim et al in [3]. Some interesting properties of those polynomials are derived from umbral calculus (see [4]). In this paper, we consider Witt-type formula for the n-th twisted q-Changhee numbers and polynomials of higher order and derive some new interesting identities and properties of those polynomials and numbers from the Witt-type formula which are related to special polynomials and numbers.

1. Introduction

Let p be an odd prime number. $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized by $|p|_p = \frac{1}{p}$. Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the *fermionic p-adic integral on* $\mathbb{Z}_p$ is defined by T.Kim to be

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \text{ (see [6, 7, 9])}. \tag{1.1}$$

Let $f_1(x) = f(x+1)$. Then, by (1.1), we get

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0), \text{ (see [6, 7])}. \tag{1.2}$$

By (1.2), we easily see that

$$q^n I_{-q} + (-1)^{n-1} I_{-q} = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l), \tag{1.3}$$

where $f_n(x) = f(x+n)$ and $n \geq 0$.

It is well known that the *twisted q-Euler polynomials* are defined by the generating function to be

$$\frac{[2]_q}{1+q\varepsilon e^t} e^{xt} = \sum_{n=0}^{\infty} E_{n,\varepsilon,q}(x) \frac{t^n}{n!}, \text{ (see [13])}. \tag{1.4}$$

When $x = 0$, $E_{n,\varepsilon,q} = E_{n,\varepsilon,q}(0)$ are called the *$n-th$ twisted q-Euler numbers.* For $\varepsilon = 1$, $E_{n,1,q}(x) = E_{n,q}(x)$ are the *n-th q-Euler polynomials*, and $x = 0$, $E_{n,1,q}(0) = E_{n,q}(0)$ are the *n-th q-Euler numbers.*

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Indeed, we note that $E_{n,1,q}(x) = H_n(x|-q)$, where $H_n(x|\lambda)$ are the Frobenius-Euler polynomials which are defined by the generating function to be

$$\frac{1-\lambda}{e^t-\lambda}e^{tx} = \sum_{n=0}^{\infty} H_n(x|\lambda)\frac{t^n}{n!}, \text{ (see [1]).}$$

Recently, the q-Changhee polynomials are defined by the generating function to be

$$\frac{[2]_q}{1+q^{\varepsilon t}}(1+t)^x = \sum_{n=0}^{\infty} Ch_{n,q}(x)\frac{t^n}{n!}, \text{ (see [10]).} \tag{1.5}$$

When $x = 0$, $Ch_{n,\varepsilon,q} = Ch_{n,\varepsilon,q}(0)$ are called the *q-Changhee numbers*, (see [3]).
The Stirling number of the first kind is defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l, \text{(see [3]).} \tag{1.6}$$

The q-Changhee numbers and polynomials are introduced by T. Kim et. al. in [3], and found interesting identities in [5, 8, 11, 12]. In this paper, we consider the twisted q-Changhee numbers and polynomials of order k which are derived from the multivariate fermionic p-adic q-integral of higher order on $\mathbb{Z}_p$, and give some relationship between twisted q-Changhee polynomials and numbers of higher-order and special polynomials and numbers.

2. Twisted q-Changhee numbers and polynomials of higher-order

For $n \in \mathbb{N}$, let T_p be the p-adic locally constant space defined by

$$T_p = \bigcup_{n\geq 1} C_{p^n} = \lim_{n\to\infty} C_{p^n},$$

where $C_{p^n} = \left\{\omega|\omega^{p^n} = 1\right\}$ is the cyclic group of order p^n.

For $\varepsilon \in T_p$, let us take $f(x) = (1+\varepsilon t)^x$ for $|t|_p < p^{-\frac{1}{p-1}}$. Then by (1.2), we get

$$\int_{\mathbb{Z}_p} (1+\varepsilon t)^x d\mu_{-q}(x) = \frac{[2]_q}{q\varepsilon t + [2]_q} = \sum_{n=0}^{\infty} Ch_{n,\varepsilon,q}\frac{t^n}{n!} \tag{2.1}$$

where $Ch_{n,\varepsilon,q}$ are called the *n-th twisted q-Changhee numbers.*
From (2.1), we can derive the following equation:

$$\int_{\mathbb{Z}_p} (1+\varepsilon t)^{x+y} d\mu_{-q}(y) = \frac{[2]_q}{q\varepsilon t + [2]_q}(1+\varepsilon t)^x = \sum_{n=0}^{\infty} Ch_{n,\varepsilon,q}(x)\frac{t^n}{n!}, \tag{2.2}$$

where $Ch_{n,\varepsilon,q}(x)$ are called the *n-th twisted q-Changhee polynomials.* Note that $Ch_{n,\varepsilon,q}(0) = Ch_{n,\varepsilon,q}$ are n-th twisted q-Changhee numbers.
Since

$$\begin{aligned}\int_{\mathbb{Z}_p} (1+\varepsilon t)^{x+y} d\mu_{-q}(y) &= \sum_{n=0}^{\infty} \varepsilon^n \int_{\mathbb{Z}_p} \binom{x+y}{n} d\mu_{-q}(y)t^n \\ &= \sum_{n=0}^{\infty} \varepsilon^n \int_{\mathbb{Z}_p} (x+y)_n d\mu_{-q}(y)\frac{t^n}{n!},\end{aligned} \tag{2.3}$$

by (2.2) and (2.3), we obtained the following theorem.

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Theorem 2.1. *For $n \geq 0$, we have*

$$Ch_{n,\varepsilon,q}(x) = \varepsilon^n \int_{\mathbb{Z}_p} (x+y)_n d\mu_{-q}(y).$$

From (2.1), we note that

$$\sum_{n=0}^{\infty} \varepsilon^n \int_{\mathbb{Z}_p} \binom{x}{n} d\mu_{-q}(x) t^n = \frac{[2]_q}{q\varepsilon t + [2]_q} = \sum_{n=0}^{\infty} \left(-\frac{q\varepsilon}{[2]_q}\right)^n t^n. \tag{2.4}$$

Thus, by comparing the coefficients on the both sides, we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$\int_{\mathbb{Z}_p} \binom{x}{n} d\mu_{-q}(x) = \left(-\frac{q}{[2]_q}\right)^n.$$

Replacing t by $\frac{e^t-1}{\varepsilon}$ in (2.2), we get

$$\sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!} = \frac{[2]_q}{qe^t - 1} e^{xt} = \sum_{n=0}^{\infty} Ch_{n,\varepsilon,q}(x) \frac{1}{n!} \left(\frac{e^t-1}{\varepsilon}\right)^n, \tag{2.5}$$

where $E_{n,q}$ is the n-th q-Euler polynomials and

$$\begin{aligned} \sum_{n=0}^{\infty} Ch_{n,\varepsilon,q}(x) \frac{1}{n!} \left(\frac{e^t-1}{\varepsilon}\right)^n &= \sum_{n=0}^{\infty} Ch_{n,\varepsilon,q}(x) \frac{1}{n!} \varepsilon^{-n} n! \left(\sum_{m=n}^{\infty} S_2(m,n) \frac{t^m}{m!}\right) \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{m} Ch_{n,\varepsilon,q}(x) S_2(m,n) \varepsilon^{-n} \frac{t^m}{m!}, \end{aligned} \tag{2.6}$$

where $S_2(m,n)$ is the Striling number of the second kind.

By comparing the coefficients on the both sides of (2.5) and (2.6), we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, we have*

$$E_{n,q}(x) = \sum_{m=0}^{n} Ch_{m,\varepsilon,q}(x) S_2(n,m) \varepsilon^{-m}.$$

By Theorem 2.1, we easily get

$$\begin{aligned} Ch_{n,\varepsilon,q}(x) =& \varepsilon^n \int_{\mathbb{Z}_p} (x+y)_n d\mu_{-q}(y) \\ =& \varepsilon^n \sum_{l=0}^{n} S_1(n,l) \int_{\mathbb{Z}_p} (x+y)^l d\mu_{-q}(y) = \varepsilon^n \sum_{l=0}^{n} S_1(n,l) E_{l,q}(x). \end{aligned} \tag{2.7}$$

Therefore, by (2.7), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$Ch_{n,\varepsilon,q}(x) = \varepsilon^n \sum_{l=0}^{n} S_1(n,l) E_{l,q}(x).$$

where $S_1(n,l)$ is the Stirling number of the first kind.

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In viewpoint of (2.3), the *$n-th$ twisted q-Changhee numbers of the first kind with order k* are defined by the generating function to be

$$Ch_{n,\varepsilon}^{(k)} = \varepsilon^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k)_n d\mu_{-q}(x_1) \ldots d\mu_{-q}(x_k), \tag{2.8}$$

where n is a positive integer.

By (2.8), we easily get

$$\begin{aligned} \sum_{n=0}^{\infty} Ch_{n,\varepsilon,q}^{(k)} \frac{t^n}{n!} &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \binom{x_1 + \cdots + x_k}{n} (\varepsilon t)^n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + \varepsilon t)^n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k). \end{aligned} \tag{2.9}$$

From (2.1) and (2.9), we have

$$\sum_{n=0}^{\infty} Ch_{n,\varepsilon,q}^{(k)} \frac{t^n}{n!} = \left(\frac{[2]_q}{q\varepsilon t + [2]_q} \right)^k, \tag{2.10}$$

and

$$\left(\frac{[2]_q}{q\varepsilon t + [2]_q} \right)^k = \sum_{n=0}^{\infty} \left(\sum_{l_1 + \cdots + l_k = n} \binom{n}{l_1, \ldots, l_k} Ch_{l_1,\varepsilon,q} \cdots Ch_{l_k,\varepsilon,q} \right) \frac{t^n}{n!}. \tag{2.11}$$

By simple calculation. we easily see that

$$\left(\frac{[2]_q}{q\varepsilon t + [2]_q} \right)^k = \sum_{n=0}^{\infty} \left(-\frac{q}{[2]_q} \right)^n n! \varepsilon^n \binom{k+n-1}{n} \frac{t^n}{n!}. \tag{2.12}$$

Thus, by (2.10) and (2.12), we get

$$\begin{aligned} [2]_q^n Ch_{n,\varepsilon,q}^{(k)} =& (-q)^n n! \varepsilon^n \binom{n+k-1}{n} \\ =& (-q)^n \varepsilon^n (k+n-1)_n \\ =& (-q)^n \varepsilon^n \sum_{l=0}^{n} S_1(n,l)(k+n-1)^l. \end{aligned} \tag{2.13}$$

Therefore, by (2.10), (2.11) and (2.13), we obtain the following theorem.

Theorem 2.5. *For $n \geq 0$, we have*

$$\begin{aligned} [2]_q^n Ck_{n,\varepsilon,q}^{(k)} =& [2]_q^n \sum_{l_1 + \cdots + l_k = n} \binom{n}{l_1, \ldots, l_k} Ch_{i_1,\varepsilon,q} \cdots Ch_{l_k,\varepsilon,q} \\ =& (-q)^n \varepsilon^n \sum_{l=0}^{n} S_1(n,l)(k+n-1)^l. \end{aligned}$$

From (2.8), we have

$$\begin{aligned} Ch_{n,\varepsilon,q}^{(k)} =& \varepsilon^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k)_n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \\ =& \varepsilon^n \sum_{l=0}^{n} S_1(n,l) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k)^l d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k). \end{aligned} \tag{2.14}$$

Now, we observe that

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(x_1+\cdots+x_k)t}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)=\left(\frac{[2]_q}{qe^t+1}\right)^k=\sum_{n=0}^{\infty}E_{n,q}^{(k)}\frac{t^n}{n!}, \quad (2.15)$$

where $E_{n,q}^{(k)}$ are the q-Euler numbers of order k.

From (2.14) and (2.15), we obtain the following theorem.

Theorem 2.6. *For $n \geq 0$, we have*

$$Ch_{n,\varepsilon,q}^{(k)}=\varepsilon^n\sum_{l=0}^{n}S_1(n,l)E_{l,q}^{(k)}.$$

Replacing t by $\frac{e^t-1}{\varepsilon}$, we get

$$\sum_{n=0}^{\infty}Ch_{n,\varepsilon,q}^{(k)}\frac{1}{n!}\left(\frac{e^t-1}{\varepsilon}\right)^n=\left(\frac{[2]_q}{qe^t+1}\right)^k=\sum_{n=0}^{\infty}E_{n,q}^{(k)}\frac{t^n}{n!}, \quad (2.16)$$

and

$$\sum_{n=0}^{\infty}Ch_{n,\varepsilon,q}^{(k)}\frac{1}{n!}\left(\frac{e^t-1}{\varepsilon}\right)^n=\sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\varepsilon^{-n}Ch_{n,\varepsilon,q}^{(k)}S_2(m,n)\right)\frac{t^m}{m!}. \quad (2.17)$$

Thus, by (2.16) and (2.17), we obtain the following theorem.

Theorem 2.7. *For $n \geq 0$, we have*

$$E_{n,q}^{(k)}=\sum_{m=0}^{n}\varepsilon^{-m}Ch_{m,\varepsilon,q}^{(k)}S_2(n,m).$$

Now we define the *twisted q-Changhee polynomials of the first kind with order k* as follows:

$$Ch_{n,\varepsilon,q}^{(k)}(x)=\varepsilon^n\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_k+x)_n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k), \quad (2.18)$$

where $n \geq 0$ and $k \in \mathbb{N}$.

From (2.18), we can derive the generating function of the twisted q-Changhee polynomials as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}Ch_{n,\varepsilon,q}^{(k)}(x)\frac{t^n}{n!}&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\varepsilon t)^{x_1+\cdots+x_k+x}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)\\&=\left(\frac{[2]_q}{q\varepsilon t+[2]_q}\right)^k(1+\varepsilon t)^x.\end{aligned} \quad (2.19)$$

It is easy to show that

$$\left(\frac{[2]_q}{q\varepsilon t+[2]_q}\right)^k(1+\varepsilon t)^x=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\varepsilon^m\binom{n}{m}(x)_m Ch_{n-m,\varepsilon,q}^{(k)}\right)\frac{t^n}{n!}. \quad (2.20)$$

By (2.20), we get

$$\begin{aligned}Ch_{n,\varepsilon,q}^{(k)}(x)&=\sum_{m=0}^{n}\varepsilon^m\binom{x}{m}\frac{n!}{(n-m)!}Ch_{n-m,\varepsilon,q}^{(k)}\\&=\sum_{m=0}^{n}\varepsilon^{n-m}\binom{x}{n-m}\frac{n!}{m!}Ch_{m,\varepsilon,q}^{(k)}.\end{aligned} \quad (2.21)$$

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From (2.18), we have

$$\begin{aligned} Ch_{n,\varepsilon,q}^{(k)}(x) =& \varepsilon^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k + x)_n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \\ =& \varepsilon^n \sum_{l=0}^{n} S_1(n,l) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k + x)^l d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \\ =& \varepsilon^n \sum_{l=0}^{n} S_1(n,l) E_{l,q}^{(k)}(x). \end{aligned} \tag{2.22}$$

Hence, by (2.22), we obtain the following theorem.

Theorem 2.8. *For $n \geq 0$, we have*

$$Ch_{n,\varepsilon,q}^{(k)}(x) = \sum_{m=0}^{n} \varepsilon^m \binom{x}{n-m} \frac{n!}{m!} Ch_{m,\varepsilon,q}^{(k)} = \varepsilon^n \sum_{l=0}^{n} S_1(n,l) E_{l,q}^{(k)}(x).$$

where $E_{l,q}^{(k)}$ are the q-Euler polynomials of order k.

Now, we consider the *twisted q-Changhee polynomials of second kind with order* k as follows:

$$\widehat{Ch}_{n,\varepsilon,q}^{(k)}(x) = \varepsilon^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-x_1 - \cdots - x_k + x)_n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x)_k. \tag{2.23}$$

By (2.23), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \widehat{Ch}_{n,\varepsilon,q}^{(k)}(x) \frac{t^n}{n!} =& \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+\varepsilon t)^{-x_1 - \cdots - x_k + x} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \\ =& \left(\frac{[2]_q}{\varepsilon t + [2]_q} \right)^k (1+\varepsilon t)^{k+x}, \end{aligned} \tag{2.24}$$

where k is positive integer.

Hence,

$$\begin{aligned} & \widehat{Ch}_{n,\varepsilon,q}^{(k)}(x) \\ =& \varepsilon^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-x_1 - \cdots - x_k + x)_n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \\ =& \varepsilon^n \sum_{l=0}^{n} S_1(n,l)(-1)^l \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k - x)^l d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \\ =& \varepsilon^n \sum_{l=0}^{n} S_1(n,l)(-1)^l E_{l,q}^{(k)}(-x). \end{aligned} \tag{2.25}$$

Therefor, by (2.25), we obtain the following theorem.

Theorem 2.9. *For $n \geq 0$, we have*

$$\widehat{Ch}_{n,\varepsilon,q}^{(k)}(x) = \varepsilon^n \sum_{l=0}^{n} S_1(n,l)(-1)^l E_{l,q}^{(k)}(-x).$$

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Now, we consider the n-th twisted q-Changhee polynomials of the first kind relate to n-th twisted q-Changhee polynomials of second kind.

$$\begin{aligned}
&\frac{(-1)^n\widehat{Ch}^{(k)}_{n,\varepsilon,q}(x)}{n!}\\
=&(-1)^n\varepsilon^n\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{-x_1-\cdots-x_k+x}{n}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)\\
=&\varepsilon^n\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{x_1+\cdots+x_k-x+n-1}{n}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)\\
=&\varepsilon^n\sum_{m=0}^{\infty}\binom{n-1}{n-m}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{x_1+\cdots+x_k-x}{m}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)\\
=&\varepsilon^n\sum_{m=1}^{n}\binom{n-1}{m-1}\frac{\varepsilon^{-m}}{m!}m!\varepsilon^m\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{x_1+\cdots+x_k-x}{m}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)\\
=&\sum_{m=1}^{n}\binom{n-1}{m-1}\varepsilon^{n-m}\frac{Ch^{(k)}_{m,\varepsilon,q}(-x)}{m!}.
\end{aligned}\tag{2.26}$$

By (2.26) and proceeding similar to (2.26), we have the following theorem.

Theorem 2.10. *For $n\geq 0$, we have*

$$\frac{(-1)^n\widehat{Ch}^{(k)}_{n,\varepsilon,q}(x)}{n!}=\sum_{m=1}^{n}\binom{n-1}{m-1}\varepsilon^{n-m}\frac{Ch^{(k)}_{m,\varepsilon,q}(-x)}{m!},$$

and

$$\frac{(-1)^nCh^{(k)}_{n,\varepsilon,q}(x)}{n!}=\sum_{m=1}^{n}\binom{n-1}{m-1}\varepsilon^{n-m}\frac{\widehat{Ch}^{(k)}_{m,\varepsilon,q}(-x)}{m!},$$

By (2.25),

$$\begin{aligned}
&\widehat{Ch}_{n,\varepsilon,q}(x)\\
=&\varepsilon^n\sum_{l=0}^{n}S_1(n,l)(-1)^l\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_k-x)^l d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)\\
=&\varepsilon^n\sum_{l=0}^{n}S_1(n,l)\sum_{m=0}^{l}(-1)^{l+m}\binom{l}{m}E^{(k)}_{l-m}x^m,
\end{aligned}$$

and thus we obtain the following theorem.

Theorem 2.11. *For $n\geq 0$, we have*

$$\widehat{Ch}_{n,\varepsilon,q}(x)=\varepsilon^n\sum_{l=0}^{n}\sum_{m=0}^{l}(-1)^{l+m}\binom{l}{m}S_1(n,l)E^{(k)}_{l-m}x^m.$$

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SOME SYMMETRY IDENTITIES FOR THE (h, q)-BERNOULLI POLYNOMIALS UNDER THE THIRD DIHEDRAL GROUP D_3 ARISING FROM q-VOLKENBORN INTEGRAL ON $\mathbb{Z}_p$

S.-H. RIM, T. G. KIM, S. H. LEE

Abstract. In this paper, we give some new identities of symmetry for the (h, q)-Bernoulli polynomials arising from q-Volkenborn integral on $\mathbb{Z}_p$.

1. Introduction

let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$ and let q be an indeterminate in $\mathbb{C}_p$ with $|1 - q|_p < p^{-\frac{1}{p-1}}$. The q-extension of x is defined by $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to1}[x]_q = x$. Suppose that f is a uniformly differentiable function on $\mathbb{Z}_p$. Then the p-adic q-Vollenborn integral is defined by Kim to be

$$\begin{aligned} I_q(f) &= \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_q(x + p^N\mathbb{Z}_p) \\ &= \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x. \end{aligned} \tag{1}$$

As is well known, Carlitz's q-Bernoulli numbers are defined by

$$\beta_{0,q} = 1, \quad q(q\beta + 1)^n - \beta_{n,q} = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1, \end{cases}$$

with the usual convention about replacing β_q^n by $\beta_{n,q}$ (see [1,8,10]).

The q-Bernoulli polynomials are given by

$$\begin{aligned} \beta_{n,q}(x) &= \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} \beta_{l,q} \\ &= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \frac{l+1}{[l+1]_q}, \quad \text{(see [10])}. \end{aligned}$$

In 1999, Kim gave the formula which is given by

$$\beta_{n,q}(x) = \int_{\mathbb{Z}_p} [x + y]_q^n d\mu_q(x), \quad (n \in \mathbb{N} \cup \{0\},) \quad \text{(see [1-15])}.$$

1

For $h \in \mathbb{Z}$, we consider (h,q)-Bernoulli polynomials as follows:

$$\beta_{n,q}^{(h)}(x) = \int_{\mathbb{Z}_p} q^{(h-1)x}[x+y]_q^n d\mu_q(x), \quad (n \in \mathbb{Z}_{\geq 0}) \tag{2}$$
$$= \frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l} q^{lx}(-1)^l \frac{h+1}{[h+l]_q}, \quad \text{(see [8,10])}.$$

When $x=0$, $\beta_{n,q}^{(h)} = \beta_{n,q}^{(h)}(0)$ are called the (h,q)-Bernoulli numbers.

In this paper, we consider the symmetric identities for the (h,q)-Bernoulli polynomials under the third Dihedral group D_3 which are derive from p-adic q-Volkenborn integral on $\mathbb{Z}_p$.

2. Symmetric identities for the (h,q)-Bernoulli polynomials

Let $w_1,\ w_2,\ w_3$ be positive integers. Then we observe that

(3)
$$\int_{\mathbb{Z}_p} q^{(h-1)w_2w_3y} e^{[w_2w_3y+w_1w_2w_3x+w_1w_3i+w_1w_2j]_q t} d\mu_{q^{w_2w_3}}(y)$$
$$= \lim_{N\to\infty}\frac{1}{[p^N]_{q^{w_2w_3}}}\sum_{y=0}^{p^N-1} q^{(h-1)w_2w_3y} e^{[w_2w_3y+w_1w_2w_3x+w_1w_3i+w_1w_2j]_q t} q^{w_2w_3y}$$
$$= \lim_{N\to\infty}\frac{1}{[w_1p^N]_{q^{w_2w_3}}}\sum_{k=0}^{w_1-1}\sum_{y=0}^{p^N-1} q^{hw_2w_3(k+w_1y)} e^{[w_2w_3(k+w_1y)+w_1w_2w_3x+w_1w_3i+w_1w_2j]_q t}.$$

By (3), we get

$$\frac{1}{[w_2w_3]_q}\sum_{i=0}^{w_2-1}\sum_{j=0}^{w_3-1} q^{(w_1w_3i+w_1w_2j)h} \tag{4}$$
$$\times \int_{\mathbb{Z}_p} q^{(h-1)w_2w_3y} e^{[w_2w_3y+w_1w_2w_3x+w_1w_3i+w_1w_2j]_q t} d\mu_{q^{w_2w_3}}(y)$$
$$= \lim_{N\to\infty}\frac{1}{[w_1w_2w_3p^N]_q}\sum_{i=0}^{w_2-1}\sum_{j=0}^{w_3-1}\sum_{k=0}^{w_1-1} q^{h(w_1w_3i+w_1w_2j+w_2w_3k)+hw_1w_2w_3y}$$
$$\times e^{[w_2w_3(k+w_1y)+w_1w_2w_3x+w_1w_3i+w_1w_2j]_q t}.$$

From (4), we note that the expression is invariant under any permutation of $w_1,\ w_2,\ w_3$ in third Dihedral group D_3. Therefore, by (4), we obtain the following theorem.

Theorem 2.1. *Let $w_1,\ w_2,\ w_3$ be positive integers. Then, the following expressions*

$$\frac{1}{[w_{\sigma(2)}w_{\sigma(3)}]_q}\sum_{i=0}^{w_{\sigma(2)}-1}\sum_{j=0}^{w_{\sigma(3)}-1} q^{h(w_{\sigma(1)}w_{\sigma(3)}i+w_{\sigma(1)}w_{\sigma(2)}j)}$$
$$\times \int_{\mathbb{Z}_p} q^{(h-1)w_{\sigma(2)}w_{\sigma(3)}y} e^{[w_{\sigma(2)}w_{\sigma(3)}y+w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}x+w_{\sigma(1)}w_{\sigma(3)}i+w_{\sigma(1)}w_{\sigma(2)}j]_q t} d\mu_{q^{w_{\sigma(2)}w_{\sigma(3)}}}(y)$$

are the same for any $\sigma \in D_3$.

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Now, we note that

(5)
$$[w_2w_3y + w_1w_2w_3x + w_1w_3i + w_1w_2j]_q = [w_2w_3]_q\left[y + w_1x + \frac{w_1}{w_2}i + \frac{w_1}{w_3}j\right]_{q^{w_2w_3}}$$

Therefore, by (2), Theorem 1 and (5), we obtain the following theorem.

Theorem 2.2. *For $w_1,\ w_2,\ w_3 \in \mathbb{N}$, the following expressions*

$$[w_{\sigma(2)}w_{\sigma(3)}]_q^{n-1}\sum_{i=0}^{w_{\sigma(2)}-1}\sum_{j=0}^{w_{\sigma(3)}-1} q^{h(w_{\sigma(1)}w_{\sigma(3)}i+w_{\sigma(1)}w_{\sigma(2)}j)} \times \beta^{(h)}_{n,q^{w_{\sigma(2)}w_{\sigma(3)}}}\big(w_{\sigma(1)}x + \frac{w_{\sigma(1)}}{w_{\sigma(2)}}i + \frac{w_{\sigma(1)}}{w_{\sigma(3)}}j\big)$$

are the same for any $\sigma \in D_3$.

It is not difficult to show that

(6)
$$\begin{aligned}\left[y + w_1x + \frac{w_1}{w_2}i + \frac{w_1}{w_3}j\right]_{q^{w_2w_3}} &= \frac{1 - q^{w_1w_3i+w_1w_2j}}{1 - q^{w_2w_3}} + q^{w_1w_3i+w_1w_2j}[y + w_1x]_{q^{w_2w_3}}\\ &= \frac{[w_1]_q}{[w_2w_3]_q}[w_3i + w_2j]_{q^{w_1}} + q^{w_1w_3i+w_1w_2j}[y + w_1x]_{q^{w_2w_3}}\end{aligned}$$

From (6), we have

(7)
$$\begin{aligned}&\int_{\mathbb{Z}_p}\left[y + w_1x + \frac{w_1}{w_2}i + \frac{w_1}{w_3}j\right]^n_{q^{w_2w_3}} q^{(h-1)w_2w_3y}d\mu_{q^{w_2w_3}}(y)\\ &= \sum_{k=0}^{n}\binom{n}{k}\left(\frac{[w_1]_q}{[w_2w_3]_q}\right)^{n-k}[w_3i + w_2j]^{n-k}_{q^{w_1}}q^{k(w_1w_3i+w_1w_2j)}\beta^{(h)}_{k,q^{w_2w_3}}(w_1x).\end{aligned}$$

Thus, by Theorem 2 and (7), we get

(8)
$$\begin{aligned}&[w_2w_3]_q^{n-1}\sum_{i=0}^{w_2-1}\sum_{j=0}^{w_3-1} q^{h(w_1w_3i+w_1w_2j)}\int_{\mathbb{Z}_p} q^{(h-1)w_2w_3y}\left[y + w_1x + \frac{w_1}{w_2}i + \frac{w_1}{w_3}j\right]^n_{q^{w_2w_3}}d\mu_{q^{w_2w_3}}(y)\\ &= \sum_{k=0}^{n}\binom{n}{k}[w_2w_3]_q^{k-1}[w_1]_q^{n-k}\beta^{(h)}_{k,q^{w_2w_3}}(w_1x)\sum_{i=0}^{w_2-1}\sum_{j=0}^{w_3-1} q^{(k+h)(w_1w_3i+w_1w_2j)}[w_3i + w_2j]^{n-k}_{q^{w_1}}\\ &= \sum_{k=0}^{n}\binom{n}{k}[w_2w_3]_q^{k-1}[w_1]_q^{n-k}\beta^{(h)}_{k,q^{w_2w_3}}(w_1x)\ T^{(h)}_{n,q^{w_1}}(w_2, w_3|k),\end{aligned}$$

where

(9)
$$T^{(h)}_{n,q}(w_1, w_2|k) = \sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1} q^{(k+h)(w_2i+w_1j)}[w_2i + w_1j]^{n-k}_q.$$

As this expression is invariant under the third Dihedral group D_3, we have the following theorem.

Theorem 2.3. *For $n \geq 0$, $w_1,\ w_2,\ w_3 \in \mathbb{N}$, the following expressions*

$$\sum_{k=0}^{n}\binom{n}{k}[w_{\sigma(2)}w_{\sigma(3)}]_q^{k-1}[w_{\sigma(1)}]_q^{n-k}\beta_{k,q^{w_{\sigma(2)}w_{\sigma(3)}}}^{(h)}(w_{\sigma(1)}x)\ T_{n,q^{w_{\sigma(1)}}}^{(h)}(w_{\sigma(2)},w_{\sigma(3)}|k)$$

are all the same for any $\sigma \in D_3$.

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SOME IDENTITIES OF BELL POLYNOMIALS ASSOCIATED WITH p-ADIC INTEGRAL ON $\mathbb{Z}_p$

SEOG-HOON RIM, HONG KYUNG PAK, J.K. KWON, AND TAE GYUN KIM

ABSTRACT. In this paper, we investigate some identities of Bell polynomials associated with special polynomials which are derived from p-adic integral on $\mathbb{Z}_p$.

1. INTRODUCTION

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$. Let q be an indeterminate in $\mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$ and let the q-extension of number x is defined as $[x]_q = \frac{1-q^x}{1-q}$. The Euler polynomials of order r are defined by the generating function to be

$$\left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!}, \quad \text{(see [1 − 18])}$$

and the higher-order Bernoulli polynomials of order r are given by

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!}, \quad \text{(see [9 − 10])}.$$

When $x = 0$, $B_n^{(r)} = B_n^{(r)}(0)$, $E_n^{(r)} = E_n^{(r)}(0)$ are called higher-order Bernoulli numbers and Euler numbers.

Let $f(x)$ be a uniformly continuous function on $\mathbb{Z}_p$. Then the bosonic p-adic integral on $\mathbb{Z}_p$ is defined by

$$\int_{\mathbb{Z}_p} f(x)d\mu_0(x) = \lim_{N\to\infty}\frac{1}{p^N}\sum_{x=0}^{p^N-1} f(x), \quad \text{(see [12])}, \tag{1}$$

and the fermieuic p-adic integral on $\mathbb{Z}_p$ is given by

$$\int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{N\to\infty}\frac{1}{p^N}\sum_{x=-1}^{p^N-1} f(x)(-1)^x, \quad \text{(see [12])}. \tag{2}$$

Thus, we have

$$\int_{\mathbb{Z}_p} f(x+1)d\mu_0(x) - \int_{\mathbb{Z}_p} f(x)d\mu_0(x) = f'(0), \tag{3}$$

and

$$\int_{\mathbb{Z}_p} f(x+1)d\mu_{-1}(x) + \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = 2f(0). \tag{4}$$

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As is well know, the higher-order Changhee polynomials are given by

$$\left(\frac{2}{t+2}\right)^r (1+t)^x = \sum_{n=0}^{\infty} Ch_n^{(r)}(x)\frac{t^n}{n!}, \quad (\text{see } [11-15]), \tag{5}$$

and the higher-order Daehee polynomials are defined by the generating function to be

$$\left(\frac{\log(1+t)}{t}\right)^r (1+t)^x = \sum_{n=0}^{\infty} D_n^{(r)}(x)\frac{t^n}{n!}, \quad (\text{see } [11-15]). \tag{6}$$

When $x = 0$, $Ch_n^{(r)} = Ch_n^{(r)}(0)$ and $D_n^{(r)} = D_n^{(r)}(0)$ are called the Changhee numbers and the Daehee numbers with order r.

Finally, we introduce the Bell polynomials which are given by the generating function to be

$$e^{(e^t-1)x} = \sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!}, \quad (\text{see } [4,14,16]). \tag{7}$$

The purpose of this paper is to given some identities of Bell polynomials associated with special polynomials arising from p-adic integral on $\mathbb{Z}_p$.

2. Some identities of Bell polynomials

From (2), we note that

$$\begin{aligned}
&\int_{\mathbb{Z}_p} e^{(e^t-1)(x+y)} d\mu_0(y) \\
&= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (x+y)^n d\mu_0(y) \frac{(e^t-1)^n}{n!} \\
&= \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} B_k(x) S_2(n,k) \Big) \frac{t^n}{n!},
\end{aligned} \tag{8}$$

where $S_2(n,k)$ is the Stirling number of the second kind. On the other hand,

$$\int_{\mathbb{Z}_p} e^{(e^t-1)(x+y)} d\mu_0(y) = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} Bel_n(x+y) d\mu_0(y) \frac{t^n}{n!}. \tag{9}$$

Thus, by (8) and (9), we get

$$\int_{\mathbb{Z}_p} Bel_n(x+y) d\mu_0(y) = \sum_{k=0}^{n} B_k(x) S_2(n,k). \tag{10}$$

By the same method as (10), we get

$$\int_{\mathbb{Z}_p} Bel_n(x+y) d\mu_{-1}(y) = \sum_{k=0}^{n} E_k(x) S_2(n,k). \tag{11}$$

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Note that

$$\begin{aligned}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{(x_1+\cdots+x_r+x)}d\mu_0(x_1)\cdots d\mu_0(x_r)&=\Big(\frac{\log(1+t)}{t}\Big)^r(1+t)^x\\&=\sum_{n=0}^{\infty}D_n^{(r)}(x)\frac{t^n}{n!}.\end{aligned}\tag{12}$$

By replacing t by $e^{e^t-1}-1$, we get

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}e^{(e^t-1)(x_1+\cdots+x_r+x)}d\mu_0(x_1)\cdots d\mu_0(x_r)\\&=\Big(\frac{e^t-1}{e^{e^t-1}-1}\Big)^r e^{(e^t-1)x}=\Big(\sum_{l=0}^{\infty}B_l^{(r)}\frac{(e^t-1)^l}{l!}\Big)\Big(\sum_{m=0}^{\infty}Bel_m(x)\frac{t^m}{m!}\Big)\\&=\Big(\sum_{l=0}^{\infty}B_l^{(r)}\sum_{k=l}^{\infty}S_2(k,l)\frac{t^k}{k!}\Big)\Big(\sum_{m=0}^{\infty}Bel_m(x)\frac{t^m}{m!}\Big)\\&=\Big(\sum_{k=0}^{\infty}\Big(\sum_{l=0}^{k}B_l^{(r)}S_2(k,l)\Big)\frac{t^k}{k!}\Big)\Big(\sum_{m=0}^{\infty}Bel_m(x)\frac{t^m}{m!}\Big)\\&=\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}\frac{Bel_m(x)n!}{m!(n-m)!}\sum_{l=0}^{n-m}B_l^{(r)}S_2(n-m,l)\Big)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}\binom{n}{m}Bel_m(x)\sum_{l=0}^{n-m}B_l^{(r)}S_2(n-m,l)\Big)\frac{t^n}{n!}.\end{aligned}\tag{13}$$

On the other hand,

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}e^{(e^t-1)(x_1+\cdots+x_r+x)}d\mu_0(x_1)\cdots d\mu_0(x_r)\\&=\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}Bel_n(x_1+\cdots+x_r+x)d\mu_0(x_1)\cdots d\mu_0(x_r)\frac{t^n}{n!}.\end{aligned}\tag{14}$$

Therefore, we obtain the following theorem.

Theorem 1. *For $n\geq 0$, we have*

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}Bel_n(x_1+\cdots+x_r+x)d\mu_0(x_1)\cdots d\mu_0(x_r)\\&=\sum_{m=0}^{n}\binom{n}{m}Bel_m(x)\sum_{l=0}^{n-m}B_l^{(r)}S_2(n-m,l).\end{aligned}$$

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From (12), we note that

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(e^t-1)(x_1+\cdots+x_r+x)}d\mu_0(x_1)\cdots d\mu_0(x_r)\\
&=\sum_{n=0}^{\infty} D_n^{(r)}(x)\frac{1}{n!}\Big(e^{(e^t-1)}-1\Big)^n=\sum_{k=0}^{\infty} D_k^{(r)}(x)\sum_{m=k}^{\infty} S_2(m,k)\frac{(e^t-1)^m}{m!}\\
&=\sum_{m=0}^{\infty}\sum_{k=0}^{m} D_k^{(r)}(x)S_2(m,k)\frac{1}{m!}(e^t-1)^m \qquad (15)\\
&=\sum_{m=0}^{\infty}\sum_{k=0}^{m} D_k^{(r)}(x)S_2(m,k)\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\Big\{\sum_{m=0}^{n}\sum_{k=0}^{m} D_k^{(r)}(x)S_2(m,k)S_2(n,m)\Big\}\frac{t^n}{n!}.
\end{aligned}$$

Therefore, by Theorem 1 and (15), we obtain the following theorem.

Theorem 2. *For $n \geq 0$, we have*

$$\begin{aligned}
&\sum_{m=0}^{n}\binom{n}{m}Bel_m(x)\sum_{l=0}^{n-m} B_l^{(r)} S_2(n-m,l)\\
&=\sum_{m=0}^{n}\sum_{k=0}^{m} D_k^{(r)}(x)S_2(m,k)S_2(n,m).
\end{aligned}$$

From (7), we note that

$$\begin{aligned}
e^{xt}&=\sum_{m=0}^{\infty} Bel_m(x)\frac{1}{m!}\Big(\log(1+t)\Big)^m\\
&=\sum_{m=0}^{\infty} Bel_m(x)\sum_{n=m}^{\infty} S_1(n,m)\frac{t^m}{m!} \qquad (16)\\
&=\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n} Bel_m(x)S_1(n,m)\Big)\frac{t^n}{n!},
\end{aligned}$$

where $S_1(n,m)$ is the Stirling number of the first kind.

Therefore, by (16), we obtain the following theorem.

Theorem 3. *For $n \geq 0$, we have*

$$x^n=\sum_{m=0}^{n} Bel_m(x)S_1(n,m).$$

It is easy to show that

$$\int_{\mathbb{Z}_p} e^{xt}d\mu_0(x)=\frac{t}{e^t-1}=\sum_{n=0}^{\infty} B_n\frac{t^n}{n!}. \qquad (17)$$

Thus, by (17), we have

$$\int_{\mathbb{Z}_p} x^n d\mu_0(x)=B_n, \quad (n\geq 0).$$

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From Theorem 3, we can derive the following equation:

$$B_n = \int_{\mathbb{Z}_p} x^n d\mu_0(x) = \sum_{m=0}^{n} S_1(n,m) \int_{\mathbb{Z}_p} Bel_m(x) d\mu_0(x), \quad (n \geq 0). \tag{18}$$

Therefore, by (10) and (18), we obtain the following theorem.

Theorem 4. *For $n \geq 0$, we have*

$$B_n = \sum_{m=0}^{n} \sum_{k=0}^{m} S_1(n,m) S_2(m,k) B_k.$$

It is not difficult to show that

$$\int_{\mathbb{Z}_p} e^{xt} d\mu_{-1}(x) = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}. \tag{19}$$

Thus, by (19), we get

$$\int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = E_n, \quad (n \geq 0). \tag{20}$$

From Theorem 3 and (20), we have

$$E_n = \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = \sum_{m=0}^{n} S_1(n,m) \int_{\mathbb{Z}_p} Bel_m(x) d\mu_{-1}(x). \tag{21}$$

Therefore, by (11) and (21), we obtain the following theorem.

Theorem 5. *For $n \geq 0$, we have*

$$E_n = \sum_{m=0}^{n} \sum_{k=0}^{m} S_1(n,m) S_2(m,k) E_k.$$

Now, we consider the following equation.

$$\begin{aligned} e^{(x+x_1+\cdots+x_r)t} &= \sum_{m=0}^{\infty} Bel_m(x_1 + \cdots + x_r + x) \frac{(\log(1+t))^m}{m!} \\ &= \sum_{m=0}^{\infty} Bel_m(x_1 + \cdots + x_r + x) \sum_{n=m}^{\infty} S_1(n,m) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \Big(\sum_{m=0}^{n} Bel_m(x_1 + \cdots + x_r + x) S_1(n,m) \Big) \frac{t^n}{n!}. \end{aligned} \tag{22}$$

Thus, by (22), we have the following theorem.

Theorem 6. *For $n \geq 0$, we have*

$$(x + x_1 + \cdots + x_r)^n = \sum_{m=0}^{n} Bel_m(x_1 + \cdots + x_r + x) S_1(n,m).$$

From (4), we can easily derive the following equation:

$$\begin{aligned}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(x_1+\cdots+x_r+x)t}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) &= \Big(\frac{2}{e^t+1}\Big)^r e^{xt}\\ &= \sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!}.\end{aligned} \tag{23}$$

Thus, by (23), we get

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r+x)^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) = E_n^{(r)}(x). \tag{24}$$

By (3), we easily get

$$\begin{aligned}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(x_1+\cdots+x_r+x)t}d\mu_0(x_1)\cdots d\mu_0(x_r) &= \Big(\frac{t}{e^t-1}\Big)^r e^{xt}\\ &= \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!}.\end{aligned} \tag{25}$$

From (25), we have

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r+x)^n d\mu_0(x_1)\cdots d\mu_0(x_r) = B_n^{(r)}(x). \tag{26}$$

From Theorem 6, (24) and (26), we have

$$B_n^{(r)}(x) = \sum_{m=0}^{n} S_1(n,m)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} Bel_m(x+x_1+\cdots+x_r)d\mu_0(x_1)\cdots d\mu_0(x_r) \tag{27}$$

and

$$E_n^{(r)}(x) = \sum_{m=0}^{n} S_1(n,m)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} Bel_m(x+x_1+\cdots+x_r)d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r). \tag{28}$$

Now, we observe that

$$\begin{aligned}&\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} Bel_n(x+x_1+\cdots+x_r)d\mu_0(x_1)\cdots d\mu_0(x_r)\frac{t^n}{n!}\\ &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(e^t-1)(x_1+\cdots+x_r+x)t}d\mu_0(x_1)\cdots d\mu_0(x_r)\\ &= \sum_{m=0}^{\infty} B_m^{(r)}(x)\frac{1}{m!}\Big(e^t-1\Big)^m\\ &= \sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n} B_m^{(r)}(x)S_2(n,m)\Big)\frac{t^n}{n!}.\end{aligned} \tag{29}$$

Thus, by (29), we get

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} Bel_n(x_1+\cdots+x_r+x)d\mu_0(x_1)\cdots d\mu_0(x_r) = \sum_{m=0}^{n} B_m^{(r)}(x)S_2(n,m). \tag{30}$$

Therefore, by (27) and (30), we obtain the following theorem.

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Theorem 7. *For $n \geq 0$, we have*

$$B_n^{(r)}(x) = \sum_{m=0}^{n} \sum_{k=0}^{m} S_1(n,m) S_2(m,k) B_k^{(r)}(x).$$

By the same method of (29), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} Bel_n(x + x_1 + \cdots + x_r) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \frac{t^n}{n!} \\
&= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(e^t-1)(x_1+\cdots+x_r+x)t} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
&= \sum_{n=0}^{\infty} \Big(\sum_{m=0}^{n} E_m^{(r)}(x) S_2(n,m) \Big) \frac{t^n}{n!}.
\end{aligned} \tag{31}$$

From (31), we have

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} Bel_n(x_1 + \cdots + x_r + x) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = \sum_{m=0}^{n} E_m^{(r)}(x) S_2(n,m). \tag{32}$$

Therefore, by Theorem 6 and (32), we obtain the following theorem.

Theorem 8. *For $n \geq 0$, we have*

$$E_n^{(r)}(x) = \sum_{m=0}^{n} \sum_{k=0}^{m} S_1(n,m) S_2(m,k) E_k^{(r)}(x).$$

From (4), we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{(x_1+\cdots+x_r+x)} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
&= \Big(\frac{2}{1+t} \Big)^r (1+t)^x = \sum_{n=0}^{\infty} Ch_n^{(r)}(x) \frac{t^n}{n!}.
\end{aligned} \tag{33}$$

By replacing t by $e^{(e^t-1)} - 1$, we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(e^t-1)(x_1+\cdots+x_r+x)} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
&= \sum_{m=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r + x) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \frac{1}{m!} (e^t - 1)^m \\
&= \sum_{m=0}^{\infty} E_m^{(r)}(x) \frac{1}{m!} (e^t - 1)^m \\
&= \sum_{m=0}^{\infty} E_m^{(r)}(x) \sum_{n=m}^{\infty} S_2(n,m) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \Big(\sum_{m=0}^{n} E_m^{(r)}(x) S_2(n,m) \Big) \frac{t^n}{n!},
\end{aligned} \tag{34}$$

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and

$$\begin{aligned} 2^r e^{-(e^t-1)r} e^{(e^t-1)x} &= 2^r \Big(\sum_{l=0}^{\infty} Bel_l(-r)\frac{t^l}{l!}\Big)\Big(\sum_{m=0}^{\infty} Bel_m(x)\frac{t^m}{m!}\Big) \\ &= 2^r \sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n} \frac{Bel_m(x)Bel_{n-m}(-r)n!}{m!(n-m)!}\Big)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\Big(2^r \sum_{m=0}^{n} \binom{n}{m} Bel_m(x)Bel_{n-m}(-r)\Big)\frac{t^n}{n!}. \end{aligned} \tag{35}$$

Therefore, by (33),(34) and (35), we obtain the following theorem.

Theorem 9. *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} E_m^{(r)}(x)S_2(n,m) = 2^r \sum_{m=0}^{n} \binom{n}{m} Bel_m(x)Bel_{n-m}(-r).$$

Now, we observe that

$$\begin{aligned} &\sum_{m=0}^{\infty} Ch_m^{(r)}(x)\frac{1}{m!}\Big(e^{(e^t-1)}-1\Big)^m = \sum_{m=0}^{\infty} Ch_m^{(r)}(x)\sum_{k=m}^{\infty} S_2(k,m)\frac{(e^t-1)^k}{k!} \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{k} Ch_m^{(r)}(x)S_2(k,m)\frac{1}{k!}(e^t-1)^k \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{k} Ch_m^{(r)}(x)S_2(k,m)\sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\sum_{m=0}^{k} Ch_m^{(r)}(x)S_2(k,m)S_2(n,k)\Big)\frac{t^n}{n!}. \end{aligned} \tag{36}$$

Therefore, by (33), (34) and (36), we obtain the following theorem.

Theorem 10. *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} E_m^{(r)}(x)S_2(n,m) = \sum_{k=0}^{n}\sum_{m=0}^{k} Ch_m^{(r)}(x)S_2(k,m), S_2(n,k).$$

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From (4), we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(e^t-1)(x_1+\cdots+x_r+x)}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\
&=\Big(\frac{2}{e^{e^t-1}+1}\Big)^r e^{(e^t-1)x}\\
&=\Big(\sum_{m=0}^{\infty}E_m^{(r)}\frac{(e^t-1)^m}{m!}\Big)\Big(\sum_{l=0}^{\infty}Bel_l(x)\frac{t^l}{l!}\Big)\\
&=\Big(\sum_{m=0}^{\infty}E_m^{(r)}\sum_{k=m}^{\infty}S_2(k,m)\frac{t^k}{k!}\Big)\Big(\sum_{l=0}^{\infty}Bel_l(x)\frac{t^l}{l!}\Big)\\
&=\Big(\sum_{k=0}^{\infty}\Big(\sum_{m=0}^{k}E_m^{(r)}S_2(k,m)\Big)\frac{t^k}{k!}\Big)\Big(\sum_{l=0}^{\infty}Bel_l(x)\frac{t^l}{l!}\Big)\\
&=\sum_{n=0}^{\infty}\Big\{\sum_{k=0}^{n}\sum_{m=0}^{k}E_m^{(r)}(x)S_2(k,m)Bel_{n-k}(x)\frac{n!}{k!(n-k)!}\Big\}\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\Big\{\sum_{k=0}^{n}\binom{n}{k}\sum_{m=0}^{k}E_m^{(r)}(x)S_2(k,m)Bel_{n-k}(x)\Big\}\frac{t^n}{n!}
\end{aligned}\tag{37}$$

Therefore, by (34) and (37), we obtain the following theorem.

Theorem 11. *For $n \geq 0$, we have*

$$\sum_{k=0}^{n}E_k^{(r)}(x)S_2(n,k)=\sum_{k=0}^{n}\binom{n}{k}\sum_{m=0}^{k}E_m^{(r)}(x)S_2(k,m)Bel_{n-k}(x).$$

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ON A PRODUCT-TYPE OPERATOR FROM WEIGHTED BERGMAN-NEVANLINNA SPACES TO WEIGHTED ZYGMUND SPACES ON THE UNIT DISK

ZHI JIE JIANG, HONG BIN BAI, AND ZUO AN LI

Abstract. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk, φ an analytic self-mapping of $\mathbb{D}$ and ψ an analytic function in $\mathbb{D}$. Let $\mathcal{D}$ be the differentiation operator and $W_{\varphi,\psi}$ the weighted composition operator. The boundedness and compactness of the product-type operator $W_{\varphi,\psi}\mathcal{D}$ from weighted Bergman-Nevanlinna spaces to weighted Zygmund spaces on $\mathbb{D}$ are characterized.

1. Introduction

Let $\mathbb{C}$ be the complex plane, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk in $\mathbb{C}$, $H(\mathbb{D})$ the class of all holomorphic functions on $\mathbb{D}$, φ a holomorphic self-mapping of $\mathbb{D}$ and $\psi \in H(\mathbb{D})$. *Weighted composition operator* $W_{\varphi,\psi}$ on $H(\mathbb{D})$ is defined by

$$W_{\varphi,\psi}f(z) = \psi(z) \cdot f(\varphi(z)),\ z \in \mathbb{D}.$$

If $\psi \equiv 1$ the operator is reduced to, so called, *composition operator* and usually denote by C_φ. If $\varphi(z) = z$, it is reduced to, so called, *multiplication operator* and usually denote by M_ψ. Standard problem is to provide function theoretic characterizations when φ and ψ induce a bounded or compact weighted composition operator. Weighted composition operators between various spaces of holomorphic functions on different domains have been studied by numerous authors, see, e.g., [1, 2, 8, 9, 11, 13–17, 19, 21, 23, 28, 34, 35, 45, 49, 50, 53] and the references therein.

Let $\mathcal{D}$ be the differentiation operator on $H(\mathbb{D})$, that is,

$$\mathcal{D}f(z) = f'(z),\ z \in \mathbb{D}.$$

The product-type operator $C_\varphi\mathcal{D}$ has been studied, for example, in [4, 18, 20, 25, 26, 29, 41, 44, 46]. In [31] Sharma has studied the following operators from Bergman-Nevanlinna spaces to Bloch-type spaces:

$$M_\psi C_\varphi \mathcal{D}f(z) = \psi(z)f'(\varphi(z)),$$

$$M_\psi \mathcal{D} C_\varphi f(z) = \psi(z)\varphi'(z)f'(\varphi(z)),$$

$$C_\varphi M_\psi \mathcal{D}f(z) = \psi(\varphi(z))f'(\varphi(z)),$$

and

$$C_\varphi \mathcal{D} M_\psi f(z) = \psi'(\varphi(z))f(\varphi(z)) + \psi(\varphi(z))f'(\varphi(z)),$$

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for $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$. These operators on weighted Bergman spaces, were also studied in [51] and [52] by Stević, Sharma and Bhat. If we consider the *product-type operator* $W_{\varphi,\psi}\mathcal{D}$, then it is clear that

$$M_\psi C_\varphi \mathcal{D} = W_{\varphi,\psi}\mathcal{D}, \ \ M_\psi \mathcal{D} C_\varphi = W_{\varphi,\psi\cdot\varphi'}\mathcal{D},$$

$$C_\varphi M_\psi \mathcal{D} = W_{\varphi,\psi\circ\varphi}\mathcal{D} \ \text{ and } C_\varphi \mathcal{D} M_\psi = W_{\varphi,\psi'\circ\varphi} + W_{\varphi,\psi\circ\varphi}\mathcal{D}.$$

Quite recently, the present author has considered operator $W_{\varphi,\psi}\mathcal{D}$ from weighted Bergman spaces to weighted Zygmund spaces in [10]. This paper is devoted to characterizing the boundedness and compactness of operator $W_{\varphi,\psi}\mathcal{D}$ from weighted Bergman-Nevanlinna spaces to weighted Zygmund spaces. It can be regarded as a continuation of the investigation of operators from weighted Bergman-Nevanlinna spaces to other spaces (see. e.g., [12] and [30]).

Next we introduce the needed spaces and some facts. Let $dA(z) = \frac{1}{\pi}dxdy$ be the normalized Lebesgue measure on $\mathbb{D}$. For $\alpha > -1$, let $dA_\alpha(z) = (\alpha + 1)(1-|z|^2)^\alpha dA(z)$ be the weighted Lebesgue measure on $\mathbb{D}$. The *weighted Bergman-Nevanlinna space* $\mathcal{A}^\alpha_{\log}$ on $\mathbb{D}$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{A}^\alpha_{\log}} = \int_{\mathbb{D}} \log(1 + |f(z)|)dA_\alpha(z) < \infty.$$

It is a Fréchet space with the translation invariant metric

$$d(f, g) = \|f - g\|_{\mathcal{A}^\alpha_{\log}}.$$

For some details of this space, see, e.g., [6], [7], [47] and [54].

For $\beta > 0$, the *weighted-type* $\mathcal{A}_\beta$ consists of all $f \in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}}(1 - |z|^2)^\beta |f(z)| < \infty.$$

This space is a non-separable Banach space with the norm defined by

$$\|f\|_{\mathcal{A}_\beta} = \sup_{z\in\mathbb{D}}(1 - |z|^2)^\beta |f(z)|.$$

The closure of the set of polynomials in $\mathcal{A}_\beta$ is denoted by $\mathcal{A}_{\beta,0}$, which is a separable Banach space and consists exactly of those functions f in $\mathcal{A}_\beta$ satisfying the next condition

$$\lim_{|z|\to 1^-}(1 - |z|^2)^\beta |f(z)| = 0.$$

For $\beta > 0$, the *weighted Bloch space* is defined by

$$\mathcal{B}_\beta = \big\{f \in H(\mathbb{D}) : \sup_{z\in\mathbb{D}}(1 - |z|^2)^\beta |f'(z)| < \infty\big\}.$$

Under the norm

$$\|f\|_{\mathcal{B}_\beta} = |f(0)| + \sup_{z\in\mathbb{D}}(1 - |z|^2)^\beta |f'(z)|,$$

it is a Banach space. For more detail on the space, see, e.g. [55]. The closure of the set of polynomials in $\mathcal{B}_\beta$ is called the *little weighted Bloch space* and is denoted by $\mathcal{B}_{\beta,0}$. For a good source for such spaces, we refer to [55].

For $\beta > 0$, the *weighted Zygmund space* $\mathcal{Z}_\beta$ consists of all $f \in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}}(1 - |z|^2)^\beta |f''(z)| < \infty.$$

It is a Banach space with the norm

$$\|f\|_{\mathcal{Z}_\beta} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1 - |z|^2)^\beta |f''(z)|.$$

The *little weighted Zygmund space* $\mathcal{Z}_{\beta,0}$ consists those functions f in $\mathcal{Z}_\beta$ satisfying

$$\lim_{|z|\to 1^-}(1-|z|^2)^\beta|f''(z)|=0,$$

and it is a closed subspace of the weighted Zygmund space.

For weighted-type spaces, weighted Bloch spaces and weighted Zygmund spaces on the unit disk, the upper half plane, the unit ball, the unit polydisk and some operators, see, e.g. [5, 11, 16, 22–24, 27, 28, 32, 33, 36–40, 42, 43, 48] and the references therein.

Since the weighted Bergman-Nevanlinna space is a Fréchet space and not a Banach space, it is necessary to introduce several definitions needed in this paper. Let X and Y be topological vector spaces whose topologies are given by translation invariant metrics d_X and d_Y, respectively, and let $L: X\to Y$ be a linear operator. It is said that L is *metrically bounded* if there exists a positive constant K such that $d_Y(Lf,0)\le Kd_X(f,0)$ for all $f\in X$. When X and Y are Banach spaces, the metrical boundedness coincides with the usual definition of bounded operators between Banach spaces. Recall that $L: X\to Y$ is *metrically compact* if it maps bounded sets into relatively compact sets. When X and Y are Banach spaces, the metrical compactness coincides with the usual definition of compact operators between Banach spaces. When $X=\mathcal{A}^\alpha_{\log}$ and Y is a Banach space, we define

$$\|L\|_{\mathcal{A}^\alpha_{\log}\to Y}=\sup_{\|f\|_{\mathcal{A}^\alpha_{\log}}\le 1}\|Lf\|_Y,$$

and we often write $\|L\|_{\mathcal{A}^\alpha_{\log}\to Y}$ by $\|L\|$.

Throughout this paper, an operator is bounded (respectively, compact), if it is metrically bounded (respectively, metrically compact). Constants are denoted by C, they are positive and may differ from one occurrence to the next. The notation $a\asymp b$ means that there exists a positive constant C such that $a/C\le b\le Ca$.

2. The operator $W_{\varphi,\psi}\mathcal{D}:\mathcal{A}^\alpha_{\log}\to\mathcal{Z}_\beta$ ($\mathcal{Z}_{\beta,0}$)

Our first lemma characterizes the compactness in terms of sequential convergence. Since the proof is standard, it is omitted (see, e.g., Proposition 3.11 in [3]).

Lemma 2.1. *Let $\alpha>-1$, $\beta>0$ and $\mathcal{Y}\in\{\mathcal{Z}_\beta,\mathcal{Z}_{\beta,0}\}$. Then the bounded operator $W_{\varphi,\psi}\mathcal{D}:\mathcal{A}^\alpha_{\log}\to\mathcal{Y}$ is compact if and only if for every bounded sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{A}^\alpha_{\log}$ such that $f_n\to 0$ uniformly on every compact subset of $\mathbb{D}$ as $n\to\infty$, it follows that*

$$\lim_{n\to\infty}\|W_{\varphi,\psi}\mathcal{D}f_n\|_{\mathcal{Y}}=0.$$

The next result can be found, for example, in [54].

Lemma 2.2. *Let $\alpha>-1$ and $n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$. Then for all $f\in\mathcal{A}^\alpha_{\log}$ and $z\in\mathbb{D}$, there exists a positive constant C independent of f such that*

$$(1-|z|^2)^n|f^{(n)}(z)|\le\exp\frac{C\|f\|_{\mathcal{A}^\alpha_{\log}}}{(1-|z|^2)^{\alpha+2}}.$$

Now we consider the boundedness of operator $W_{\varphi,\psi}\mathcal{D}:\mathcal{A}^\alpha_{\log}\to\mathcal{Z}_\beta$.

Theorem 2.3. *Let $\alpha > -1$, $\beta > 0$, φ be an analytic self-map of $\mathbb{D}$ and $\psi \in H(\mathbb{D})$. Then for all $c > 0$, the following statements are equivalent:*
(i) The operator $W_{\varphi,\psi}\mathcal{D} : \mathcal{A}^{\alpha}_{\log} \to \mathcal{Z}_\beta$ is bounded.
(ii) The operator $W_{\varphi,\psi}\mathcal{D} : \mathcal{A}^{\alpha}_{\log} \to \mathcal{Z}_\beta$ is compact.
(iii) $\psi \in \mathcal{Z}_\beta$,

$$M_0 = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\psi(z)||\varphi'(z)|^2 < \infty,$$

$$M_1 = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)| < \infty,$$

$$\lim_{\varphi(z)\to\partial\mathbb{D}} \frac{(1-|z|^2)^\beta}{1-|\varphi(z)|^2}|\psi''(z)| \exp\frac{c}{(1-|\varphi(z)|^2)^{\alpha+2}} = 0,$$

$$\lim_{\varphi(z)\to\partial\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^2}\big|\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)\big| \exp\frac{c}{(1-|\varphi(z)|^2)^{\alpha+2}} = 0,$$

and

$$\lim_{\varphi(z)\to\partial\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^3}|\psi(z)||\varphi'(z)|^2 \exp\frac{c}{(1-|\varphi(z)|^2)^{\alpha+2}} = 0.$$

Proof. Suppose that (i) holds. Take the functions $f(z) = z$ and $f(z) = z^2$, respectively. Since the operator $W_{\varphi,\psi}\mathcal{D} : \mathcal{A}^{\alpha}_{\log} \to \mathcal{Z}_\beta$ is bounded, we have

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\psi''(z)| \le \|W_{\varphi,\psi}\mathcal{D}z\|_{\mathcal{Z}_\beta} \le C\|W_{\varphi,\psi}\mathcal{D}\| \tag{1}$$

and

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\big|\psi''(z)\varphi(z) + 2\psi'(z)\varphi'(z) + \psi(z)\varphi''(z)\big| \le C\|W_{\varphi,\psi}\mathcal{D}\|. \tag{2}$$

Inequality (1) shows that $\psi \in \mathcal{Z}_\beta$. Also by (1) and the boundedness of φ,

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\psi''(z)||\varphi(z)| < \infty. \tag{3}$$

Then by (2), (3) and the boundedness of φ,

$$M_1 = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)| < \infty. \tag{4}$$

Let the function $f(z) = z^3$. Then

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\big|\psi''(z)\varphi(z)^2 + 2\psi(z)\varphi'(z)^2 + 4\psi'(z)\varphi'(z)\varphi(z) + 2\psi(z)\varphi''(z)\varphi(z)\big| \le C\|W_{\varphi,\psi}\mathcal{D}\|. \tag{5}$$

By (1), (4) and (5),

$$M_0 = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\psi(z)||\varphi'(z)|^2 \le C\|W_{\varphi,\psi}\mathcal{D}\| < \infty. \tag{6}$$

For $w \in \mathbb{D}$, we choose the function

$$\begin{aligned} f_1(z) = {} & c_1\frac{(1-|\varphi(w)|^2)^{\alpha+2}}{(1-\overline{\varphi(w)}z)^{2(\alpha+2)}} + c_2\frac{(1-|\varphi(w)|^2)^{\alpha+4}}{(1-\overline{\varphi(w)}z)^{2(\alpha+2)+2}} \\ & + c_3\frac{(1-|\varphi(w)|^2)^{\alpha+5}}{(1-\overline{\varphi(w)}z)^{2(\alpha+2)+3}} - \frac{(1-|\varphi(w)|^2)^{\alpha+6}}{(1-\overline{\varphi(w)}z)^{2(\alpha+2)+4}} \end{aligned}$$

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where

$$c_2 = -\frac{48\alpha^3 + 460\alpha^2 + 1398\alpha + 1340}{24\alpha^3 + 214\alpha^2 + 655\alpha + 682},$$

$$c_3 = \frac{16\alpha^2 + 104\alpha + 164}{6\alpha^2 + 37\alpha + 62},$$

and

$$c_1 = 1 - c_2 - c_3.$$

We also choose the function

$$g_1(z) = \frac{2\alpha+7}{4\alpha+8}\frac{(1-|\varphi(w)|^2)^{\alpha+2}}{(1-\overline{\varphi(w)}z)^{2(\alpha+2)}} - \frac{6\alpha+21}{4\alpha+12}\frac{(1-|\varphi(w)|^2)^{\alpha+4}}{(1-\overline{\varphi(w)}z)^{2(\alpha+2)+2}} + \frac{(1-|\varphi(w)|^2)^{\alpha+5}}{(1-\overline{\varphi(w)}z)^{2(\alpha+2)+3}}.$$

For the functions f_1 and g_1, we have

$$f_1(\varphi(w)) = f_1''(\varphi(w)) = f_1'''(\varphi(w)) = 0 \tag{7}$$

and

$$g_1'(\varphi(w)) = g_1''(\varphi(w)) = 0. \tag{8}$$

Consequently, (7) and (8) make the function $f(z) = f_1(z)\exp cg_1(z)$ to satisfy

$$f''(\varphi(w)) = f'''(\varphi(w)) = 0$$

and

$$f'(\varphi(w)) = C\frac{\overline{\varphi(w)}}{(1-|\varphi(w)|^2)^{\alpha+3}}\exp\frac{c}{(1-|\varphi(w)|^2)^{\alpha+2}},$$

where

$$C = 2c_2 + 3c_3 - 4.$$

By the boundedness of the operator $W_{\varphi,\psi}\mathcal{D} : \mathcal{A}^\alpha_{\log} \to \mathcal{Z}_\beta$, we find

$$\frac{|\varphi(w)|(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^{\alpha+3}}\left|\psi''(w)\right|\exp\frac{c}{(1-|\varphi(w)|^2)^{\alpha+2}} \le C.$$

Thus

$$\lim_{\varphi(w)\to\partial\mathbb{D}}\frac{(1-|w|^2)^\beta}{1-|\varphi(w)|^2}\left|\psi''(w)\right|\exp\frac{c}{(1-|\varphi(w)|^2)^{\alpha+2}} = 0.$$

For $w \in \mathbb{D}$, we choose the functions

$$f_2(z) = \frac{3\alpha+8}{3\alpha+10}\frac{(1-|\varphi(w)|^2)^{\alpha+2}}{(1-\overline{\varphi(w)}z)^{2(\alpha+2)}} - \frac{6\alpha+22}{3\alpha+10}\frac{(1-|\varphi(w)|^2)^{\alpha+4}}{(1-\overline{\varphi(w)}z)^{2(\alpha+2)+2}} + \frac{6\alpha+24}{3\alpha+10}\frac{(1-|\varphi(w)|^2)^{\alpha+5}}{(1-\overline{\varphi(w)}z)^{2(\alpha+2)+3}} - \frac{(1-|\varphi(w)|^2)^{\alpha+6}}{(1-\overline{\varphi(w)}z)^{2(\alpha+2)+4}},$$

and

$$g_2(z) = \frac{\alpha+3}{\alpha+2}\frac{(1-|\varphi(w)|^2)^{\alpha+2}}{(1-\overline{\varphi(w)}z)^{2(\alpha+2)}} - \frac{(1-|\varphi(w)|^2)^{\alpha+4}}{(1-\overline{\varphi(w)}z)^{2(\alpha+2)+2}}.$$

Then

$$f_2(\varphi(w)) = f_2'(\varphi(w)) = f_2'''(\varphi(w)) = 0 \tag{9}$$

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and $g_2'(\varphi(w)) = 0$. From this and (9), for the function $g(z) = f_2(z) \exp cg_2(z)$ we have

$$g'(\varphi(w)) = g'''(\varphi(w)) = 0$$

and

$$g''(\varphi(w)) = C\frac{\overline{\varphi(w)}^2}{(1-|\varphi(w)|^2)^{\alpha+4}} \exp\frac{c}{(1-|\varphi(w)|^2)^{\alpha+2}},$$

where

$$C = -\frac{24\alpha+120\alpha+141}{3\alpha+10}.$$

By the boundedness of $W_{\varphi,\psi}\mathcal{D} : \mathcal{A}^{\alpha}_{\log} \to \mathcal{Z}_\beta$,

$$\|W_{\varphi,\psi}\mathcal{D}g\|_{\mathcal{Z}_\beta} \le C\|W_{\varphi,\psi}\mathcal{D}\|,$$

and from which we obtain

$$\frac{|\varphi(w)|^2(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^{\alpha+4}}\big|\psi(w)\varphi''(w) + 2\psi'(w)\varphi'(w)\big| \exp\frac{c}{(1-|\varphi(w)|^2)^{\alpha+2}} \le C.$$

This shows that

$$\lim_{\varphi(w)\to\partial\mathbb{D}} \frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^2}\big|\psi(w)\varphi''(w) + 2\psi'(w)\varphi'(w)\big| \exp\frac{c}{(1-|\varphi(w)|^2)^{\alpha+2}} = 0.$$

For $w \in \mathbb{D}$, we choose the functions

$$\begin{aligned} f_3(z) = {} & \frac{1}{3}\frac{(1-|\varphi(w)|^2)^{\alpha+2}}{(1-\overline{\varphi(w)}z)^{2(\alpha+2)}} - 2\frac{(1-|\varphi(w)|^2)^{\alpha+4}}{(1-\overline{\varphi(w)}z)^{2(\alpha+2)+2}} \\ & + \frac{8}{3}\frac{(1-|\varphi(w)|^2)^{\alpha+5}}{(1-\overline{\varphi(w)}z)^{2(\alpha+2)+3}} - \frac{(1-|\varphi(w)|^2)^{\alpha+6}}{(1-\overline{\varphi(w)}z)^{2(\alpha+2)+4}} \end{aligned}$$

and

$$g_3(z) = \frac{(1-|\varphi(w)|^2)^{\alpha+2}}{(1-\overline{\varphi(w)}z)^{2(\alpha+2)}}.$$

From a calculation, we obtain

$$f_3(\varphi(w)) = f_3'(\varphi(w)) = f_3''(\varphi(w)) = 0. \tag{10}$$

Define the function $h(z) = f_3(z) \exp cg_3(z)$. Then by (10),

$$h'(\varphi(w)) = h''(\varphi(w)) = 0,$$

and by a direct calculation,

$$h'''(\varphi(w)) = C\frac{\overline{\varphi(w)}^3}{(1-|\varphi(w)|^2)^{\alpha+5}} \exp\frac{c}{(1-|\varphi(w)|^2)^{\alpha+2}},$$

where $C = -30(\alpha+2)^2 - 8$. Since $W_{\varphi,\psi}\mathcal{D} : \mathcal{A}^{\alpha}_{\log} \to \mathcal{Z}_\beta$ is bounded, we have

$$\|W_{\varphi,\psi}\mathcal{D}h\|_{\mathcal{Z}_\beta} \le C\|W_{\varphi,\psi}\mathcal{D}\|,$$

and so

$$(1-|z|^2)^\beta |(W_{\varphi,\psi}\mathcal{D}h)''(z)| \le C\|W_{\varphi,\psi}\mathcal{D}\|, \tag{11}$$

for all $z \in \mathbb{D}$. Letting $z = w$ in (11) yields to

$$\frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^{\alpha+5}}|\psi(w)||\varphi'(w)|^2|\varphi(w)|^3 \exp\frac{c}{(1-|\varphi(w)|^2)^{\alpha+2}} \le C\|W_{\varphi,\psi}\mathcal{D}\|.$$

Thus

$$\frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^3}|\psi(w)||\varphi'(w)|^2 \exp\frac{c}{(1-|\varphi(w)|^2)^{\alpha+2}} \le \frac{C(1-|\varphi(w)|^2)^{\alpha+2}}{|\varphi(w)|^3}. \quad (12)$$

Taking limit as $\varphi(w)\to\partial\mathbb{D}$ in (12) gives

$$\lim_{\varphi(w)\to\partial\mathbb{D}} \frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^3}|\psi(w)||\varphi'(w)|^2 \exp\frac{c}{(1-|\varphi(w)|^2)^{\alpha+2}} = 0.$$

The proof of the implication is finished.

$(iii) \Rightarrow (ii)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}^\alpha_{\log}$ with $\sup_{n\in\mathbb{N}}\|f_n\|_{\mathcal{A}^\alpha_{\log}} \le M$ and $f_n \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $n\to\infty$. We have that for the constant C in Lemma 2.2, for any $\varepsilon>0$ there exits a constant $\delta\in(0,1)$ such that whenever $\delta<|\varphi(z)|<1$, it follows that

$$\frac{(1-|z|^2)^\beta}{1-|\varphi(z)|^2}|\psi''(z)| \exp\frac{C}{(1-|\varphi(z)|^2)^{\alpha+2}} < \varepsilon,$$

$$\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^2}\big|\psi(z)\varphi''(z)+2\psi'(z)\varphi'(z)\big| \exp\frac{C}{(1-|\varphi(z)|^2)^{\alpha+2}} < \varepsilon,$$

and

$$\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^3}|\psi(z)||\varphi'(z)|^2 \exp\frac{C}{(1-|\varphi(z)|^2)^{\alpha+2}} < \varepsilon.$$

Then by Lemma 2.2, for a fixed $\delta\in(0,1)$ we have

$$\begin{aligned}
\|W_{\varphi,\psi}\mathcal{D}f_n\|_{\mathcal{Z}_\beta} &= \big|(\psi\cdot f_n'\circ\varphi)(0)\big| + \big|(\psi\cdot f_n'\circ\varphi)'(0)\big| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\big|(\psi(z)f_n'(\varphi(z))''\big| \\
&= \big|\psi(0)\big|\big|f_n'(\varphi(0))\big| + \Big|\psi'(0)f_n'(\varphi(0)) + \psi(0)f_n''(\varphi(0))\varphi(0)\Big| \\
&\quad + \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\Big|\psi''(z)f_n'(\varphi(z)) + \Big(\psi(z)\varphi''(z)+2\psi'(z)\varphi'(z)\Big)f_n''(\varphi(z)) + \psi(z)\varphi'(z)^2 f_n'''(\varphi(z))\Big| \\
&\le \Big(\big|\psi(0)\big| + \big|\psi'(0)\big|\Big)\big|f_n'(\varphi(0))\big| + \big|\varphi(0)\big|\big|\psi(0)\big|\big|f_n''(\varphi(0))\big| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\big|\psi''(z)\big|\big|f_n'(\varphi(z))\big| \\
&\quad + \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\Big|\psi(z)\varphi''(z)+2\psi'(z)\varphi'(z)\Big|\big|f_n''(\varphi(z))\big| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\big|\psi(z)\big|\big|\varphi'(z)\big|^2\big|f_n'''(\varphi(z))\big| \\
&\le \Big(\big|\psi(0)\big| + \big|\psi'(0)\big|\Big)\big|f_n'(\varphi(0))\big| + \big|\varphi(0)\big|\big|\psi(0)\big|\big|f_n''(\varphi(0))\big| \\
&\quad + \sup_{|\varphi(z)|\le\delta}(1-|z|^2)^\beta\big|\psi''(z)\big|\big|f_n'(\varphi(z))\big| + \sup_{\delta<|\varphi(z)|<1}(1-|z|^2)^\beta\big|\psi''(z)\big|\big|f_n'(\varphi(z))\big| \\
&\quad + \sup_{|\varphi(z)|\le\delta}(1-|z|^2)^\beta\Big|\psi(z)\varphi''(z)+2\psi'(z)\varphi'(z)\Big|\big|f_n''(\varphi(z))\big| \\
&\quad + \sup_{\delta<|\varphi(z)|<1}(1-|z|^2)^\beta\Big|\psi(z)\varphi''(z)+2\psi'(z)\varphi'(z)\Big|\big|f_n''(\varphi(z))\big| \\
&\quad + \sup_{|\varphi(z)|\le\delta}(1-|z|^2)^\beta\big|\psi(z)\big|\big|\varphi'(z)\big|^2\big|f_n'''(\varphi(z))\big| \\
&\quad + \sup_{\delta<|\varphi(z)|<1}(1-|z|^2)^\beta\big|\psi(z)\big|\big|\varphi'(z)\big|^2\big|f_n'''(\varphi(z))\big| \\
&\le \Big(\big|\psi(0)\big| + \big|\psi'(0)\big|\Big)\big|f_n'(\varphi(0))\big| + \big|\varphi(0)\big|\big|\psi(0)\big|\big|f_n''(\varphi(0))\big| + \|\psi\|_{\mathcal{Z}_\beta}\sup_{|\varphi(z)|\le\delta}\big|f_n'(\varphi(z))\big|
\end{aligned}$$

$$+\sup_{\delta<|\varphi(z)|<1}\frac{(1-|z|^2)^\beta}{1-|\varphi(z)|^2}|\psi''(z)|\exp\frac{C}{(1-|\varphi(z)|^2)^{\alpha+2}}+M_1\sup_{|\varphi(z)|\le\delta}\big|f_n''(\varphi(z))\big|$$
$$+\sup_{\delta<|\varphi(z)|<1}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^2}\big|\psi(z)\varphi''(z)+2\psi'(z)\varphi'(z)\big|\exp\frac{C}{(1-|\varphi(z)|^2)^{\alpha+2}}$$
$$+M_0\sup_{|\varphi(z)|\le\delta}\big|f_n'''(\varphi(z))\big|+\sup_{\delta<|\varphi(z)|<1}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^3}|\psi(z)||\varphi'(z)|^2\exp\frac{C}{(1-|\varphi(z)|^2)^{\alpha+2}}.$$

By Cauchy's estimation, if $(f_n)_{n\in\mathbb{N}}$ converges to zero on each compact subset of $\mathbb{D}$, then $(f_n')_{n\in\mathbb{N}}$, $(f_n'')_{n\in\mathbb{N}}$ and $(f_n''')_{n\in\mathbb{N}}$ also do as $n\to\infty$. From this, and since both $\{z\in\mathbb{D}:|z|\le\delta\}$ and $\{0\}$ are compact subset of $\mathbb{D}$, there exists a natural number N such that whenever $n>N$, it follows that

$$\Big(\big|\psi(0)\big|+\big|\psi'(0)\big|\Big)\big|f_n'(\varphi(0))\big|+\big|\varphi(0)\big|\big|\psi(0)\big|\big|f_n''(\varphi(0))\big|<\varepsilon$$

and

$$\sup_{|\varphi(z)|\le\delta}\big|f_n^{(i)}(\varphi(z))\big|<\varepsilon,$$

where $i=1,2,3$. Consequently, for all $n>N$ it follows that

$$\|W_{\varphi,\psi}\mathcal{D}f_n\|_{\mathcal{Z}_\beta}\le(4+\|\psi\|_{\mathcal{Z}_\beta}+M_0+M_1)\varepsilon,$$

which shows that the operator $W_{\varphi,\psi}\mathcal{D}:\mathcal{A}^\alpha_{\log}\to\mathcal{Z}_\beta$ is compact.

$(ii)\Rightarrow(i)$. This implication is obvious. The proof is finished.

Now, we consider the boundedness of operator $W_{\varphi,\psi}\mathcal{D}:\mathcal{A}^\alpha_{\log}\to\mathcal{Z}_{\beta,0}$. We first have the following result.

Lemma 2.4. *Let $\alpha>-1$, $\beta>0$, φ be an analytic self-map of $\mathbb{D}$ and $\psi\in H(\mathbb{D})$. Then for all $c>0$, the following statements are equivalent:*

(i)

$$\lim_{z\to\partial\mathbb{D}}\frac{(1-|z|^2)^\beta}{1-|\varphi(z)|^2}\big|\psi(z)\varphi''(z)+2\psi'(z)\varphi'(z)\big|\exp\frac{c}{(1-|\varphi(z)|^2)^{\alpha+2}}=0.$$

(ii)

$$\lim_{\varphi(z)\to\partial\mathbb{D}}\frac{(1-|z|^2)^\beta}{1-|\varphi(z)|^2}\big|\psi(z)\varphi''(z)+2\psi'(z)\varphi'(z)\big|\exp\frac{c}{(1-|\varphi(z)|^2)^{\alpha+2}}=0,$$

and $\psi\varphi''+2\psi'\varphi'\in\mathcal{A}_{\beta,0}$.

Proof. Suppose that (i) holds. Since

$$\frac{1}{1-|\varphi(z)|^2}\exp\frac{c}{(1-|\varphi(z)|^2)^{\alpha+2}}\ge1$$

for all $z\in\mathbb{D}$, we have

$$\begin{aligned}&(1-|z|^2)^\beta|\psi(z)\varphi''(z)+2\psi'(z)\varphi'(z)|\\&\le\frac{(1-|z|^2)^\beta}{1-|\varphi(z)|^2}\big|\psi(z)\varphi''(z)+2\psi'(z)\varphi'(z)\big|\exp\frac{c}{(1-|\varphi(z)|^2)^{\alpha+2}}\\&\to0,\end{aligned}$$

as $z\to\partial\mathbb{D}$. Hence $\psi\varphi''+2\psi'\varphi'\in\mathcal{A}_{\beta,0}$. Since $\varphi(z)\to\partial\mathbb{D}$ implies $z\to\partial\mathbb{D}$, it follows that the first assertion in (ii) holds.

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Now suppose that (ii) holds, but (i) is not true. Then there exist constants $c_0 > 0$, $\varepsilon_0 > 0$ and a sequence $\{z_n\}$ tending to $\partial\mathbb{D}$ as $n \to \infty$ such that

$$\frac{(1-|z_n|^2)^\beta}{1-|\varphi(z_n)|^2}\big|\psi(z_n)\varphi''(z_n)+2\psi'(z_n)\varphi'(z_n)\big|\exp\frac{c}{(1-|\varphi(z_n)|^2)^{\alpha+2}} \geq \varepsilon_0. \tag{13}$$

Since $\psi\varphi''+2\psi'\varphi' \in \mathcal{A}_{\beta,0}$, it follows from (13) that the sequence $(z_n)_{n\in\mathbb{N}}$ has a subsequence $(z_{n_k})_{k\in\mathbb{N}}$ with $\varphi(z_{n_k}) \to \partial\mathbb{D}$. Therefore, applying $(z_{n_k})_{k\in\mathbb{N}}$ to the first assertion in (ii), we arrive a contradiction to (13), finishing the proof.

By Lemma 2.4, the following result follows similar to the proof of Theorem 2.3. Hence, the proof is omitted.

Theorem 2.5. *Let $\alpha > -1$, $\beta > 0$, φ be an analytic self-mapping of $\mathbb{D}$ and $\psi \in H(\mathbb{D})$. Then for all $c > 0$, the following statements are equivalent:*
(i) The operator $W_{\varphi,\psi}\mathcal{D} : \mathcal{A}^\alpha_{\log} \to \mathcal{Z}_{\beta,0}$ is bounded.
(ii) The operator $W_{\varphi,\psi}\mathcal{D} : \mathcal{A}^\alpha_{\log} \to \mathcal{Z}_{\beta,0}$ is compact.
(iii)

$$\psi'',\ \psi\varphi'^2,\ \psi\varphi''+2\psi'\varphi' \in \mathcal{A}_{\beta,0},$$

$$\lim_{\varphi(z)\to\partial\mathbb{D}}\frac{(1-|z|^2)^\beta}{1-|\varphi(z)|^2}|\psi''(z)|\exp\frac{c}{(1-|\varphi(z)|^2)^{\alpha+2}}=0,$$

$$\lim_{\varphi(z)\to\partial\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^3}|\psi(z)||\varphi'(z)|^2\exp\frac{c}{(1-|\varphi(z)|^2)^{\alpha+2}}=0,$$

and

$$\lim_{\varphi(z)\to\partial\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^2}\big|\psi(z)\varphi''(z)+2\psi'(z)\varphi'(z)\big|\exp\frac{c}{(1-|\varphi(z)|^2)^{\alpha+2}}=0.$$

(iv)

$$\lim_{z\to\partial\mathbb{D}}\frac{(1-|z|^2)^\beta}{1-|\varphi(z)|^2}|\psi''(z)|\exp\frac{c}{(1-|\varphi(z)|^2)^{\alpha+2}}=0,$$

$$\lim_{z\to\partial\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^3}|\psi(z)||\varphi'(z)|^2\exp\frac{c}{(1-|\varphi(z)|^2)^{\alpha+2}}=0,$$

and

$$\lim_{z\to\partial\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^2}\big|\psi(z)\varphi''(z)+2\psi'(z)\varphi'(z)\big|\exp\frac{c}{(1-|\varphi(z)|^2)^{\alpha+2}}=0.$$

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Hesitant fuzzy Maclaurin symmetric mean operators and their application in multiple attribute decision making

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Abstract: The Maclaurin symmetric mean (MSM), originally introduced by Maclaurin, can capture the interrelationship among the multi-input arguments. It plays an important role in many multiple attribute decision making problems. In this paper, we first extend MSM operator to deal with hesitant fuzzy information and propose some new hesitant fuzzy aggregation operators, such as the hesitant fuzzy Maclaurin symmetric mean ($HFMSM$) and the weighted hesitant fuzzy Maclaurin symmetric mean ($WHFMSM$). Then, we further investigate some desirable properties and special cases of those operators in detail. Finally, we develop an approach to hesitant fuzzy multiple attribute decision making problems based on the proposed operators. A practical example is given to illustrate the practicality and effectiveness of the proposed method.

Keywords: fuzzy set; hesitant fuzzy set; aggregation operator; Maclaurin symmetric mean; multiple attribute decision making

1 Introduction

Multiple attribute decision making is one of the most significant human activities in many fields including social science, economics, medical science, engineering, environmental science and so on. The purpose of a decision making is to find a desirable solution from a finite alternatives. In order to obtain a desirable solution, the decision information provided by decision makers always need to be aggregated into an overall one by using a proper aggregation technique. Therefore, the research on information aggregation method is an important topic in multiple attribute decision making. In the past few decades, a variety of aggregation operators have been developed and applied to multiple attribute decision making with different decision information, such as accurate numbers, fuzzy numbers, intuitionistic fuzzy numbers, trapezoidal fuzzy numbers and so on [1–4].

Recently, Torra introduced the hesitant fuzzy set (HFS) [5], which allows membership degree to have a set of possible values. Therefore, it is an efficient tool in the situation where the evaluation of an alternative under each attribute is represented by several possible values. Since its appearance, HFS has attracted more and more attention from researchers [6–8]. Hesitant fuzzy information aggregation has become a hot topic in the hesitant fuzzy set theory and lots of hesitant fuzzy aggregation operators have been developed [9–17]. For example, Xia and Xu [11] first presented some hesitant fuzzy operational laws, based on which they proposed a series of aggregation operators, such as hesitant fuzzy weighted averaging ($HFWA$) operator, hesitant fuzzy weighted geometric ($HFWG$) operator and so on. Xia et al. [17] developed some confidence induced aggregation operators for hesitant fuzzy information. Xia et al. [12] gave several series of hesitant fuzzy aggregation operators with the help of quasi-arithmetic means. Wei [10] explored some hesitant fuzzy prioritized aggregation operators and applied them to hesitant fuzzy decision making problems in which the attributes are in different priority levels. Zhang [14] extended the power aggregation operator to the hesitant fuzzy power aggregation operators, whose weighting vectors depend upon the input arguments and allow values being aggregated to support and reinforce each other. Zhu et al. [16] extended Bonferroni mean to deal with hesitant fuzzy information and get the hesitant fuzzy Bonferroni mean operator. By combining

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the Bonferroni mean and the geometric mean, Zhu et al. [15] further investigated the geometric Bonferroni mean under hesitant fuzzy environment.

The Maclaurin symmetric mean (MSM) was originally proposed by Maclaurin [18] and many important results on the MSM have been obtained [19–22]. It is worth noting that the MSM has desirable properties capturing the interrelationships among multi-input arguments. The BM also can capture the interrelationships among arguments, but it only reflect the interrelationships between two arguments whereas the MSM can reflect the interrelationships among multi-input arguments. Furthermore, for the same collection of arguments, the MSM is monotonically decreasing with respect to the parameter, which make the decision makers can select easily the parameter value according to their risk preferences in decision making progress. Therefore, the MSM is more flexible and robust such that it is more adequate to solve multiple attribute decision making problem where the attributes are independent. So far, the MSM has been used successful to deal with not only the crisp values but also the intuitionistic fuzzy values [23]. But we have not seen any work based on the MSM for aggregating hesitant fuzzy information. Thus, it is meaningful to use the MSM to develop the aggregation techniques under hesitant fuzzy environment. In this paper, motivated by Qin [23], we develop some new hesitant fuzzy aggregation operators based on the MSM, and apply them to multiple attribute decision making under hesitant fuzzy environment.

The rest of this paper is organized as follows. In Section 2, we review the notions of HFS and the MSM. In Section 3, we introduce the hesitant fuzzy Maclaurin symmetric mean ($HFMSM$) operator and discuss some desirable properties and special cases of the proposed operator. In Section 4, we further develop the weighted forms of the previous operator and apply them to hesitant fuzzy decision making. Finally, conclusions are stated in Section 5.

2 Preliminaries

In this section, we recall briefly the necessary notations on HFS and MSM. We also present the dual Maclaurin symmetric mean based on the MSM.

2.1 Hesitant fuzzy set

Torra and Narukawa [5] extended the fuzzy set to the hesitant fuzzy set (HFS), shown as follows:

Definition 2.1. *Let X be a reference set, an HFS on X is in terms of a function that when applied to X returns a subset of* $[0,1]$.

To be easily understood, Xia and Xu [11] expressed the HFS by mathematical symbol

$$H = \left\{ \frac{h_H(x)}{x} | x \in X \right\},$$

where $h_H(x)$ is a set of some values in $[0,1]$, denoting the possible membership degrees of the element $x \in X$ to the set H. For convenience, Xu and Xia [7] called $h_H(x)$ an hesitant fuzzy element (HFE).

Let h_1 and h_2 be $HFEs$, the union, intersection and complement of them are definded by Torra and Narukawa [5] as:

(1) $h_1 \cup h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} max\{\gamma_1, \gamma_2\}$;
(2) $h_1 \cap h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} min\{\gamma_1, \gamma_2\}$;
(3) $h_1^c = \cup_{\gamma_1 \in h_1} \{1 - \gamma_1\}$.

Let $\alpha > 0$, h_1 and h_2 be two $HFEs$, Xu and Xia [11] defined some operations on the $HFEs$ h_1 and h_2 as follows:

(5) $h_1 \oplus h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$
(6) $h_1 \otimes h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1\gamma_2\}$
(7) $\alpha h = \cup_{\gamma \in h} \{\gamma^\alpha\}$
(8) $h^\alpha = \cup_{\gamma \in h} \{1 - (1-\gamma)^\alpha\}$

In [11], Xia and Xu defined the score function of $HFEs$ and gave the comparison laws.

Definition 2.2. *Let h be an HFE, $s(h) = \frac{1}{n(h)} \sum_{\gamma \in h} \gamma$ is called the score function of h, where $n(h)$ is the number of values of h. For two $HFEs$ h_1 and h_2, if $s(h_1) > s(h_2)$, then $h_1 > h_2$; if $s(h_1) = s(h_2)$, then $h_1 = h_2$.*

Xia and Xu [11, 12] further gave some hesitant fuzzy aggregation operators as follows:

Let $h_j (j = 1, 2, \cdots, n)$ be a collection of HFEs, $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ be the weight vector of $h_j (j = 1, 2, \cdots, n)$ with $\omega_j \in [0,1]$ and $\sum\limits_{j=1}^{n} \omega_j = 1$, then

(1) The hesitant fuzzy weighted averaging ($HFWA$) operator

$$HFWA(h_1,h_2,\cdots,h_n)=\bigoplus_{j=1}^{n}(\omega_j h_j)=\bigcup_{\substack{\gamma_j\in h_j,\\ i=1,\cdots,n}}\left\{1-\prod_{j=1}^{n}(1-\gamma_j)^{\omega_j}\right\}$$

Especially, if $\omega=(1/n,1/n,\cdots,1/n)^T$, then the $HFWA$ operator reduces to the hesitant fuzzy averaging (HFA) operator

$$HFA(h_1,h_2,\cdots,h_n)=\bigcup_{\substack{\gamma_j\in h_j,\\ i=1,\cdots,n}}\left\{1-\prod_{j=1}^{n}(1-\gamma_j)^{1/n}\right\} \tag{1}$$

(2) The hesitant fuzzy weighted geometric ($HFWG$) operator

$$HFWG(h_1,h_2,\cdots,h_n)=\bigotimes_{j=1}^{n}{h_j}^{\omega_j}=\bigcup_{\substack{\gamma_j\in h_j,\\ i=1,\cdots,n}}\left\{\prod_{j=1}^{n}{\gamma_j}^{\omega_j}\right\}$$

Especially, if $\omega=(1/n,1/n,\cdots,1/n)^T$, then the $HFWG$ operator becomes to the hesitant fuzzy geometric (HFG) operator

$$HFG(h_1,h_2,\cdots,h_n)=\bigcup_{\substack{\gamma_j\in h_j,\\ i=1,\cdots,n}}\left\{\prod_{j=1}^{n}{\gamma_j}^{1/n}\right\} \tag{2}$$

2.2 Maclaurin symmetric mean

The MSM introduced by Maclaurin [18] is a useful technique characterized by the ability to capture the interrelationship among the multi-input arguments. The definition of MSM is given as follows.

Definition 2.3. *[18] Let $a_i (i=1,2,\cdots,n)$ be a collection of nonnegative real numbers and $r=1,2,\cdots,n$. If*

$$MSM^{(r)}(a_1,a_2,\cdots,a_n)=\left(\frac{\sum\limits_{1\le i_1<\cdots<i_r\le i_n}\prod\limits_{j=1}^{r}a_{i_j}}{C_n^r}\right)^{\frac{1}{r}}$$

then $MSM^{(r)}$ is called the Maclaurin symmetric mean, where $(i_1,i_2,\cdots,i_r)$ traversal all the r-tuple combination of $(1,2,\cdots,n)$, C_n^r is the binomial coefficient.

It is clear that the $MSM^{(r)}$ have the following properties:

$(1) MSM^{(r)}(0,0,\cdots,0)=0;$

$(2) MSM^{(r)}(a,a,\cdots,a)=a;$

$(3) MSM^{(r)}(a_1,a_2,\cdots,a_n)\le MSM^{(r)}(b_1,b_2,\cdots,b_n),\ if\ a_i\le b_i\, for\ all\ i;$

$(4) \min\limits_i\{a_i\}\le MSM^{(r)}(a_1,a_2,\cdots,a_n)\le \max\limits_i\{a_i\}.$

3 Hesitant fuzzy MSM operator

In this section, we shall extend MSM to aggregate hesitant fuzzy information and obtain a hesitant fuzzy Maclaurin symmetric mean operator. We also investigate a variety of desirable properties and some special cases.

Definition 3.1. *Let $h_i (i=1,2,\cdots,n)$ be a collection of $HFEs$ and $r=1,2,\cdots,n$. If*

$$HFMSM^{(r)}(h_1,h_2,\cdots,h_n)=\left(\frac{\bigoplus\limits_{\substack{1\le i_1<\cdots\\ <i_r\le i_n}}\bigotimes\limits_{j=1}^{r}h_{i_j}}{C_n^r}\right)^{\frac{1}{r}} \tag{3}$$

then $HFMSM^{(r)}$ is called the hesitant fuzzy Maclaurin symmetric mean ($HFMSM$), where $(i_1, i_2, \cdots, i_r)$ traversal all the r-tuple combination of $(1, 2, \cdots, n)$, C_n^r is the binomial coefficient.

Based on the operations of $HFEs$ described in Section 2, we can derive the following Theorem 3.2.

Theorem 3.2. *Let $h_i (i = 1, 2, \cdots, n)$ be a collection of $HFEs$ and $r = 1, 2, \cdots, n$. Then the aggregated value by using the $HFMSM^{(r)}$ is also an HFE, and*

$$HFMSM^{(r)}(h_1, h_2, \cdots, h_n) = \bigcup_{\substack{\gamma_i \in h_i,\\ i=1,\cdots,n}} \left\{ \left(1 - \left(\prod_{\substack{1 \le i_1 < \cdots \\ < i_r \le n}} \left(1 - \prod_{j=1}^{r} \gamma_{i_j} \right) \right)^{\frac{1}{C_n^r}} \right)^{\frac{1}{r}}, \right\} \tag{4}$$

Proof. By the operational laws (5)-(8) described in Section 2, we have

$$\bigotimes_{j=1}^{r} h_{i_j} = \bigcup_{\substack{\gamma_i \in h_i,\\ i=1,\cdots,n}} \left\{ \prod_{j=1}^{r} \gamma_{i_j} \right\}$$

and

$$\bigoplus_{\substack{1 \le i_1 < \cdots \\ < i_r \le n}} \bigotimes_{j=1}^{r} h_{i_j} = \bigcup_{\substack{\gamma_i \in h_i,\\ i=1,\cdots,n}} \left\{ 1 - \prod_{\substack{1 \le i_1 < \cdots \\ < i_r \le n}} \left(1 - \prod_{j=1}^{r} \gamma_{i_j} \right) \right\}$$

then we obtain

$$\frac{1}{C_n^r} \left(\bigoplus_{\substack{1 \le i_1 < \cdots \\ < i_r \le n}} \bigotimes_{j=1}^{r} h_{i_j} \right) = \bigcup_{\substack{\gamma_i \in h_i,\\ i=1,\cdots,n}} \left\{ 1 - \left(\prod_{\substack{1 \le i_1 < \cdots \\ < i_r \le n}} \left(1 - \prod_{j=1}^{r} \gamma_{i_j} \right) \right)^{\frac{1}{C_n^r}} \right\}$$

Thus

$$HFMSM^{(r)}(h_1, h_2, \cdots, h_n) = \bigcup_{\substack{\gamma_i \in h_i,\\ i=1,\cdots,n}} \left\{ \left(1 - \left(\prod_{\substack{1 \le i_1 < \cdots \\ < i_r \le n}} \left(1 - \prod_{j=1}^{r} \gamma_{i_j} \right) \right)^{\frac{1}{C_n^r}} \right)^{\frac{1}{r}} \right\},$$

which completes the proof of Theorem 3.2. □

In the following, we shall study some desirable properties of $HFMSM$.

Theorem 3.3. *Let $h_i (i = 1, 2, \cdots, n)$ be a collection of $HFEs$. If $h_i = h = \{\gamma\}$ for all $i \in \{1, 2, \cdots, n\}$, then*

$$HFMSM^{(r)}(h_1, h_2, \cdots, h_n) = h$$

Proof. Let $h_i = \{\gamma_i\}$, then $\gamma_i = \gamma (i = 1, 2, \cdots, n)$. By Theorem 3.2, we have

$$\begin{aligned}
&HFMSM^{(r)}(h_1, h_2, \cdots, h_n) \\
&= \bigcup_{\substack{\gamma_i \in h_i, \\ i=1,\cdots,n}} \left\{ \left(1 - \left(\prod_{\substack{1 \le i_1 < \cdots \\ < i_r \le n}} \left(1 - \prod_{j=1}^{r} \gamma_i \right) \right)^{\frac{1}{C_n^r}} \right)^{\frac{1}{r}} \right\} \\
&= \bigcup_{\substack{\gamma_i \in h_i, \\ i=1,\cdots,n}} \left\{ \left(1 - \left(\prod_{\substack{1 \le i_1 < \cdots \\ < i_r \le n}} (1 - \gamma_i^r) \right)^{\frac{1}{C_n^r}} \right)^{\frac{1}{r}} \right\} \\
&= \bigcup_{\substack{\gamma_i \in h_i, \\ i=1,\cdots,n}} \left\{ \left(1 - \left((1 - \gamma_i^r)^{C_n^r} \right)^{\frac{1}{C_n^r}} \right)^{\frac{1}{r}} \right\} \\
&= \left\{ (1 - (1 - \gamma^r))^{\frac{1}{r}} \right\} \\
&= \{\gamma\} = h.
\end{aligned}$$

□

Corollary 3.4. *Let $h_i (i = 1, 2, \cdots, n)$ be a collection of $HFEs$.*
(1) If $h_i = h = \{0\}$ for all i, then $HFMSM^{(r)}(h_1, h_2, \cdots, h_n) = \{0\}$;
(2) If $h_i = h = \{1\}$ for all i, then $HFMSM^{(r)}(h_1, h_2, \cdots, h_n) = \{1\}$.

Theorem 3.5. *Let $h_i (i = 1, 2, \cdots, n)$ be a collection of $HFEs$, and $h_i^{'} (i = 1, 2, \cdots, n)$ be any permutation of $h_i (i = 1, 2, \cdots, n)$, then*

$$HFMSM^{(r)}(h_1, h_2, \cdots, h_n) = HFMSM^{(r)}(h_1^{'}, h_2^{'}, \cdots, h_n^{'})$$

Proof. Since $h_i^{'} (i = 1, 2, \cdots, n)$ is any permutation of $h_i (i = 1, 2, \cdots, n)$, by Definition 3.1, we have

$$HFMSM^{(r)}(h_1, h_2, \cdots, h_n) = \left(\frac{\bigoplus_{\substack{1 \le i_1 < \cdots \\ < i_r \le i_n}} \bigotimes_{j=1}^{r} h_{i_j}}{C_n^r} \right)^{\frac{1}{r}}$$

$$= \left(\frac{\bigoplus_{\substack{1 \le i_1 < \cdots \\ < i_r \le i_n}} \bigotimes_{j=1}^{r} h_{i_j}^{'}}{C_n^r} \right)^{\frac{1}{r}} = HFMSM^{(r)}(h_1^{'}, h_2^{'}, \cdots, h_n^{'}).$$

□

Theorem 3.6. *Let $h_\alpha = \{h_{\alpha_1}, \cdots, h_{\alpha_n}\}$ and $h_\beta = \{h_{\beta_1}, \cdots, h_{\beta_n}\}$ be two collections of $HFEs$. If for any $\gamma_{\alpha_i} \in h_{\alpha_i}$ and $\gamma_{\beta_i} \in h_{\beta_i}$, we have $\gamma_{\alpha_i} \le \gamma_{\beta_i}$ for all $i (i = 1, \cdots, n)$, then*

$$HFMSM^{(r)}(h_{\alpha_1}, h_{\alpha_2}, \cdots, h_{\alpha_n}) \le HFMSM^{(r)}(h_{\beta_1}, h_{\beta_2}, \cdots, h_{\beta_n})$$

Proof. Since $\gamma_{\alpha_i} \leq \gamma_{\beta_i}$ for all i, $i = 1, \cdots, n$, we have

$$1 - \prod_{j=1}^{r} \gamma_{\alpha_i} \geq 1 - \prod_{j=1}^{r} \gamma_{\beta_i}$$

$$\Longrightarrow \left(\prod_{\substack{1 \leq i_1 < \cdots \\ < i_r \leq n}} \left(1 - \prod_{j=1}^{r} \gamma_{\alpha_i} \right) \right)^{\frac{1}{C_n^r}} \geq \left(\prod_{\substack{1 \leq i_1 < \cdots \\ < i_r \leq n}} \left(1 - \prod_{j=1}^{r} \gamma_{\beta_i} \right) \right)^{\frac{1}{C_n^r}}$$

$$\Longrightarrow \left(1 - \left(\prod_{\substack{1 \leq i_1 < \cdots \\ < i_r \leq n}} \left(1 - \prod_{j=1}^{r} \gamma_{\alpha_i} \right) \right)^{\frac{1}{C_n^r}} \right)^{\frac{1}{r}} \leq \left(1 - \left(\prod_{\substack{1 \leq i_1 < \cdots \\ < i_r \leq n}} \left(1 - \prod_{j=1}^{r} \gamma_{\beta_i} \right) \right)^{\frac{1}{C_n^r}} \right)^{\frac{1}{r}}$$

According to Definition 2.2 and Eq. (4), we can complete the proof of Theorem 3.6. □

Theorem 3.7. *Let $h_i (i = 1, 2, \cdots, n)$ be a collection of HFEs, $h^-_{min} = \min_i \{h^-_i | h^-_i = \min\{\gamma_i \in h_i\}\}$, and $h^+_{max} = \max_i \{h^+_i | h^+_i = \max\{\gamma_i \in h_i\}\}$. Then*

$$h^-_{min} \leq HFMSM^{(r)}(h_1, h_2, \cdots, h_n) \leq h^+_{max}$$

Proof. Since $h^-_{min} \leq h^-_i \leq \gamma_i \leq h^+_i \leq h^+_{max}$ for any $\gamma_i \in h_i (i = 1, 2, \cdots, n)$, then we have

$$(h^-_{min})^r \leq \bigcup_{\substack{\gamma_i \in h_i, \\ i=1,\cdots,n}} \left\{ \prod_{j=1}^{r} \gamma_{i_j} \right\} \leq (h^+_{max})^r$$

$$\Longrightarrow 1 - (h^-_{min})^r \geq \bigcup_{\substack{\gamma_i \in h_i, \\ i=1,\cdots,n}} \left\{ \left(\prod_{\substack{1 \leq i_1 < \cdots \\ < i_r \leq n}} \left(1 - \prod_{j=1}^{r} \gamma_{i_j} \right) \right)^{\frac{1}{C_n^r}} \right\} \geq 1 - (h^+_{max})^r$$

$$\Longrightarrow h^-_{min} \leq \bigcup_{\substack{\gamma_i \in h_i, \\ i=1,\cdots,n}} \left\{ \left(1 - \left(\prod_{\substack{1 \leq i_1 < \cdots \\ < i_r \leq n}} \left(1 - \prod_{j=1}^{r} \gamma_{i_j} \right) \right)^{\frac{1}{C_n^r}} \right)^{\frac{1}{r}} \right\} \leq h^+_{max}.$$

Thus the proof is completed. □

Next, we present some special cases of the $HFMSM^{(r)}$ operator by changing the parameter r.

Theorem 3.8. *If $r = 1$, then $HFMSM^{(r)}$ operator reduces to the hesitant fuzzy averaging (HFA) operator* (i.e., Eq. (1)).

Proof. By the definition of $HFMSM^{(r)}$, we have

$$\begin{aligned}
&HFMSM^{(1)}(h_1, h_2, \cdots, h_n) \\
&= \bigcup_{\substack{\gamma_i \in h_i, \\ i=1,\cdots,n}} \left\{ \left(1 - \left(\prod_{1 \leq i_1 \leq n} \left(1 - \prod_{j=1}^{1} \gamma_{i_j} \right) \right)^{\frac{1}{C_n^1}} \right)^{\frac{1}{1}} \right\} \\
&= \bigcup_{\substack{\gamma_i \in h_i, \\ i=1,\cdots,n}} \left\{ 1 - \left(\prod_{1 \leq i_1 \leq n} (1 - \gamma_{i_1}) \right)^{\frac{1}{n}} \right\} (let\ i_1 = i) \\
&= \bigcup_{\substack{\gamma_i \in h_i, \\ i=1,\cdots,n}} \left\{ 1 - \prod_{i=1}^{n} (1 - \gamma_i)^{\frac{1}{n}} \right\} \\
&= HFA(h_1, h_2, \cdots, h_n)
\end{aligned}$$

□

Theorem 3.9. *If $r = 2$, then $HFMSM^{(r)}$ operator reduces to the hesitant fuzzy interrelated square Bonferroni mean ($HFBM^{1,1}$) which was introduced by Zhu et al. in* [16].

Proof. Let $\rho_{i,j,i\neq j} = h_i \otimes h_j = \bigcup_{\gamma_i\in h_i,\gamma_j\in h_j,i\neq j} \{1-\gamma_i\gamma_j\} = \bigcup_{\delta_{i,j}\in\rho_{i,j}} \{1-\delta_{i,j}\}$, then by the definition of $HFMSM^{(r)}$, we have

$$\begin{aligned}
&HFMSM^{(2)}(h_1,h_2,\cdots,h_n)\\
&=\bigcup_{\substack{\gamma_i\in h_i,\\ i=1,\cdots,n}}\left\{\left(1-\left(\prod_{\substack{1\le i_1<\\ i_2\le n}}\left(1-\prod_{j=1}^{2}\gamma_{i_j}\right)\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{2}}\right\}\\
&=\bigcup_{\substack{\gamma_i\in h_i,\\ i=1,\cdots,n}}\left\{\left(1-\left(\prod_{\substack{1\le i_1<\\ i_2\le n}}(1-\gamma_{i_1}\gamma_{i_2})\right)^{\frac{2}{n(n-1)}}\right)^{\frac{1}{2}}\right\}(let\ i_1=i,i_2=j)\\
&=\bigcup_{\substack{\gamma_i\in h_i,\gamma_j\in h_j,\\ i,j=1,\cdots,n,i<j}}\left\{\left(1-\prod_{\substack{i,j=1\\ i<j}}^{n}(1-\gamma_i\gamma_j)^{\frac{2}{n(n-1)}}\right)^{\frac{1}{2}}\right\}\\
&=\bigcup_{\substack{\delta_{i,j}\in\rho_{i,j},\\ i,j=1,\cdots,n,i\neq j}}\left\{\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}(1-\delta_{i,j})^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\right\}\\
&=HFB^{1,1}(h_1,h_2,\cdots,h_n)
\end{aligned}$$

□

Theorem 3.10. *If $r = n$, then $HFMSM$ operator reduces to the hesitant fuzzy geometric (IFG) operator* (i.e., Eq. (2)).

Proof. By the definition of $HFMSM$, we have

$$\begin{aligned}
&HFMSM^{(1)}(h_1,h_2,\cdots,h_n)\\
&=\bigcup_{\substack{\gamma_i\in h_i,\\ i=1,\cdots,n}}\left\{\left(1-\left(\prod_{\substack{1=i_1<\cdots\\ <i_n=n}}\left(1-\prod_{j=1}^{n}\gamma_{i_j}\right)\right)^{\frac{1}{C_n^n}}\right)^{\frac{1}{n}}\right\}\\
&=\bigcup_{\substack{\gamma_i\in h_i,\\ i=1,\cdots,n}}\left\{\left(1-\left(1-\prod_{j=1}^{n}\gamma_{i_j}\right)^{\frac{1}{1}}\right)^{\frac{1}{n}}\right\}(let\ i_j=j)\\
&=\bigcup_{\substack{\gamma_j\in h_j,\\ j=1,\cdots,n}}\left\{\left(\prod_{j=1}^{n}\gamma_j\right)^{\frac{1}{n}}\right\}\\
&=HFG(h_1,h_2,\cdots,h_n)
\end{aligned}$$

□

Theorem 3.8-3.10 show that some exiting hesitant fuzzy aggregation operators are the special cases of the $HFMSM$ oprator.

4 The weighted hesitant fuzzy operator and its application in decision making

In many practical applications, the weights of attributes should be taken into account. Especially for multiple attribute decision making problems, the considered attributes usually are of different importance. To overcome the limitations of the $HFMSM$ operator defined in the previous section, in this section, we shall introduce the weighted hesitant fuzzy Maclaurin symmetric mean ($WHFMSM$) operator and apply it to solve multiple attribute decision making problems.

4.1 $WHFMSM$ operator

we first introduce the definition of $WHFMSM$ operator as follows.

Definition 4.1. *Let $h_i(i=1,2,\cdots,n)$ be a collection of $HFEs$, $r=1,2,\cdots,n$, $w=(w_1,w_2,\cdots,w_n)^T$ is the weight vector of $h_i(i=1,2,\cdots,n)$ with $w_i\in[0,1]$ and $\sum\limits_{i=1}^{n}=1$. If*

$$WHFMSM_w^{(r)}(h_1,h_2,\cdots,h_n)=\left(\frac{\bigoplus\limits_{\substack{1\le i_1<\cdots\\<i_r\le i_n}}\bigotimes\limits_{j=1}^{r}w_{i_j}h_{i_j}}{C_n^r}\right)^{\frac{1}{r}} \tag{5}$$

then $WHFMSM_w^{(r)}$ is called the weighted hesitant fuzzy Maclaurin symmetric mean, where $(i_1,i_2,\cdots,i_r)$ traversal all the r-tuple combination of $(1,2,\cdots,n)$, C_n^r is the binomial coefficient.

According to the operations of $HFEs$ described in Section 2, we can derive the following Theorem 4.2.

Theorem 4.2. *Let $h_i(i=1,2,\cdots,n)$ be a collection of $HFEs$ and $r=1,2,\cdots,n$. Then the aggregated value, by using the $WHFMSM^{(r)}$, is also an HFE, and*

$$WHFMSM_w^{(r)}(h_1,h_2,\cdots,h_n)=\bigcup_{\substack{\gamma_i\in h_i,\\ i=1,\cdots,n}}\left\{\left(1-\left(\prod_{\substack{1\le i_1<\cdots\\<i_r\le n}}\left(1-\prod_{j=1}^{r}\left(1-(1-\gamma_{i_j})^{w_{i_j}}\right)\right)\right)^{\frac{1}{C_n^r}}\right)^{\frac{1}{r}}\right\}$$

Proof. The proof is similar to one of Theorem 3.2. ☐

4.2 An application to multiple attribute decision making

Based on $WHFMSM$ operator, below we develop an approach to multiple attribute decision making under hesitant fuzzy environment.

For a multiple attribute decision making problem, let $Y=\{Y_1,Y_2,\cdots,Y_m\}$ be a discrete set of alternatives, $A=\{A_1,A_2,\cdots,A_n\}$ be a collection of attributes, whose weight vector is $w=(w_1,w_2,\cdots,w_n)^T$, satisfying $w_i\in[0,1]$ and $\sum_{i=1}^{n}=1$, where w_i represents the importance degree of the attribute A_i. The decision makers provide several values for the alternative $Y_i(i=1,2,\cdots,m)$ under the attribute $A_j(j=1,2,\cdots,n)$ with anonymity, these values can be considered as an HFE $h_{ij}=\cup_{\gamma_{ij}\in h_{ij}}\{\gamma_{ij}\}$. All elements $h_{ij}(i=1,2,\cdots,m,j=1,2,\cdots,n)$ construct a hesitant fuzzy decision matrix the decision matrix $H=(h_{ij})_{m\times n}$.

Then, we use the $WHFMSM$ operator to develop an approach to multiple attribute decision making problems with hesitant fuzzy information, which can be described as follows:

Step1. According to the decision information provided by the decision makers, construct the hesitant fuzzy decision matrix $H=(h_{ij})_{m\times n}$. If there are some cost attributes in decision making problems, then we need to transform the decision matrix $H=(h_{ij})_{m\times n}$ into a normalization matrix $P=(p_{ij})_{m\times n}$, where

$$p_{ij}=\begin{cases}p_{ij}, & \text{for benefit attribute } A_{ij},\\ p_{ij}^c, & \text{for for cost attribute } A_{ij}.\end{cases}$$

Here $p_{ij}=\cup_{\gamma_{ij}\in p_{ij}}\{\gamma_{ij}\}$, p_{ij}^c is the complement of p_{ij} and $p_{ij}^c=\cup_{\gamma_{ij}\in p_{ij}}\{1-\gamma_{ij}\}$.

Step2. Utilize the $WHFMSM$ operator

$$p_i=WHFMSM_w^{(r)}(p_{i1},p_{i2},\cdots,p_{in})$$

to aggregate all the performance values p_{ij} $(j = 1, 2, \cdots, n)$ of the ith line and get the overall performance value p_i corresponding to the alternative Y_i $(i = 1, 2, \cdots, m)$.
Step3. Calculate the score values $s(p_i)$ of the overall preference value p_i $(i = 1, 2, \cdots, m)$.
Step4. Rank all the alternatives Y_i $(i = 1, 2, \cdots, m)$ according to $s(p_i)$ in descending order, and then select the best one.

4.3 Illustrative example

Let us consider a Management School in a Chinese university, which wants to introduce a teacher (adapted from [24]). There is a panel with five possible alternatives. A set of four factors are considered: $A = \{A_1, A_2, A_3, A_4\}$={morality, research capability, teaching skill, education background}, whose weight vector is $w = (0.3, 0.2, 0.1, 0.4)^T$. The experts evaluate four alternatives $Y_i(i = 1, 2, \cdots, 4)$ in relation to the factors $A = \{A_1, A_2, A_3, A_4\}$. The evaluation information on the four alternatives $Y_i(i = 1, 2, \cdots, 4)$ under the factors $A = \{A_1, A_2, A_3, A_4\}$ are represented by the $HFEs$.

Step1. Construct the hesitant fuzzy decision matrix $H = (h_{ij})_{5\times 4}$, which is listed in Table 1. Considering that all the attributes $A_j(j = 1, 2, 3, 4)$ are the benefit type attributes, the performance values of the alternatives $Y_i(j = 1, 2, \cdots, 5)$ do not need normalization.

Step2. Utilize the $WHFMSM$ operator aggregate all the performance values $h_{ij}(j = 1, 2, 3, 4)$ of the ith line and obtain the overall preference value h_i corresponding to the alternative Y_i. Take alternative Y_1 for an example, and let $r = 2$, we have

$$\begin{aligned} h_1 &= WHFDMSM_w^{(r)}(h_{11}, h_{12}, \cdots, h_{14}) \\ &= \bigcup_{\substack{\gamma_i \in h_{1i},\\ i=1,\cdots,n}} \left\{ \left(1 - \left(\prod_{\substack{1\le i_1<\cdots\\<i_r\le n}} \left(1 - \prod_{j=1}^{r} \gamma_{i_j}^{w_{i_j}} \right) \right)^{\frac{1}{C_n^r}} \right)^{\frac{1}{r}} \right\}_, \\ &= \{0.814019, 0.816764, 0.819298, 0.818635, 0.821303, 0.823767, 0.823095, 0.82569, 0.828086, \\ &\quad 0.820225, 0.822867, 0.825307, 0.824669, 0.827238, 0.829611, 0.828963, 0.831463, 0.833771, \\ &\quad 0.826476, 0.829016, 0.831362, 0.830748, 0.833218, 0.8355, 0.834877, 0.837281, 0.839501\}. \end{aligned}$$

As the parameter r changes we can get different results for each alternative, here we will not list them for vast amounts of data.

Step3. Compute the score values $s(h_i)(i = 1, 2, \cdots, 5)$ of $h_i(i = 1, 2, 3, 4)$ by Definition 2.2. The score values for the alternatives are listed in Table 2.

Step4. By ranking $s(h_i)(i = 1, 2, \cdots, 5)$, we can get the priorities of the alternatives $Y_i(i = 1, 2, \cdots, 5)$ as the parameter r changes, which are shown in Table 2.

From Table 2, it can be seen that the ranking results are slightly different when the parameter change, which indicates the parameter can reflect the decision maker's risk preferences. Furthermore, we can find that the score values obtained by the $WHFMSM$ operator become smaller when the parameter r increases for the same aggregation arguments. Therefore, the decision makers can choose a proper value of the parameter r according to their risk preferences in real practical decision making process.

Table 1: Hesitant fuzzy decision making matrix H

	A_1	A_2	A_3	A_4
Y_1	{{0.4, 0.5,0.6}	{0.7}	{0.2}	{0.4}
Y_2	{0.2, 0.5, 0.8}	{(0.5, 0.7}	{0.6, 0.8}	{0.6, 0.7}
Y_3	{0.8, 0.9}	{0.4, 0.6}	{0.3, 0.4}	{0.1, 0.3}
Y_4	{0.3, 0.4, 0.6, 0.8}	{0.4, 0.8}	{0.3, 0.5}	{0.5, 0.6}
Y_5	{0.3, 0.5, 0.7}	{0.2, 0.4}	{0.6, 0.7}	{0.4, 0.6}

Table 2: Score values obtained by the $WHFMSM$ and the rankings of alternatives

	Y_1	Y_2	Y_3	Y_4	Y_5	Rankings
$r=1$	0.156157	0.218865	0.194147	0.188703	0.158	$Y_2 \succ Y_3 \succ Y_4 \succ Y_5 \succ Y_1$
$r=2$	0.14665	0.205581	0.15069	0.175264	0.147543	$Y_2 \succ Y_4 \succ Y_3 \succ Y_5 \succ Y_1$
$r=3$	0.135688	0.198208	0.132396	0.163811	0.140404	$Y_2 \succ Y_4 \succ Y_5 \succ Y_1 \succ Y_3$
$r=4$	0.113003	0.185361	0.116269	0.14538	0.130822	$Y_2 \succ Y_4 \succ Y_5 \succ Y_3 \succ Y_1$

Remark 4.3. To demonstrate the advantages of our method, in the following, we compare our method with the existing methods, such as the $HFWA$ and $HFWG$ operators introduced by Xia and Xu [11], and the weighted hesitant fuzzy Bonferroni mean ($WHFB_w^{p,q}$) and weighted hesitant fuzzy geometric Bonferroni mean ($WHFGB_w^{p,q}$) proposed by [15,16]. The rankings obtained by different aggregation operators are listed in Table 3.

From Table 3, we can see that i) when $r=1$, the $WHFMSM$ and $HFWA$ operators have the same rankings; ii) when $r=n$, the $WHFMSM$ and $HFWG$ operators have the same rankings; iii) when $r=2$, the $WHFMSM$, $WHFB_w^{p,q}$ and $WHFGB_w^{p,q}$ operators have the same rankings. It verifies the proposed method is reasonable and validity.

(1) Compare with the $HFWA$ and $HFWG$ operators. Our method can deal with the multiple attribute decision making problems where the attributes are independent, whereas the $HFWA$ and $HFWG$ operators can not do them. In addition, the $WHFMSM$ has an alterable parameter, With the change of the parameter, the proposed operator can be evolved into lots of different aggregation operators, which make decision making more flexible and can meet the needs of different types of decision makers. But the $HFWA$ (or $HFWG$) operator has not alterable parameter, so they can only satisfy the demand of a type of decision makers.

(2) Compare with the $WHFB_w^{p,q}$ and $WHFGB_w^{p,q}$ operators. The main advantage of the proposed method is that it can capture the interrelationship among the multi-input arguments, while the $WHFB_w^{p,q}$ and $WHFGB_w^{p,q}$ operators can only capture the interrelationship between two arguments. That is to say, our method is more general. In addition, the $WHFB_w^{p,q}$ and $WHFGB_w^{p,q}$ operators consider two parameters, while our method only needs to take one parameter. Therefore, the computational complexity of the $WHFB_w^{p,q}$ and $WHFGB_w^{p,q}$ operators are much higher than our method. Moreover, the $WHFMSM$ has a desirable property that the score values are more smaller when the parameter r increases, which indicates the decision makers can select easily a proper value for the parameter r according to their risk preferences. But the $WHFB_w^{p,q}$ and $WHFGB_w^{p,q}$ operators do not have the property. It follows that they are difficult to determine the values of the parameters p and q to reflect the decision makers' risk preferences in real practical decision making process.

According to the comparisons and analysis above, it is clear that our method is more flexible and robust to aggregate hesitant fuzzy information. Therefore, It is more suitable than the exiting aggregation operators to solve hesitant fuzzy multiple attribute decision making problems in which the attributes are independent.

Table 3: Comparisons with the exiting aggregation operators

Aggregation operator	Rankings	Aggregation operator	Rankings
$WHFMSM_w^{(1)}$	$Y_2 \succ Y_3 \succ Y_4 \succ Y_5 \succ Y_1$	$WHFMSM_w^{(2)}$	$Y_2 \succ Y_4 \succ Y_3 \succ Y_5 \succ Y_1$
$HFWA$	$Y_2 \succ Y_3 \succ Y_4 \succ Y_5 \succ Y_1$	$WHFB_w^{1,1}$	$Y_2 \succ Y_4 \succ Y_3 \succ Y_5 \succ Y_1$
$WHFMSM_w^{(n)}$	$Y_2 \succ Y_4 \succ Y_5 \succ Y_3 \succ Y_1$	$WHFGB_w^{1,1}$	$Y_2 \succ Y_4 \succ Y_3 \succ Y_5 \succ Y_1$
$HFWG$	$Y_2 \succ Y_4 \succ Y_5 \succ Y_3 \succ Y_1$		

5 Conclusions

The MSM is a classical averaging mean operator, which has been widely used in information fusion. However, it can not deal with the hesitant fuzzy information. To fill this gap, in this paper, we have extended the MSM to hesitant fuzzy environment, and defined a hesitant fuzzy Maclaurin symmetric mean. Some desirable properties and special cases have been discussed in detail. Considering the weight vector of the arguments, we have further developed a weighted hesitant fuzzy Maclaurin symmetric mean which can consider the importance of each attribute and the interrelationship among multi-input arguments. We also have proposed a method to solve hesitant fuzzy multiple attribute decision making problems. The illustrative example has shown that the proposed method is not only reasonable and validity but also more suitable to deal with multiple attribute decision making problems in which the attributes are independent under hesitant fuzzy environment.

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A NOTE ON THE GENERALIZED q-CHANGHEE NUMBERS OF HIGHER ORDER

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Abstract. Recently, Changhee numbers and polynomials are introduced by T. Kim et al in [3]. In this paper, we consider the generalized q-Changhee polynomials and numbers of higher order by using the fermionic p-adic q-integral and give some relations between the generalized q-Changhee numbers of higher order and special numbers.

1. Introduction

Let d be fixed odd positive integer and let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will respectively denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completions of algebraic closure of $\mathbb{Q}_p$. The p-adic norm is defined $|p|_p = \frac{1}{p}$.

We set

$$X = X_d = \varprojlim_{N} \mathbb{Z}/dp^N\mathbb{Z},\ X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p),$$

$$a + dp^N\mathbb{Z}_p = \left\{x \in X | x \equiv a \pmod{dp^N}\right\},$$

where $a \in \mathbb{Z}$ and $0 \leq a < dp^n$.

When one talks of q-extension, q is various considered as an indeterminate, a complex $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. Throughout this paper, we use the notation :

$$[x]_{-q} = \frac{1-(-q)^x}{1-(-q)} \text{ and } [x]_q = \frac{1-q^x}{1-q}.$$

Hence, $\lim_{q\to1}[x]_q = x$ for each $x \in \mathbb{Z}_p$.

Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the *fermionic p-adic q-integral on $\mathbb{Z}_p$* is defined by T. Kim as follows :

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \text{ (see [4, 5]).} \tag{1.1}$$

Then, by (1.1), we can get the following well-known integral identity

$$I_{-q}(f_1) + I_{-q}(f) = [2]_q f(0), \tag{1.2}$$

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where $f_1(x) = f(x+1)$ (see [1, 4, 5, 6]).

Recently, q-Changhee numbers and polynomials are introduced by Kim et. al. in [9], and have been studied by many mathematicians, and possess many interesting properties (see [3, 7, 9, 10]). In this paper, we consider the generalized q-Changhee polynomials and numbers of higher order by using the fermionic p-adic q-integral and give some relations between the generalized q-Changhee numbers of higher order and special numbers.

2. The generalized q-Daehee numbers attached to χ

Let χ be the Dirichlet character with conductor $d \in \mathbb{N} = \{1, 2, \ldots\}$ with $d \equiv 1 \pmod 2$. Then the *generalized q-Changhee numbers $Ch_{n,\chi,q}$ attached to χ* are defined by the generating function to be

$$\frac{[2]_q}{1+q^d(1+t)^d}\sum_{a=0}^{d-1}(-1)^a\chi(a)q^a(1+t)^a = \sum_{n=0}^{\infty} Ch_{n,\chi,q}\frac{t^n}{n!}, \tag{2.1}$$

where $t \in \mathbb{C}_p$ and $|t|_p < p^{-\frac{1}{p-1}}$.

As is well known, the *generalized q-Euler numbers $E_{n,\chi,q}$ attached to χ* are defined by the generating function to be

$$\frac{[2]_q}{1+q^d e^{dt}}\sum_{a=0}^{d-1}(-1)^a\chi(a)q^a e^{at} = \sum_{n=0}^{\infty} E_{n,\chi,q}\frac{t^n}{n!}, \text{ (see [12]).}$$

The Stirling numbers of the first kind is given by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l \ (x \geq 0),$$

and the Stirling numbers of the second kind is defined by the generating function to be

$$(e^t-1)^n = n!\sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!}$$

(see [2, 11]).

By replacing t by $e^t - 1$ in (2.1), we can have

$$\sum_{n=0}^{\infty} Ch_{n,\chi,q}\frac{(e^t-1)^n}{n!} = \frac{[2]_q}{1+q^d e^{dt}}\sum_{a=0}^{d-1}(-1)^a\chi(a)q^a e^{at} = \sum_{m=0}^{\infty} E_{m,\chi,q}\frac{t^m}{m!}, \tag{2.2}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} Ch_{n,\chi,q}\frac{(e^t-1)^n}{n!} &= \sum_{n=0}^{\infty}\frac{Ch_{n,\chi,q}}{n!}n!\sum_{m=n}^{\infty} S_2(m,n)\frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} Ch_{n,\chi,q}S_2(m,n)\right)\frac{t^m}{m!}.\end{aligned} \tag{2.3}$$

Therefore, by (2.2) and (2.3), we obtain the following theorem.

Theorem 2.1. *For $m \geq 0$, we have*

$$E_{m,\chi,q} = \sum_{n=0}^{m} Ch_{n,\chi,q}S_2(m,n).$$

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Now, we define the *generalized q-Changhee polynomials* $Ch_{n,\chi,q}(x)$ as follows:

$$\frac{[2]_q}{1+q^d(1+t)^d}\sum_{a=0}^{d-1}(-1)^a\chi(a)q^a(1+t)^{a+x}=\sum_{n=0}^{\infty}Ch_{n,\chi,q}(x)\frac{t^n}{n!}, \tag{2.4}$$

where $t\in\mathbb{C}_p$ and $|t|_p<p^{-\frac{1}{p-1}}$.

Note that, in the special case, $x=0$, $Ch_{n,\chi,q}(0)=Ch_{n,\chi,q}$ are generalized q-Changhee numbers.

From (1.2), we can derive the following equation.:

$$q^nI_{-q}(f_n)+(-1)^{n-1}I_{-q}(f)=[2]_q\sum_{l=0}^{n-1}(-1)^{n-1-l}q^lf(l), \tag{2.5}$$

where $f_n(x)=f(x+n)$ and $n\geq 0$.

If taking $f(x)=\chi(x)(1+t)^x$ in (2.5), we can have

$$\begin{aligned}&q^d\int_X\chi(x)(1+t)^{x+d}d\mu_{-q}(x)+\int_X\chi(x)(1+t)^xd\mu_{-q}(x)\\=&[2]_q\sum_{a=0}^{d-1}(-1)^a\chi(a)q^a(1+t)^a.\end{aligned} \tag{2.6}$$

By (2.6), we can easily have

$$\begin{aligned}\int_X\chi(x)(1+t)^xd\mu_{-q}(x)&=\frac{[2]_q}{1+q^d(1+t)^d}\sum_{a=0}^{d-1}(-1)^a\chi(a)q^a(1+t)^a\\&=\sum_{n=0}^{\infty}Ch_{n,\chi,q}\frac{t^n}{n!},\end{aligned} \tag{2.7}$$

and

$$\int_X\chi(x)(1+t)^xd\mu_{-q}(x)=\sum_{n=0}^{\infty}\left(\int_X\chi(x)(x)_nd\mu_{-q}(x)\right)\frac{t^n}{n!}. \tag{2.8}$$

Therefore, by (2.7) and (2.8), we obtain the following theorem.

Theorem 2.2. *For $n\geq 0$, we have*

$$\frac{Ch_{n,\chi,q}}{n!}=\int_X\chi(x)\binom{x}{n}d\mu_{-q}(x).$$

By (2.4), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}Ch_{n,\chi,q}(x)\frac{t^n}{n!}=&\frac{[2]_q}{1+q^d(1+t)^d}\sum_{a=0}^{d-1}(-1)^a\chi(a)q^a(1+t)^{a+x}\\=&\left(\sum_{m=0}^{\infty}Ch_{m,\chi,q}\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}\binom{x}{l}t^l\right)\\=&\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{x}{n-m}\frac{Ch_{m,\chi,q}}{m!}t^n.\end{aligned} \tag{2.9}$$

So, by (2.9), we can have

$$\frac{Ch_{n,\chi,q}(x)}{n!} = \sum_{m=0}^{n} \binom{x}{n-m} \frac{Ch_{m,\chi,q}}{m!}. \tag{2.10}$$

From Theorem 2.2 and (2.10), we can derive the equations

$$\begin{aligned} \frac{Ch_{n,\chi,q}(x)}{n!} &= \sum_{m=0}^{n} \binom{x}{n-m} \frac{1}{m!} \int_X \chi(y) \binom{y}{m} d\mu_{-q}(y) \\ &= \int_X \chi(y) \binom{x+y}{n} d\mu_{-q}(y). \end{aligned} \tag{2.11}$$

Therefore, by (2.11), we obtain the following corollary.

Corollary 2.3. *For $n \geq 0$, we have*

$$\frac{Ch_{n,\chi,q}(x)}{n!} = \int_X \chi(y) \binom{x+y}{n} d\mu_{-q}(y).$$

For $r \in \mathbb{N}$, let us consider the *generalized q-Changhee numbers of order r attached to χ* as follows:

$$\begin{aligned} &\left(\sum_{a=0}^{d-1} \frac{[2]_q}{1+q^d(1+t)^d} (-1)^a \chi(a) q^a (1+t)^a \right)^r \\ =& \sum_{a_1,\dots,a_r=0}^{d-1} \left(\frac{[2]_q}{1+q^d(1+t)^d} \right)^r (-1)^{a_1+\cdots+a_r} \chi(a_1)\cdots\chi(a_r) q^{a_1+\cdots+a_r} (1+t)^{a_1+\cdots+a_r} \\ =& \sum_{n=0}^{\infty} Ch_{n,\chi,q}^{(r)} \frac{t^n}{n!}. \end{aligned} \tag{2.12}$$

By (2.7), we can see that

$$\begin{aligned} &\int_X \cdots \int_X \chi(x_1)\cdots\chi(x_r)(1+t)^{x_1+\cdots+x_r} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) \\ =& \sum_{a_1,\dots,a_r=0}^{d-1} \left(\frac{[2]_q}{1+q^d(1+t)^d} \right)^r (-1)^{a_1+\cdots+a_r} \chi(a_1)\cdots\chi(a_r) q^{a_1+\cdots+a_r} (1+t)^{a_1+\cdots+a_r}. \end{aligned} \tag{2.13}$$

Thus, by (2.12) and (2.13), we get

$$Ch_{n,\chi,q}^{(r)} = \int_X \cdots \int_X \chi(x_1)\cdots\chi(x_r)(x_1+\cdots+x_r)_n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r). \tag{2.14}$$

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From (2.14) and Theorem 2.2, we can drive

$$\begin{aligned}
&\frac{Ch^{(r)}_{n,\chi,q}}{n!} \\
=&\int_X \cdots \int_X \chi(x_1)\cdots\chi(x_r)\binom{x_1+\cdots+x_r}{n} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) \\
=&\int_X \cdots \int_X \chi(x_1)\cdots\chi(x_r)\sum_{l_1=0}^{n}\binom{x_1}{l_1}\sum_{l_2=0}^{n-l_1}\binom{x_2}{l_2}\cdots\sum_{l_{r-1}=0}^{n-l_1\cdots-l_{r-2}}\binom{x_{r-1}}{l_{r-1}} \\
&\times\binom{x_r}{n-l_1-\cdots-l_{r-1}} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) \\
=&\sum_{l_1=0}^{n}\sum_{l_2=0}^{n-l_1}\cdots\sum_{l_{r-1}=0}^{n-l_1\cdots-l_{r-2}}\frac{Ch_{l_1,\chi,q}Ch_{l_2,\chi,q}\cdots Ch_{l_{r-1},\chi,q}Ch_{n-l_1-\cdots-l_{r-1},\chi,q}}{l_1!l_2!\cdots l_{r-1}!(n-l_1-l_2-\cdots-l_{r-1})!}.
\end{aligned} \tag{2.15}$$

Therefore, by (2.13), (2.14) and (2.15), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$\begin{aligned}
Ch^{(r)}_{n,\chi,q} =& \sum_{l_1=0}^{n}\sum_{l_2=0}^{n-l_1}\cdots\sum_{l_{r-1}=0}^{n-l_1\cdots-l_{r-2}}\binom{n}{l_1,l_2,\cdots,l_{r-1},n-l_1-\cdots-l_{r-1}} \\
&\times Ch_{l_1,\chi,q}Ch_{l_2,\chi,q}\cdots Ch_{l_{r-1},\chi,q}Ch_{n-l_1-\cdots-l_{r-1},\chi,q}
\end{aligned}$$

where $\binom{n}{l_1,l_2,\cdots,l_r} = \frac{n!}{l_1!l_2!\cdots l_r!}$.

From (2.14), we note that

$$\begin{aligned}
&Ch^{(r)}_{n,\chi,q} \\
=&\int_X \cdots \int_X \chi(x_1)\cdots\chi(x_r)(x_1+\cdots+x_r)_n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) \\
=&\sum_{l=0}^{n} S_1(n,l)\int_X \cdots \int_X \chi(x_1)\cdots\chi(x_r)(x_1+\cdots+x_r)^l d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) \\
=&\sum_{l=0}^{n} S_1(n,l)E^{(r)}_{l,\chi,q},
\end{aligned} \tag{2.16}$$

where $E^{(r)}_{l,\chi,q}$ are the l-th generalized q-Euler numbers of order r attached to χ, which given by

$$\left(\frac{[2]_q}{1+q^d e^{dt}}\sum_{a=0}^{d-1}(-1)^a\chi(a)q^a e^{at}\right)^r = \sum_{n=0}^{\infty} E^{(r)}_{n,\chi,q}\frac{t^n}{n!}, \text{ (see [8]).}$$

Therefore, by (2.16), we obtain the following theorem.

Theorem 2.5. *For $n \geq 0$, we have*

$$Ch^{(r)}_{n,\chi,q} = \sum_{l=0}^{n} S_1(n,l)E^{(r)}_{l,\chi,q}.$$

By replacing t by $e^t - 1$ in (2.12), we can get

$$\begin{aligned}
&\sum_{n=0}^{\infty} Ch_{n,\chi,q}^{(r)} \frac{(e^t-1)^n}{n!} \\
&= \sum_{a_1,\ldots,a_r=0}^{d-1} \left(\frac{[2]_q}{1+q^d e^{dt}}\right)^r (-1)^{a_1+\cdots+a_r} \chi(a_1)\cdots\chi(a_r) q^{a_1+\cdots+a_r} e^{(a_1+\cdots+a_r)t} \quad (2.17)\\
&= \sum_{m=0}^{\infty} E_{m,\chi,q}^{(r)} \frac{t^m}{m!},
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty} Ch_{n,\chi,q}^{(r)} \frac{(e^t-1)^n}{n!} &= \sum_{n=0}^{\infty} \frac{Ch_{n,\chi,q}^{(r)}}{n!} n! \sum_{m=n}^{\infty} S_2(m,n) \frac{t^m}{m!} \\
&= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} Ch_{n,\chi,q}^{(r)} S_2(m,n) \right) \frac{t^m}{m!}.
\end{aligned} \tag{2.18}$$

Therefore, by (2.17) and (2.18), we obtain the following theorem.

Theorem 2.6. *For $n \geq 0$, we have*

$$E_{m,\chi,q}^{(r)} = \sum_{n=0}^{m} Ch_{n,\chi,q}^{(r)} S_2(m,n).$$

From (2.12), we can consider the *generalized q-Changhee polynomials of order r attached to χ* as follows:

$$\begin{aligned}
&\left(\sum_{a=0}^{d-1} \frac{[2]_q}{1+q^d(1+t)^d} (-1)^a \chi(a) q^a (1+t)^a \right)^r (1+t)^x \\
&= \sum_{a_1,\ldots,a_r=0}^{d-1} \left(\frac{[2]_q}{1+q^d(1+t)^d}\right)^r (-1)^{a_1+\cdots+a_r} \chi(a_1)\cdots\chi(a_r) q^{a_1+\cdots+a_r} (1+t)^{a_1+\cdots+a_r+x} \\
&= \sum_{n=0}^{\infty} Ch_{n,\chi,q}^{(r)}(x) \frac{t^n}{n!}.
\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}
&\int_X \cdots \int_X \chi(x_1)\cdots\chi(x_r)(1+t)^{x_1+\cdots+x_r+x} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) \\
&= \sum_{a_1,\ldots,a_r=0}^{d-1} \left(\frac{[2]_q}{1+q^d(1+t)^d}\right)^r (-1)^{a_1+\cdots+a_r} \chi(a_1)\cdots\chi(a_r) q^{a_1+\cdots+a_r} (1+t)^{a_1+\cdots+a_r+x}.
\end{aligned}$$

Thus, we get

$$Ch_{n,\chi,q}^{(r)}(x) = \int_X \cdots \int_X \chi(x_1)\cdots\chi(x_r)(x_1+\cdots+x_r+x)_n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r). \tag{2.20}$$

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From (2.20), we have

$$\begin{aligned}
&Ch^{(r)}_{n,\chi,q}(x)\\
&=\int_X \cdots \int_X \chi(x_1)\cdots\chi(x_r)(x_1+\cdots+x_r+x)_n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\\
&=\sum_{l=0}^{n} S_1(n,l)\int_X \cdots \int_X \chi(x_1)\cdots\chi(x_r)(x_1+\cdots+x_r+x)^l d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\\
&=\sum_{l=0}^{n} S_1(n,l)E^{(r)}_{l,\chi,q}(x).
\end{aligned} \tag{2.21}$$

Therefore, by (2.21), we obtain the following theorem.

Theorem 2.7. *For $n \geq 0$, we have*

$$Ch^{(r)}_{n,\chi,q}(x) = \sum_{l=0}^{n} S_1(n,l)E^{(r)}_{l,\chi,q}(x).$$

In (2.19), by replacing t by e^t-1, we can get

$$\begin{aligned}
\sum_{n=0}^{\infty} Ch^{(r)}_{n,\chi,q}(x)\frac{(e^t-1)^n}{n!} &= \left(\sum_{a=0}^{d-1}\frac{[2]_q}{1+q^d e^{dt}}(-1)^a\chi(a)q^a e^{at}\right)^r e^{xt}\\
&=\sum_{m=0}^{\infty} E^{(r)}_{m,\chi,q}(x)\frac{t^m}{m!}.
\end{aligned} \tag{2.22}$$

and

$$\sum_{n=0}^{\infty} Ch^{(r)}_{n,\chi,q}(x)\frac{(e^t-1)^n}{n!} = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} Ch^{(r)}_{n,\chi,q}(x)S_2(m,n)\right)\frac{t^m}{m!}. \tag{2.23}$$

Therefore, by (2.22) and (2.23), we obtain the following theorem.

Theorem 2.8. *For $n \geq 0$, we have*

$$E^{(r)}_{m,\chi,q}(x) = \sum_{n=0}^{m} Ch^{(r)}_{n,\chi,q}(x)S_2(m,n).$$

As is well-known, the rising factorial is given by

$$(x)^{(n)} = x(x+1)\cdots(x+n-1) = (-1)^n(-x)_n = \sum_{l=0}^{n}(-1)^{n-l}S_1(n,l)x^l, \tag{2.24}$$

where $n \geq 0$ (see [2, 11]).

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Next, we consider the *generalized q-Changhee numbers of order r attached to* χ of the second kind as follows:

$$\begin{aligned}
&\widehat{Ch}_{n,\chi,q}^{(r)}\\
=&\int_X \cdots \int_X \chi(x_1)\cdots\chi(x_r)(-x_1-\cdots-x_r)_n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\\
=&\sum_{l=0}^{n}(-1)^l S_1(n,l)\int_X \cdots \int_X \chi(x_1)\cdots\chi(x_r)(x_1+\cdots+x_r)^l d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\\
=&\sum_{l=0}^{n}(-1)^l S_1(n,l)E_{l,\chi,q}^{(r)}.
\end{aligned}$$

The generating function of $\widehat{Ch}_{n,\chi,q}^{(r)}$ is given by

$$\begin{aligned}
&\sum_{n=0}^{\infty}\widehat{Ch}_{n,\chi,q}^{(r)}\frac{t^n}{n!}\\
=&\int_X \cdots \int_X \chi(x_1)\cdots\chi(x_r)(1+t)_n^{-x_1-\cdots-x_r} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) \\
=&\left(\sum_{a=0}^{d-1}\frac{[2]_q}{1+q^d(1+t)^d}(-1)^a\chi(a)q^a(1+t)^a\right)^r(1+t)^r.
\end{aligned}\tag{2.25}$$

Now, we can observe that

$$\begin{aligned}
&\left(\sum_{a=0}^{d-1}\frac{[2]_q}{1+q^d(1+t)^d}(-1)^a\chi(a)q^a(1+t)^a\right)^r(1+t)^r\\
=&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{r}{m}Ch_{n-m,\chi,q}^{(r)}\frac{n!}{(n-m)!}\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.26}$$

Thus, by (2.25) and (2.26), we get

$$\widehat{Ch}_{n,\chi,q}^{(r)}=\sum_{m=0}^{n}m!\binom{r}{m}\binom{n}{m}Ch_{n-m,\chi,q}^{(r)}.\tag{2.27}$$

Therefore, by (2.27), we obtain the following theorem.

Theorem 2.9. *For $n\geq 0$, we have*

$$\widehat{Ch}_{n,\chi,q}^{(r)}=\sum_{m=0}^{n}m!\binom{r}{m}\binom{n}{m}Ch_{n-m,\chi,q}^{(r)}.$$

In (2.25), by replacing t by e^t-1, we can get

$$\begin{aligned}
\sum_{n=0}^{\infty}\widehat{Ch}_{n,\chi,q}^{(r)}\frac{(e^t-1)^n}{n!}&=\left(\sum_{a=0}^{d-1}\frac{[2]_q}{1+q^de^{dt}}(-1)^a\chi(a)q^ae^{at}\right)^r e^{rt}\\
&=\sum_{m=0}^{\infty}E_{m,\chi,q}^{(r)}(r)\frac{t^m}{m!}.
\end{aligned}\tag{2.28}$$

and

$$\sum_{n=0}^{\infty}\widehat{Ch}_{n,\chi,q}^{(r)}\frac{(e^t-1)^n}{n!}=\sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\widehat{Ch}_{n,\chi,q}^{(r)}S_2(m,n)\right)\frac{t^m}{m!}.\tag{2.29}$$

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Therefore, by (2.28) and (2.29), we obtain the following theorem.

Theorem 2.10. *For $n \geq 0$, we have*

$$E_{m,\chi,q}^{(r)}(r) = \sum_{n=0}^{m} \widehat{Ch}_{n,\chi,q}^{(r)} S_2(m,n).$$

Now, we define the *generalized q-Changhee polynomials of order r attached to χ* of the second kind as follows:

$$\widehat{Ch}_{n,\chi,q}^{(r)}(x) = \int_X \cdots \int_X \chi(x_1)\cdots\chi(x_r)(-x_1 - \cdots - x_r + x)_n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r). \tag{2.30}$$

Thus, by (2.30), we get

$$\begin{aligned} &\sum_{n=0}^{\infty} \widehat{Ch}_{n,\chi,q}^{(r)}(x)\frac{t^n}{n!} \\ =&\int_X \cdots \int_X \chi(x_1)\cdots\chi(x_r)(1+t)_n^{-x_1-\cdots-x_r+x} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) \\ =&\left(\sum_{a=0}^{d-1} \frac{[2]_q}{1+q^d(1+t)^d}(-1)^a\chi(a)q^a(1+t)^a\right)^r (1+t)^{x+r}. \end{aligned} \tag{2.31}$$

It is easy to show that

$$\begin{aligned} &\left(\sum_{a=0}^{d-1} \frac{[2]_q}{1+q^d(1+t)^d}(-1)^a\chi(a)q^a(1+t)^a\right)^r (1+t)^{x+r} \\ =&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} m!\binom{x}{m}\binom{n}{m}Ch_{n-m,\chi,q}^{(r)}\right)\frac{t^n}{n!}. \end{aligned} \tag{2.32}$$

Therefore, by (2.31) and (2.32), we obtain the following theorem.

Theorem 2.11. *For $n \geq 0$, we have*

$$\widehat{Ch}_{n,\chi,q}^{(r)}(x) = \sum_{m=0}^{n} m!\binom{x}{m}\binom{n}{m}Ch_{n-m,\chi,q}^{(r)}.$$

By (2.30), we get

$$\begin{aligned} &\widehat{Ch}_{n,\chi,q}^{(r)}(x) \\ =&\sum_{l=0}^{n}(-1)^l S_1(n,l)\int_X \cdots \int_X \chi(x_1)\cdots\chi(x_r)(x_1+\cdots+x_r-x)^l d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) \\ =&\sum_{l=0}^{n}(-1)^l S_1(n,l)E_{l,\chi,q}^{(r)}(-x). \end{aligned}$$

In (2.32), by replacing t by $e^t - 1$, we can get

$$\begin{aligned} \sum_{n=0}^{\infty} \widehat{Ch}_{n,\chi,q}^{(r)}(x)\frac{(e^t-1)^n}{n!} &= \left(\sum_{a=0}^{d-1} \frac{[2]_q}{1+q^d e^{dt}}(-1)^a\chi(a)q^a e^{at}\right)^r e^{(x+r)t} \\ &= \sum_{m=0}^{\infty} E_{m,\chi,q}^{(r)}(x+r)\frac{t^m}{m!}. \end{aligned} \tag{2.33}$$

and

$$\sum_{n=0}^{\infty} \widehat{Ch}_{n,\chi,q}^{(r)}(x)\frac{(e^t-1)^n}{n!} = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\widehat{Ch}_{n,\chi,q}^{(r)}(x)S_2(m,n)\right)\frac{t^m}{m!}. \tag{2.34}$$

Therefore, by (2.33) and (2.34), we obtain the following theorem.

Theorem 2.12. *For $n \geq 0$, we have*

$$E_{m,\chi,q}^{(r)}(x+r) = \sum_{n=0}^{m}\widehat{Ch}_{n,\chi,q}^{(r)}(x)S_2(m,n).$$

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An Investigation of the Certain Class of Multivalent Harmonic Mappings

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The main purpose of the present paper is to investigate some properties of the certain class of sense-preserving p-valent harmonic mappings in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} \| z| < 1\}$.

1 Introduction

Let Ω_1 be the family of functions $\varphi(z)$ which are analytic in the open unit disc $\mathbb{D}$, and satisfying the condition $|\varphi(z)| < 1$ for all $z \in \mathbb{D}$, and let Ω_2 be the family of functions $\phi(z)$ which are regular in $\mathbb{D}$ and satisfying the conditions $\phi(0) = 0$ and $|\phi(z)| < 1$ for every $z \in \mathbb{D}$. Denote by $\mathcal{P}(p, n)$, $p \geq 1$, $n \geq 1$ the family of functions $p(z) = p + p_1 z + \cdots$ which are regular in $\mathbb{D}$ and satisfying the condition $\mathrm{Re}p(z) > 0$. Let $s_1(z) = z + d_2 z^2 + \cdots$ and $s_2(z) = z + e_2 z^2 + \cdots$ be analytic functions in $\mathbb{D}$. If there exists $\phi(z) \in \Omega_2$ such that $s_1(z) = s_2(\phi(z))$ for every $z \in \mathbb{D}$, then we say that $s_1(z)$ is subordinate to $s_2(z)$ and we write $s_1 \prec s_2$. Specially, if $s_2(z)$ is univalent in $\mathbb{D}$, then $s_1 \prec s_2$ if and only if $s_1(\mathbb{D}) \subset s_2(\mathbb{D})$, and $s_1(0) = s_2(0)$ implies $s_1(\mathbb{D}_r) \subset s_2(\mathbb{D}_r)$, where $\mathbb{D}_r = \{z \| z| < r, 0 < r < 1\}$ (see [1], [4]).

We denote by $\mathcal{S}(p, n)$ ($p \geq 1$ and $n \geq 1$, integers) the class of all regular and p-valent functions in $\mathbb{D}$, having the series expansion of the form

$$s(z) = z^p + c_{np+1} z^{np+1} + c_{np+2} z^{np+2} + c_{np+3} z^{np+3} + \cdots + c_{np+m} z^{np+m} + \cdots \quad (1)$$

for all $z \in \mathbb{D}$. It is clear that $\mathcal{S}(p, 1) \supset \mathcal{S}(p, 2) \supset \mathcal{S}(p, 3) \supset \cdots \mathcal{S}(p, m) \supset \cdots$. Let $\mathcal{S}^*(p, n)$ ($p \geq 1$ and $n \geq 1$ integers) denote the class of functions of the form (1) which are regular in $\mathbb{D}$ and satisfying

$$\mathrm{Re}\left(z \frac{s'(z)}{s(z)}\right) > 0 \quad (2)$$

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and

$$\int_0^{2\pi} \operatorname{Re}\left(z\frac{s'(z)}{s(z)}\right) d\theta = 2pn\pi \tag{3}$$

for every $z \in \mathbb{D}$. A member of $\mathcal{S}^*(p,n)$ is called p-valent starlike function in the unit disc $\mathbb{D}$.

Finally, a planar harmonic mapping in the open unit disc $\mathbb{D}$ is a complex-valued harmonic function f, which maps $\mathbb{D}$ onto the some planar domain $f(\mathbb{D})$. Since $\mathbb{D}$ is a simply connected domain, the mapping f has a canonical decomposition $f = h + \overline{g}$, where $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$ and have the following power series expansion

$$h(z) = z^p + a_{np+1}z^{np+1} + a_{np+2}z^{np+2} + \cdots + a_{np+m}z^{np+m} + \cdots$$

and

$$g(z) = b_{np}z^{np} + b_{np+1}z^{np+1} + b_{np+2}z^{np+2} + \cdots + b_{np+m}z^{np+m} + \cdots$$

where $|b_{np}| < 1$, $p \geq 1$ and $n \geq 1$ integers, $a_{np+m}, b_{np+m} \in \mathbb{C}$ and every $z \in \mathbb{D}$. As usual, we call $h(z)$ the analytic part and $g(z)$ the co-analytic part of f, respectively, and let the class of such harmonic mappings is denoted by $\mathcal{SH}(p,n)$. Lewy (see [2]) proved in 1936 that the harmonic mapping f is locally univalent in $\mathbb{D}$ if and only if its Jacobian $J_f = |h'(z)|^2 - |g'(z)|^2$ is strictly positive in $\mathbb{D}$. In view of this result, locally univalent harmonic mappings in the open unit disc are either sense-reversing if $|g'(z)| > |h'(z)|$ or sense-preserving if $|g'(z)| < |h'(z)|$ in $\mathbb{D}$. Throughout this paper, we restrict ourselves to the study of sense-preserving harmonic mappings. We also note that an elegant and complete treatment theory of the harmonic mapping is given Duren's monograph (see [2]).

The main aim of this paper is to investigate the some properties of the following class

$$\mathcal{S}^*\mathcal{H}(p,n) = \Big\{ f = h + \overline{g} \in \mathcal{SH}(p,n) | w(z) = \frac{g'(z)}{h'(z)} \prec b_{np}\frac{1+\phi(z)}{1-\phi(z)},$$
$$\phi(z) = z^n\psi(z),\ \psi(z) \in \Omega_1,\ h(z) \in \mathcal{S}^*(p,n),\ z \in \mathbb{D} \Big\}$$

and for this aim we need the following lemma

Lemma 1.1 *([3]) Let $w(z) = a_nz^n + a_{n+1}z^{n+1} + a_{n+1}z^{n+2} + \cdots$ $(a_n \neq 0,\ n \geq 1)$ be analytic in $\mathbb{D}$. If the maximum value of $|w(z)|$ on the circle $|z| = r < 1$ is attained at $z = z_0$, then we have $z_0w'(z_0) = pw(z_0)$ where $p \geq n$ and every $z \in \mathbb{D}$.*

2 Main Results

Lemma 2.1 *If $p(z) \in \mathcal{P}(p,n)$ then*

$$p(z) = p\frac{1 + z^n\psi(z)}{1 - z^n\psi(z)},\ z \in \mathbb{D} \tag{4}$$

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where $\psi(z) \in \Omega_1$.

Proof. Consider the function $H(z)$ such that

$$H(z) = \frac{p(z)}{p}, \; z \in \mathbb{D} \tag{5}$$

where $p(z) \in \mathcal{P}(p,n)$. So, that $H(z)$ is regular and satisfies the conditions $\mathrm{Re}H(z) > 0$ and $H(0) = 1$ in $\mathbb{D}$. Let $\varphi(z) = (1 + H(z))/(1 - H(z))$, then $\varphi(z)$ is regular and $|\varphi(z)| < 1$ in the unit disc $\mathbb{D}$, and also $\varphi(z)$ has n$^{\text{th}}$ order zero at the origin. Hence, $\varphi(z) = z^n\psi(z)$ where $\psi(z)$ in Ω_1 for all $z \in \mathbb{D}$. Expressing $H(z)$ in terms of $\varphi(z)$ we have

$$H(z) = \frac{1+\varphi(z)}{1-\varphi(z)}, \; z \in \mathbb{D}. \tag{6}$$

Thus,

$$H(z) = \frac{p(z)}{p} = \frac{1+\varphi(z)}{1-\varphi(z)} = \frac{1+z^n\psi(z)}{1-z^n\psi(z)}$$

or

$$p(z) = p\frac{1+z^n\psi(z)}{1-z^n\psi(z)}$$

for all $z \in \mathbb{D}$.

Lemma 2.2 *Let* $f = h + \overline{g}$ *be an element of* $\mathcal{S}^*\mathcal{H}(p,n)$, *then*

$$\left| w(z) - \frac{b_{np}(1-r^{2m})}{1-|b_{np}|^2 r^{2m}} \right| \le \frac{(1-|b_{np}|^2)r^m}{1-|b_{np}|^2 r^{2m}}, \; |z| = r < 1 \tag{7}$$

where $m = np - p + 1$.

Proof. Since $f = h + \overline{g} \in \mathcal{S}^*\mathcal{H}(p,n)$, then

$$\begin{aligned} w(z) = \frac{g'(z)}{h'(z)} &= \frac{(b_{np}z^p + b_{np+1}z^{np+1} + b_{np+2}z^{np+2} + \cdots)'}{(z^p + a_{np+1}z^{np+1} + a_{np+2}z^{np+2} + \cdots)'} \\ &= \frac{b_{np} + \frac{(np+1)b_{np+1}}{p} z^{np+1-p} + \cdots}{1 + \frac{(np+1)a_{np+1}}{p} z^{np+1-p} + \cdots} \end{aligned}$$

so that $w(0) = b_{np}$. On the other hand, because of the sense-preserving property we have that $|w(z)| < 1$ for every $z \in \mathbb{D}$. Thus, the function defined by

$$\phi(z) = \frac{w(z) - w(0)}{1 - \overline{w(0)}w(z)}, \; z \in \mathbb{D}$$

satisfies the conditions of Schwarz Lemma (see [1]). Therefore, we have the following subordination relation

$$w(z) = \frac{b_{np} + \phi(z)}{1 + \overline{b_{np}}\phi(z)} \text{ if and only if } w(z) \prec \frac{b_{np} + z^m}{1 + \overline{b_{np}}z^m} \; z \in \mathbb{D}. \tag{8}$$

It is easy to see that the linear transformation $\frac{b_{np}+z^m}{1+\overline{b_{np}}z^m}$ maps $|z|=r$ onto the circle with the center $C(r)=\left(\frac{\alpha_1(1-r^{2m})}{1-|b_{np}|^2r^{2m}},\frac{\alpha_2(1-r^{2m})}{1-|b_{np}|^2r^{2m}}\right)$ and having the radius $\rho(r)=\frac{(1-|b_{np}|^2)r^m}{1-|b_{np}|^2r^{2m}}$, where $\alpha_1=$Reb_{np} and $\alpha_2=$Imb_{np}, then we can write

$$\left|w(z)-\frac{b_{np}(1-r^{2m})}{1-|b_{np}|^2r^{2m}}\right|\leq\frac{(1-|b_{np}|^2)r^m}{1-|b_{np}|^2r^{2m}}$$

for all $|z|=r<1$. As a simple consequence of Lemma 2.2, we give the following corollary.

Corollary 2.3 *If $f=h(z)+\overline{g(z)}\in\mathcal{S}^*\mathcal{H}(p,n)$, then*

$$\frac{|b_{np}|-r^n}{1-|b_{np}|r^n}\leq|w(z)|\leq\frac{|b_{np}|+r^n}{1+|b_{np}|r^n},$$

$$\frac{(1-r^n)(1-|b_{np}|)}{1+|b_{np}|r^n}\leq1-|w(z)|\leq\frac{(1+r^n)(1-|b_{np}|)}{1+|b_{np}|r^n}$$

$$\frac{(1-r^n)(1+|b_{np}|)}{1-|b_{np}|r^n}\leq1+|w(z)|\leq\frac{(1+r^n)(1+|b_{np}|)}{1+|b_{np}|r^n}$$

and

$$\frac{(1-|b_{np}|^2)(1-r^{2n})}{(1+|b_{np}|r^n)^2}\leq1-|w(z)|^2\leq\frac{(1-|b_{np}|^2)(1-r^{2n})}{(1-|b_{np}|r^n)^2},$$

for all $|z|=r<1$.

Theorem 2.4 *Let $s(z)$ be an element of $\mathcal{S}^*(p,n)$, then the inequalities*

$$\frac{r^p}{(1+r^n)^{2p/n}}\leq|s(z)|\leq\frac{r^p}{(1-r^n)^{2p/n}}\tag{9}$$

and

$$\frac{pr^{p-1}(1-r^n)}{(1+r^n)^{(2p/n)+1}}\leq|s'(z)|\leq\frac{pr^{p-1}(1+r^n)}{(1-r^n)^{(2p/n)+1}}\tag{10}$$

hold for every $|z|=r<1$.

Proof. Since $f=h(z)+\overline{g(z)}\in\mathcal{S}^*\mathcal{H}(p,n)$ then we have $z\frac{s'(z)}{s(z)}\prec p\frac{1+z^n}{1-z^n}$ for all z in $\mathbb{D}$. Therefore, the inequality $\left|z\frac{s'(z)}{s(z)}\frac{p(1+r^{2n})}{1-r^{2n}}\right|\leq\frac{2pr^n}{1-r^{2n}}$ holds for every $|z|=r<1$. Thus, we have

$$\frac{p(1-r^n)}{1+r^n}\leq\left|z\frac{s'(z)}{s(z)}\right|\leq\frac{p(1+r^n)}{1-r^n}\tag{11}$$

or

$$\frac{p(1-r^n)}{1+r^n}\leq\text{Re}\left(z\frac{s'(z)}{s(z)}\right)\leq\frac{p(1+r^n)}{1-r^n}\tag{12}$$

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for all $|z| = r < 1$. It is fact that

$$\operatorname{Re}\left(z\frac{s'(z)}{s(z)}\right) = r\frac{\partial}{\partial r}\log|s(z)| \tag{13}$$

true for every $|z| = r < 1$. Considering (12) and (13) together we obtain

$$\frac{p(1-r^n)}{r(1+r^n)} \leq \frac{\partial}{\partial r}\log|s(z)| \leq \frac{p(1+r^n)}{r(1-r^n)},\ |z| = r < 1. \tag{14}$$

Integrating (14), we get (9). On the other hand the inequality (11) can be written in the form

$$\frac{p(1-r^n)}{r(1+r^n)}|s(z)| \leq |s'(z)| \leq \frac{p(1+r^n)}{r(1-r^n)}|s(z)|, |z| = r < 1. \tag{15}$$

Using (9) in (15) we get (10).

Theorem 2.5 *Let $f = h(z) + \overline{g(z)}$ be an element of $\mathcal{S}^*\mathcal{H}(p,n)$, then*

$$\frac{g(z)}{h(z)} = b_{np}\frac{1+\phi(z)}{1-\phi(z)}$$

where $|b_{np}| < 1$, $\phi(z) = z^n\psi(z)$ and $\psi(z) \in \Omega_1$ for every $z \in \mathbb{D}$.

Proof. Since $f = h(z) + \overline{g(z)} \in \mathcal{S}^*\mathcal{H}(p,n)$, we can write

$$w(\mathbb{D}_r) = \left\{z \in \mathbb{C} : \left|\frac{g'(z)}{h'(z)} - b_{np}\frac{1+r^{2n}}{1-r^{2n}}\right| \leq \frac{2|b_{np}|r^n}{1-r^{2n}},\ |z| = r < 1\right\}. \tag{16}$$

On the other hand, since $h(z)$ is an element of $\mathcal{S}^*(p,n)$, the value of $h(z)/(zh'(z))$ at a point z_1 on the circle $|z| = r$ is

$$\frac{h(z_1)}{z_1h'(z_1)} = \frac{1}{p}\frac{1-r^n}{1+r^n}. \tag{17}$$

Now, we define the function

$$\frac{g(z)}{h(z)} = \frac{1+\phi(z)}{1-\phi(z)}, \tag{18}$$

where $\phi(z) = z^n\psi(z)$, $\psi(z) \in \Omega_1$ and $z \in \mathbb{D}$, then $\phi(z)$ analytic in $\mathbb{D}$ and $\phi(0) = 0$. We need to show that $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. Assume to the contrary, that there exists a $z_1 \in \mathbb{D}$ such that $|\phi(z_1)| = 1$. If we take the derivative of (18) and after simple calculations we get

$$w(z) = \frac{g'(z)}{h'(z)} = b_{np}\left(\frac{1+\phi(z)}{1-\phi(z)} + \frac{2z\phi'(z)}{(1-\phi(z))^2}\frac{h(z)}{zh'(z)}\right), z \in \mathbb{D}.$$

Considering (12), (13), (15) and Lemma 1.1 together we obtain that

$$w(z_1) = \frac{g'(z_1)}{h'(z_1)} = b_{np}\left(\frac{1+\phi(z_1)}{1-\phi(z_1)} + \frac{2p\phi'(z_1)}{(1-\phi(z_1))^2}\frac{1}{p}\frac{1-r^n}{1+r^n}\right) \notin w(\mathbb{D}_r), |z| = r.$$

But this is a contradiction, therefore, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. Thus, for a function $f = h(z) + \overline{g(z)}$ in $\mathcal{S}^*\mathcal{H}(p,n)$ we have

$$\frac{g(z)}{h(z)} = b_{np}\frac{1+\phi(z)}{1-\phi(z)}, z \in \mathbb{D}.$$

Corollary 2.6 *Let $f = h(z) + \overline{g(z)}$ be an element of $\mathcal{S}^*\mathcal{H}(p,n)$, then*

$$\frac{p|b_{np}|r^{p-1}(1-r^n)^2}{(1+r^n)^{\frac{2p}{n}+2}} \le |g'(z)| \le \frac{p|b_{np}|r^{p-1}(1+r^n)^2}{(1-r^n)^{\frac{2p}{n}+2}}, \tag{19}$$

and

$$\frac{|b_{np}|r^p(1-r^n)}{(1+r^n)^{\frac{2p}{n}+1}} \le |g(z)| \le \frac{|b_{np}|r^p(1+r^n)}{(1-r^n)^{\frac{2p}{n}+1}}, \tag{20}$$

for every $|z| = r < 1$.

Proof. Using the definition of the class $\mathcal{S}^*\mathcal{H}(p,n)$ and Theorem 2.5, we obtain

$$\frac{|b_{np}|(1-r^n)}{1+r^n}|h'(z)| \le |g'(z)| \le \frac{|b_{np}|(1+r^n)}{1-r^n}|h'(z)|$$

and

$$\frac{|b_{np}|(1-r^n)}{1+r^n}|h(z)| \le |g(z)| \le \frac{|b_{np}|(1+r^n)}{1-r^n}|h(z)|$$

for all z in $\mathbb{D}$. If we use Theorem 2.4 in the last inequalities we obtain (19) and (20).

Corollary 2.7 *If $f = h(z) + \overline{g(z)} \in \mathcal{S}^*\mathcal{H}(p,n)$, then*

$$\frac{p^2r^{2(p-1)}(1-r^n)^3(1+|b_{np}|^2)}{(1+r^n)^{\frac{4p}{n}+1}(1+|b_{np}|r^n)^2} \le J_f \le \frac{p^2r^{2(p-1)}(1+r^n)^3(1-|b_{np}|^2)}{(1-r^n)^{\frac{4p}{n}+1}(1-|b_{np}|r^n)^2},\ |z| = r < 1.$$

This corollary is a simple consequence of Corollary 2.3, Theorem 2.4 and the following equalities

$$J_f = |h'(z)|^2 - |g'(z)|^2 = |h'(z)|^2(1-|w(z)|^2), z \in \mathbb{D}.$$

Corollary 2.8 *Let $f = h(z) + \overline{g(z)}$ be an element of $\mathcal{S}^*\mathcal{H}(p,n)$, then*

$$p(1-|b_{np}|)\int \frac{r^{p-1}(1-r^n)^2}{(1+r^n)^{\frac{2p}{n}+1}(1+|b_{np}|r^n)}dr \le |f|$$
$$\le p(1+|b_{np}|)\int \frac{r^{p-1}(1+r^n)^2}{(1-r^n)^{\frac{2p}{n}+1}(1+|b_{np}|r^n)}dr$$

This corollary is a simple consequence of Corollary 2.3, Theorem 2.4 and the following inequalities

$$|h'(z)|(1-|w(z)|)|dz| \le |df| \le |h'(z)|(1+|w(z)|)|dz|, z \in \mathbb{D}.$$

6

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Robust Stabilization Based on Periodic Observers for LDP Systems *

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Abstract

In this paper, the problem of robust stabilization based on observers for linear discrete-time periodic (LDP) systems is studied. It is proofed that principle of separating exists in this type of systems. Based on this, periodic controllers and periodic state observers can be builded independently. Utilizing parametric poles assignment algorithm and robust performance index, an algorithm of robust stabilization based on periodic observers is proposed. A numerical example is employed to verify the effectiveness of the presented approaches.

Keywords: Robust stabilization; Periodic observers; Principle of separating; LDP systems.

1 Introduction

The analysis and control of linear discrete periodic (LDP) systems have long been interesting problems in the control fields, because LDP systems, such as cyclostationary process, and multirate digital control which occur in control systems, arise often in nature and in engineering ([1]). Thus, this type of systems have been widely researched (see [2]-[8] and references therein). The lifting technique and the cyclic technique are used to carry out such analysis studies, since they can preserve the system's algebraic structure and norms. Based on their lifted LTI reformulation, structural properties such as observability, reachability, detectability, and stabilizability are analyzed [9].

Periodic linear systems have received renewed interested in recent years. For example, semi-global stabilization of discrete-time periodic systems subject to actuator saturation is investigated in [10] by solutions to a parametric periodic Lyapunov equation, stability and stabilization of discrete-time periodic linear systems with actuator saturation is studied in [11] via periodic invariant set, stabilization of continuous-time periodic linear systems is solved in [12] via a periodic Lyapunov equation based approach, L_∞ and L_2 semi-global stabilization of continuous-time periodic linear systems with bounded controls is studied in [13], and stabilization of periodic systems with input and output delays is investigated in [14]. For more related recent work on the control of periodic systems, interested readers may refer to the references cited in [10, 11, 12] and [13].

In engineering, it is usually required to stabilize an unstable periodic motion or a critically stable periodic motion by using proper control. The stabilization problem has a fundamental importance in engineering, and hence the stabilization of periodic motions of dynamic systems has drawn much attention over the past years (see [11]-[15] and references therein). Observers can extract real-time information of a plant's internal state from its input-output data. Therefore, Observer-based control has been widely investigated (e.g., [16]-[21]).

In this paper, we investigate the problem of robust stabilization for uncertain LDP systems. On the problem of observer based control without robustness considerations, a trivial result has been present at [22]. According to the principle of separating, the problem of stabilization based on observer is converted into problems of stabilizing an argumented system and designing a periodic observer respectively. By adopting parametric poles assignment approach combined with a sensitivity index, robust stabilization problem is solved. Two

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detailed algorithms are presented and an example is utilized to illustrate the design procedures proposed in this paper.

Notation 1 *The superscripts "T" and "-1" stand for matrix transposition and matrix inverse, respectively;* $\mathbb{R}^n$ *denotes the n-dimensional Euclidean space;* $\overline{i,j}$ *represents the integer set* $\{i, i+1, \ldots, j-1, j\}$*; For a square time-varying matrix* $A(t), t = 0, 1, \cdots$ *,we denote* $\Phi_{\mathrm{A}}(j,i) = A(j-1)A(j-2)\cdots A(i)$ *for* $j > i$ *and* $\Phi_{\mathrm{A}}(i,i) = I$*; The notation* $\|\cdot\|_{\mathrm{F}}$ *is Frobenius norm.*

2 Preliminaries

Consider LDP systems with the following state space representation

$$\begin{cases} x(t+1) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) \end{cases} \tag{1}$$

where $t \in \mathbb{Z}$, the set of integers, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^r$ and $y(t) \in \mathbb{R}^m$ are respectively the state vector, the input vector and the output vector, $A(t)$, $B(t)$, $C(t)$ are matrices of compatible dimensions satisfying

$$A(t+T) = A(t), \quad B(t+T) = B(t), C(t+T) = C(t).$$

In case that the state of system (1) can be measured, by periodic feedback control law

$$u(t) = K(t)x(t) + v(t), \quad K(t+T) = K(t), \quad K(t) \in \mathbb{R}^{r\times n} \tag{2}$$

where $v(t)$ is the reference input, we can obtain the following combined system with period T

$$\begin{cases} x(t+1) = (A(t) + B(t)K(t))x(t) + B(t)v(t) \\ y(t) = C(t)x(t) \end{cases} \tag{3}$$

When there exists some restrictions in practice, the state of system (1) can not be gotten by hardware, but the input $u(t)$ and the output $y(t)$ can be measured. In this case, we need build another periodic system giving an asymptotic estimation of system states. The system with the following form can be adopted:

$$\begin{cases} \hat{x}(t+1) = A(t)\hat{x}(t) + B(t)u(t) + L(t)(C(t)\hat{x} - y(t)) \\ \hat{x}(0) = \hat{x}_0 \end{cases} \tag{4}$$

where $\hat{x} \in \mathbb{R}^n$ and $L(t) \in \mathbb{R}^{n\times m}$, $t \in \mathbb{Z}$ are real matrices of period T.

Utilizing observer (4), we can build a periodic control law based on the observed states as

$$u(t) = K(t)\hat{x}(t) + v(t) \tag{5}$$

such that the combined system meets some control aims, e.g., stability.

Similar to its LTI counterpart, for LDP systems, we present a simple existence condition of observers and omit its proof.

Proposition 1 *There exist matrices* $L(t)$, $t \in \overline{0, T-1}$ *such that system (4) becomes a full order state observer of system (1) if and only if periodic matrix pairs* $(A(t), C(t))$ *are detectable. In this case, we only need to choose* $L(t)$, $t \in \overline{0, T-1}$ *such that matrix*

$$\Phi_{\mathrm{A+LC}}(T,0) = (A(T-1) + L(T-1)C(T-1))(A(T-2) + L(T-2)C(T-2))\cdots(A(0) + L(0)C(0))$$

is stable.

Plugging (5) into (4) gives

$$\begin{cases} \hat{x}(t+1) = (A(t) + L(t)C(t))\hat{x}(t) - L(t)y(t) + B(t)u(t) \\ u(t) = K(t)\hat{x}(t) + v(t) \end{cases} \tag{6}$$

Integrating control law (6) into system (1), we can get

$$\left\{\begin{array}{l}\left[\begin{array}{c}x(t+1)\\ \hat{x}(t+1)\end{array}\right]=\left[\begin{array}{cc}A(t) & B(t)K(t)\\ -L(t)C(t) & F(t)+B(t)K(t)\end{array}\right]\left[\begin{array}{c}x(t)\\ \hat{x}(t)\end{array}\right]+\left[\begin{array}{c}B(t)\\ B(t)\end{array}\right]v(t)\\ y(t)=\left[\begin{array}{cc}C(t) & 0\end{array}\right]\left[\begin{array}{c}x(t)\\ \hat{x}(t)\end{array}\right]\end{array}\right. \tag{7}$$

where $F(t)=A(t)+L(t)C(t)$.

With the preparation, the stabilization problem of system (1) based on observers can be formed as the following:

Problem 1 *Given a completely observable and reachable LDP system (1), find matrices* $L(t)\in\mathbf{R}^{n\times m}, t\in\overline{0,T-1}$ *and* $K(t)\in\mathbf{R}^{r\times n}, t\in\overline{0,T-1}$*, such that the augmented system (7) is stable.*

Because of the inaccuracy of modelling and the influence of their internal perturbation and external disturbance from environment, unavoidably, system model has uncertainties, leading to the necessity of the study of robustness for LDP systems. Robust stabilization of system (1) based on observers can be formed as follows:

Problem 2 *Given a completely observable and reachable LDP system (1), find matrices* $L(t)\in\mathbf{R}^{n\times m}, t\in\overline{0,T-1}$ *and* $K(t)\in\mathbf{R}^{r\times n}, t\in\overline{0,T-1}$*, such that the augmented system (7) is stable and as insensitive as possible to small changes of system data.*

3 Main result

Consider the following LDP system

$$\left\{\begin{array}{l}x(t+1)=\tilde{A}(t)x(t)+\tilde{B}(t)u(t)\\ y(t)=\tilde{C}(t)x(t)\end{array}\right. \tag{8}$$

where the system data possess the same dimensions with that of system (1).

Lemma 1 *Given two LDP systems (1) and (8). If there exists a nonsingular matrix P satisfying*

$$\tilde{A}(t)=PA(t)P^{-1},\quad \tilde{B}(t)=PB(t),\quad \tilde{C}(t)=C(t)P^{-1}, \tag{9}$$

then the lifted systems of this two systems are equivalent.

Proof. Lifting system (1) gives the following LTI system

$$\left\{\begin{array}{l}x^{\mathrm{L}}(t+1)=A^{\mathrm{L}}x^{\mathrm{L}}(t)+B^{\mathrm{L}}u^{\mathrm{L}}(t)\\ y^{\mathrm{L}}(t)=C^{\mathrm{L}}x^{\mathrm{L}}(t)\end{array}\right., \tag{10}$$

where

$$A^{\mathrm{L}}=A(T-1)A(T-2)\cdots A(0)$$

$$B^{\mathrm{L}}=\left[\begin{array}{cccc}A(T-1)A(T-2)\cdots A(1)B(0) & \cdots & A(T-1)B(T-2) & B(T-1)\end{array}\right]$$

$$C^{\mathrm{L}}=\left[\begin{array}{c}C(0)\\ C(1)A(0)\\ \vdots\\ C(T-1)A(T-2)\cdots A(0)\end{array}\right]$$

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Lifting system (8) gives the following LTI system

$$\begin{cases} x^{\mathrm{L}}(t+1) = \tilde{A}^{\mathrm{L}} x^{\mathrm{L}}(t) + \tilde{B}^{\mathrm{L}} u^{\mathrm{L}}(t) \\ y^{\mathrm{L}}(t) = \tilde{C}^{\mathrm{L}} x^{\mathrm{L}}(t) \end{cases} \tag{11}$$

where

$$\tilde{A}^{\mathrm{L}} = \tilde{A}(T-1)\tilde{A}(T-2)\cdots\tilde{A}(0),$$

$$\tilde{B}^{\mathrm{L}} = \begin{bmatrix} \tilde{A}(T-1)\tilde{A}(T-2)\cdots\tilde{A}(1)\tilde{B}(0) & \cdots & \tilde{A}(T-1)\tilde{B}(T-2) & \tilde{B}_{T-1} \end{bmatrix},$$

$$\tilde{C}^{\mathrm{L}} = \begin{bmatrix} \tilde{C}(0) \\ \tilde{C}(1)\tilde{A}(0) \\ \vdots \\ \tilde{C}(T-1)\tilde{A}(T-2)\cdots\tilde{A}(0) \end{bmatrix}.$$

According to (9), we get

$$\begin{aligned} \tilde{A}^{\mathrm{L}} &= \tilde{A}(T-1)\tilde{A}(T-2)\cdots\tilde{A}(0) \\ &= PA(T-1)P^{-1}PA(T-2)P^{-1}\cdots PA(0)P^{-1} \\ &= PA^{\mathrm{L}}P^{-1}, \end{aligned}$$

$$\begin{aligned} \tilde{B}^{\mathrm{L}} &= [\ \tilde{A}(T-1)\tilde{A}(T-2)\cdots\tilde{A}(1)\tilde{B}(0) \quad \cdots \\ &\qquad \tilde{A}(T-1)\tilde{B}(T-2) \quad \tilde{B}(T-1)\ ] \\ &= [\ PA(T-1)A(T-2)\cdots A(1)B(0) \quad \cdots \\ &\qquad PA(T-1)B(T-2) \quad PB(T-1)\ ] \\ &= PB^{\mathrm{L}}, \end{aligned}$$

$$\begin{aligned} \tilde{C}^{\mathrm{L}} &= \begin{bmatrix} \tilde{C}(0) \\ \tilde{C}_1\tilde{A}(0) \\ \vdots \\ \tilde{C}(T-1)\tilde{A}(T-2)\cdots\tilde{A}(0) \end{bmatrix} \\ &= \begin{bmatrix} C(0)P^{-1} \\ C_1A(0)P^{-1} \\ \vdots \\ C(T-1)A(T-2)\cdots A(0)P^{-1} \end{bmatrix} \\ &= C^{\mathrm{L}}P^{-1}. \end{aligned}$$

Thus, we can see the lifted systems (10) and (11) are algebraically equivalent, which means the equivalence between system (1) and system (8). ■

By virtue of this conclusion, we can form the following Theorem.

Theorem 1 *Consider systems (3) and (7). The eigenvalue set of system (7) are composed by sets* $\sigma(\Phi_{\mathrm{A+BK}}(T,0))$ *and* $\sigma(\Phi_{\mathrm{F}}(T,0))$ *corresponding to systems (3) and (4), respectively.*

Proof. Let

$$P = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}.$$

It is easily computed that

$$P^{-1} = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}.$$

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Therefore,

$$P\begin{bmatrix} A(t) & B(t)K(t) \\ -L(t)C(t) & F(t)+B(t)K(t) \end{bmatrix}P^{-1} = \begin{bmatrix} A(t)+B(t)K(t) & B(t)K(t) \\ 0 & F(t) \end{bmatrix},$$

$$P\begin{bmatrix} B(t) \\ B(t) \end{bmatrix} = \begin{bmatrix} B(t) \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} C(t) & 0 \end{bmatrix}P^{-1} = \begin{bmatrix} C(t) & 0 \end{bmatrix}.$$

By lemma 1, system (7) and the following system have equivalent lifted systems

$$\left(\begin{bmatrix} A(t)+B(t)K(t) & B(t)K(t) \\ 0 & F(t) \end{bmatrix}, \begin{bmatrix} B(t) \\ 0 \end{bmatrix}, \begin{bmatrix} C(t) & 0 \end{bmatrix}\right). \tag{12}$$

Thus, all the eigenvalues of the two lifted systems are the same. Since eigenvalues of LDP systems are defined to be eigenvalues of their lifted system, the proof is completed. ■

We call the above result as principle of separating for LDP systems. It is shown that the introduction of full order state observers has no influence on the stability of the close-loop system by state feedback law (2). At the same time, the introduction of state feedback has no influence on the designed observers. By this theorem, when discussing the problem of stabilizing LDP systems based on observers, periodic control laws and periodic observers can be designed independently. The work remaining is to find matrices $K(t)$ and $L(t)$ such that matrix $\Phi_{\mathrm{A+BK}}(T,0)$ and matrix $\Phi_{\mathrm{F}}(T,0)$ are stable respectively. Here, we adopt poles assignment approach.

Let A^{L} and B^{L} denote the lifted system matrices corresponding to periodic matrix pair $(A(\cdot),C(\cdot))$, A^{LT} and C^{LT} denote the lifted system matrices corresponding to periodic matrix pair $(A^{\mathrm{T}}(\cdot),C^{\mathrm{T}}(\cdot))$, and matrices F and G are real matrices possessing the desired pole set of matrices $\Phi_{\mathrm{A+BK}}(T,0)$ and matrix $\Phi_{\mathrm{F}}(T,0)$ respectively. Introducing the following polynomial matrix factorizations:

$$(zI - A^{\mathrm{L}})^{-1}B^{\mathrm{L}} = N(z)D^{-1}(z) \tag{13}$$

$$(zI - A^{\mathrm{LT}})^{-1}C^{\mathrm{LT}} = H(z)L^{-1}(z) \tag{14}$$

where $N(z) \in \mathbb{R}^{n\times Tr}$, $D(z) \in \mathbb{R}^{Tr\times Tr}$ are right coprime matrix polynomials in z, and $H(z) \in \mathbb{R}^{n\times Tm}$, $L(z) \in \mathbb{R}^{Tm\times Tm}$ are the same. If we denote

$$D(z) = [d_{ij}(z)]_{Tr\times Tr}\,, N(z) = [n_{ij}(z)]_{n\times Tr}$$

$$H(z) = [h_{ij}(z)]_{Tm\times Tm}\,, L(z) = [l_{ij}(z)]_{n\times Tm}$$

and $\alpha = \max\{\alpha_1,\alpha_2\}\,, \beta = \max\{\beta_1,\beta_2\}\,,$ where

$$\alpha_1 = \max_{i,j\in\overline{1,Tr}}\{\deg(d_{ij}(z))\}$$
$$\alpha_2 = \max_{i\in\overline{1,n},j=\overline{1,Tr}}\{\deg(n_{ij}(z))\}$$

$$\beta_1 = \max_{i,j\in\overline{1,Tm}}\{\deg(h_{ij}(z))\}$$
$$\beta_2 = \max_{i\in\overline{1,n},j=\overline{1,Tm}}\{\deg(l_{ij}(z))\}$$

then $N(z)$ and $D(z)$ can be rewritten as

$$\begin{cases} N(z) = \sum\limits_{i=0}^{\alpha} N_i z^i, N_i \in \mathbb{C}^{n\times Tr} \\ D(z) = \sum\limits_{i=0}^{\alpha} D_i z^i, D_i \in \mathbb{C}^{Tr\times Tr} \end{cases} \tag{15}$$

$H(z)$ and $L(z)$ can be rewritten as

$$\begin{cases} H(z) = \sum\limits_{i=0}^{\beta} H_i z^i, H_i \in \mathbb{C}^{n\times Tm} \\ L(z) = \sum\limits_{i=0}^{\beta} L_i z^i, L_i \in \mathbb{C}^{Tm\times Tm} \end{cases} \tag{16}$$

Denote

$$\begin{cases} V_{\mathrm{K}}(Z_1) = N_0 Z_1 + N_1 Z_1 F + \cdots + N_\alpha Z_1 F^\alpha \\ W_{\mathrm{K}}(Z_1) = D_0 Z_1 + D_1 Z_1 F + \cdots + D_\alpha Z_1 F^\alpha \end{cases} \tag{17}$$

$$\begin{cases} V_{\mathrm{L}}(Z_2) = H_0 Z_2 + H_1 Z_2 G + \cdots + H_\beta Z_2 G^\beta \\ W_{\mathrm{L}}(Z_2) = L_0 Z + L_1 Z_2 G + \cdots + L_\beta Z_2 G^\beta \end{cases} \tag{18}$$

and

$$\mathcal{Z}_1 = \left\{ Z_1 \middle| \det\left(\sum_{i=0}^{\alpha} N_i Z_1 F^i\right) \neq 0 \right\} \tag{19}$$

$$\mathcal{Z}_2 = \left\{ Z_2 \middle| \det\left(\sum_{i=0}^{\beta} H_i Z_2 G^i\right) \neq 0 \right\} \tag{20}$$

where Z_1 and Z_2 are arbitrary parameter matrices with compatible dimension.

Let

$$X(Z_1) = W_{\mathrm{K}}(Z_1) V_{\mathrm{K}}^{-1}(Z_1) \triangleq \begin{bmatrix} X_0^{\mathrm{T}} & X_1^{\mathrm{T}} & \cdots & X_{T-1}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \tag{21}$$

$$Y(Z_2) = W_{\mathrm{L}}(Z_2) V_{\mathrm{L}}^{-1}(Z_2) \triangleq \begin{bmatrix} Y_0^{\mathrm{T}} & Y_1^{\mathrm{T}} & \cdots & Y_{T-1}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \tag{22}$$

where $Z_1 \in \mathcal{Z}_1$ and $Z_2 \in \mathcal{Z}_2$.

According to theorem 1 in this paper and the theorem 1 of literature [23], we have the following conclusion.

Theorem 2 *For given LDP system (1) and stable real constant matrices F, G with compatible dimensions and the desired poles, if $V_{\mathrm{K}}(Z_1)$ and $W_{\mathrm{K}}(Z_1)$ are given by (17), $V_{\mathrm{L}}(Z_2)$ and $W_{\mathrm{L}}(Z_2)$ are given by (18), X_i, $i \in \overline{0, T-1}$ and Y_i, $i \in \overline{0, T-1}$ are given by (21) and (22) respectively, then the whole set of solutions to Problem 1 can be given by (23) and (24).*

$$\mathcal{K} = \left\{ \begin{pmatrix} K(0) \\ K(1) \\ \vdots \\ K(T-1) \end{pmatrix} \middle| \begin{cases} X(Z) = W_{\mathrm{K}}(Z_1) V_{\mathrm{K}}^{-1}(Z_1), Z_1 \in \mathcal{Z}_1 \\ K(0) = [X_1]^{\mathrm{T}}, \\ K(t) = \left[X_{t+1} \prod\limits_{j=0}^{i-1} (A(j) + B(j)K(j))^{-1} \right]^{\mathrm{T}}, \ t \in \overline{1, T-1} \end{cases} \right\} \tag{23}$$

$$\mathcal{L} = \left\{ \begin{pmatrix} L(0) \\ L(1) \\ \vdots \\ L(T-1) \end{pmatrix} \middle| \begin{cases} Y(Z) = W_{\mathrm{L}}(Z_2) V_{\mathrm{L}}^{-1}(Z_2), Z_2 \in \mathcal{Z}_2 \\ L(0) = [Y_1]^{\mathrm{T}}, \\ L(t) = \left[Y_{t+1} \prod\limits_{j=0}^{i-1} \left(A^{\mathrm{T}}(j) + C^{\mathrm{T}}(j) L^{\mathrm{T}}(j)\right)^{-1} \right]^{\mathrm{T}}, \ t \in \overline{1, T-1} \end{cases} \right\} \tag{24}$$

Based upon theorem 2, an algorithm for solving problem 1 follows.

Algorithm 1 *(Stabilization of LDP systems)*

1. *Select constant matrices F and G such that all of their poles lie in the unit circle.*
2. *Solve the right coprime polynomial matrices $N(z), D(z)$ satisfying factorization (13) and the right coprime polynomial matrices $H(z), L(z)$ satisfying factorization (14).*

3. *According to formulae (15), compute matrices* $N_i, D_i, i \in \overline{0,\alpha}$*; According to formulae (16), compute matrices* $H_i, L_i, i \in \overline{0,\beta}$.
4. *Compute* $V_{\mathrm{K}}(Z_1)$ *and* $W_{\mathrm{K}}(Z_1)$ *by formulae (17); Compute* $V_{\mathrm{L}}(Z_2)$ *and* $W_{\mathrm{L}}(Z_2)$ *by formulae (18).*
5. *According to formulaes (21) and (23), compute periodic state feedback matrices* $K(t)$, $t \in \overline{0,T-1}$*; According to formulaes (22) and (24), compute periodic observer gains* $L(t)$, $t \in \overline{0,T-1}$.

Because of the arbitrariness of the choose of parameter matrices Z_1 and Z_2 in the design process, the above parametric design algorithm can provide numerous solutions to problem 1. This makes multi-object design possible for LDP systems. Here, we only consider robustness. According to literature [23], the following robustness performance index can be adopted:

$$J_1(Z_1) \triangleq \kappa_{\mathrm{F}}(V_{\mathrm{K}}) \sum_{t=0}^{T-1} \|A(t) + B(t)K(t)\|_{\mathrm{F}}^{T-1}, \tag{25}$$

$$J_2(Z_2) \triangleq \kappa_{\mathrm{F}}(V_{\mathrm{L}}) \sum_{t=0}^{T-1} \|A(t) + L(t)C(t)\|_{\mathrm{F}}^{T-1}, \tag{26}$$

where $\kappa_{\mathrm{F}}(V_{\mathrm{K}}) \triangleq \left\|V_{\mathrm{K}}^{-1}\right\|_{\mathrm{F}} \|V_{\mathrm{K}}\|_{\mathrm{F}}$ and $\kappa_{\mathrm{F}}(V_{\mathrm{L}}) \triangleq \left\|V_{\mathrm{L}}^{-1}\right\|_{\mathrm{F}} \|V_{\mathrm{L}}\|_{\mathrm{F}}$ are the Frobenius-norm conditional numbers of matrix V_{K} and matrix V_{L} respectively. Thus, we can summarize the robust stabilization algorithm based on observers as follows.

Algorithm 2 *(Robust stabilization algorithm of LDP systems)*

1. *Select constant matrices* F *and* G *such that all of their poles lie in the unit circle.*
2. *Solve the right coprime polynomial matrices* $N(z), D(z)$ *satisfying factorization (13) and the right coprime polynomial matrices* $H(z), L(z)$ *satisfying factorization (14).*
3. *According to formulae (15), compute matrices* $N_i, D_i, i \in \overline{0,\alpha}$*; According to formulae (16), compute matrices* $H_i, L_i, i \in \overline{0,\beta}$.
4. *Construct general expressions for matrices* V_{K} *and* $K(t)$, $t \in \overline{0,T-1}$ *according to formulaes (17), (21) and (23), construct general expressions for matrices* V_{L} *and* $L(t)$, $t \in \overline{0,T-1}$ *according to formulaes (18), (22) and (24).*
5. *Solving optimization problems*
$$\mathrm{Minimize} J_1(Z_1)$$
and
$$\mathrm{Minimize} J_2(Z_2)$$
by using gradient based searching method. The optimal decision matrix is denoted by Z_1^{opt} *and* Z_2^{opt} *respectively.*
6. *Compute matrices* $K^{\mathrm{opt}}(t), t \in \overline{0,T-1}$ *according to (17), (21) and (23) by using optimal decision matrix* Z_1^{opt}*; Compute matrices* $L^{\mathrm{opt}}(t), t \in \overline{0,T-1}$ *according to (18), (22) and (24) by using optimal decision matrix* Z_2^{opt}.

4 Numerical example

Consider LDP system (1) with parameters as follows:

$$\begin{aligned}
&A(0) = \begin{bmatrix} -4.5 & -1 \\ 2.5 & 0.5 \end{bmatrix}, \quad A(1) = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \\
&A(2) = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \ B(0) = B(1) = B(2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\
&C(0) = \begin{bmatrix} 2 & 1 \end{bmatrix}, \quad C(1) = \begin{bmatrix} -1 & 1 \end{bmatrix}, \\
&C(2) = \begin{bmatrix} 0 & 1 \end{bmatrix}.
\end{aligned}$$

It is an oscillation system possessing performances of complete reachability and complete observability. In the following, we will design a robust stabilization law for this system.

For convenience, we can choose matrices F and G as

$$F = \begin{bmatrix} -0.3 & 0 \\ 0 & 0.3 \end{bmatrix}, G = \begin{bmatrix} -0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

According to algorithm 1, by randomly choosing parameter matrices Z_1 and Z_2, we obtain a group of solutions as follows:

$$\begin{aligned} K_{\mathrm{rand}}(0) &= \begin{bmatrix} 0.7900 & 0.3400 \end{bmatrix}, \\ K_{\mathrm{rand}}(1) &= \begin{bmatrix} 2.0000 & 2.2857 \end{bmatrix}, \\ K_{\mathrm{rand}}(2) &= \begin{bmatrix} -0.6667 & -1.2593 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} L_{\mathrm{rand}}(0) &= \begin{bmatrix} 1.6377 \\ -0.8841 \end{bmatrix}, \\ L_{\mathrm{rand}}(1) &= \begin{bmatrix} 0.3871 \\ -0.0645 \end{bmatrix}, \\ L_{\mathrm{rand}}(2) &= \begin{bmatrix} -33.0000 \\ -67.3333 \end{bmatrix}. \end{aligned}$$

Applying algorithm 2 gives solutions to problem 2 with the following gains:

$$\begin{aligned} K_{\mathrm{robu}}(0) &= \begin{bmatrix} 1.0448 & 0.0428 \end{bmatrix}, \\ K_{\mathrm{robu}}(1) &= \begin{bmatrix} -0.6217 & -1.3782 \end{bmatrix}, \\ K_{\mathrm{robu}}(2) &= \begin{bmatrix} -0.6217 & -1.6218 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} L_{\mathrm{robu}}(0) &= \begin{bmatrix} 2.0876 \\ -0.8805 \end{bmatrix}, \\ L_{\mathrm{robu}}(1) &= \begin{bmatrix} -0.7062 \\ -0.3082 \end{bmatrix}, \\ L_{\mathrm{robu}}(2) &= \begin{bmatrix} -2.4536 \\ -1.3617 \end{bmatrix}. \end{aligned}$$

Denote

$$\begin{aligned} K_{\mathrm{rand}} &= (K_{\mathrm{rand}}(0),\ K_{\mathrm{rand}}(1),\ K_{\mathrm{rand}}(2)) \\ L_{\mathrm{rand}} &= (L_{\mathrm{rand}}(0),\ L_{\mathrm{rand}}(1),\ L_{\mathrm{rand}}(2)) \\ K_{\mathrm{robu}} &= (K_{\mathrm{robu}}(0),\ K_{\mathrm{robu}}(1),\ K_{\mathrm{robu}}(2)) \\ L_{\mathrm{robu}} &= (L_{\mathrm{robu}}(0),\ L_{\mathrm{robu}}(1),\ L_{\mathrm{robu}}(2)) \end{aligned}$$

Choose the sine signal $v(t) = 0.1 * \sin(t + \pi/2)$ as reference input and $x_0 = \begin{bmatrix} -1 & 1 \end{bmatrix}^{\mathrm{T}}$, $\hat{x}_0 = \begin{bmatrix} 0 & 0 \end{bmatrix}^{\mathrm{T}}$ as the initial states of systems (1) and (4), respectively. We give the state histories of the system (1) in Figure. 1. With $(K_{\mathrm{rand}}, L_{\mathrm{rand}})$, Figure. 2 shows the state $x(t)$ of system (7). From this figure, we can see the good control effectiveness of Algorithm 1 when there is no uncertainty in system data.

To verify the effectiveness of the robust controller, let the system matrices be perturbed as follows:

$$\begin{aligned} A(t) &\mapsto A(t) + \mu\Delta_{at}, t \in \overline{0,2} \\ B(t) &\mapsto B(t) + \mu\Delta_{bt}, t \in \overline{0,2} \\ C(t) &\mapsto C(t) + \mu\Delta_{ct},\ t \in \overline{0,2} \end{aligned}$$

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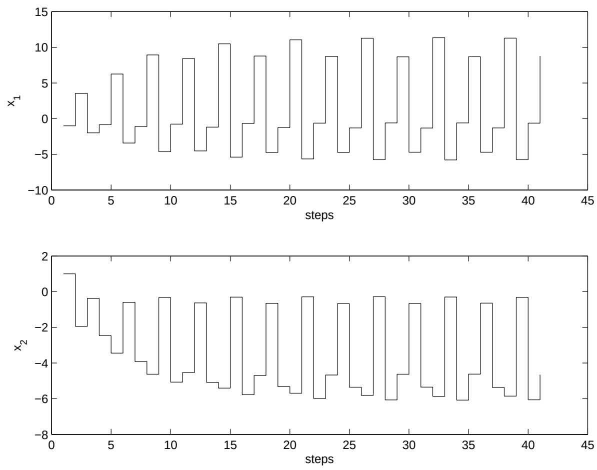

Figure 1: State $x(t)$ of the original system

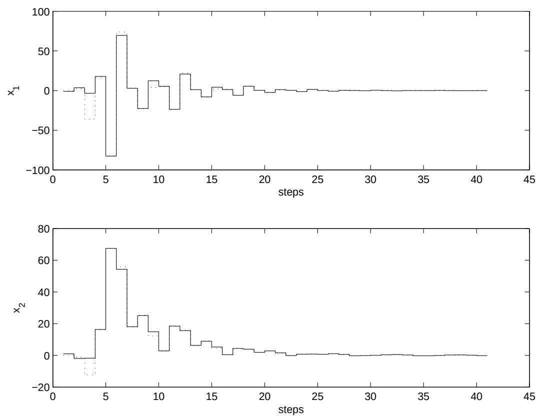

Figure 2: $x(t)$ and $\hat{x}(t)$ with $(K_{\text{rand}}, L_{\text{rand}})$ when $\mu = 0$

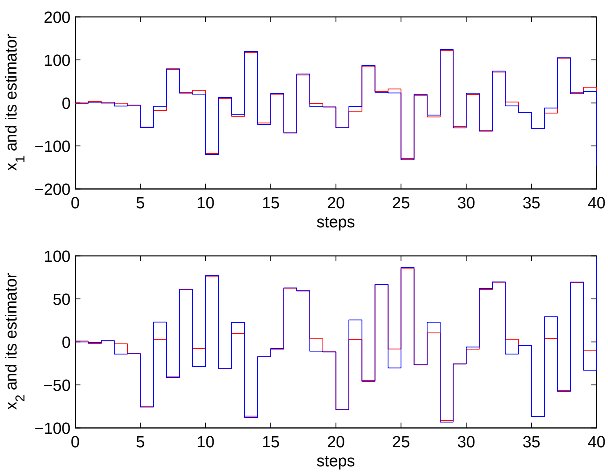

Figure 3: $x(t)$ with $(K_{\text{rand}}, L_{\text{rand}})$ when $\mu = 0.015$

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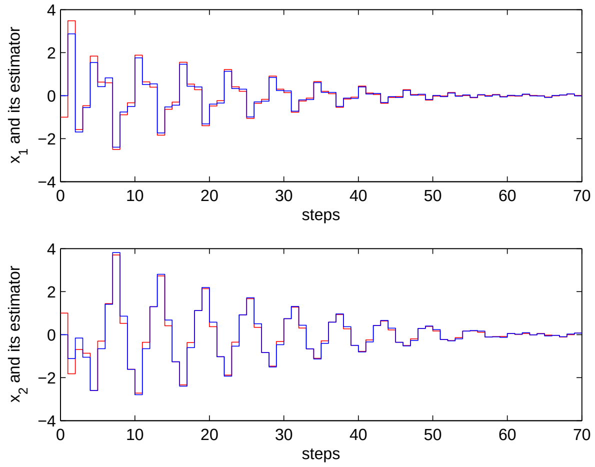

Figure 4: $x(t)$ and $\hat{x}(t)$ with $(K_{\text{robu}}, L_{\text{robu}})$ when $\mu = 0.35$

where $\Delta_{at} \in \mathbb{R}^{2\times 2}$, $\Delta_{bt} \in \mathbb{R}^{2\times 1}$, $\Delta_{ct} \in \mathbb{R}^{1\times 2}, t \in \overline{0,2}$ are random perturbations normalized such that $\|\Delta_{at}\|_{\text{F}} = 1, \|\Delta_{bt}\|_{\text{F}} = 1, \|\Delta_{ct}\|_{\text{F}} = 1$, $t \in \overline{0,2}$ and $\mu > 0$ is a parameter controlling the level of perturbations. Let $\mu = 0.015$, we depict the response histories of $x(t)$ and $\hat{x}(t)$ with gains $(K_{\text{rand}}, L_{\text{rand}})$ in figure. 3, where the solid line denotes $x(t)$ and the dotted line denotes $\hat{x}(t)$. It is obvious that system (7) with gains $(K_{\text{rand}}, L_{\text{rand}})$ is not stable even the perturbation level is reduced to $\mu = 0.015$. To measure robustness of the designed robust controller based on periodic observers, we continuously increase the perturbation controlling level until $\mu = 0.35$ and depict the results in figure. 4. From simulation results, we can see the designed robust controller has strong anti-interference ability. In addition, we notice that $(K_{\text{robu}}, L_{\text{robu}})$ has a very small norm compared with $(K_{\text{rand}}, L_{\text{rand}})$. This means that the robust controllers and observers can possess less energy consumption, since small gains lead to small control signals.

From the simulation results, we can see the approaches proposed in this paper are very effective.

5 Conclusion

In this paper, the observer-based robust stabilization problem for LDP systems is considered. It is proofed that the principle of separating exists in this type of systems. Thus, periodic controllers and periodic observers can be designed separately. By using poles assignment technique, numerous periodic controllers and observers are obtained in the form of iteration and parametrization. Combined with our recent result about robustness, robust stabilization problem based on observers is solved. Two detailed algorithms are presented. The proposed approaches are checked by a numerical example and the simulation results are of great satisfaction. A possible future study is to combine the developed approach with the truncated predictor feedback [14, 24, 25] and constrained control theory [26, 27] to investigate the observer-based robust stabilization problem for LDP systems with time delays and input saturation.

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Embedding relations of Besov classes under GBV

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Abstract. In this paper, we strengthen some of Leindler's results from [L. Leindler. Embedding relations of Besov classes. Acta Sci. Math. (Szeged), 73(2007)133-149.] under GBV condition. First, we discuss embedding relations between two Besov classes. Next, we give an equivalent estimate for the k-order modulus of continuity of $f(x)$ in L^p norm under GBV condition. Finally, we give the condition to ensure a function $f \in L^p$ have Fourier coefficients of GBV belongs to the Besov class.

Keywords. GBV, Besov classes, embedding relations, Fourier coefficients.
2010 Mathematics Subject Classification. 26A15, 42A16

1 Introduction

Many classical results in Fourier analysis have been generalized by weakening the condition imposed on the coefficients of trigonometric series from MS to RBVS, GBVS and, Finally, to MVBVS(see [26] for more details). In [15], Leindler defined the class of sequences of rest bounded variation, in symbol: RBVS, and showed that it is not comparable to the classical quasi monotone sequences, in symbol: CQMS. In [6], Le and Zhou defined the class GBVS containing both RBVS and CQMS. In [10], Leindler introduced a new class of sequences, the class γRBVS.

Definition 1.1. *Let $\gamma := \{\gamma_n\}$ be a positive sequence. A null-sequence $A := \{a_n\}(a_n \to 0)$ of real number satisfying the inequalities*

$$\sum_{i=n}^{\infty} |\Delta a_i| \leq K(\mathrm{A})\gamma_n \quad (\Delta a_i := a_i - a_{i+1}), \quad n = 1, 2, \cdots \tag{1.1}$$

with a positive constant $K(\mathrm{A})$ is said to be a sequence of γ rest bounded variation, in symbol: A$\in \gamma$RBVS.

If $\gamma \equiv$A and $a_n > 0$, then γRBVS $\equiv$ RBVS. It is easy to see that if A$\in$ RBVS, then it is also almost monotone, in symbol: A$\in$ AMS, that is for all $n \geq m$, we have

$$a_n \leq K(\mathrm{A}) a_m.$$

In [11] and [10], Leindler introduced the class of mean rest bounded variation sequences, where γ is defined by a certain arithmetical mean of the coefficients, e.g.,

$$\gamma_n^* := \frac{1}{n} \sum_{i \geq n/2}^{n} a_i \quad or \quad \overline{\gamma}_n := \frac{1}{n} \sum_{i=n}^{2n-1} a_i.$$

It is easy to see that the class γ^*MRBVS includes the class RBVS, consequently the almost monotone and monotone sequences, too; but $\overline{\gamma}$MRBVS does not, in general. In [21], B. Szal proved that RBVS $\neq \gamma^*$MRBVS. Namely, he showed that the sequence

$$d_n := \begin{cases} 1, & \text{if} \quad n = 1, \\ \frac{1+m+(-1)^m}{(2^{\mu_m})^2 m}, & \text{if} \quad \mu_m \leq n < \mu_{m+1} \end{cases}$$

where $\mu_m = 2^m$ for $m = 1, 2, 3 \cdots$, belongs to the class γ^*MRBVS but it does not belong to the class RBVS. In [23], B. Szal showed that $\overline{\gamma}$MRBVS $\subset \gamma^*$MRBVS and $\overline{\gamma}$MRBVS $\neq \gamma^*$MRBVS. Namely, he showed that the above sequence d_n belongs to the class γ^*MRBVS but it does not belong to the class $\overline{\gamma}$MRBVS. In [22], B. Szal introduced the class of infinity mean rest bounded variation, briefly A $\in$ IMRBVS, if $\sum\limits_{n=1}^{\infty} \dfrac{a_n}{n} < \infty$ and $\gamma_n = \sum\limits_{i=n}^{\infty} \dfrac{a_i}{i}$. Moreover, he showed that $\overline{\gamma}$MRBVS $\neq$ IMRBVS and γ^*MRBVS $\neq$ IMRBVS.

In [6], Le and Zhou first defined the class GBVS as follow:

Definition 1.2. *A positive sequence* A $:= \{a_n\}_{n=1}^{\infty}$ *satisfying the inequalities*

$$\sum_{i=n}^{2n-1} |\Delta a_i| \leq K(A) a_n, \; n = 1, 2, \cdots$$

with a positive constant $K(A)$ is said to be a sequence of group bounded variation, in symbol: A $\in$ GBVS.

Moreover, they proved that RBVS $\subseteq$ GBVS. If A$\in$ GBVS, then for all $m \leq n \leq 2m$, we have $a_n \leq K(\mathrm{A}) a_m$. Thus, GBVS also named general monotone sequences in [16] and [24] (in symbol: GMS). In [11], Leindler proved that MRBVS $\not\subset$ GBVS.

Many classical theorems were generalized under RBV condition or GBV condition in [9], [5], [8], [7] and so on .The properties of the Besov classes have been studied by many authors (see [22], [10], [14], [18], [19]). Their major work studied three theorems in connection with Besov classes of functions $f \in L^p_{[-\pi,\pi]}$ under coefficient sequence satisfying restricted condition. In [22], [23], [10], [14], [18] and [19] studied them under IMRBV condition, γ^*MRBV condition, $\overline{\gamma}$MRBV condition, RBV condition, M condition, respectively. In view of the relation between GBVS and other RBVS, we make further efforts to generalize the three theorems under GBV.

The rest of the paper is organized as follows. In Section 2 we give notions and notations used in the paper. In Section 3 we give our main results. In Section 4 we introduce some lemmas to prove our results. In Section 5 we prove the main results.

2 Notions and notations

Let $L^p_{[-\pi,\pi]} (1 \leq p \leq \infty)$ be the space of all p-power integrable real functions of period 2π with the norms

$$\|f\|_p := \begin{cases} \left(\int_{-\pi}^{\pi} |f(t)|^p \mathrm{d}t\right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \operatorname*{ess\,sup}_{x \in [-\pi,\pi]} |f(t)|, & p = \infty. \end{cases}$$

The best trigonometric approximation $E_n(f)_p$ and the modulus of smoothness $\omega_k(f; \delta)_p$ are defined as follows:

$$E_n(f)_p = \min\Big(\|f - T\|_p : T \in \mathbf{T}_n\Big), \; \mathbf{T}_n = \operatorname{span}\{\cos mx, \sin mx : |m| \leq n\}$$

and

$$\omega_k(f;\delta)_p = \sup_{|h|<\delta} \left\| \Delta_h^k f(x) \right\|_p$$

$$\Delta_h^k f(x) = \Delta_h^{k-1}(\Delta_h f(x))\ \Delta_h f(x) = f(x+h) - f(x),$$

respectively.

A function $\alpha(t)$ is called σ-type if it is measurable on $[0,1]$, integrable on $[\delta,1]$ for every $\delta \in (0,1)$, and there exist positive constants C_1 and C_2 such that

(i) $\alpha(t) \geq C_1$ for all $t \in [0,1]$,

(ii) $\int_0^\delta \alpha(t)t^\sigma \mathrm{d}t \leq C_2\delta^\sigma \int_\delta^{2\delta} \alpha(t)\mathrm{d}t$ for all $\delta \in (0,\delta_0)$, where $0 < \delta_0 \leq \frac{1}{2}$ is given.

A positive function $\alpha(t)$ is said to satisfy the λ-condition, $\lambda > 0$, if there exists a positive constant C_3 such that

$$\int_{2\delta}^1 \alpha(t)t^\lambda \mathrm{d}t \leq C_3\delta^\lambda \int_\delta^{2\delta} \alpha(t)\mathrm{d}t, \text{ for all } \delta \in (0,\delta_0).$$

We say that $f \in B(p,\gamma,\alpha)$ if

(i) $f \in L^p_{[-\pi,\pi]}$,

(ii) $0 < \gamma < \infty$,

(iii) $\alpha(t)$ is σ-type,

(iv) $\int_0^1 \omega_k^\gamma(f;t)_p \,\mathrm{d}t < \infty$, $k \geq \frac{\sigma}{\gamma}$.

We use the notation $L \ll R$ at inequalities if there exists a positive constant K such that $L \leq KR$; and if $L \ll R$ and $R \ll L$ hold simultaneously, then we shall write $L \asymp R$.

3 Main results

We formulate our results as follow:

Theorem 3.1. *If $1 < p < q \leq \infty$, the function $\alpha(t)$ satisfies λ-condition with*

$$\lambda = \left(\frac{1}{p} - \frac{1}{q}\right)\gamma,\ 0 < \gamma < \infty,\ \alpha^*(t) := \alpha(t)t^\lambda,$$

$A := \{a_n\}_{n=1}^\infty \in GBVS$, and f has the Fourier expansion

$$f(x) \sim \sum_{n=1}^\infty a_n \cos nx, \tag{3.1}$$

then the Besov classes $B(p,\gamma,\alpha)$ and $B(p,\gamma,\alpha^*)$ coincide. Furthermore, for any

$$k_1 \geq \frac{\sigma}{\gamma}, k_2 \geq \frac{\sigma^*}{\gamma}, k_3 \geq \frac{\sigma^*}{\gamma}, \sigma^* = \sigma - \lambda,$$

we have

$$\int_0^1 \alpha^*(t)\omega_{k_2}^\gamma(f;t)_q \mathrm{d}t \ll \int_0^1 \alpha(t)\omega_{k_1}^\gamma(f;t)_q \mathrm{d}t \ll \int_0^1 \alpha^*(t)\omega_{k_3}^\gamma(f;t)_q \mathrm{d}t. \tag{3.2}$$

Theorem 3.2. *If $f \in L^p_{[-\pi,\pi]}$, $1 < p < \infty$, f has the Fourier expansion* (3.1) *with $A := \{a_n\} \in GBVS$, then*

$$S(A,p,k,n) \ll \omega_k\left(f;\frac{1}{n}\right)_p \ll S(A,p,k,n), \tag{3.3}$$

where

$$S(A,q,k,n) := \begin{cases} a_n, & if\ q = 1, \\ n^{-k}\left(\sum\limits_{i=1}^{n} a_i^q i^{(k+1)q-2}\right)^{1/q} + \left(\sum\limits_{i=n+1}^{\infty} a_i^q i^{q-2}\right)^{1/q}, & if\ 1 < q < \infty, \\ n^{-k}\sum\limits_{i=1}^{n} a_i i^k + \sum\limits_{i=n+1}^{\infty} a_i, & if\ q = \infty. \end{cases}$$

Theorem 3.3. *If $f \in L^p_{[-\pi,\pi]}$, $1 < p < \infty$, f has the Fourier expansion* (3.1) *with $A := \{a_n\} \in GBVS$, $\alpha(t) = t^{-r\gamma-1}$ and $k > r$. If $\gamma \geq 1$, then $f \in B(p,\gamma,\alpha)$ if and only if*

$$J_1 := \sum_{n=1}^{\infty} a_n^{\gamma} n^{r\gamma+\gamma-\frac{\gamma}{p}-1} < \infty. \tag{3.4}$$

If $0 < \gamma \leq 1$, then a sufficient condition for $f \in B(p,\gamma,\alpha)$ is

$$J_2 := \sum_{n=1}^{\infty} a_n^{\gamma} n^{r\gamma-\gamma/p} < \infty \tag{3.5}$$

and a necessary condition is

$$J_1 := \sum_{n=1}^{\infty} a_n^{\gamma} n^{r\gamma+\gamma-\frac{\gamma}{p}-1} < \infty. \tag{3.6}$$

4 Auxiliary lemmas

In order to verify our theorems we need several lemmas: most of them are the analogues of the lemmas used in the proofs of the theorems with monotone coefficients or other conditions.

Lemma 4.1. ([*13*], Corollary 1) *If $\lambda_n > 0$ and $a_n \geq 0$, then*

$$\sum_{n=1}^{\infty} \lambda_n \left(\sum_{k=1}^{n} a_k\right)^p \leq p^p \sum_{n=1}^{\infty} \lambda_n^{1-p} a_n^p \left(\sum_{k=n}^{\infty} \lambda_k\right)^p \tag{4.1}$$

$$\sum_{n=1}^{\infty} \lambda_n \left(\sum_{k=n}^{\infty} a_k\right)^p \leq p^p \sum_{n=1}^{\infty} \lambda_n^{1-p} a_n^p \left(\sum_{k=1}^{n} \lambda_k\right)^p \tag{4.2}$$

hold for any $p \geq 1$; while if $0 < p < 1$, then the inequality in (4.1) *and* (4.2) *hold with opposite direction.*

Lemma 4.2. ([*2*], Theorem 19) *If $a_n \geq 0$ and $0 < p_1 < p_2 < \infty$, then*

$$\left(\sum_{n=1}^{\infty} a_n^{p_2}\right)^{\frac{1}{p_2}} \leq \left(\sum_{n=1}^{\infty} a_n^{p_1}\right)^{\frac{1}{p_1}}. \tag{4.3}$$

Lemma 4.3. ([*1*], p. 293) *If $f \in L^{\infty}_{[-\pi,\pi]} \equiv C_{[-\pi,\pi]}$ and $a_n \geq 0$,*

$$f(x) = \sum_{n=1}^{\infty} a_n \cos nx, x \in [-\pi,\pi],$$

then

$$\sum_{k=2n}^{\infty} a_k \leq 4E_n(f)_C.$$

Lemma 4.4. [25] *If $f \in L^p_{[-\pi,\pi]}$, $1 < p \le 2$, then*

$$\omega_k\left(f; \frac{1}{n}\right)_p \ll n^{-k}\left(\sum_{i=1}^{n} i^{kp-1}E_i^p(f)_p\right)^{\frac{1}{p}};$$

while if $p > 2$, then the reverse inequality holds.

Lemma 4.5. ([19], pp. 847 − 848) *If $f \in L^p_{[-\pi,\pi]}$, $1 \le p \le \infty$, $0 < \gamma < \infty$, α is a σ-type function and $k \ge \frac{\sigma}{\gamma}$, then*

$$E_0^r(f)_p + E_1^r(f)_p + \sum_{i=1}^{\infty}\mu(i)E_{2^i}^{\gamma}(f)_p \asymp \int_0^1 \alpha(t)\omega_k^{\gamma}(f;t)_p \mathrm{d}t,$$

where

$$\mu(n) := \int_{2^{-n}}^{2^{-n+1}} \alpha(t)\mathrm{d}t, n \ge 1 \text{ and } \mu(0) = 1.$$

Lemma 4.6. ([23], Lemma 6) *If α is a σ-type function, then*

$$\mu(n+1) \ll \mu(n) \tag{4.4}$$

hold for all n.

Lemma 4.7. ([20], Theorem 1) *If $f \in L^p_{[-\pi,\pi]}$, $1 \le p \le \infty$, f has the Fourier expansion* (3.1)*, and $P_1 := \min\{2, p\}$, $P_2 := \max\{2, p\}$, then*

$$S(A, P_1, k, n) \ll \omega_k\left(f; \frac{1}{n}\right)_p \ll S(A, P_2, k, n).$$

Lemma 4.8. ([18], Theorem 1) *If $f \in B(p,\gamma,\alpha)$, $1 < p < q \le \infty$ and α satisfies λ-condition with $\lambda = \left(\frac{1}{p} - \frac{1}{q}\right)\gamma$, then $f \in B(q,\gamma,\alpha^*)$, where*

$$\alpha^*(t) := \alpha(t)t^{\lambda}, \text{ that is, } B(p,\gamma,\alpha) \subset B(q,\gamma,\alpha^*);$$

furthermore,

$$\int_0^1 \alpha^*(t)\omega_{k_2}^{\gamma}(f;t)_q \mathrm{d}t \ll \int_0^1 \alpha(t)\omega_{k_1}^{\gamma}(f;t)_p \mathrm{d}t$$

for any

$$k_1 \ge \frac{\sigma}{\gamma},\ k_2 \ge \frac{\sigma^*}{\gamma} \text{ and } \sigma^* := \sigma - \left(\frac{1}{p} - \frac{1}{q}\right) + \varepsilon,\ \varepsilon > 0.$$

Lemma 4.9. [6] *Let $\{a_n\} \in GBVS$, then for all $n \ge 1$, the following inequalities hold*

$$\sum_{i=1}^{\infty} a_{2^i n} \ll \sum_{i=n}^{\infty} \frac{a_i}{i}. \tag{4.5}$$

$$a_{n+1} \ll \sum_{i=[n/2]+1}^{2n} \frac{a_i}{i}. \tag{4.6}$$

Lemma 4.10. [7] *If $1 < p < \infty$, and f has the Fourier expansion* (3.1) *with $\{a_n\} \in GBVS$, then $f \in L^p_{[-\pi,\pi]}$ if and only if*

$$\sum_{n=1}^{\infty} n^{p-2}a_n^p < \infty \tag{4.7}$$

or, more precisely

$$\|f\|_p^p \asymp \sum_{n=1}^{\infty} n^{p-2}a_n^p. \tag{4.8}$$

Lemma 4.11. [3] *Assume that f has the Fourier expansion* (3.1) *with $\{a_n\} \in GBVS$. If* $1 < p < \infty$ *and* (4.7) *holds, then*

$$E_n(f)_p \ll a_{n+1}(n+1)^{1-\frac{1}{p}} + \left(\sum_{i=n+1}^{\infty} i^{p-2} a_i^p\right)^{\frac{1}{p}}. \tag{4.9}$$

Lemma 4.12. ([4], Theorem 5) *If $f \in L^p_{[-\pi,\pi]}$, $1 < p < \infty$, and f has the Fourier expansion* (3.1) *with $a_n \geq 0$, then for $\eta > \frac{1}{p}$*

$$\sum_{i=n}^{\infty} \frac{a_i}{i^\eta} \leq n^{-\eta+\frac{1}{p}} E_n(f)_p.$$

Lemma 4.13. *If $f \in L^p_{[-\pi,\pi]}$, $1 < p < \infty$, f has the Fourier expansion* (3.1) *with $\{a_n\} \in GBVS$, then*

$$E_n^p(f)_p \gg \sum_{i=2n}^{\infty} a_i^p i^{p-2}.$$

Proof. We want to apply Lemma 4.10 to the following function:

$$f_0(x) := f(x) - \sum_{i=1}^{2n-1} a_i \cos ix + a_{2n} \sum_{i=1}^{2n-1} \cos ix.$$

First, we show that the $\mathrm{A}^0 := \{a_n^0\}$ of coefficients of f_0 belongs to GBVS, that is, that

$$\sum_{i=m}^{2m-1} \left|\Delta a_i^0\right| \ll a_m^0, m = 1, 2, \cdots . \tag{4.10}$$

We consider three cases:

(i) If $m \geq 2n$, then $a_i^0 = a_i$ for all $i \geq m$, we easily know

$$\sum_{i=m}^{2m-1} \left|\Delta a_i^0\right| = \sum_{i=m}^{2m-1} |\Delta a_i| \ll a_m = a_m^0.$$

(ii) If $m \leq n$, then $a_i^0 = a_{2n}$ for all $1 \leq i \leq 2m$, we easily know

$$\sum_{i=m}^{2m-1} \left|\Delta a_i^0\right| = 0 < a_m^0.$$

(iii) If $n < m < 2n$, then $a_i^0 = a_{2n}$ for all $m \leq i \leq 2n$ and $a_i^0 = a_k$ for all $i \geq 2n$, we easily know

$$\sum_{i=m}^{2m-1} \left|\Delta a_i^0\right| = \sum_{i=m}^{2n-1} \left|\Delta a_i^0\right| + \sum_{i=2n}^{2m-1} \left|\Delta a_i^0\right| < 0 + \sum_{i=2n}^{4n-1} |\Delta a_i| \ll a_{2n} = a_m^0.$$

That means $\mathrm{A}^0 \in$ GBVS, we can apply Lemma 4.10 to f_0, thus we obtain

$$\|f - S_{2n-1}(f)\|_p^p + a_{2n}^p \left\|\sum_{i=1}^{2n-1} \cos ix\right\|_p^p \gg \|f_0\|_p^p \gg \sum_{i=2n}^{\infty} a_i^p i^{p-2}.$$

Since

$$\begin{aligned}\left\|\sum_{i=1}^{2n-1}\cos ix\right\|_p^p &= 2\int_0^{\pi}\left|\sum_{i=1}^{2n-1}\cos ix\right|^p \mathrm{d}x\\ &= 2\left(\int_0^{\frac{\pi}{n}}+\int_{\frac{\pi}{n}}^{\pi}\right)\left|\frac{\cos nx\sin\frac{2n-1}{2}x}{\sin\frac{x}{2}}\right|^p \mathrm{d}x\\ &\ll n^p\int_0^{\frac{\pi}{n}}\mathrm{d}x+\int_{\frac{\pi}{n}}^{\pi}\frac{1}{x^p}\mathrm{d}x \ll n^{p-1},\end{aligned}$$

by a theorem of M. Riesz ([17], Theorem 3, p. 221), we obtain

$$\sum_{i=2n}^{\infty} a_i^p i^{p-2} \ll E_{2n-1}^p(f)_p + a_{2n}^p n^{p-1} < E_n^p(f)_p + a_{2n}^p n^{p-1}. \tag{4.11}$$

Applying Lemma 4.12 with $\eta = 1$ and (4.5), we obtain

$$a_{2n}^p n^{p-1} \le n^{p-1}\left(\sum_{i=1}^{\infty} a_{2^i n}\right)^p \ll n^{p-1}\left(\sum_{i=n}^{\infty}\frac{a_i}{i}\right)^p \ll E_n^p(f)_p. \tag{4.12}$$

The inequalities (4.11) and (4.12) imply the assertion. □

Lemma 4.14. *If $f \in L^p_{[-\pi,\pi]}$, $1 < p < q \le \infty$, and f has the Fourier expansion* (3.1) *with* $\mathrm{A} := \{a_n\}_{n=1}^{\infty} \in GBVS$. *If $q < \infty$, then*

$$S_1 := \sum_{i=8n}^{\infty} i^{\frac{q}{p}-2} E_i^q(f)_p \ll E_n^q(f)_q;$$

while if $q = \infty$, then

$$S_2 := \sum_{i=8n}^{16n} i^{\frac{1}{p}-2} E_i(f)_p \ll E_n(f)_q.$$

Proof. By Lemma 4.11, we have

$$S_1 \ll \sum_{i=8n}^{\infty} i^{\frac{q}{p}-2} a_{i+1}^q (i+1)^{q\left(1-\frac{1}{p}\right)} + \sum_{i=8n}^{\infty} i^{\frac{q}{p}-2}\left(\sum_{l=i+1}^{\infty} l^{p-2} a_l^p\right)^{\frac{q}{p}}$$

Using the inequalities of Lemma 4.1 and Lemma 4.13, we obtain

$$\begin{aligned}S_1 &\ll \sum_{i=8n}^{\infty} a_{i+1}^q (i+1)^{q-2} + \sum_{i=8n}^{\infty} a_{i+1}^q (i+1)^{q-2+\frac{q}{p}-\left(\frac{q}{p}\right)^2}\left(\sum_{l=1}^{i+1} l^{\frac{q}{p}-2}\right)^{\frac{q}{p}}\\ &\le \sum_{i=8n}^{\infty} a_i^q i^{q-2} \ll E_n^q(f)_q.\end{aligned}$$

To estimate S_2, we apply Lemma 4.11 again. Thus

$$\begin{aligned}S_2 &\ll \sum_{i=8n}^{\infty} i^{\frac{1}{p}-1} a_{i+1}(i+1)^{1-\frac{1}{p}} + \sum_{i=8n}^{\infty} i^{\frac{1}{p}-1}\left(\sum_{l=i+1}^{\infty} a_l^p l^{p-2}\right)^{\frac{1}{p}}\\ &: = S_{21} + S_{22}.\end{aligned}$$

First, we you

$$S_{21} \ll \sum_{i=2n}^{\infty} a_{i+1} < \sum_{i=2n}^{\infty} a_i \ll E_n(f)_q$$

and since A $\in$ GBVS, for all $m \le i \le 2m$, we have $a_i \ll a_m$, if $2^j n \le i < 2^{j+1}n$,

$$a_i \ll a_{2^j n} \ll \sum_{i=2^j n}^{\infty} |\Delta a_i| \ll \sum_{v=j}^{\infty} a_{2^v n} \ll \sum_{v=2^{j-1}n}^{\infty} \frac{a_v}{v} \ll \sum_{v=[i/4]}^{\infty} \frac{a_v}{v+1}$$

we obtain

$$\begin{aligned}
S_{22} &\le \sum_{i=8n}^{16n} i^{\frac{1}{p}-1} \left(\sum_{l=i}^{\infty} a_l^p l^{p-2} \right)^{\frac{1}{p}} \ll \sum_{i=8n}^{16n} i^{\frac{1}{p}-1} \left(\sum_{l=i}^{\infty} l^{p-2} \left(\sum_{v=[l/4]}^{\infty} \frac{a_v}{v+1} \right)^p \right)^{\frac{1}{p}} \\
&\ll \sum_{i=8n}^{16n} i^{\frac{1}{p}-1} \left(\sum_{l=i}^{\infty} l^{p-2} \left(\sum_{v=[l/4]}^{\infty} \frac{a_v}{v+1} \right)^p \right)^{\frac{1}{p}} \ll \sum_{i=8n}^{16n} i^{\frac{1}{p}-1} \left(\sum_{l=i}^{\infty} l^{-2} \left(\sum_{v=[l/4]}^{\infty} a_v \right)^p \right)^{\frac{1}{p}} \\
&\ll \sum_{i=2n}^{\infty} a_i \sum_{i=8n}^{16n} i^{\frac{1}{p}-1} \left(\sum_{l=i}^{\infty} l^{-2} \right)^{\frac{1}{p}} \ll \sum_{i=2n}^{\infty} a_i \sum_{i=8n}^{16n} i^{-1} \ll \sum_{i=2n}^{\infty} a_i.
\end{aligned}$$

Collecting our estimates, by Lemma 4.3, we obtain that $S_2 \ll E_n(f)_\infty$, herewith the proof of lemma is complete. □

5 Proofs of the theorems

5.1 Proof of Theorem 3.1

By Lemma 4.8 the first inequality in (3.2) is proved, whence

$$B(p, \gamma, \alpha) \subset B(q, \gamma, \alpha^*) \tag{5.1}$$

also holds. To prove the second inequality of (3.2), we use Lemma 4.5, Assume $f \in B(q, \gamma, \alpha^*)$, then

$$I_q := E_0^\gamma(f)_q + E_1^\gamma(f)_q + \sum_{n=1}^{\infty} \mu^*(n) E_{2^n}^\gamma(f)_q \ll \int_0^1 \alpha^*(t) \omega_{k_3}^\gamma(f;t)_q \mathrm{d}t < \infty,$$

where $k_3 \ge \frac{\sigma^*}{\gamma}$ and

$$\mu^*(n) := \int_{2^{-n}}^{2^{1-n}} \alpha^*(t)\mathrm{d}t, n > 1 \textit{ and } \mu^*(0) = 1.$$

Since $1 < p < q$, by Lemma 4.5 and Lemma 4.6, we have

$$\mu(n) \ll \mu^*(n) 2^{n(1/p-1/q)\gamma},\ \mu(4) \ll \mu(3) \ll \mu(2) \ll \mu(1) \ll 1 \text{ and } \mu(n+4) \ll \mu(n).$$

It is clear that

$$\begin{aligned} I_p := &E_0^\gamma(f)_q + E_1^\gamma(f)_q + \sum_{n=1}^{\infty}\mu(n)E_{2^n}^\gamma(f)_q \\ \ll &E_0^\gamma(f)_q + E_1^\gamma(f)_q + \sum_{n=1}^{4}\mu(n)E_{2^n}^\gamma(f)_q + \sum_{n=1}^{\infty}\mu(n+4)E_{2^{n+4}}^\gamma(f)_q \\ \ll &E_0^\gamma(f)_q + E_1^\gamma(f)_q + \sum_{n=1}^{\infty}\mu(n)E_{2^{n+4}}^\gamma(f)_q \\ \ll &E_0^\gamma(f)_q + E_1^\gamma(f)_q + \sum_{n=1}^{\infty}\mu^*(n)2^{n(1/p-1/q)\gamma}E_{2^{n+4}}^\gamma(f)_q \\ \ll &E_0^\gamma(f)_q + E_1^\gamma(f)_q + \sum_{n=1}^{\infty}\mu^*(n)\left(2^{n(1/p-1/q)}E_{2^{n+4}}(f)_q\right)^\gamma \\ \ll &E_0^\gamma(f)_q + E_1^\gamma(f)_q + \sum_{n=1}^{\infty}\mu^*(n)\left(\sum_{i=2^{n+3}}^{2^{n+4}} i^{(1/p-1/q)-1}E_i(f)_q\right)^\gamma. \end{aligned}$$

Hence, if $q = \infty$, by Lemma 4.14, we obtain

$$I_p \ll E_0^\gamma(f)_q + E_1^\gamma(f)_q + \sum_{n=1}^{\infty}\mu^*(n)E_{2^n}^\gamma(f)_q$$

and immediately $I_p \ll I_q$. If $1 < q < \infty$, then applying Hölder's inequality and Lemma 4.14, we have

$$\begin{aligned} \sum_{i=2^{n+3}}^{2^{n+4}} i^{(1/p-1/q)-1}E_i(f)_q &= \sum_{i=2^{n+3}}^{2^{n+4}} i^{1/p-2/q}E_i(f)_q i^{1/q-1} \\ &\le \left(\sum_{i=2^{n+3}}^{2^{n+4}} i^{q/p-2}E_i^q(f)_q\right)^{1/q}\left(\sum_{i=2^{n+3}}^{2^{n+4}}\left(i^{1/q-1}\right)^{q/(q-1)}\right)^{1-1/q} \\ &\ll \left(\sum_{i=2^{n+3}}^{2^{n+4}} i^{q/p-2}E_i^q(f)_q\right)^{1/q}. \end{aligned}$$

From this and Lemma 4.5, we can obtain

$$\begin{aligned} I_p &\ll E_0^\gamma(f)_q + E_1^\gamma(f)_q + \left(\sum_{i=2^{n+3}}^{2^{n+4}} i^{q/p-2}E_i^q(f)_q\right)^{\gamma/q} \\ &\ll E_0^\gamma(f)_q + E_1^\gamma(f)_q + \sum_{n=1}^{\infty}\mu^*(n)E_{2^n}^\gamma(f)_q \end{aligned}$$

then by Lemma 4.14, $I_p \ll I_q$ is visible.

Finally, by Lemma 4.5, we obtain that

$$\int_0^1 \alpha(t)\omega_{k_1}^\gamma(f;t)_p\mathrm{d}t \ll I_p \ll I_q \ll \int_0^1 \alpha^*(t)\omega_{k_3}^\gamma(f;t)_p\mathrm{d}t < \infty$$

follows with $k_1 \ge \frac{\sigma}{\gamma}$.

This proves the second inequality of (3.2), consequently

$$B(q,\gamma,\alpha^*) \subset B(p,\gamma,\alpha). \tag{5.2}$$

Thus, (5.1) and (5.2) completes the proof of Theorem 3.1 with $\{a_n\} \in$ GBVS. ■

5.2 Proof of Theorem 3.2

First, we prove $S(A,p,k,n) \ll \omega_k\left(f;\frac{1}{n}\right)_p$. We separate two cases:

(i) If $1 < p \le 2$, by Lemma 4.7, we easily know $S(A,p,k,n) \ll \omega_k\left(f;\frac{1}{n}\right)_p$ holds.

(ii) If $p \ge 2$, then by Lemma 4.13, Jackson's theorem and the properties of $\omega_k(f;\delta)_p$, we obtain

$$\left(\sum_{i=n+1}^{\infty} a_i^p i^{p-2}\right)^{1/p} \ll E_{n^*}(f)_p \ll \omega_k\left(f;\frac{1}{n}\right)_p, \tag{5.3}$$

where

$$n^* = \begin{cases} m, & \text{if } n = 2m, \\ m, & \text{if } n = 2m-1. \end{cases}$$

By (4.6) and Lemma 4.13, we easily obtain

$$\begin{aligned} a_i^p i^{p-1} &\ll \left(\sum_{j=[(i-1)/2]+1}^{2(i-1)} \frac{a_j}{j}\right)^p i^{p-1} \ll \left(\sum_{j=[i/4]+1}^{2i} \frac{a_j}{j}\right)^p i^{p-1} \ll \left(\sum_{j=[i/4]+1}^{2i} a_j\right)^p i^{-1} \ll \left(\sum_{j=[i/4]+1}^{2i} j^{-1/p} a_j\right)^p \\ &\ll \sum_{j=[i/4]+1}^{2i} a_j^p \left(\sum_{j=[i/4]+1}^{2i} j^{-1/(p-1)}\right)^{p-1} \ll i^{p-2} \sum_{j=[i/4]+1}^{2i} a_j^p \ll \sum_{j=[i/4]+1}^{2i} j^{p-2} a_j^p \ll E_{[i/8]}^p(f)_p. \end{aligned}$$

Putting this into the following sum and applying Lemma 4.4, we find the following estimates:

$$\begin{aligned} n^{-k}\left(\sum_{i=1}^{n} a_i^p i^{(k+1)p-2}\right)^{1/p} &\ll n^{-k}\left(\sum_{i=1}^{n} E_{[i/8]}^p(f)_p i^{kp-1}\right)^{1/p} \\ &\ll n^{-k}\left(\sum_{i=1}^{n} i^{kp-1} E_i^p(f)_p\right)^{1/p} \quad \ll \omega_k\left(f;\frac{1}{n}\right)_p. \end{aligned} \tag{5.4}$$

The inequalities (5.3) and (5.4) verify $S(A,p,k,n) \ll \omega_k\left(f;\frac{1}{n}\right)_p$ for $2 \le p < \infty$, thus it is proved for any $1 < p < \infty$.

Next, we prove that $\omega_k\left(f;\frac{1}{n}\right)_p \ll S(A,p,k,n)$. We consider two cases:

(i) If $2 \le p < \infty$, by Lemma 4.7, we easily know $\omega_k\left(f;\frac{1}{n}\right)_p \ll S(A,p,k,n)$ holds.

(ii) If $1 < p \le 2$, then we use Lemma 4.4 and Lemma 4.2, thus an elementary calculation, we obtain that

(5.5)

$$\begin{aligned} \omega_k\left(f;\frac{1}{n}\right)_p &\ll n^{-k}\left(\sum_{i=1}^{n} i^{kp-1} E_n^p(f)_p\right)^{1/p} \\ &\ll n^{-k}\left(\sum_{i=1}^{n} i^{kp-1} a_{i+1}^p (i+1)^{p-1} + \sum_{i=1}^{n} i^{kp-1} \sum_{j=i+1}^{\infty} j^{p-2} a_j^p\right)^{1/p} \\ &\ll n^{-k}\left(\sum_{i=1}^{n} i^{(k+1)p-2} a_{i+1}^p + \sum_{i=1}^{n} i^{kp-1} \sum_{j=i+1}^{n} j^{p-2} a_j^p + \sum_{i=1}^{n} i^{kp-1} \sum_{j=n+1}^{\infty} j^{p-2} a_j^p\right)^{1/p} \\ &\ll n^{-k}\left(\sum_{i=1}^{n} i^{(k+1)p-2} a_i^p + \sum_{j=2}^{n} j^{p-2} a_j^p \sum_{i=1}^{j} i^{kp-1} + (n+1)^{(k+1)p-2} a_{n+1}^p + n^{kp} \sum_{j=n+1}^{\infty} j^{p-2} a_j^p\right)^{1/p} \\ &\ll n^{-k}\left(\sum_{i=1}^{n} i^{(k+1)p-2} a_i^p + n^{kp} \sum_{j=n+1}^{\infty} j^{p-2} a_j^p\right)^{1/p} \\ &\ll n^{-k}\left(\sum_{i=1}^{n} i^{(k+1)p-2} a_i^p\right)^{1/p} + \left(\sum_{j=n+1}^{\infty} j^{p-2} a_j^p\right)^{1/p} = S(A,p,k,n). \end{aligned}$$

This proves $\omega_k\left(f;\frac{1}{n}\right)_p \ll S(A,p,k,n)$ for $1<p\le 2$, and consequently for any $1<p<\infty$.

Herewith the proof of Theorem 3.2 is complete. ■

5.3 Proof of Theorem 3.3

By the following inequality

$$J := \int_0^1 t^{-r\gamma-1}\omega_k^\gamma(f;t)_p\,\mathrm{d}t \asymp \sum_{n=1}^{\infty} n^{r\gamma-1}\omega_k^\gamma\left(f;\frac{1}{n}\right)_p \tag{5.6}$$

and Theorem 3.2, we can obtain

$$\begin{aligned} J &\ll \sum_{n=1}^{\infty} n^{r\gamma-1}\omega_k^\gamma\left(f;\frac{1}{n}\right)_p \\ &\ll \sum_{n=1}^{\infty} n^{r\gamma-1}\left(n^{-k}\left(\sum_{i=1}^{n} i^{(k+1)p-2}a_i^p\right)^{1/p}+\left(\sum_{j=n+1}^{\infty} j^{p-2}a_j^p\right)^{1/p}\right)^\gamma \\ &\ll \sum_{n=1}^{\infty} n^{(r-k)\gamma-1}\left(\sum_{i=1}^{n} i^{(k+1)p-2}a_i^p\right)^{\gamma/p}+\sum_{n=1}^{\infty} n^{r\gamma-1}\left(\sum_{j=n+1}^{\infty} j^{p-2}a_j^p\right)^{\gamma/p}. \end{aligned} \tag{5.7}$$

Case (i): $\gamma\ge 1$

Sufficiency. We distinguish two cases listed under (A) and (B):

Case (A): $\gamma/p\ge 1$, by Lemma 4.1, we can obtain

$$\begin{aligned} J &\ll \sum_{n=1}^{\infty} n^{r\gamma+\gamma-\frac{\gamma}{p}-1-\frac{(r-k)\gamma^2}{p}}a_n^\gamma\left(\sum_{i=n}^{\infty} i^{(r-k)\gamma-1}\right)^{\gamma/p}+\sum_{n=1}^{\infty} n^{r\gamma+\gamma-\frac{\gamma}{p}-1-\frac{r\gamma^2}{p}}a_n^\gamma\left(\sum_{i=1}^{n} i^{r\gamma-1}\right)^{\gamma/p} \\ &\ll \sum_{n=1}^{\infty} n^{r\gamma+\gamma-\gamma/p-1}a_n^\gamma. \end{aligned} \tag{5.8}$$

From the above estimate we get that $J\ll J_1$. under $\gamma/p\ge 1$.

Case (B): $\gamma/p<1$, by (5.6), we can yields that

$$J \asymp \sum_{n=0}^{\infty} 2^{nr\gamma}\omega_k^\gamma\left(f;\frac{1}{2^n}\right)_p \tag{5.9}$$

then using again Theorem 3.2, we obtain that

$$\begin{aligned} J &\ll \sum_{n=0}^{\infty} 2^{n(r-k)\gamma}\left(\sum_{i=1}^{2^n} i^{(k+1)p-2}a_i^p\right)^{\gamma/p}+\sum_{n=0}^{\infty} 2^{nr\gamma}\left(\sum_{i=2^n+1}^{\infty} i^{p-2}a_i^p\right)^{\gamma/p} \\ :&= J_{11}+J_{12}. \end{aligned} \tag{5.10}$$

Applying Lemma 4.1, Lemma 4.2, Lemma 4.9 and Hölder's inequality, we can obtain that

(5.11)

$$\begin{aligned} J_{11} &= \sum_{n=0}^{\infty} 2^{n(r-k)\gamma}\left(\sum_{i=1}^{2^n} i^{(k+1)p-2}a_i^p\right)^{\gamma/p} \ll \sum_{n=0}^{\infty} 2^{n(r-k)\gamma}\left(\sum_{j=0}^{n}\sum_{i=2^j}^{2^{j+1}} i^{(k+1)p-2}\left(\sum_{t=[i/4]+1}^{2i}\frac{a_t}{t}\right)^p\right)^{\gamma/p} \\ &\ll \sum_{n=0}^{\infty} 2^{n(r-k)\gamma}\left(\sum_{j=0}^{n} 2^{j((k+1)p-1)}\left(\frac{1}{2^j}\sum_{t=[2^{j-2}]+1}^{2^{j+1}} a_t\right)^p\right)^{\gamma/p} \ll \sum_{n=0}^{\infty} 2^{n(r-k)\gamma}\sum_{j=0}^{n} 2^{j((k+1)\gamma-\gamma/p)}\left(\frac{1}{2^j}\sum_{t=[2^{j-2}]+1}^{2^{j+1}} a_t\right)^\gamma \\ &\ll \sum_{n=0}^{\infty} 2^{n(r-k)\gamma}\sum_{j=0}^{n} 2^{j((k+1)\gamma-\gamma/p-1)}\sum_{t=[2^{j-2}]+1}^{2^{j+1}} a_t^\gamma \ll \sum_{j=0}^{\infty} 2^{j((k+1)\gamma-\gamma/p-1)}\sum_{t=[2^{j-2}]+1}^{2^{j+1}} a_t^\gamma\sum_{n=j}^{\infty} 2^{n(r-k)\gamma} \\ &\ll \sum_{j=0}^{\infty} 2^{j((r+1)\gamma-\gamma/p-1)}\sum_{t=[2^{j-2}]+1}^{2^{j+1}} a_t^\gamma \ll \sum_{j=0}^{\infty}\sum_{t=[2^{j-2}]+1}^{2^{j+1}} t^{(r+1)\gamma-\gamma/p-1}a_t^\gamma \ll \sum_{n=1}^{\infty} n^{r\gamma+\gamma-\gamma/p-1}a_n^\gamma \end{aligned}$$

and

$$
\begin{aligned}
J_{12} &= \sum_{n=0}^{\infty} 2^{nr\gamma}\left(\sum_{i=2^n+1}^{\infty} i^{p-2}a_i^p\right)^{\gamma/p} \ll \sum_{n=0}^{\infty} 2^{nr\gamma}\left(\sum_{i=2^n}^{\infty} i^{p-2}a_{i+1}^p\right)^{\gamma/p} \\
&\ll \sum_{n=0}^{\infty} 2^{nr\gamma}\left(\sum_{j=n}^{\infty}\sum_{i=2^j}^{2^{j+1}} i^{p-2}\left(\sum_{t=[i/2]+1}^{2^i}\frac{a_t}{t}\right)^p\right)^{\gamma/p} \ll \sum_{n=0}^{\infty} 2^{nr\gamma}\left(\sum_{j=n}^{\infty} 2^{jp-1}\left(\sum_{t=[2^{j-1}]+1}^{2^{j+1}}\frac{a_t}{t}\right)^p\right)^{\gamma/p} \\
&\ll \sum_{n=0}^{\infty} 2^{nr\gamma}\sum_{j=n}^{\infty} 2^{j\gamma-\gamma/p}\left(\sum_{t=[2^{j-1}]+1}^{2^{j+1}}\frac{a_t}{t}\right)^{\gamma} \ll \sum_{j=0}^{\infty} 2^{j\gamma-\gamma/p}\left(\sum_{t=[2^{j-1}]+1}^{2^{j+1}}\frac{a_t}{t}\right)^{\gamma}\sum_{n=0}^{j} 2^{nr\gamma} \\
&\ll \sum_{j=0}^{\infty} 2^{jr\gamma+\gamma-\gamma/p-1}\sum_{t=[2^{j-1}]+1}^{2^{j+1}} a_t^{\gamma} \ll \sum_{n=1}^{\infty} n^{r\gamma+\gamma-\gamma/p-1}a_n^{\gamma}.
\end{aligned}
\tag{5.12}
$$

The inequalities (5.11) and (5.12) verify $J \ll J_1$ for $\gamma/p \leq 1$, and consequently we complete the proof of sufficiency under $\gamma \geq 1$.

Necessity. Now, we prove that $J \gg J_1$, we start again with (5.9) and use Theorem 3.2, thus we get that

$$
\begin{aligned}
J &\gg \sum_{n=0}^{\infty} 2^{n(r-k)\gamma}\left(\sum_{i=1}^{2^n} i^{(k+1)p-2}a_i^p\right)^{\gamma/p} + \sum_{n=0}^{\infty} 2^{nr\gamma}\left(\sum_{i=2^n+1}^{\infty} i^{p-2}a_i^p\right)^{\gamma/p} \\
&:= J_{21} + J_{22}
\end{aligned}
\tag{5.13}
$$

Similarly, we distinguish two cases listed under (C) and (D):

Case (C): If $\gamma/p \geq 1$, since A$\in$ GBVS, we know that $a_n \leq a_m$ when $m \leq n \leq 2m$. From this the property, we can deduce that

$$2^j a_{2^j+1}^{\gamma} \gg \sum_{i=2^j+1}^{2^{j+1}} a_i^{\gamma}.$$

Combining Lemma 4.1, Lemma 4.2, Lemma 4.9 and Hölder's inequality, we can obtain that

$$
\begin{aligned}
J_{21} &\gg \sum_{n=0}^{\infty} 2^{n(r-k)\gamma}\left(\sum_{i=1}^{2^n} i^{(k+1)p-2}a_i^p\right)^{\gamma/p} \gg \sum_{n=0}^{\infty} 2^{n(r-k)\gamma}\left(\sum_{j=0}^{n-1}\sum_{i=2^j}^{2^{j+1}} i^{(k+1)p-2}a_i^p\right)^{\gamma/p} \\
&\gg \sum_{n=0}^{\infty} 2^{n(r-k)\gamma}\left(\sum_{j=0}^{n-1}\sum_{i=[2^{j-1}]+1}^{2^{j+1}} i^{(k+1)p-2}a_i^p\right)^{\gamma/p} \gg \sum_{n=0}^{\infty} 2^{n(r-k)\gamma}\left(\sum_{j=0}^{n-1} 2^{j((k+1)p-1)}\frac{1}{2^j}\sum_{i=[2^{j-1}]+1}^{2^{j+1}} a_i^p\right)^{\gamma/p} \\
&\gg \sum_{n=0}^{\infty} 2^{n(r-k)\gamma}\sum_{j=0}^{n-1} 2^{j((k+1)r-\gamma/p)}\left(\frac{1}{2^j}\sum_{i=[2^{j-1}]+1}^{2^{j+1}} a_i^p\right)^{\gamma/p} \gg \sum_{n=0}^{\infty} 2^{n(r-k)\gamma}\sum_{j=0}^{n-1} 2^{j((k+1)r-\gamma/p)}\left(\frac{1}{2^j}\sum_{i=[2^{j-1}]+1}^{2^{j+1}} a_i\right)^{\gamma} \\
&\gg \sum_{n=0}^{\infty} 2^{n(r-k)\gamma}\sum_{j=0}^{n-1} 2^{j((k+1)r-\gamma/p)}a_{2^j+1}^{\gamma} \gg \sum_{j=0}^{\infty} 2^{j((k+1)r-\gamma/p)}a_{2^j+1}^{\gamma}\sum_{n=j}^{\infty} 2^{n(r-k)\gamma} \gg \sum_{j=0}^{\infty} 2^{j(r\gamma+r-\gamma/p-1)}2^j a_{2^j+1}^{\gamma} \\
&\gg \sum_{j=0}^{\infty} 2^{j(r\gamma+r-\gamma/p-1)}\sum_{i=2^j+1}^{2^{j+1}} a_i^{\gamma} \gg \sum_{j=0}^{\infty}\sum_{i=2^j+1}^{2^{j+1}} i^{(r\gamma+r-\gamma/p-1)}a_i^{\gamma} \gg \sum_{n=1}^{\infty} n^{r\gamma+\gamma-\gamma/p-1}a_n^{\gamma}.
\end{aligned}
$$

Similarly, we can get that

$$J_{22} \gg \sum_{n=1}^{\infty} n^{r\gamma+\gamma-\gamma/p-1}a_n^{\gamma}. \tag{5.14}$$

From the above two estimates we get that $J \gg J_1$ under $\gamma/p \geq 1$.

Case (D): If $\gamma/p < 1$, using (5.6), Theorem 3.2 and Lemma 4.1, we can obtain that

$$
\begin{aligned}
J &\gg \sum_{n=1}^{\infty} n^{(r-k)\gamma-1}\left(\sum_{i=1}^{n} i^{(k+1)p-2}a_i^p\right)^{\gamma/p} + \sum_{n=1}^{\infty} n^{r\gamma-1}\left(\sum_{j=n+1}^{\infty} j^{p-2}a_j^p\right)^{\gamma/p} \\
&\gg \sum_{n=1}^{\infty} n^{r\gamma+\gamma-\frac{\gamma}{p}-1-\frac{(r-k)\gamma^2}{p}} a_n^{\gamma}\left(\sum_{i=n}^{\infty} i^{(r-k)\gamma-1}\right)^{\gamma/p} + \sum_{n=1}^{\infty} n^{r\gamma+\gamma-\frac{\gamma}{p}-1-\frac{r\gamma^2}{p}} a_n^{\gamma}\left(\sum_{i=1}^{n} i^{r\gamma-1}\right)^{\gamma/p} \\
&\gg \sum_{n=1}^{\infty} n^{r\gamma+\gamma-\gamma/p-1}a_n^{\gamma} = J_1.
\end{aligned} \tag{5.15}
$$

The inequality (5.15) verify $J \gg J_1$ for $\gamma/p < 1$, and consequently we complete the proof of necessity under $\gamma \geq 1$.

Case (ii): $0 < \gamma < 1$, in this case, we easily know that $\gamma/p < 1$.

Necessity. Necessity can be proved by (5.15).

Sufficiency. Applying (5.10), Lemma 4.1, Lemma 4.2 and Lemma 4.9, we can obtain that

$$
\begin{aligned}
J &\ll \sum_{n=0}^{\infty} 2^{n(r-k)\gamma}\left(\sum_{i=1}^{2^n} i^{(k+1)p-2}a_i^p\right)^{\gamma/p} + \sum_{n=0}^{\infty} 2^{nr\gamma}\left(\sum_{i=2^n+1}^{\infty} i^{p-2}a_i^p\right)^{\gamma/p} \\
&\ll \sum_{n=0}^{\infty} 2^{n(r-k)\gamma}\left(\sum_{j=0}^{n}\sum_{i=2^j}^{2^{j+1}} i^{(k+1)p-2}a_i^p\right)^{\gamma/p} + \sum_{n=0}^{\infty} 2^{nr\gamma}\left(\sum_{j=n}^{\infty}\sum_{i=2^j}^{2^{j+1}} i^{p-2}a_i^p\right)^{\gamma/p} \\
&\ll \sum_{n=0}^{\infty} 2^{n(r-k)\gamma}\left(\sum_{j=0}^{n}\sum_{i=2^j}^{2^{j+1}} i^{(k+1)p-2}\left(\sum_{l=[i/4]+1}^{2i}\frac{a_l}{l}\right)^p\right)^{\gamma/p} + \sum_{n=0}^{\infty} 2^{nr\gamma}\left(\sum_{j=n}^{\infty}\sum_{i=2^j}^{2^{j+1}} i^{p-2}\left(\sum_{l=[i/4]+1}^{2i}\frac{a_l}{l}\right)^p\right)^{\gamma/p} \\
&\ll \sum_{n=0}^{\infty} 2^{n(r-k)\gamma}\left(\sum_{j=0}^{n}\sum_{i=2^j}^{2^{j+1}} i^{kp-2}\left(\sum_{l=[2^{j-2}]+1}^{2^{j+2}} a_l\right)^p\right)^{\gamma/p} + \sum_{n=0}^{\infty} 2^{nr\gamma}\left(\sum_{j=n}^{\infty}\sum_{i=2^j}^{2^{j+1}} i^{-2}\left(\sum_{l=[2^{j-2}]+1}^{2^{j+2}} a_l\right)^p\right)^{\gamma/p} \\
&\ll \sum_{n=0}^{\infty} 2^{n(r-k)\gamma}\left(\sum_{j=0}^{n} 2^{j(kp-1)}\left(\sum_{l=[2^{j-2}]+1}^{2^{j+2}} a_l\right)^p\right)^{\gamma/p} + \sum_{n=0}^{\infty} 2^{nr\gamma}\left(\sum_{j=n}^{\infty} 2^{-j}\left(\sum_{l=[2^{j-2}]+1}^{2^{j+2}} a_l\right)^p\right)^{\gamma/p} \\
&\ll \sum_{n=0}^{\infty} 2^{n(r-k)\gamma}\sum_{j=0}^{n} 2^{j(k\gamma-\gamma/p)}\left(\sum_{l=[2^{j-2}]+1}^{2^{j+2}} a_l\right)^{\gamma} + \sum_{n=0}^{\infty} 2^{nr\gamma}\sum_{j=n}^{\infty} 2^{-j\gamma/p}\left(\sum_{l=[2^{j-2}]+1}^{2^{j+2}} a_l\right)^{\gamma} \\
&\ll \sum_{n=0}^{\infty} 2^{n(r-k)\gamma}\sum_{j=0}^{n} 2^{j(k\gamma-\gamma/p)}\sum_{l=[2^{j-2}]+1}^{2^{j+2}} a_l^{\gamma} + \sum_{n=0}^{\infty} 2^{nr\gamma}\sum_{j=n}^{\infty} 2^{-j\gamma/p}\sum_{l=[2^{j-2}]+1}^{2^{j+2}} a_l^{\gamma} \\
&\ll \sum_{j=0}^{\infty} 2^{j(k\gamma-\gamma/p)}\sum_{l=[2^{j-2}]+1}^{2^{j+2}} a_l^{\gamma}\sum_{n=j}^{\infty} 2^{n(r-k)\gamma} + \sum_{j=0}^{\infty} 2^{-j\gamma/p}\sum_{l=[2^{j-2}]+1}^{2^{j+2}} a_l^{\gamma}\sum_{n=0}^{j} 2^{nr\gamma} \\
&\ll \sum_{j=0}^{\infty} 2^{j(r\gamma-\gamma/p)}\sum_{l=[2^{j-2}]+1}^{2^{j+2}} a_l^{\gamma} \ll \sum_{n=0}^{\infty}\sum_{i=[2^{n-2}]+1}^{2^{n+2}} i^{(r\gamma-\gamma/p)}a_i^{\gamma} \\
&\ll \sum_{n=1}^{\infty} n^{r\gamma-\gamma/p}a_n^{\gamma} \ll J_2.
\end{aligned}
$$

This ends our proof of Theorem 3.3. ■

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Existence and uniqueness results for a nonlocal q-fractional integral boundary value problem of sequential orders

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Abstract

In this paper, we discuss the existence of solutions for a new boundary value problem of nonlinear q-fractional integral equations involving fractional orders $0 < \beta \leq 1$, $1 < \gamma \leq 2$ and nonlocal q-integral boundary conditions. Our results rely on classical tools of fixed point theory. We demonstrate the application of our work with the aid of an example.

Key words and phrases: Sequential; fractional integro-differential equations; boundary conditions; existence; fixed point.
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1 Introduction

Fractional calculus has developed into a useful mathematical tool for modelling of several real world phenomena occurring in applied and technical sciences ([1]-[3]). As a matter of fact, fractional-order models are replacing their integer-order counterparts due to the ability of fractional-order operators to describe the hereditary properties of processes and phenomena involved in the models under consideration. For examples and details, we refer to a series of papers [4]-[10]) and the references cited therein.

Motivated by the popularity of fractional differential equations, q-difference equations of fractional-order are also attracting a considerable attention. Fractional q-difference equations may be regarded as fractional analogue of q-difference equations. For earlier work on the topic, we refer to ([11]-[12]), while some recent development of fractional q-difference equations, for instance, can be found in ([13]-[21]). The basic concepts of q-fractional calculus can be found in a recent text [22].

In this paper, we consider a nonlocal fractional q-difference integral boundary value problem of sequential orders given by

$$^cD_q^{\beta}(^cD_q^{\gamma} + \lambda)x(t) = pf(t, x(t)) + kI_q^{\xi}g(t, x(t)), \quad 0 \leq t \leq 1, \ 0 < q < 1, \tag{1}$$

$$\begin{cases} x(0) = aI_q^{\alpha-1}x(\eta) = a\displaystyle\int_0^{\eta} \frac{(\eta - qs)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)}x(s)d_qs, \\ x(1) = bI_q^{\alpha-1}x(\sigma) = b\displaystyle\int_0^{\sigma} \frac{(\sigma - qs)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)}x(s)d_qs, \ \alpha > 2, \ 0 < \eta, \sigma < 1, \\ D_qx(1) = 0, \end{cases} \tag{2}$$

where $^cD_q^{\beta}$ and $^cD_q^{\gamma}$ denote the fractional q-derivative of the Caputo type, $0 < \beta \leq 1$, $1 < \gamma \leq 2$, $I_{q,0}^{\xi}(.) = I_q^{\xi}(.)$ denotes Riemann-Liouville integral with $0 < \xi < 1$, f, g are given continuous functions, and λ, p, k are real constants.

The paper is organized as follows. Section 2 contains some necessary background material on the topic, while the main results are presented in Section 3. We make use of Banach's contraction principle, Krasnoselskii's fixed point theorem and Leray-Schauder nonlinear alternative to establish the existence results for the problem at hand. Although these tools are standard, yet their exposition in the framework of the present problem is new.

2 Preliminaries on fractional q-calculus

This section is devoted to the notations of and basic concepts of q-fractional calculus [23]-[24].

A q-real number for a real parameter $q \in \mathbb{R}^+ \setminus \{1\}$, denoted by $[u]_q$, is defined by

$$[u]_q = \frac{1-q^u}{1-q}, \quad u \in \mathbb{R}.$$

The q-analogue of the Pochhammer symbol (q-shifted factorial) is defined as

$$(u;q)_0 = 1, \quad (u;q)_k = \prod_{i=0}^{k-1}(1-uq^i), \; k \in \mathbb{N} \cup \{\infty\}.$$

The q-analogue of the exponent $(u-v)^k$ is

$$(u-v)^{(0)} = 1, \quad (u-v)^{(k)} = \prod_{j=0}^{k-1}(u-vq^j), \; k \in \mathbb{N}, \; u, v \in \mathbb{R}.$$

The q-gamma function $\Gamma_q(u)$ is defined as

$$\Gamma_q(u) = \frac{(1-q)^{(u-1)}}{(1-q)^{u-1}},$$

where $u \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$. Observe that $\Gamma_q(v+1) = [v]_q \Gamma_q(v)$.

Definition 2.1 *([23]) Let f be a function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type of order $\beta \geq 0$ is $(I_q^0 f)(t) = f(t)$ and*

$$I_q^\beta f(t) := \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} f(s) d_q s = t^\beta (1-q)^\beta \sum_{k=0}^{\infty} q^k \frac{(q^\beta;q)_n}{(q;q)_n} f(tq^k), \quad \beta > 0, \quad t \in [0,1].$$

Observe that $\beta = 1$ in the Definition 2.1 yields q-integral

$$I_q f(t) := \int_0^t f(s) d_q s = t(1-q) \sum_{k=0}^{\infty} q^k f(tq^k).$$

For more details on q-integral and fractional q-integral, see Section 1.3 and Section 4.2 respectively in [22].

Remark 2.2 *The q-fractional integration possesses the semigroup property (Proposition 4.3 [22]):*

$$I_q^\gamma I_q^\beta f(t) = I_q^{\beta+\gamma} f(t); \; \gamma, \beta \in \mathbb{R}^+.$$

Further, it has been shown in Lemma 6 of [24] that

$$I_q^\beta (x)^{(\sigma)} = \frac{\Gamma_q(\sigma+1)}{\Gamma_q(\beta+\sigma+1)} (x)^{(\beta+\sigma)}, \; 0 < x < a, \beta \in \mathbb{R}^+, \sigma \in (-1, \infty).$$

Before giving the definition of fractional q-derivative, we recall the concept of q-derivative. We know that the q-derivative of a function $f(t)$ is defined as

$$(D_qf)(t) = \frac{f(t)-f(qt)}{t-qt},\ t \neq 0,\ (D_qf)(0) = \lim_{t\to 0}(D_qf)(t).$$

Furthermore,

$$D_q^0 f = f,\ \ D_q^n f = D_q(D_q^{n-1}f),\ \ n = 1,2,3,\ldots. \tag{3}$$

Definition 2.3 *([22]) The Caputo fractional q-derivative of order $\beta > 0$ is defined by*

$${}^cD_q^\beta f(t) = I_q^{\lceil\beta\rceil-\beta} D_q^{\lceil\beta\rceil} f(t),$$

where $\lceil\beta\rceil$ is the smallest integer greater than or equal to β.

Next we recall some properties involving Riemann-Liouville q-fractional integral and Caputo fractional q-derivative (Theorem 5.2 [22]):

$$I_q^\beta\, {}^cD_q^\beta f(t) = f(t) - \sum_{k=0}^{\lceil\beta\rceil-1} \frac{t^k}{\Gamma_q(k+1)}(D_q^k f)(0^+),\ \forall\, t \in (0,a],\ \beta > 0; \tag{4}$$

$${}^cD_q^\beta\, I_q^\beta f(t) = f(t), \quad \forall\, t \in (0,a],\ \beta > 0. \tag{5}$$

In order to define the solution for the problem (1)-(2), we need the following lemma.

Lemma 2.4 *For a given $h \in C([0,1],\mathbb{R})$, the unique solution of the linear boundary value problem:*

$${}^cD_q^\beta({}^cD_q^\gamma + \lambda)x(t) = h(t), \quad 0 \le t \le 1,\ 0 < q < 1, \tag{6}$$

$$\begin{cases} x(0) = aI_q^{\alpha-1}x(\eta) = a\displaystyle\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}x(s)d_qs, \\ x(1) = bI_q^{\alpha-1}x(\sigma) = b\displaystyle\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}x(s)d_qs,\ \alpha > 2,\ 0 < \eta,\sigma < 1, \\ D_qx(1) = 0, \end{cases} \tag{7}$$

is given by

$$\begin{aligned} x(t) \ &= \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(I_q^\beta h(u) - \lambda x(u)\Big)d_qu \\ &+aA(t)\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(I_q^\beta h(u) - \lambda x(u)\Big)d_qu\Big)d_qs \\ &-bB(t)\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(I_q^\beta h(u) - \lambda x(u)\Big)d_qu\Big)d_qs \\ &+B(t)\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(I_q^\beta h(u) - \lambda x(u)\Big)d_qu \\ &-C(t)\int_0^1 \frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\Big(I_q^\beta h(u) - \lambda x(u)\Big)d_qu\Big)d_qs, \end{aligned} \tag{8}$$

where

$$A(t) = \frac{1}{\Delta}\Big[\Big(\mu_5[\gamma-2]_q - \mu_6[\gamma-1]_q\Big)t^\gamma - \Big(\mu_4[\gamma-2]_q - \mu_6[\gamma]_q\Big)t^{\gamma-1} + \Big(\mu_4[\gamma-1]_q - \mu_5[\gamma]_q\Big)t^{\gamma-2}\Big], \tag{9}$$

$$B(t) = \frac{1}{\Delta}\Big[\Big(\mu_2[\gamma-2]_q - \mu_3[\gamma-1]_q\Big)t^\gamma - \Big(\mu_1[\gamma-2]_q - \mu_3[\gamma]_q\Big)t^{\gamma-1} + \Big(\mu_1[\gamma-1]_q - \mu_2[\gamma]_q\Big)t^{(\gamma-2)}\Big], \tag{10}$$

$$C(t) = \frac{1}{\Delta}\Big[\Big(\mu_3\mu_5 - \mu_2\mu_6\Big)t^{\gamma} - \Big(\mu_3\mu_4 - \mu_1\mu_6\Big)t^{\gamma-1} + \Big(\mu_2\mu_4 - \mu_1\mu_5\Big)t^{\gamma-2}\Big], \tag{11}$$

$$\mu_1 = \Big(\frac{a\eta^{(\gamma+\alpha-1)}\Gamma_q(\gamma+1)}{\Gamma_q(\gamma+\alpha)}\Big), \quad \mu_2 = \Big(\frac{a\eta^{(\gamma+\alpha-2)}\Gamma_q(\gamma)}{\Gamma_q(\gamma+\alpha-1)}\Big),$$

$$\mu_3 = \Big(\frac{a\eta^{(\gamma+\alpha-3)}\Gamma_q(\gamma-1)}{\Gamma_q(\gamma+\alpha-2)}\Big), \quad \mu_4 = \Big(\frac{b\sigma^{(\gamma+\alpha-1)}\Gamma_q(\gamma+1)}{\Gamma_q(\gamma+\alpha)} - 1\Big),$$

$$\mu_5 = \Big(\frac{b\sigma^{(\gamma+\alpha-2)}\Gamma_q(\gamma)}{\Gamma_q(\gamma+\alpha-1)} - 1\Big), \quad \mu_6 = \Big(\frac{b\sigma^{(\gamma+\alpha-3)}\Gamma_q(\gamma-1)}{\Gamma_q(\gamma+\alpha-2)} - 1\Big),$$

$$\Delta = (\mu_1\mu_5 - \mu_2\mu_4)[\gamma-2]_q + (\mu_3\mu_4 - \mu_1\mu_6)[\gamma-1]_q + (\mu_2\mu_6 - \mu_3\mu_5)[\gamma]_q \neq 0.$$

Proof. Using (4), the solution $x(t)$ of (6) can be written as

$$x(t) = \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(I_q^{\beta}h(u) - \lambda x(u)\Big)d_qu - \frac{t^{\gamma}}{\Gamma_q(\gamma+1)}c_0 - t^{\gamma-1}c_1 - t^{\gamma-2}c_2. \tag{12}$$

q-differentiating both sides of (12), we obtain

$$\begin{aligned} D_qx(t) = &\int_0^t \frac{(t-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\Big(I_q^{\beta}h(u) - \lambda x(u)\Big)d_qu \\ &- \frac{[\gamma]_q t^{\gamma-1}}{\Gamma_q(\gamma+1)}c_0 - [\gamma-1]_q t^{\gamma-2}c_1 - [\gamma-2]_q t^{\gamma-3}c_2, \quad t \in [0,1]. \end{aligned} \tag{13}$$

Using the boundary conditions (7) in (12), we have

$$\begin{aligned} &\frac{1}{\Gamma_q(\gamma+1)}\Big(\frac{a\eta^{(\gamma+\alpha-1)}\Gamma_q(\gamma+1)}{\Gamma_q(\gamma+\alpha)}\Big)c_0 + \Big(\frac{a\eta^{(\gamma+\alpha-2)}\Gamma_q(\gamma)}{\Gamma_q(\gamma+\alpha-1)}\Big)c_1 + \Big(\frac{a\eta^{(\gamma+\alpha-3)}\Gamma_q(\gamma-1)}{\Gamma_q(\gamma+\alpha-2)}\Big)c_2 \\ &= a\int_0^{\eta} \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(I_q^{\beta}h(u) - \lambda x(u)\Big)d_qu\Big)d_qs, \end{aligned}$$

$$\begin{aligned} &\frac{1}{\Gamma_q(\gamma+1)}\Big(\frac{b\sigma^{(\gamma+\alpha-1)}\Gamma_q(\gamma+1)}{\Gamma_q(\gamma+\alpha)} - 1\Big)c_0 + \Big(\frac{b\sigma^{(\gamma+\alpha-2)}\Gamma_q(\gamma)}{\Gamma_q(\gamma+\alpha-1)} - 1\Big)c_1 \\ &+ \Big(\frac{b\sigma^{(\gamma+\alpha-3)}\Gamma_q(\gamma-1)}{\Gamma_q(\gamma+\alpha-2)} - 1\Big)c_2 \\ &= b\int_0^{\sigma} \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(I_q^{\beta}h(u) - \lambda x(u)\Big)d_qu\Big)d_qs \\ &- \int_0^1 \frac{(1-qu)^{\gamma-1}}{\Gamma_q(\gamma)}\Big(I_q^{\beta}h(u) - \lambda x(u)\Big)d_qu, \end{aligned}$$

$$\frac{1}{\Gamma_q(\gamma+1)}[\gamma]_q c_0 + [\gamma-1]_q c_1 + [\gamma-2]_q c_2 = \int_0^1 \frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\Big(I_q^{\beta}h(u) - \lambda x(u)\Big)d_qu.$$

Solving the above system of equations for c_0, c_1, c_2, we get

$$\begin{aligned} c_0 = &\frac{\Gamma_q(\gamma+1)}{\Delta}\Bigg[\Big(\mu_5[\gamma-2]_q - \mu_6[\gamma-1]_q\Big)a\int_0^{\eta} \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \\ &\times\Big(I_q^{\beta}h(u) - \lambda x(u)\Big)d_qu\Big)d_qs \\ &-\Big(\mu_2[\gamma-2]_q - \mu_3[\gamma-1]_q\Big)b\int_0^{\sigma} \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \\ &\times\Big(I_q^{\beta}h(u) - \lambda x(u)\Big)d_qu\Big)d_qs \end{aligned}$$

$$
\begin{aligned}
&+\Big(\mu_2[\gamma-2]_q-\mu_3[\gamma-1]_q\Big)\int_0^1\frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(I_q^{\beta}h(u)-\lambda x(u)\Big)d_qu\\
&-\Big(\mu_3\mu_5-\mu_2\mu_6\Big)\int_0^1\frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\Big(I_q^{\beta}h(u)-\lambda x(u)\Big)d_qu\Bigg],\\
c_1 \;=\;& \frac{-1}{\Delta}\Bigg[\Big(\mu_4[\gamma-2]_q-\mu_6[\gamma]_q\Big)a\int_0^{\eta}\frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^{s}\frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\\
&\times\Big(I_q^{\beta}h(u)-\lambda x(u)\Big)d_qu\Big)d_qs\\
&-\Big(\mu_1[\gamma-2]_q-\mu_3[\gamma]_q\Big)b\int_0^{\sigma}\frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^{s}\frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\\
&\times\Big(I_q^{\beta}h(u)-\lambda x(u)\Big)d_qu\Big)d_qs\\
&+\Big(\mu_1[\gamma-2]_q-\mu_3[\gamma]_q\Big)\int_0^1\frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(I_q^{\beta}h(u)-\lambda x(u)\Big)d_qu\\
&-\Big(\mu_3\mu_4-\mu_1\mu_6\Big)\int_0^1\frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\Big(I_q^{\beta}h(u)-\lambda x(u)\Big)d_qu\Bigg],\\
c_2 \;=\;& \frac{1}{\Delta}\Bigg[\Big(\mu_4[\gamma-1]_q-\mu_5[\gamma]_q\Big)a\int_0^{\eta}\frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^{s}\frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\\
&\times\Big(I_q^{\beta}h(u)-\lambda x(u)\Big)d_qu\Big)d_qs\\
&-\Big(\mu_1[\gamma-1]_q-\mu_2[\gamma]_q\Big)b\int_0^{\sigma}\frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^{s}\frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\\
&\times\Big(I_q^{\beta}h(u)-\lambda x(u)\Big)d_qu\Big)d_qs\\
&+\Big(\mu_1[\gamma-1]_q-\mu_2[\gamma]_q\Big)\int_0^1\frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(I_q^{\beta}h(u)-\lambda x(u)\Big)d_qu\\
&-\Big(\mu_2\mu_4-\mu_1\mu_5\Big)\int_0^1\frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\Big(I_q^{\beta}h(u)-\lambda x(u)\Big)d_qu\Bigg].
\end{aligned}
$$

Substituting the values of c_0, c_1 and c_2 in (12) yields the solution (8). This completes the proof. □

3 Main results

Let $\mathcal{C} = C([0,1],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ endowed with the usual norm defined by $\|x\| = \sup\{|x(t)|, t \in [0,1]\}$.

In the sequel, we need the following assumptions:

(A_1) $f, g : [0,1]\times\mathbb{R} \to \mathbb{R}$ are continuous functions such that $|f(t,x) - f(t,y)| \le L_1|x-y|$ and $|g(t,x) - g(t,y)| \le L_2|x-y|$, $\forall t \in [0,1],\ L_1,\ L_2 > 0,\ x, y \in \mathbb{R}$;

(A_2) there exist $\delta_1, \delta_2 \in C([0,1],\mathbb{R}^+)$ with $|f(t,x)| \le \delta_1(t)$, $|g(t,x)| \le \delta_2(t)$, $\forall (t,x) \in [0,1]\times\mathbb{R}$, where $\sup_{t\in[0,1]}|\delta_i(t)| = \|\delta_i\|,\ i = 1,2$.

For the sake of computational convenience, let us set the following notations:

$$
\omega_1 = \frac{(1+B_1)}{\Gamma_q(\beta+\gamma+1)} + \frac{1}{\Gamma_q(\beta+\gamma+\alpha)}\Big(|a|A_1\eta^{(\beta+\gamma+\alpha-1)} + |b|B_1\sigma^{(\beta+\gamma+\alpha-1)}\Big) + \frac{C_1}{\Gamma_q(\beta+\gamma)}, \tag{14}
$$

$$\omega_2 = \frac{(1+B_1)}{\Gamma_q(\beta+\xi+\gamma+1)} + \frac{1}{\Gamma_q(\beta+\xi+\gamma+\alpha)}\Big(|a|A_1\eta^{(\beta+\xi+\gamma+\alpha-1)} + |b|B_1\sigma^{(\beta+\xi+\gamma+\alpha-1)}\Big) + \frac{C_1}{\Gamma_q(\beta+\xi+\gamma)}, \tag{15}$$

$$\omega_3 = \frac{(1+B_1)}{\Gamma_q(\gamma+1)} + \frac{1}{\Gamma_q(\gamma+\alpha)}\Big(|a|A_1\eta^{(\gamma+\alpha-1)} + |b|B_1\eta^{(\gamma+\alpha-1)}\Big) + \frac{C_1}{\Gamma_q(\gamma)}, \tag{16}$$

$$\begin{aligned}\Omega = {} & L\Big[|p|\Big(\frac{1}{\Gamma_q(\beta+\gamma+\alpha)}\Big(|a|A_1\eta^{(\beta+\gamma+\alpha-1)} + |b|B_1\sigma^{(\beta+\gamma+\alpha-1)}\Big)\\ & + \frac{B_1}{\Gamma_q(\beta+\gamma+1)} + \frac{C_1}{\Gamma_q(\beta+\gamma)}\Big)\\ & + |k|\Big(\frac{1}{\Gamma_q(\beta+\xi+\gamma+\alpha)}\Big(|a|A_1\eta^{(\beta+\xi+\gamma+\alpha-1)} + |b|B_1\sigma^{(\beta+\xi+\gamma+\alpha-1)}\Big)\\ & + \frac{B_1}{\Gamma_q(\beta+\xi+\gamma+1)} + \frac{C_1}{\Gamma_q(\beta+\xi+\gamma)}\Big)\Big]\\ & + |\lambda|\Big[\frac{1}{\Gamma_q(\gamma+\alpha)}\Big(|a|A_1\eta^{(\gamma+\alpha-1)} + |b|B_1\sigma^{(\gamma+\alpha-1)}\Big) + \frac{B_1}{\Gamma_q(\gamma+1)} + \frac{C_1}{\Gamma_q(\gamma)}\Big],\end{aligned} \tag{17}$$

where $A_1 = \max_{t\in[0,1]}|A(t)|,\ B_1 = \max_{t\in[0,1]}|B(t)|,\ C_1 = \max_{t\in[0,1]}|C(t)|,\ L = \max\{L_1, L_2\}$ and $A(t), B(t), C(t)$ are respectively given by (9), (10) and (11).

In view of Lemma 2.4, we define an operator $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ as

$$\begin{aligned}& (\mathcal{F}x)(t)\\ & = \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(p\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} f(m,x(m))d_qm\\ & + k\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} g(m,x(m))d_qm - \lambda x(u)\Big)d_qu\\ & + aA(t)\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(p\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} f(m,x(m))d_qm\\ & + k\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} g(m,x(m))d_qm - \lambda x(u)\Big)d_qu\Big)d_qs\\ & - bB(t)\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(p\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} f(m,x(m))d_qm\\ & + k\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} g(m,x(m))d_qm - \lambda x(u)\Big)d_qu\Big)d_qs\\ & + B(t)\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(p\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} f(m,x(m))d_qm\\ & + k\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} g(m,x(m))d_qm - \lambda x(u)\Big)d_qu\\ & - C(t)\int_0^1 \frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\Big(p\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} f(m,x(m))d_qm\\ & + k\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} g(m,x(m))d_qm - \lambda x(u)\Big)d_qu.\end{aligned} \tag{18}$$

Observe that problem (1)-(2) has solutions only if the operator equation $x = \mathcal{F}x$ has fixed points.

Our first existence result is based on Krasnoselskii's fixed point theorem.

Lemma 3.1 *(Krasnoselskii) [25]. Let Y be a closed, convex, bounded and nonempty subset of a Banach space X. Let Q_1, Q_2 be the operators such that (a) $Q_1x + Q_2y \in Y$ whenever $x, y \in Y$; (b) Q_1 is compact and continuous and (c) Q_2 is a contraction mapping. Then there exists $z \in Y$ such that $z = Q_1z + Q_2z$.*

Theorem 3.2 *Let $f, g : [0,1]\times\mathbb{R}\to\mathbb{R}$ be continuous functions satisfying the assumption $(A_1)-(A_2)$. Furthermore $\Omega < 1$, where Ω is given by (17) Then the problem (1)-(2) has at least one solution on* $[0,1]$.

Proof. Let us fix

$$\varepsilon \quad \geq \quad \frac{|p|\|\delta_1\|\omega_1 + |k|\|\delta_2\|\omega_2}{1-|\lambda|\omega_3},$$

where $\omega_1,\ \omega_2,\ \omega_3$ are respectively given by (14), (15), (16), and consider $B_\varepsilon = \{x\in\mathcal{C} : \|x\|\leq\varepsilon\}$. We define operators $\mathcal{S}_1$ and $\mathcal{S}_2$ on B_ε as

$$\begin{aligned}
& (\mathcal{S}_1x)(t) \\
= & \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(p\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} f(m,x(m))d_qm \\
+ & \ k\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} g(m,x(m))d_qm - \lambda x(u)\Big)d_qu, \quad t\in[0,1],
\end{aligned}$$

$$\begin{aligned}
& (\mathcal{S}_2x)(t) \\
= & \ aA(t)\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(p\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} f(m,x(m))d_qm \\
+ & \ k\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} g(m,x(m))d_qm - \lambda x(u)\Big)d_qu\Big)d_qs \\
- & \ bB(t)\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(p\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} f(m,x(m))d_qm \\
+ & \ k\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} g(m,x(m))d_qm - \lambda x(u)\Big)d_qu\Big)d_qs \\
+ & \ B(t)\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(p\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} f(m,x(m))d_qm \\
+ & \ k\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} g(m,x(m))d_qm - \lambda x(u)\Big)d_qu \\
- & \ C(t)\int_0^1 \frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\Big(p\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} f(m,x(m))d_qm \\
+ & \ k\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} g(m,x(m))d_qm - \lambda x(u)\Big)d_qu, \quad t\in[0,1].
\end{aligned}$$

For $x, y\in B_\varepsilon$, we find that

$$\|\mathcal{S}_1x+\mathcal{S}_2y\| \quad \leq |p|\|\delta_1\|\omega_1 + |k|\|\delta_2\|\omega_2 + |\lambda|\varepsilon\omega_3 \leq \varepsilon.$$

Thus, $\mathcal{S}_1x+\mathcal{S}_2y\in B_\varepsilon$. Continuity of f and g imply that the operator $\mathcal{S}_1$ is continuous. Also, $\mathcal{S}_1$ is uniformly bounded on B_ε as

$$\|\mathcal{S}_1x\| \leq \frac{|p|\|\delta_1\|}{\Gamma_q(\beta+\gamma+1)} + \frac{|k|\|\delta_2\|}{\Gamma_q(\beta+\xi+\gamma+1)} + \frac{|\lambda|\varepsilon}{\Gamma_q(\gamma+1)}.$$

Now we prove the compactness of the operator $\mathcal{S}_1$. In view of (A_1), we define

$$\sup_{(t,x)\in[0,1]\times B_\varepsilon} |f(t,x)| = \overline{f}, \qquad \sup_{(t,x)\in[0,1]\times B_\varepsilon} |g(t,x)| = \overline{g}.$$

Consequently we have

$$\begin{aligned}
& \|(\mathcal{S}_1x)(t_2)-(\mathcal{S}_1x)(t_1)\| \\
\leq & \int_0^{t_1} \frac{(t_2-qu)^{(\gamma-1)}-(t_1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big[|p|\overline{f}\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}d_qm \\
+ & \ |k|\overline{g}\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}d_qm + |\lambda|\varepsilon\Big]d_qu
\end{aligned}$$

$$+ \quad \int_{t_1}^{t_2} \frac{(t_2-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big[|p|\overline{f}\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}d_qm + |k|\overline{g}\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}d_qm + |\lambda|\varepsilon\Big]d_qu,$$

which is independent of x and tends to zero as $t_2 \to t_1$. Thus, $\mathcal{S}_1$ is relatively compact on B_ε. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{S}_1$ is compact on B_ε. Now, we shall show that $\mathcal{S}_2$ is a contraction.
From (A_1) and for $x, y \in B_\varepsilon$, we have

$$\begin{aligned}
&\|\mathcal{S}_2x - \mathcal{S}_2y\| \\
&\le \sup_{t\in[0,1]}\Bigg\{|a||A(t)|\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \\
&\times\Big|f(m,x(m))-f(m,y(m))\Big|d_qm \\
&+|k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\Big|g(m,x(m))-g(m,y(m))\Big|d_qm + |\lambda||x(u)-y(u)|\Big)d_qu\Big)d_qs \\
&+|b||B(t)|\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \\
&\times\Big|f(m,x(m))-f(m,y(m))\Big|d_qm \\
&+|k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\Big|g(m,x(m))-g(m,y(m))\Big|d_qm + |\lambda||x(u)-y(u)|\Big)d_qu\Big)d_qs \\
&+|B(t)|\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\Big|f(m,x(m))-f(m,y(m))\Big|d_qm \\
&+|k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\Big|g(m,x(m))-g(m,y(m))\Big|d_qm + |\lambda||x(u)-y(u)|\Big)d_qu \\
&+|C(t)|\int_0^1 \frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\Big|f(m,x(m))-f(m,y(m))\Big|d_qm \\
&+|k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\Big|g(m,x(m))-g(m,y(m))\Big|d_qm + |\lambda||x(u)-y(u)|\Big)d_qu\Bigg\} \\
&\le \sup_{t\in[0,1]}\Bigg\{|a||A(t)|\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}L_1\Big|x(m)-y(m)\Big|d_qm \\
&+|k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}L_2\Big|x(m)-y(m)\Big|d_qm + |\lambda||x(u)-y(u)|\Big)d_qu\Big)d_qs \\
&+|b||B(t)|\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}L_1\Big|x(m)-y(m)\Big|d_qm \\
&+|k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}L_2\Big|x(m)-y(m)\Big|d_qm + |\lambda||x(u)-y(u)|\Big)d_qu\Big)d_qs \\
&+|B(t)|\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}L_1\Big|x(m)-y(m)\Big|d_qm \\
&+|k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}L_2\Big|x(m)-y(m)\Big|d_qm + |\lambda||x(u)-y(u)|\Big)d_qu \\
&+|C(t)|\int_0^1 \frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}L_1\Big|x(m)-y(m)\Big|d_qm
\end{aligned}$$

$$+|k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}L_2\Big|x(m)-y(m)\Big|d_qm+|\lambda||x(u)-y(u)|\Big)d_qu\Big\}$$
$$\le \Bigg[L\Big[|p|\Big(\frac{1}{\Gamma_q(\beta+\gamma+\alpha)}\Big(|a|A_1\eta^{(\beta+\gamma+\alpha-1)}+|b|B_1\sigma^{(\beta+\gamma+\alpha-1)}\Big)+\frac{B_1}{\Gamma_q(\beta+\gamma+1)}+\frac{C_1}{\Gamma_q(\beta+\gamma)}\Big)$$
$$+|k|\Big(\frac{1}{\Gamma_q(\beta+\xi+\gamma+\alpha)}\Big(|a|A_1\eta^{(\beta+\xi+\gamma+\alpha-1)}+|b|B_1\sigma^{(\beta+\xi+\gamma+\alpha-1)}\Big)$$
$$+\frac{B_1}{\Gamma_q(\beta+\xi+\gamma+1)}+\frac{C_1}{\Gamma_q(\beta+\xi+\gamma)}\Big)\Big]$$
$$+|\lambda|\Big[\frac{1}{\Gamma_q(\gamma+\alpha)}\Big(|a|A_1\eta^{(\gamma+\alpha-1)}+|b|B_1\sigma^{(\gamma+\alpha-1)}\Big)+\frac{B_1}{\Gamma_q(\gamma+1)}+\frac{C_1}{\Gamma_q(\gamma)}\Big]\Bigg]\|x-y\|=\Omega\|x-y\|,$$

where we have used (17). Since $\Omega<1$ by our assumption, therefore $\mathcal{S}_2$ is a contraction. Thus all the assumptions of Lemma 3.1 are satisfied. So, by the conclusion of Lemma 3.1, the problem $(1)-(2)$ has at least one solution on $[0,1]$. □

In the next result, we make use of Leray-Schauder Alternative.

Lemma 3.3 *(Nonlinear alternative for single valued maps)[26]. Let E be a Banach space, C a closed, convex subset of E, W an open subset of C and $0\in W$. Suppose that $\mathcal{F}:\overline{W}\to C$ is a continuous, compact (that is, $\mathcal{F}(\overline{W})$ is a relatively compact subset of C) map. Then either*

- *(i) $\mathcal{F}$ has a fixed point in $\overline{W}$, or*
- *(ii) there is a $x\in\partial W$ (the boundary of W in C) and $\tau\in(0,1)$ with $x=\tau\mathcal{F}(x)$.*

Theorem 3.4 *Let $f,g:[0,1]\times\mathbb{R}\to\mathbb{R}$ be continuous functions and the following assumptions hold:*

- (A_3) *there exist functions $\phi_1,\phi_2\in C([0,1],\mathbb{R}^+)$, and nondecreasing functions $\Psi_1,\Psi_2:\mathbb{R}^+\to\mathbb{R}^+$ such that $|f(t,x)|\le\phi_1(t)\Psi_1(\|x\|)$, $|g(t,x)|\le\phi_2(t)\Psi_2(\|x\|)$, $\forall(t,x)\in[0,1]\times\mathbb{R}$.*
- (A_4) *There exists a constant $H>0$ such that*
$$H>\frac{|p|\|\phi_1\|\Psi_1(H)\omega_1+|k|\|\phi_2\|\Psi_2(H)\omega_2}{1-|\lambda|\omega_3},$$
where $|\lambda|<\frac{1}{\omega_3}$.

Then the boundary value problem $(1)-(2)$ has at least one solution on $[0,1]$.

Proof. Consider the operator $\mathcal{F}:\mathcal{C}\to\mathcal{C}$ defined by (18). The proof consists of several steps.

- (i) $\mathcal{F}$ is continuous.
  It is easy to show that $\mathcal{F}$ is continuous.
- (ii) $\mathcal{F}$ maps bounded sets into bounded sets in $C([0,1]\times\mathbb{R})$.
  For a positive number $\overline{r}$, let $B_{\overline{r}}=\{x\in\mathcal{C}:\|x\|\le\overline{r}\}$ be a bounded set in $C([0,1]\times\mathbb{R})$ and $x\in B_{\overline{r}}$. Then, we have

$$\begin{aligned}&\|(\mathcal{F}x)\|\\ \le\ &\sup_{t\in[0,1]}\Bigg\{\int_0^t\frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u\frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}|f(m,x(m))|d_qm\\ +\ &|k|\int_0^u\frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}|g(m,x(m))|d_qm+|\lambda||x(u)|\Big)d_qu\\ +\ &|a||A(t)|\int_0^\eta\frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s\frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u\frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}|f(m,x(m))|d_qm\end{aligned}$$

$$
\begin{aligned}
&+ \quad |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}|g(m,x(m))|d_qm+|\lambda||x(u)|\Big)d_qu\Big)d_qs\\
&+ \quad |b||B(t)|\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}|f(m,x(m))|d_qm\\
&+ \quad |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}|g(m,x(m))|d_qm+|\lambda||x(u)|\Big)d_qu\Big)d_qs\\
&+ \quad |B(t)|\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}|f(m,x(m))|d_qm\\
&+ \quad |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}|g(m,x(m))|d_qm+|\lambda||x(u)|\Big)d_qu\\
&+ \quad |C(t)|\int_0^1 \frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}|f(m,x(m))|d_qm\\
&+ \quad |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}|g(m,x(m))|d_qm+|\lambda||x(u)|\Big)d_qu\Bigg\}\\
&\leq \quad \sup_{t\in[0,1]}\Bigg\{\int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\phi_1(m)\Psi_1(\|x\|)d_qm\\
&+ \quad |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\phi_2(m)\Psi_2(\|x\|)d_qm+|\lambda||x(u)|\Big)d_qu\\
&+ \quad |a||A(t)|\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\phi_1(m)\Psi_1(\|x\|)d_qm\\
&+ \quad |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\phi_2(m)\Psi_2(\|x\|)d_qm+|\lambda||x(u)|\Big)d_qu\Big)d_qs\\
&+ \quad |b||B(t)|\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\phi_1(m)\Psi_1(\|x\|)d_qm\\
&+ \quad |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\phi_2(m)\Psi_2(\|x\|)d_qm+|\lambda||x(u)|\Big)d_qu\Big)d_qs\\
&+ \quad |B(t)|\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\phi_1(m)\Psi_1(\|x\|)d_qm\\
&+ \quad |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\phi_2(m)\Psi_2(\|x\|)d_qm+|\lambda||x(u)|\Big)d_qu\\
&+ \quad |C(t)|\int_0^1 \frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\phi_1(m)\Psi_1(\|x\|)d_qm\\
&+ \quad |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\phi_2(m)\Psi_2(\|x\|)d_qm+|\lambda||x(u)|\Big)d_qu\Bigg\}\\
&\leq \quad |p|\|\phi_1\|\Psi_1(\|x\|)\sup_{t\in[0,1]}\Bigg\{\int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big[\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}d_qm\Big]d_qu\\
&+ \quad |a||A(t)|\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big[\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}d_qm\Big)d_qu\Big]d_qs
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad |b||B(t)|\int_0^{\sigma}\frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big[\int_0^s\frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u\frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}d_qm\Big)d_qu\Big]d_qs\\
&+ \quad |B(t)|\int_0^1\frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big[\int_0^u\frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}d_qm\Big]d_qu\\
&+ \quad |C(t)|\int_0^t\frac{(t-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\Big[\int_0^u\frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}d_qm\Big]d_qu\Bigg\}\\
&+ \quad |k|\|\phi_2\|\Psi_2(\|x\|)\sup_{t\in[0,1]}\Bigg\{\int_0^t\frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big[\int_0^u\frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}d_qm\Big]d_qu\\
&+ \quad |a||A(t)|\int_0^{\eta}\frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big[\int_0^s\frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u\frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}d_qm\Big)d_qu\Big]d_qs\\
&+ \quad |b||B(t)|\int_0^{\sigma}\frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big[\int_0^s\frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u\frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}d_qm\Big)d_qu\Big]d_qs\\
&+ \quad |B(t)|\int_0^1\frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big[\int_0^u\frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}d_qm\Big]d_qu\\
&+ \quad |C(t)|\int_0^t\frac{(t-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\Big[\int_0^u\frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}d_qm\Big]d_qu\Bigg\}\\
&+ \quad |\lambda|\|x\|\sup_{t\in[0,1]}\Bigg\{\int_0^t\frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}d_qu+|a||A(t)|\int_0^{\eta}\frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big[\int_0^s\frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}d_qu\Big]d_qs\\
&+ \quad |b||B(t)|\int_0^{\sigma}\frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big[\int_0^s\frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}d_qu\Big]d_qs+|B(t)|\int_0^1\frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}d_qu\\
&+ \quad |C(t)|\int_0^t\frac{(t-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}d_qu\Bigg\}\\
&\leq \quad |p|\|\phi_1\|\Psi_1(\|x\|)\omega_1+|k|\|\phi_2\|\Psi_2(\|x\|)\omega_2+|\lambda|\|x\|\omega_3.
\end{aligned}
$$

(*iii*) $\mathcal{F}$ maps bounded sets into equicontinuous sets of $C([0,1]\times\mathbb{R})$.
Let $t_1,t_2\in[0,1]$ with $t_1<t_2$ and $x\in B_{\overline{r}}$, where $B_{\overline{r}}$ is a bounded set of $C([0,1],\mathbb{R})$. Then, we obtain

$$
\begin{aligned}
&\|(\mathcal{F}x)(t_2)-(\mathcal{F}x)(t_1)\|\\
&\leq \quad \Bigg|\int_0^{t_1}\frac{(t_2-qu)^{(\gamma-1)}-(t_1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big[|p|\int_0^u\frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\phi_1(m)\Psi_1(\overline{r})d_qm\\
&+ \quad |k|\int_0^u\frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\phi_2(m)\Psi_2(\overline{r})d_qm+|\lambda|\overline{r}\Big]d_qu\\
&+ \quad \int_{t_1}^{t_2}\frac{(t_2-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big[|p|\int_0^u\frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\phi_1(m)\Psi_1(\overline{r})d_qm\\
&+ \quad |k|\int_0^u\frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\phi_2(m)\Psi_2(\overline{r})d_qm+|\lambda|\overline{r}\Big]d_qu\Bigg|\\
&+ \quad \frac{|a|}{|\Delta|}\Bigg[\Big|\mu_5[\gamma-2]_q-\mu_6[\gamma-1]_q\Big|\Big|t_2^{\gamma}-t_1^{\gamma}\Big|+\Big|\mu_4[\gamma-2]_q-\mu_6[\gamma-1]_q\Big|\Big|t_2^{\gamma-1}-t_1^{\gamma-1}\Big|\\
&+ \quad \Big|\mu_4[\gamma-1]_q-\mu_5[\gamma]_q\Big|\Big|t_2^{\gamma-2}-t_1^{\gamma-2}\Big|\Bigg]\int_0^{\eta}\frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s\frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\\
&\times \quad \Big(|p|\int_0^u\frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\phi_1(m)\Psi_1(\overline{r})d_qm\\
&+ \quad |k|\int_0^u\frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\phi_2(m)\Psi_2(\overline{r})d_qm+|\lambda|\overline{r}\Big)d_qu\Big)d_qs
\end{aligned}
$$

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$$
\begin{aligned}
&+\ \frac{|b|}{|\Delta|}\bigg[\Big|\mu_2[\gamma-2]_q-\mu_3[\gamma-1]_q\Big|\Big|t_2^{\gamma}-t_1^{\gamma}\Big|+\Big|\mu_1[\gamma-2]_q-\mu_3[\gamma-1]_q\Big|\Big|t_2^{\gamma-1}-t_1^{\gamma-1}\Big|\\
&+\ \Big|\mu_1[\gamma-1]_q-\mu_2[\gamma]_q\Big|\Big|t_2^{\gamma-2}-t_1^{\gamma-2}\Big|\bigg]\int_0^{\sigma}\frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s\frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\\
&\times\ \Big(|p|\int_0^u\frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\phi_1(m)\Psi_1(\bar r)d_qm\\
&+\ |k|\int_0^u\frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\phi_2(m)\Psi_2(\bar r)d_qm+|\lambda|\bar r\Big)d_qu\Big)d_qs\\
&+\ \frac{1}{|\Delta|}\bigg[\Big|\mu_2[\gamma-2]_q-\mu_3[\gamma-1]_q\Big|\Big|t_2^{\gamma}-t_1^{\gamma}\Big|+\Big|\mu_1[\gamma-2]_q-\mu_3[\gamma-1]_q\Big|\Big|t_2^{\gamma-1}-t_1^{\gamma-1}\Big|\\
&+\ \Big|\mu_1[\gamma-1]_q-\mu_2[\gamma]_q\Big|\Big|t_2^{\gamma-2}-t_1^{\gamma-2}\Big|\bigg]\int_0^{1}\frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u\frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\\
&\times\ \phi_1(m)\Psi_1(\bar r)d_qm+|k|\int_0^u\frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\phi_2(m)\Psi_2(\bar r)d_qm+|\lambda||x(u)|\Big)d_qu\\
&+\ \frac{1}{|\Delta|}\bigg[\Big|\mu_3\mu_5-\mu_2\mu_6\Big|\Big|t_2^{\gamma}-t_1^{\gamma}\Big|+\Big|\mu_3\mu_4-\mu_1\mu_6\Big|\Big|t_2^{\gamma-1}-t_1^{\gamma-1}\Big|+\Big|\mu_2\mu_4-\mu_1\mu_5\Big|\Big|t_2^{\gamma-2}-t_1^{\gamma-2}\Big|\bigg]\\
&\times\ \int_0^1\frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\Big(|p|\int_0^u\frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\phi_1(m)\Psi_1(\bar r)d_qm\\
&+\ |k|\int_0^u\frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\phi_2(m)\Psi_2(\bar r)d_qm+|\lambda|\bar r\Big)d_qu.
\end{aligned}
$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_{\bar r}$ as $t_2-t_1 \to 0$. As $\mathcal{F}$ satisfies the above assumptions, therefore it follows by the Arzelá-Ascoli theorem that $\mathcal{F}: \mathcal{C} \to \mathcal{C}$ is completely continuous.

(*iv*) Let x be a solution and $x = \tau\mathcal{F}x$ for $\tau \in (0,1)$. Then, for $t \in [0,1]$, and using the computations in proving that $\mathcal{F}$ is bounded, we have

$$|x(t)| = |\tau(\mathcal{F}x)(t)| \le |p|\|\phi_1\|\Psi_1(\|x\|)\omega_1 + |k|\|\phi_2\|\Psi_2(\|x\|)\omega_2 + |\lambda|\|x\|\omega_3,$$

which implies that

$$\|x\| \le \frac{|p|\|\phi_1\|\Psi_1(\|x\|)\omega_1 + |k|\|\phi_2\|\Psi_2(\|x\|)\omega_2}{1-|\lambda|\omega_3}.$$

In view of (A_4), there exists H such that $\|x\| \ne H$. Let us set

$$W = \{x \in \mathcal{C} : \|x\| < H\}.$$

Note that the operator $\mathcal{F}: \overline{W} \to C([0,1],\mathbb{R})$ is continuous and completely continuous. From the choice of W, there is no $x \in \partial W$ such that $x = \tau\mathcal{F}(x)$ for some $\tau \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3.3), we deduce that $\mathcal{F}$ has a fixed point $x \in \overline{W}$ which is a solution of the problem $(1)-(2)$. This completes the proof. □

The third existence result is based on Banach's contraction principle (Banach fixed point theorem).

Theorem 3.5 *Suppose that the assumption (A_1) holds and that*

$$\bar\Omega = (L\Omega_1 + |\lambda|\omega_3) < 1, \quad \Omega_1 = |p|\omega_1 + |k|\omega_2, \tag{19}$$

where ω_1, ω_2, ω_3 are respectively given by (14), (15), (16), and $L = \max\{L_1,\ L_2\}$. Then the problem (1)-(2) has a unique solution on $[0,1]$.

Proof. Let us define $M = \max\{M_1,\ M_2\}$, where $M_1,\ M_2$ are finite numbers given by $\sup_{t\in[0,1]}|f(t,0)| = M_1,\ \sup_{t\in[0,1]}|g(t,0)| = M_2$. Selecting $\varepsilon \geq \dfrac{M\Omega_1}{1-\bar{\Omega}}$, we show that $\mathcal{F}B_\varepsilon \subset B_\varepsilon$, where $B_\varepsilon = \{x \in \mathcal{C} : \|x\| \leq \varepsilon\}$ For $x \in B_\varepsilon$, we have

$$
\begin{aligned}
&\|(\mathcal{F}x)\| \\
\leq\ & \sup_{t\in[0,1]}\Bigg\{\int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}|f(m,x(m))|d_qm \\
+\ & |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}|g(m,x(m))|d_qm + |\lambda||x(u)|\Big)d_qu \\
+\ & |a||A(t)|\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}|f(m,x(m))|d_qm \\
+\ & |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}|g(m,x(m))|d_qm + |\lambda||x(u)|\Big)d_qu\Big)d_qs \\
+\ & |b||B(t)|\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}|f(m,x(m))|d_qm \\
+\ & |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}|g(m,x(m))|d_qm + |\lambda||x(u)|\Big)d_qu\Big)d_qs \\
+\ & |B(t)|\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}|f(m,x(m))|d_qm \\
+\ & |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}|g(m,x(m))|d_qm + |\lambda||x(u)|\Big)d_qu \\
+\ & |C(t)|\int_0^1 \frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}|f(m,x(m))|d_qm \\
+\ & |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}|g(m,x(m))|d_qm + |\lambda||x(u)|\Big)d_qu\Bigg\} \\
\leq\ & \sup_{t\in[0,1]}\Bigg\{\int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\Big(|f(m,x(m))-f(m,0)| + |f(m,0)|\Big)d_qm \\
+\ & |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\Big(|g(m,x(m))-g(m,0)| + |g(m,0)|\Big)d_qm + |\lambda||x(u)|\Big)d_qu \\
+\ & |a||A(t)|\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\Big(|f(m,x(m))-f(m,0)| \\
+\ & |f(m,0)|\Big)d_qm + |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\Big(|g(m,x(m))-g(m,0)| + |g(m,0)|\Big)d_qm \\
+\ & |\lambda||x(u)|\Big)d_qu\Big)d_qs \\
+\ & |b||B(t)|\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\Big(|f(m,x(m))-f(m,0)| \\
+\ & |f(m,0)|\Big)d_qm + |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\Big(|g(m,x(m))-g(m,0)| + |g(m,0)|\Big)d_qm
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad |\lambda||x(u)|\Big)d_qu\Big)d_qs \\
&+ \quad |B(t)|\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\Big(|f(m,x(m))-f(m,0)|+|f(m,0)|\Big)d_qm \\
&+ \quad |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\Big(|g(m,x(m))-g(m,0)|+|g(m,0)|\Big)d_qm+|\lambda||x(u)|\Big)d_qu \\
&+ \quad |C(t)|\int_0^1 \frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\Big(|f(m,x(m))-f(m,0)|+|f(m,0)|\Big)d_qm \\
&+ \quad |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\Big(|g(m,x(m))-g(m,0)|+|g(m,0)|\Big)d_qm+|\lambda||x(u)|\Big)d_qu\Bigg\} \\
&\le \quad |p|(L_1\varepsilon+M_1)\sup_{t\in[0,1]}\Bigg\{\int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big[\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}d_qm\Big]d_qu \\
&+ \quad |a||A(t)|\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big[\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}d_qm\Big)d_qu\Big]d_qs \\
&+ \quad |b||B(t)|\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big[\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}d_qm\Big)d_qu\Big]d_qs \\
&+ \quad |B(t)|\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big[\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}d_qm\Big)d_qu \\
&+ \quad |C(t)|\int_0^t \frac{(t-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\Big[\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}d_qm\Big]d_qu\Bigg\} \\
&+ \quad |k|(L_2\varepsilon+M_2)\sup_{t\in[0,1]}\Bigg\{\int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big[\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}d_qm\Big]d_qu \\
&+ \quad |a||A(t)|\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big[\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}d_qm\Big)d_qu\Big]d_qs \\
&+ \quad |b||B(t)|\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big[\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}d_qm\Big)d_qu\Big]d_qs \\
&+ \quad |B(t)|\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big[\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}d_qm\Big)d_qu \\
&+ \quad |C(t)|\int_0^t \frac{(t-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\Big[\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}d_qm\Big]d_qu\Bigg\} \\
&+ \quad |\lambda|\varepsilon\sup_{t\in[0,1]}\Bigg\{\int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}d_qu+|a||A(t)|\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big[\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}d_qu\Big]d_qs \\
&+ \quad |b||B(t)|\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big[\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}d_qu\Big]d_qs+|B(t)|\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}d_qu \\
&+ \quad |C(t)|\int_0^t \frac{(t-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}d_qu\Bigg\} \\
&\le \quad (L\varepsilon+M)\Omega_1+|\lambda|\varepsilon\omega_3\le\varepsilon.
\end{aligned}
$$

This shows that $\mathcal{F}B_\varepsilon \subset B_\varepsilon$. Now, for $x, y \in \mathcal{C}$, we obtain

$$
\begin{aligned}
& \quad \|\mathcal{F}x-\mathcal{F}y\| \\
\le & \quad \sup_{t\in[0,1]}\Bigg\{\int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}|f(m,x(m))-f(m,y(m))|d_qm \\
+ & \quad |k|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}|g(m,x(m))-g(m,y(m))|d_qm+|\lambda||x(u)-y(u)|\Big)d_qu
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad |a||A(t)|\int_0^{\eta}\frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s\frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u\frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}|f(m,x(m))-f(m,y(m))|d_qm \\
&+ \quad |k|\int_0^u\frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}|g(m,x(m))-g(m,y(m))|d_qm+|\lambda||x(u)-y(u)|\Big)d_qu\Big)d_qs \\
&+ \quad |b||B(t)|\int_0^{\sigma}\frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s\frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u\frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}|f(m,x(m))-f(m,y(m))|d_qm \\
&+ \quad |k|\int_0^u\frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}|g(m,x(m))-g(m,y(m))|d_qm+|\lambda||x(u)-y(u)|\Big)d_qu\Big)d_qs \\
&+ \quad |B(t)|\int_0^1\frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|p|\int_0^u\frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}|f(m,x(m))-f(m,y(m))|d_qm \\
&+ \quad |k|\int_0^u\frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}|g(m,x(m))-g(m,y(m))|d_qm+|\lambda||x(u)-y(u)|\Big)d_qu \\
&+ \quad |C(t)|\int_0^1\frac{(1-qu)^{(\gamma-2)}}{\Gamma_q(\gamma-1)}\Big(|p|\int_0^u\frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}|f(m,x(m))-f(m,y(m))|d_qm \\
&+ \quad |k|\int_0^u\frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}|g(m,x(m))-g(m,y(m))|d_qm+|\lambda||x(u)-y(u)|\Big)d_qu\Big\} \\
&\leq \quad \bar{\Omega}\|x-y\|,
\end{aligned}
$$

which shows that $\mathcal{F}$ is a contraction as $\bar{\Omega} < 1$ by the given assumption. Therefore, it follows by Banach's contraction principle that the problem (1)-(2) has a unique solution. □

Example. Consider a boundary value problem of integro-differential equations of fractional order given by

$$
\left\{\begin{array}{c} {}^cD_q^{1/2}({}^cD_q^{1/2}+\frac{1}{5})x(t)=\frac{1}{6}f(t,x(t))+\frac{1}{9}I_q^{1/2}g(t,x(t)),\quad 0\le t\le 1,\ 0<q<1, \\ x(0)=I_q^{\alpha-1}x(1/3),\quad x(1)=1/2I_q^{\alpha-1}x(2/3),\quad D_qx(1)=0, \end{array}\right. \tag{20}
$$

Here $f(t,x)=\frac{1}{(4+t^2)^2}\Big(\sin t+\frac{|x|}{1+|x|}+|x|\Big)$, $g(t,x)=\frac{1}{2}\tan^{-1}x+t^3$. Clearly

$$|f(t,x)-f(t,y)|\le\frac{1}{8}|x-y|,\quad |g(t,x)-g(t,y)|\le\frac{1}{2}|x-y|.$$

With $\beta=\xi=1/2,\gamma=3/2,\ \lambda=1/5,p=1/6,k=1/9,q=1/2,L_1=1/8,\ L_2=1/2$, we find that

$$\bar{\Omega}\ =L(|p|\omega_1+|k|\omega_2)+|\lambda|\omega_3\simeq\ 0.49725\ <1.$$

Clearly $L=\max\{L_1,L_2\}=1/2$. Thus all the assumptions of Theorem 3.5 are satisfied. Hence, by the conclusion of Theorem 3.5, the problem (20) has a unique solution.

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Reconstruction of bivariate functions by sparse sine coefficients *

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Abstract. In application, one often expands the functions f on $[0,1]^2$ into Fourier sine series and uses few Fourier sine coefficients to reconstruct f. In this paper, we give a decomposition formula of Fourier sine coefficients. Based on it, we discuss hyperbolic cross approximations of Fourier sine series and Fourier sine expansion with simple polynomial factors. In the end of this paper, we consider the three-dimensional case.

1. Introduction

In application, one often expands the functions f on $[0,1]^2$ into Fourier sine series and uses few Fourier sine coefficients to reconstruct f. But the precise representation of Fourier sine coefficients does not available. In Section 2, we will give the following decomposition of Fourier sine coefficients.

Suppose that f is a bivariate function with $\frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y) \in C([0,1]^2)$. For its Fourier sine coefficients, we have

$$c_{n_1,n_2}(f)$$

$$= 4\int_{[0,1]^2} f(x,y)\sin(\pi n_1 x)\,\sin(\pi n_2 y)\,\mathrm{d}x\mathrm{d}y$$

$$= \frac{4}{\pi^2 n_1 n_2}\left(J_{n_1,n_2} - \frac{1}{\pi n_1}(c_{n_1}(g_1) - (-1)^{n_2+1}c_{n_2}(g_2)) - \frac{1}{\pi n_2}(c_{n_2}(g_3) + (-1)^{n_1+1}c_{n_2}(g_4)) + \frac{1}{\pi^2 n_1 n_2}c_{n_1,n_2}(h)\right),$$

where

$$J_{n_1,n_2} = f(0,0) + (-1)^{n_1+1}f(1,0) + (-1)^{n_2+1}f(0,1) + (-1)^{n_1+n_2}f(1,1)$$

is an algebraic sum of values of f at vertexes of the square $[0,1]^d$ and

$$g_1(t) = \frac{\partial^2 f}{\partial t^2}(t,0), \qquad g_2(t) = \frac{\partial^2 f}{\partial t^2}(t,1),$$

$$g_3(t) = \frac{\partial^2 f}{\partial t^2}(0,t), \qquad g_4(t) = \frac{\partial^2 f}{\partial t^2}(1,t)$$

are the second-order derivatives of f on boundary of $[0,1]^2$ and

$$c_n(g_i) = 2\int_0^1 g_i(t)\,\sin(\pi n t)\,\mathrm{d}t$$

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is Fourier sine coefficients of univariate functions g_i, and $h = \frac{\partial^4 f}{\partial x^2 \partial y^2}$ and

$$c_{n_1,n_2}(h) = 4 \int_{[0,1]^2} h(x,y)\, \sin(\pi n_1 x) \sin(\pi n_2 y)\, \mathrm{d}x\mathrm{d}y$$

is the Fourier sine coefficient of bivariate function $h(x,y)$.

It is well known that in order to reconstruct f by using fewer Fourier sine coefficients, we should replace full grid approximation by sparse grid approximation [1,3,4]. In Section 3, based on this decomposition, we prove that for the hyperbolic cross truncations

$$S_N^{(h)}(f;x,y) = \sum_{\substack{1\le n_1,n_2\le N-1 \\ 1\le n_1 n_2 \le N-1}} c_{n_1,n_2}(f)\, \sin(\pi n_1 x) \sin(\pi n_2 y)$$

of Fourier sine series of f, the approximation errors satisfy

$$\begin{aligned} &\| f - S_N^{(h)}(f) \|_2^2 \\ &= \tfrac{4}{\pi^4}(f^2(0,0) + f^2(0,1) + f^2(1,0) + f^2(1,1)) \tfrac{\log N}{N} + O\left(\tfrac{1}{N}\right). \end{aligned}$$

Since the number of coefficients in $S_N^{(h)}(f)$ is $N_c \sim N \log N$. When we use the hyperbolic cross truncations to reconstruct f, we need fewer Fourier sine coefficients than that by partial sums of Fourier sine series.

To obtain these results, we need to use a decomposition of bivariate functions in [8].

Suppose that f is a second-order continuously differentiable on $[0,1]^2$, denote by $f \in W^{(2,2)}([0,1]^2)$. Let

$$P(x,y) = f(0,0)(1-x)(1-y) + f(0,1)(1-x)y + f(1,0)x(1-y) + f(1,1)xy \tag{1.1}$$

which is a bivariate polynomial determined by the values of f at vertexes of $[0,1]^2$, and let

$$Q(x,y) = f_1(0,y)(1-x) + f_1(1,y)x + f_1(x,0)(1-y) + f_1(x,1)y \qquad (f_1 = f - P). \tag{1.2}$$

The bivariate function $Q(x,y)$ is a sum of products of separated variable types. Denote the residual

$$R = f - P - Q. \tag{1.3}$$

It is easy to check that

$$\begin{aligned} &R(x,y) = 0 \qquad ((x,y) \in \partial([0,1]^2)), \\ &\frac{\partial^2 R}{\partial x^2}(x,y) = \frac{\partial^2 R}{\partial x^2}(x,y) - \frac{\partial^2 f_1}{\partial x^2}(x,0)(1-y) - \frac{\partial^2 f_1}{\partial x^2}(x,1)y. \end{aligned}$$

So it follows that

$$\begin{aligned} &R(x,0) = R(x,1) = 0 \qquad (0 \le x \le 1), \\ &R(0,y) = R(1,y) = 0 \qquad (0 \le y \le 1), \\ &\frac{\partial^2 R}{\partial x^2}(x,1) = \frac{\partial^2 R}{\partial x^2}(x,0) = 0, \end{aligned} \tag{1.4}$$

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and we have a decomposition formula:

$$f(x,y) = P(x,y) + Q(x,y) + R(x,y), \tag{1.5}$$

where P, Q, and R are stated in (1.1)-(1.3).

In Section 4, by using the decomposition (1.5), we expand f into Fourier sine series with simple polynomial factors whose hyperbolic cross truncation can reconstruct f by using fewest Fourier sine coefficients. In order to extend the above results to stochastic processes in Section 5, we need some concepts in Calculus of stochastic processes [2,7].

If $\{\xi_n\}_1^\infty$ is a sequence of stochastic variables and ξ is a stochastic variable, if the expectation

$$E[|\xi_n - \xi|^2] \to 0 \qquad (n \to \infty),$$

we say ξ is the limit of the sequence $\{\xi_n\}_1^\infty$. Based on this concept, one defines concepts of continuity, derivatives, and integrals. If $f(\mathbf{t})$ is a stochastic variable for each $\mathbf{t} \in [0,1]^d$, we say $f(\mathbf{t})$ is a stochastic process on $[0,1]^d$. If $f(\mathbf{t})$ is a stochastic process on $[0,1]^d$ and $E\left[\int_{[0,1]^d} f^2(\mathbf{t})\,\mathrm{d}\mathbf{t}\right] < \infty$, then f can be expanded a Fourier sine series:

$$f(\mathbf{t}) = \sum_{\mathbf{n} \in \mathbb{Z}_+^d} c_{\mathbf{n}}(f) \left(\prod_{k=1}^d \sin(\pi n_k t_k) \right),$$

where the coefficients:

$$c_{\mathbf{n}}(f) = 2^d \int_{[0,1]^d} f(t_1, ..., t_d) \left(\prod_{k=1}^d \sin(\pi n_k t_k) \right) \mathrm{d}t_1 \cdots \mathrm{d}t_d.$$

For convenience, the notation $f \in W^{(l_1,\dots,l_d)}([0,1]^d)$ means $\frac{\partial^{l_1+\cdots+l_d}}{\partial t_1^{l_1} \partial t_2^{l_2} \cdots \partial t_d^{l_d}} f \in C([0,1]^d)$, and the notation $\alpha_{n_1,\dots,n_d} = o(1)$ means that $\alpha_{n_1,\dots,n_d} \to 0$ as $n_1^2 + \cdots + n_d^2 \to \infty$.

At the end of this paper (i.e. Section 6), we consider the three-dimensional case.

2. Fourier sine coefficient decomposition

From this decomposition formula (1.5), it follows that the Fourier sine coefficients of f satisfy

$$c_{n_1,n_2}(f) = c_{n_1,n_2}(P) + c_{n_1,n_2}(Q) + c_{n_1,n_2}(R).$$

Suppose that $f \in W^{(2,2)}([0,1]^2)$. Then

(i)

$$c_{n_1,n_2}(P) = \frac{4}{\pi^2 n_1 n_2} J_{\mathbf{n}},$$

where

$$J_{n_1,n_2} = f(0,0) + (-1)^{n_1+1} f(1,0) + (-1)^{n_2+1} f(0,1) + (-1)^{n_1+n_2} f(1,1). \tag{2.1}$$

(ii)

$$c_{n_1,n_2}(Q) = \frac{4}{\pi n_1} \int_0^1 F_1(y) \sin(\pi n_2 y)\,\mathrm{d}y + \frac{4}{\pi n_2} \int_0^1 F_2(x) \sin(\pi n_1 x)\,\mathrm{d}x,$$

where

$$F_1(y) = f_1(0,y) + f_1(1,y)(-1)^{n_1+1},$$

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$$F_2(x) = f_1(x,0) + f_1(x,1)(-1)^{n_2+1}.$$

By $f_1 = f - P$, we have

$$F_1(0) = F_1(1) = F_2(0) = F_2(1) = 0.$$

Since $\frac{\partial^2 P}{\partial x^2} = \frac{\partial^2 P}{\partial y^2} = 0$, we have

$$F_1''(y) = \frac{\partial^2 f}{\partial y^2}(0,y) + \frac{\partial^2 f}{\partial y^2}(1,y)(-1)^{n_1+1},$$

$$F_2''(x) = \frac{\partial^2 f}{\partial x^2}(x,0) + \frac{\partial^2 f}{\partial x^2}(x,1)(-1)^{n_2+1}.$$

Let

$$g_1(t) = \frac{\partial^2 f}{\partial t^2}(t,0), \qquad g_2(t) = \frac{\partial^2 f}{\partial t^2}(t,1),$$

$$g_3(t) = \frac{\partial^2 f}{\partial t^2}(0,t), \qquad g_4(t) = \frac{\partial^2 f}{\partial t^2}(1,t). \tag{2.2}$$

Then

$$F_1''(y) = g_3(y) + (-1)^{n_1+1} g_4(y),$$

$$F_2''(x) = g_1(x) + (-1)^{n_2+1} g_2(x).$$

From this, we deduce that

$$\begin{aligned} 2\int_0^1 F_1(y)\, \sin(\pi n_1 y)\,\mathrm{d}y &= -\tfrac{2}{(\pi n_1)^2}\int_0^1 F_1''(y)\,\sin(\pi n_1 y)\,\mathrm{d}y \\ &= -\tfrac{c_{n_1}(g_3)+(-1)^{n_1+1}c_{n_1}(g_4)}{(\pi n_1)^2}, \\ 2\int_0^1 F_2(x)\,\sin(\pi n_2 x)\,\mathrm{d}x &= -\tfrac{2}{(\pi n_2)^2}\int_0^1 F_2''(x)\,\sin(\pi n_2 x)\,\mathrm{d}x \\ &= -\tfrac{c_{n_2}(g_1)+(-1)^{n_2+1}c_{n_2}(g_2)}{(\pi n_2)^2}, \end{aligned}$$

where $c_n(g_i) = 2\int_0^1 g_i(x)\,\sin(\pi n x)\,\mathrm{d}x$, and so

$$\begin{aligned} c_{n_1,n_2}(Q) &= -\tfrac{1}{\pi^3 n_1 n_2}\left(\tfrac{c_{n_1}(g_3)+(-1)^{n_1+1}c_{n_1}(g_4)}{n_1} + \tfrac{c_{n_2}(g_1)+(-1)^{n_2+1}c_{n_2}(g_2)}{n_2}\right) \\ &= \tfrac{1}{n_1 n_2}\left(o\left(\tfrac{1}{n_1}\right) + o\left(\tfrac{1}{n_2}\right)\right). \end{aligned}$$

(iii)

$$\frac{1}{4}c_{n_1,n_2}(R) = \int_0^1\int_0^1 R(x,y)\,\sin(\pi n_1 x)\,\sin(\pi n_2 y)\,\mathrm{d}x\mathrm{d}y.$$

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Since $R \in W^{(2.,2)}([0,1]^2)$, using integration by parts, it follows by (1.4) that

$$\int_0^1 R(x,y)\,\sin(\pi n_1 x)\,\mathrm{d}x$$

$$= -\tfrac{R(x,y)}{\pi n_1}\,\cos(\pi n_1 x)\Big|_0^1 + \tfrac{1}{\pi n_1}\int_0^1 \tfrac{\partial R}{\partial x}(x,y)\,\cos(\pi n_1 x)\,\mathrm{d}x$$

$$= \tfrac{1}{\pi n_1}\left(\tfrac{1}{\pi n_1}\,\tfrac{\partial R}{\partial x}(x,y)\,\sin(\pi n_1 x)\big|_0^1 - \tfrac{1}{\pi n_1}\int_0^1 \tfrac{\partial^2 R}{\partial x^2}(x,y)\,\sin(\pi n_1 x)\,\mathrm{d}x\right)$$

$$= -\tfrac{1}{(\pi n_1)^2}\int_0^1 \tfrac{\partial^2 R}{\partial x^2}(x,y)\,\sin(\pi n_1 x)\,\mathrm{d}x.$$

So

$$\frac{1}{4}c_{n_1,n_2}(R) = \frac{1}{(\pi n_1)^2}\int_0^1 \sin(\pi n_1 x)\left(\int_0^1 \frac{\partial^2 R}{\partial x^2}(x,y)\,\sin(\pi n_2 y)\,\mathrm{dy}\right)\mathrm{d}x.$$

By (1.4), we get

$$\int_0^1 \frac{\partial^2 R}{\partial x^2}(x,y)\,\sin(\pi n_2 y)\,\mathrm{dy} = -\frac{1}{(\pi n_2)^2}\int_0^1 \frac{\partial^4 R}{\partial x^2 \partial y^2}(x,y)\,\sin(\pi n_2 y)\,\mathrm{dy}.$$

From this, we get

$$\begin{aligned} c_{n_1,n_2}(R) &= \tfrac{4}{\pi^4 n_1^2 n_2^2}\int_0^1\int_0^1 \tfrac{\partial^4 f}{\partial x^2 \partial y^2}(x,y)\,\sin(\pi n_1 x)\,\sin(\pi n_2 y)\,\mathrm{d}x\mathrm{dy} \\ &= \tfrac{c_{n_1,n_2}(\frac{\partial^4 f}{\partial x^2 \partial y^2})}{\pi^4 n_1^2 n_2^2} = o\left(\tfrac{1}{n_1^2 n_2^2}\right). \end{aligned} \tag{2.3}$$

Summarizing up all results, we get the following theorem.

Theorem 2.1. Let $f \in W^{(2,2)}([0,1]^2)$. Then its Fourier sine coefficients have the decomposition formula:

$$\begin{aligned} & c_{n_1,n_2}(f) \\ &= \tfrac{4}{\pi^2 n_1 n_2}\left(J_{n_1,n_2} - \tfrac{c_{n_1}(g_1)+(-1)^{n_2+1}c_{n_1}(g_2)}{\pi n_1} - \tfrac{c_{n_2}(g_3)+(-1)^{n_1+1}(g_4)}{\pi n_2} + \tfrac{c_{n_1,n_2}(\frac{\partial^4 f}{\partial x^2 \partial y^2})}{\pi^2 n_1 n_2}\right), \end{aligned} \tag{2.4}$$

where J_{n_1,n_2} is stated in (2.1) and $g_i\,(i=1,2,3,4)$ are stated in (2.2), and

$$c_n(g_i) = 2\int_0^1 g_i(t)\,\sin(\pi n t)\,\mathrm{d}t, \qquad n \in \mathbb{Z}_+ \quad (i=1,2,3,4).$$

By the Riemann-Lebesgue lemma [5],

$$c_{n_1,n_2}(f) = \frac{4}{\pi^2 n_1 n_2}\left(J_{n_1,n_2} + o\left(\frac{1}{n_1}\right) + o\left(\frac{1}{n_2}\right)\right). \tag{2.5}$$

In detail, we have the following asymptotic formulas:

$$c_{2n_1,2n_2}(f) = \tfrac{1}{\pi^2 n_1 n_2}\left(f(0,0) - f(0,1) - f(1,0) + f(1,1) + o\left(\tfrac{1}{n_1}\right) + o\left(\tfrac{1}{n_2}\right)\right),$$

$$c_{2n_1+1,2n_2+1}(f) = \tfrac{1}{\pi^2 n_1 n_2}\left(f(0,0) + f(0,1) + f(1,0) + f(1,1) + o\left(\tfrac{1}{n_1}\right) + o\left(\tfrac{1}{n_2}\right)\right),$$

$$c_{2n_1+1,2n_2}(f) = \tfrac{1}{\pi^2 n_1 n_2}\left(f(0,0) - f(0,1) + f(1,0) - f(1,1) + o\left(\tfrac{1}{n_1}\right) + o\left(\tfrac{1}{n_2}\right)\right),$$

$$c_{2n_1,2n_2+1}(f) = \tfrac{1}{\pi^2 n_1 n_2}\left(f(0,0) + f(0,1) - f(1,0) - f(1,1) + o\left(\tfrac{1}{n_1}\right) + o\left(\tfrac{1}{n_2}\right)\right).$$

Consider the sum of their squares:

$$\sum_{i,j=0}^{1} c^2_{2n_1+i,2n_2+j}(f) = \frac{4}{\pi^4 n_1^2 n_2^2}\left(f^2(0,0) + f^2(0,1) + f^2(1,0) + f^2(1,1) + o\left(\frac{1}{n_1}\right) + o\left(\frac{1}{n_2}\right)\right). \tag{2.6}$$

This implies that the equality:

$$\sum_{i,j=0}^{1} c^2_{2n_1+i,2n_2+j}(f) = o\left(\frac{1}{n_1}\right) + o\left(\frac{1}{n_2}\right)$$

holds if and only if

$$f(0,0) = f(0,1) = f(1,0) = f(1,1) = 0. \tag{2.7}$$

This is equivalent to that the equality:

$$c_{n_1,n_2}(f) = o\left(\frac{1}{n_1}\right) + o\left(\frac{1}{n_2}\right)$$

holds if and only if (2.7) holds. However, similar to an argument of (2.3), we can derive that $c_{n_1,n_2} = o\left(\frac{1}{n_1 n_2}\right)$ if and only if $f(x,y) = 0\,((x,y) \in \partial([0,1]^2))$.

If $f \in W^{(1,2)}([0,1]^2)$, $f \in W^{(2,1)}([0,1]^2)$, and $f \in W^{(1,1)}([0,1]^2)$, then we have the corresponding results.

Theorem 2.2. Let $f \in W^{(l_1,l_2)}([0,1]^2)$. Then Fourier sine coefficients of f satisfy asymptotic formulas:

(i) $c_{n_1,n_2}(f) = \frac{4}{\pi^2 n_1 n_2}\left(J_{n_1,n_2} + o\left(\frac{1}{n_1}\right) + o\left(\frac{1}{n_2}\right)\right)$, where $l_1 = l_2 = 2$;

(ii) $c_{n_1,n_2}(f) = \frac{4}{\pi^2 n_1 n_2}(J_{n_1,n_2} + \eta_1 + \eta_2)$, where $l_1 = l_2 = 1$;

(iii) $c_{n_1,n_2}(f) = \frac{4}{\pi^2 n_1 n_2}\left(J_{n_1,n_2} + \eta_1 + o\left(\frac{1}{n_2}\right)\right)$, where $l_1 = 1$, $l_2 = 2$;

(iv) $c_{n_1,n_2}(f) = \frac{4}{\pi^2 n_1 n_2}\left(J_{n_1,n_2} + o\left(\frac{1}{n_1}\right) + \eta_2\right)$, where $l_1 = l_2 = 1$.

Here J_{n_1,n_2} is stated in (2.1) and $\eta_i \to 0$ as $n_i \to \infty$.

3. Approximation of hyperbolic cross truncations

Suppose that $f \in W^{(2,2)}([0,1]^2)$. We expand it into a Fourier sine series:

$$f(x,y) = \sum_{n\in\mathbb{Z}^2} c_{n_1,n_2}(f)\,\sin(\pi n_1 x)\sin(\pi n_2 y) \qquad (L^2).$$

Consider its hyperbolic cross truncations:

$$S_N^{(h)}(f;\,x,y)=\sum_{\substack{1\le n_1,n_2\le N-1\\ 1\le n_1n_2\le N-1}} c_{n_1,n_2}(f)\ \sin(\pi n_1x)\sin(\pi n_2y).$$

Then

$$f(x,y)-S_N^{(h)}(f;\,x,y)=\left(\sum_{n_1=1}^{N-1}\ \sum_{n_2=\left[\frac{N-1}{n_1}\right]+1}^{\infty}+\sum_{n_1=N}^{\infty}\sum_{n_2=1}^{\infty}\right)c_{n_1,n_2}(f)\ \sin(\pi n_1x)\sin(\pi n_2y).$$

By the Parseval identity,

$$4\parallel f-S_N^{(h)}(f)\parallel_2^2=\sum_{n_1=1}^{N}\ \sum_{n_2=\left[\frac{N}{n_1}\right]}^{\infty}|c_{n_1,n_2}(f)|^2+\sum_{n_1=N}^{\infty}\sum_{n_2=1}^{\infty}|c_{n_1,n_2}(f)|^2=I_N^{(1)}+I_N^{(2)}. \tag{3.1}$$

By (2.5),

$$|c_{n_1,n_2}(f)|^2=\frac{16}{\pi^4n_1^2n_2^2}\left(J_{n_1,n_2}^2+o\left(\frac{1}{n_1}\right)+o\left(\frac{1}{n_2}\right)\right).$$

So

$$I_N^{(2)}=O\left(\frac{1}{N}\right)$$

and

$$\begin{aligned} I_N^{(1)} &= \tfrac{16}{\pi^4}\sum_{n_1=1}^{N}\ \sum_{n_2=\left[\frac{N-1}{n_1}\right]+1}^{\infty}\frac{J_{n_1,n_2}^2}{n_1^2n_2^2}\\ &+ o(1)\sum_{n_1=1}^{N}\ \sum_{n_2=\left[\frac{N-1}{n_2}+1\right]}^{\infty}\frac{1}{n_1^3n_2^2}+o(1)\sum_{n_1=1}^{N}\ \sum_{n_2=\left[\frac{N-1}{n_1}\right]+1}^{\infty}\frac{1}{n_1^2n_2^3} \qquad (3.2)\\ &= \tfrac{16}{\pi^4}\sum_{n_1=1}^{N-1}\ \sum_{n_2=\left[\frac{N-1}{n_1}\right]+1}^{\infty}\frac{J_{n_1,n_2}^2}{n_1^2n_2^2}+o\left(\tfrac{1}{N}\right). \end{aligned}$$

By (3.1), we get

$$4\parallel f-S_N^{(h)}(f)\parallel_2^2=K_N+o\left(\frac{1}{N}\right), \tag{3.3}$$

where

$$K_N=\frac{16}{\pi^4}\sum_{n_1=1}^{N-1}\ \sum_{n_2=\left[\frac{N}{n_1}\right]}^{\infty}\frac{J_{n_1,n_2}^2}{n_1^2n_2^2}.$$

A direct computation shows that

$$\begin{aligned} K_N &= \tfrac{1}{\pi^4}\sum_{n_1=1}^{\left[\frac{N-1}{2}\right]}\ \sum_{n_2=\left[\frac{N}{4n_1}\right]}^{\infty}\tfrac{1}{n_1^2n_2^2}\left(J_{2n_1,2n_2}^2+J_{2n_1-1,2n_2}^2+J_{2n_1,2n_2-1}^2+J_{2n_1-1,2n_2-1}^2\right)+O\left(\tfrac{1}{N}\right)\\ &= \tfrac{4M}{\pi^4}\sum_{n_1=1}^{\left[\frac{N-1}{2}\right]}\ \sum_{n_2=\left[\frac{N}{4n_1}\right]}^{\infty}\tfrac{1}{n_1^2n_2^2}+O\left(\tfrac{1}{N}\right), \end{aligned}$$

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where $M = f^2(0,0) + f^2(0,1) + f^2(1,0) + f^2(1,1)$. Notice that

$$\sum_{n_2=\left[\frac{N}{4n_1}\right]} \frac{1}{n_1^2 n_2^2} = \frac{1}{n_1^2}\left(\int_{\frac{N}{4n_1}}^{\infty} \frac{\mathrm{d}t}{t^2} + O\left(\frac{n_1^2}{N^2}\right)\right).$$

Then

$$K_N = \frac{16M}{\pi^4 N} \sum_{n_1=1}^{\left[\frac{N-1}{2}\right]} \frac{1}{n_1} + O\left(\frac{1}{N}\right) = \frac{16M \log N}{\pi^4 N} + O\left(\frac{1}{N}\right).$$

From this and (3.1)-(3.3), it follows that

$$\| f - S_N^{(h)}(f) \|_2^2 = \frac{4M}{\pi^4}\frac{\log N}{N} + O\left(\frac{1}{N}\right). \tag{3.4}$$

The number of Fourier sine coefficients in the Nth hyperbolic cross truncation $S_N^{(h)}(f)$ is

$$N_c = \sum_{n_1=1}^{N-1} \sum_{n_2=1}^{\left[\frac{N-1}{n_1}\right]} 1 = \sum_{n_1=1}^{N-1} \left[\frac{N-1}{n_1}\right] = \int_1^N \frac{N}{t}\,\mathrm{d}t + O(N) = N\log N + O(N).$$

Then (3.4) can be written into

$$\| f - S_N^{(h)}(f) \|_2^2 = \frac{4M}{\pi^4}\frac{\log^2 N_c}{N_c} + O\left(\frac{\log N_c}{N_c}\right).$$

Theorem 3.1. Let $f \in W^{(l_1,l_2)}([0,1]^2)$. Then the hyperbolic cross truncations of Fourier sine series of f satisfy the asymptotic formulas:

(i) $\| f - S_N^{(h)}(f) \|_2^2 = \frac{4M}{\pi^4}\frac{\log N}{N} + O\left(\frac{1}{N}\right)$ $(l_1 = l_2 = 2)$;

(ii) $\| f - S_N^{(h)}(f) \|_2^2 = \frac{4M}{\pi^4}\frac{\log N}{N} + o\left(\frac{\log N}{N}\right)$ $(l_1 = l_2$ or $l_1 = 2,\ l_2 = 1$ or $l_1 = 1,\ l_2 = 2,$

where the constant $M = f^2(0,0) + f^2(0,1) + f^2(1,0) + f^2(1,1)$.

4. Fourier sine expansion with polynomial factors

Suppose that $f \in W^{(2,2)}([0,1]^2)$. Then, by decomposition formula:

$$f(x,y) = P(x,y) + f_1(0,y)(1-x) + f_1(1,y)x + f_1(x,0)(1-y) + f_1(x,1)y + R(x,y),$$

denote

$$\alpha_1(y) = f_1(0,y), \qquad \alpha_2(y) = f_1(1,y),$$

$$\alpha_3(x) = f_1(x,0), \qquad \alpha_4(x) = f_1(x,1),$$

then $\alpha_i(0) = \alpha_i(1) = 0$ and $\alpha_i \in W([0,1])\ (i = 1,2,3,4)$.

Expanding each α_i into a univariate Fourier sine series and $R(x,y)$ into a bivariate Fourier sine series, we

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get a Fourier sine expansion of f with polynomial factors:

$$\begin{aligned} f(x,y) &= P(x,y) \\ &+ (1-x)\sum_1^\infty c_m(\alpha_1)\sin(\pi m y) + x\sum_1^\infty c_m(\alpha_2)\sin(\pi m y) \\ &+ (1-y)\sum_1^\infty c_m(\alpha_3)\sin(\pi m x) + y\sum_1^\infty c_m(\alpha_4)\sin(\pi m x) \\ &+ \sum_{n_1,n_2=1}^\infty c_{n_1,n_2}(R)\sin(\pi n_1 x)\sin(\pi n_2 y), \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} c_m(\alpha_i) &= 2\int_0^1 \alpha_i(t)\,\sin(\pi m t)\,\mathrm{d}t \\ &= -\tfrac{2}{(\pi m)^2}\int_0^1 \alpha_i''(t)\,\sin(\pi m t)\,\mathrm{d}t \\ &= -\tfrac{1}{(\pi m)^2} c_m(\alpha_i'') \quad (i=1,2,3,4), \end{aligned} \tag{4.2}$$

$$c_{n_1,n_2}(R) = \frac{c_{n_1,n_2}(\frac{\partial^4 f}{\partial x^2 \partial y^2})}{\pi^4 n_1^2 n_2^2}.$$

By the definition of f_1, we have $\alpha_i''(t) = h_i''(t)\,(i=1,2,3,4)$, where

$$h_1(t) = f(0,t), \qquad h_2(t) = f(1,t),$$

$$h_3(t) = f(t,0), \qquad h_4(t) = f(t,1).$$

Consider the hyperbolic cross truncations of the series (4.1):

$$\begin{aligned} \widetilde{S}_N^{(h)}(x,y) &= P(x,y) \\ &+ (1-x)\sum_1^{N-1} c_n(\alpha_1)\sin(\pi m y) + x\sum_1^{N-1} c_n(\alpha_2)\sin(\pi m y) \\ &+ (1-y)\sum_1^{N-1} c_n(\alpha_3)\sin(\pi m x) + y\sum_1^{N-1} c_n(\alpha_4)\sin(\pi m x) \\ &+ \sum_{\substack{1\le n_1,n_2\le N-1 \\ 1\le n_1 n_2\le N-1}} c_{n_1,n_2}(R)\sin(\pi n_1 x)\sin(\pi n_2 y). \end{aligned}$$

By using Parseval identity, it follows from (4.1) and (4.3) that

$$\| f - \widetilde{S}_N^{(h)}(f) \|_2^2 = O(1)\left(\sum_{i=1}^4\sum_{m=N}^\infty |c_n(\alpha_i)|^2 + \left(\sum_{n_1=1}^{N-1}\sum_{n_2=\left[\frac{N}{n_1}\right]}^\infty + \sum_{n_1=N}^\infty\sum_{n_2=1}^\infty\right)|c_{n_1,n_2}(R)|^2\right).$$

Finally, by the estimates (4.2) and (2.3), we get

$$\| f - \widetilde{S}_N^{(h)}(f) \|_2^2 = O\left(\frac{\log N}{N^3}\right).$$

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The number of Fourier sine coefficients in the series (4.3) satisfies $N_c \sim N \log N$. Therefore,

$$\| f - \widetilde{S}_N^{(h)}(f) \|_2^2 = O\left(\frac{\log^4 N_c}{N_c^3}\right).$$

Theorem 4.1. Let $f \in W^{(l_1,l_2)}([0,1]^2)\,(l_1 = 1 \,\mathrm{or}\, 2,\ l_2 = 1 \,\mathrm{or}\, 2)$. Then the hyperbolic cross truncations of the series (4.1) satisfy

$$\| f - \widetilde{S}_N^{(h)}(f) \|_2^2 = O\left(\frac{\log N}{N^3}\right).$$

5. Uncertainty analysis

Suppose that f is a stochastic process and $f \in W^{(2,2)}([0,1]^2)$. Then the coefficient decomposition formula still holds:

$$\begin{aligned} & c_{n_1,n_2}(f) \\ & = \tfrac{4}{\pi^2 n_1 n_2} J_{n_1,n_2} - \tfrac{2}{\pi^3 n_1 n_2}\left(\tfrac{c_{n_1}(g_3)+(-1)^{n_1+1}c_{n_1}(g_4)}{n_1} + \tfrac{c_{n_2}(g_1)+(-1)^{n_2+1}c_{n_2}(g_2)}{n_2}\right) + r_{\mathbf{n}}, \end{aligned} \tag{5.1}$$

where the error r_{n_1,n_2} is equal to

$$r_{n_1,n_2} = \frac{c_{n_1,n_2}(\frac{\partial^4 f}{\partial x^2 \partial y^2})}{\pi^4 n_1^2 n_2^2} = \frac{4}{\pi^4 n_1^2 n_2^2}\int_{[0,1]^2} \frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y)\, \sin(\pi n_1 x)\sin(\pi n_2 y)\,\mathrm{d}x\mathrm{d}y. \tag{5.2}$$

Consider the expectation of r_{n_1,n_2}. The expectation and integral can be exchanged, so

$$E[r_{n_1,n_2}] = \frac{4}{\pi^4 n_1^2 n_2^2}\int_{[0,1]^2} E\left[\frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y)\right]\sin(\pi n_1 x)\sin(\pi n_2 y)\,\mathrm{d}x\mathrm{d}y. \tag{5.3}$$

The expectation and limit can be exchanged, so it follows from $\frac{\partial^4 f}{\partial x^2 \partial y^2} \in C([0,1]^2)$ that $E\left[\frac{\partial^4 f}{\partial x^2 \partial y^2}\right] \in C([0,1]^2)$. By the Riemann-Lebesgue lemma,

$$E[r_{n_1,n_2}] = \frac{c_{n_1,n_2}(E\left[\frac{\partial^4 f}{\partial x^2 \partial y^2}\right])}{\pi^4 n_1^2 n_2^2} = o\left(\frac{1}{n_1^2 n_2^2}\right). \tag{5.4}$$

Consider the variance of $r_{\mathbf{n}}$. By (5.2), we have

$$\begin{aligned} & r_{n_1,n_2}^2 \\ & = \tfrac{16}{\pi^8 n_1^4 n_2^4}\int_{[0,1]^4} \tfrac{\partial^4 f}{\partial x^2 \partial y^2}(x,y)\,\tfrac{\partial^4 f}{\partial t^2 \partial s^2}(t,s)\, \sin(\pi n_1 x)\sin(\pi n_2 y)\sin(\pi n_1 t)\sin(\pi n_2 s)\,\mathrm{d}x\mathrm{d}y\mathrm{d}t\mathrm{d}s. \end{aligned}$$

From this and (5.4),

$$\begin{aligned} & E[r_{n_1,n_2}^2] \\ & = \tfrac{16}{\pi^8 n_1^4 n_2^4}\int_{[0,1]^4} E[\tfrac{\partial^4 f}{\partial x^2 \partial y^2}(x,y)\,\tfrac{\partial^4 f}{\partial t^2 \partial s^2}(t,s)]\, \sin(\pi n_1 x)\sin(\pi n_1 t)\sin(\pi n_2 x)\sin(\pi n_2 s)\,\mathrm{d}x\mathrm{d}y\mathrm{d}t\mathrm{d}s \\ & = \tfrac{c_{n_1,n_2,n_1,n_2}(E[\frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y)\frac{\partial^4 f}{\partial t^2 \partial s^2}(t,s)])}{\pi^8 n_1^4 n_2^4} = o\left(\tfrac{1}{n_1^4 n_2^4}\right). \end{aligned} \tag{5.5}$$

By (5.3),

$$(E[r_{n_1,n_2}])^2$$

$$= \tfrac{16}{\pi^8 n_1^4 n_2^4} \int_{[0,1]^4} E\left[\tfrac{\partial^4 f}{\partial x^2 \partial y^2}(x,y)\right] E\left[\tfrac{\partial^4 f}{\partial t^2 \partial s^2}(t,s)\right] \sin(\pi n_1 x)\sin(\pi n_2 y)\sin(\pi n_1 t)\sin(\pi n_2 s)\,\mathrm{d}x\mathrm{d}y\mathrm{d}t\mathrm{d}s.$$

Notice that

$$\mathrm{Var}(r_{n_1,n_2}) = E[r_{n_1,n_2}^2] - (E[r_{n_1,n_2}])^2,$$

$$\mathrm{Cov}\left(\frac{\partial^4 f}{\partial x^2 \partial y^2},\ \frac{\partial^4 f}{\partial t^2 \partial s^2}\right) = E\left[\frac{\partial^4 f}{\partial x^2 \partial y^2}\frac{\partial^4 f}{\partial t^2 \partial s^2}\right] - E\left[\frac{\partial^4 f}{\partial x^2 \partial y^2}\right] E\left[\frac{\partial^4 f}{\partial t^2 \partial s^2}\right].$$

Then, by (5.4) and (5.5), the variance of $r_{\mathbf{n}}$:

$$\mathrm{Var}(r_{n_1,n_2})$$

$$= \tfrac{16}{\pi^8 n_1^4 n_2^4} \int\limits_{[0,1]^4} \mathrm{Cov}\left(\tfrac{\partial^4 f}{\partial x^2 \partial y^2}(x,y),\ \tfrac{\partial^4 f}{\partial t^2 \partial s^2}(t,s)\right) \sin(\pi n_1 x)\sin(\pi n_2 y)\sin(\pi n_1 t)\sin(\pi n_2 s)\,\mathrm{d}x\mathrm{d}y\mathrm{d}t\mathrm{d}s$$

$$= \frac{c_{n_1,n_2,n_1,n_2}\left(\mathrm{Cov}(\frac{\partial^4 f}{\partial x^2 \partial y^2},\ \frac{\partial^4 f}{\partial t^2 \partial s^2})\right)}{\pi^8 n_1^4 n_2^4} = o\left(\tfrac{1}{n_1^4 n_2^4}\right).$$

Similarly, for $i = 1,2,3$, as $n_i \to \infty$, we have

$$E[c_{n_i}(g_i)] = c_{n_i}(E[g_i]) = o(1);$$

$$E[c_{n_i}^2(g_i)] = c_{n_i,n_i}(E[g_i(x)g_i(y)]) = o(1);$$

$$\mathrm{Var}(c_{n_i}(g_i)) = c_{n_i,n_i}(\mathrm{Cov}(g_i(x),\ g_i(y))) = o(1).$$

By (5.1), we get

$$E[c_{n_1,n_2}(f)] = \frac{4}{\pi^2 n_1 n_2}\left(E[J_{n_1,n_2}] + o\left(\frac{1}{n_1}\right) + o\left(\frac{1}{n_2}\right)\right).$$

For convenience, denote

$$\tau_{n_1,n_2} = -\frac{2}{\pi^3 n_1 n_2}\left(\frac{c_{n_2}(g_1) + (-1)^{n_2+1}(g_2)}{n_1} + \frac{c_{n_1}(g_3) + (-1)^{n_1+1}(g_4)}{n_2}\right)$$

$$\mu_{n_1,n_2} = \frac{4}{\pi^2 n_1 n_2} J_{n_1,n_2}.$$

So $c_{n_1,n_2}(f) = \mu_{n_1,n_2} + \tau_{n_1,n_2} + r_{n_1,n_2}$, and so

$$E[c_{n_1,n_2}^2(f)] = E[\mu_{n_1,n_2}^2] + A_{n_1,n_2},$$

where

$$A_{n_1,n_2}$$

$$= E[\tau_{n_1,n_2}^2] + E[r_{n_1,n_2}^2] + 2E[\mu_{n_1,n-2}\tau_{n_1,n_2}] + 2E[\mu_{n_1,n_2}r_{n_1,n_2}] + 2E[\tau_{n_1,n_2}r_{n_1,n_2}]$$

$$= \tfrac{1}{n_1^2 n_2^2}\left(o\left(\tfrac{1}{n_1}\right) + o\left(\tfrac{1}{n_2}\right)\right).$$

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Therefore,

$$E[c^2_{n_1,n_2}(f)] = \frac{16}{\pi^4 n_1^2 n_2^2}\left(E[J^2_{n_1,n_2}] + o\left(\frac{1}{n_1}\right) + o\left(\frac{1}{n_2}\right)\right). \tag{5.6}$$

Denote

$$\widetilde{M} = E[f^2(0,0)] + E[f^2(0,1)] + E[f^2(1,0)] + E[f^2(1,1)].$$

Similar to the argument from (3.1) to (3.4), we can deduce from (5.6) that

$$E\left[\| f - S_N^{(h)}(f) \|_2^2\right] = \frac{4\widetilde{M}}{\pi^4}\frac{\log N}{N} + O\left(\frac{1}{N}\right).$$

Theorem 5.1. Let f be a stochastic process and $f \in W^{(2,2)}([0,1]^2)$. Then

(i) Fourier sine coefficients of f satisfy

$$\begin{aligned} & c_{n_1,n_2}(f) \\ = & \tfrac{4}{\pi^2 n_1 n_2} J_{n_1,n_2} - \tfrac{2}{\pi^3 n_1 n_2}\left(\tfrac{c_{n_2}(g_1)+(-1)^{n_2+1}c_{n_2}(g_2)}{n_1} + \tfrac{c_{n_1}(g_3)+(-1)^{n_1+1}(g_4)}{n_2}\right) + r_{n_1,n_2}, \end{aligned}$$

where

$$E[r_{n_1,n_2}] = \frac{c_{n_1,n_2}\left(E[\frac{\partial^4 f}{\partial x^2 \partial y^2}]\right)}{\pi^4 n_1^2 n_2^2},$$

$$\mathrm{Var}(r_{n_1,n_2}) = \frac{c_{n_1,n_2,n_1,n_2}\left(\mathrm{Cov}\left(\frac{\partial^4 f}{\partial x^2\partial y^2},\ \frac{\partial^4 f}{\partial t^2 \partial s^2}\right)\right)}{\pi^8 n_1^4 n_2^4},$$

where $c_{n_1,n_2,n_1,n_2}(\mathrm{Cov}\left(\frac{\partial^4 f}{\partial x^2\partial y^2},\ \frac{\partial^4 f}{\partial t^2 \partial s^2}\right))$ is the four-variate Fourier sine coefficient of the covariance of $\frac{\partial^4 f}{\partial x^2\partial y^2}$ and $\frac{\partial^4 f}{\partial t^2 \partial s^2}$ at $\mathbf{n} = (n_1, n_2, n_1, n_2)$.

(ii) the hyperbolic cross truncations of Fourier sine series of f satisfy

$$E\left[\| f - S_N^{(h)}(f) \|_2^2\right] = \frac{4\widetilde{M}}{\pi^4}\frac{\log N}{N} + O\left(\frac{1}{N}\right),$$

where $\widetilde{M} = E[f^2(0,0)] + E[f^2(0,1)] + E[f^2(1,0)] + E[f^2(1,1)]$.

6. The three-dimensional case

For a three-dimensional function f on $[0,1]^3$, we can decompose f as follows:

$$f(x,y,z) = P(x,y,z) + Q(x,y,z) + R(x,y,z) + T(x,y,z), \tag{6.1}$$

where

$$\begin{aligned} P(x,y,z) &= f(0,0,0)(1-x)(1-y)(1-z) + f(0,1,0)(1-x)y(1-z) \\ &+ f(0,1,1)(1-x)yz + f(0,0,1)(1-x)(1-y)z \\ &+ f(1,0,0)x(1-y)(1-z) + f(1,1,0)xy(1-z) \\ &+ f(1,1,1)xyz + f(1,0,1)x(1-y)z \end{aligned} \tag{6.2}$$

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is a three-variate polynomial;

$$\begin{aligned} Q(x,y,z) &= f_1(x,0,0)(1-y)(1-z) + f_1(x,0,1)(1-y)z \\ &+ f_1(x,1,0)y(1-z) + f_1(x,1,1)yz \\ &+ f_1(1,y,0)(1-x)(1-z) + f_1(0,y,1)(1-x)z \\ &+ f_1(1,y,0)x(1-z) + f_1(1,y,1)xz \\ &+ f_1(0,0,z)(1-x)(1-y) + f_1(0,1,z)(1-x)y \\ &+ f_1(1,0,z)x(1-y) + f(1,1,z)xy \qquad (f_1 = f - P) \end{aligned} \tag{6.3}$$

is a sum of products of separated variable types, where one factor is the restriction of f_1 is each edge, the other factor is a bivariate polynomial;

$$\begin{aligned} R(x,y,z) &= f_2(x,y,0)(1-z) + f_2(x,0,z)(1-y) \\ &+ f_2(0,y,z)(1-x) + f_2(x,y,1)z \\ &+ f_2(x,1,z)y + f_2(1,y,z)x \qquad (f_2 = f - P - Q) \end{aligned} \tag{6.4}$$

is a sum of products of separated variable types, where one factor is the restriction of f_2, the other factor is a univariate polynomial and

$$T(x,y,z) = f(x,y,z) - P(x,y,z) - Q(x,y,z) - R(x,y,z).$$

It is easy to check the following proposition.

Proposition 6.1. $f_1(x,y,z) = 0$ for each vertex of $[0,1]^3$ and $f_2(x,y,z) = 0$ for each edge of $[0,1]^3$, and $T(x,y,z) = 0$ for each face of $[0,1]^3$.

Consider the Fourier sine coefficients $c_{n_1,n_2,n_3}(f)$. From the decomposition formula, it follows that

$$c_{n_1,n_2,n_3}(f) = c_{n_1,n_2,n_3}(P) + c_{n_1,n_2,n_3}(Q) + c_{n_1,n_2,n_3}(R) + c_{n_1,n_2,n_3}(T).$$

Since the Fourier sine coefficients:

$$c_{n_1,n_2,n_3}(f) = 8\int_{[0,1]^3} f(x,y,z)\ \sin(\pi n_1 x)\sin(\pi n_2 y)\sin(\pi n_3 z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

we obtain that

(i)

$$c_{n_1,n_2,n_3}(P) = \frac{8U_{n_1,n_2,n_3}}{\pi^3 n_1 n_2 n_3},$$

where

$$\begin{aligned} U_{n_1,n_2,n_3} &= f(0,0,0) + (-1)^{n_2+1}f(0,1,0) + (-1)^{n_2+n_3}f(0,1,1) \\ &+ (-1)^{n_3+1}f(0,0,1) + (-1)^{n_1+1}f(1,0,0) + (-1)^{n_1+n_2}f(1,1,0) \\ &+ (-1)^{n_1+n_2+n_3+1}f(1,1,1) + (-1)^{n_1+n_3}f(1,0,1); \end{aligned} \tag{6.5}$$

(ii)

$$c_{n_1,n_2,n_3}(Q) = -\frac{V^{(1)}_{n_1,n_2,n_3}}{\pi^4 n_1^2 n_2 n_3} - \frac{V^{(2)}_{n_1,n_2,n_3}}{\pi^4 n_1 n_2^2 n_3} - \frac{V^{(3)}_{n_1,n_2,n-3}}{\pi^4 n_1 n_2 n_3^2},$$

where

$$\begin{aligned} V^{(1)}_{n_1,n_2,n_3} &= c_{n_1}\left(\tfrac{\partial^2 f}{\partial x^2}(\cdot,0,0)\right) + (-1)^{n_3+1} c_{n_1}\left(\tfrac{\partial^2 f}{\partial x^2}(\cdot,0,1)\right) \\ &+ (-1)^{n_2+1} c_{n_1}\left(\tfrac{\partial^2 f}{\partial x^2}(\cdot,1,0)\right) + (-1)^{n_2+n_3} c_{n_1}\left(\tfrac{\partial^2 f}{\partial x^2}(\cdot,1,1)\right), \\ V^{(2)}_{n_1,n_2,n_3} &= c_{n_2}\left(\tfrac{\partial^2 f}{\partial y^2}(0,\cdot,0)\right) + (-1)^{n_3+1} c_{n_2}\left(\tfrac{\partial^2 f}{\partial y^2}(0,\cdot,1)\right) \\ &+ (-1)^{n_1+1} c_{n_2}\left(\tfrac{\partial^2 f}{\partial y^2}(1,\cdot,0)\right) + (-1)^{n_1+n_3} c_{n_2}\left(\tfrac{\partial^2 f}{\partial y^2}(1,\cdot,1)\right), \\ V^{(3)}_{n_1,n_2,n_3} &= c_{n_3}\left(\tfrac{\partial^2 f}{\partial z^2}(0,0,\cdot)\right) + (-1)^{n_2+1} c_{n_3}\left(\tfrac{\partial^2 f}{\partial z^2}(0,1,\cdot)\right) \\ &+ (-1)^{n_1+1} c_{n_3}\left(\tfrac{\partial^2 f}{\partial z^2}(1,0,\cdot)\right) + (-1)^{n_1+n_2} c_{n_3}\left(\tfrac{\partial^2 f}{\partial z^2}(1,1,\cdot)\right); \end{aligned}$$

(iii)

$$c_{n_1,n_2,n_3}(R) = \frac{M^{(1)}_{n_1,n_2,n_3}}{\pi^5 n_1^2 n_2^2 n_3} + \frac{M^{(2)}_{n_1,n_2,n_3}}{\pi^5 n_1^2 n_2 n_3^2} + \frac{M^{(3)}_{n_1,n_2,n_3}}{\pi^5 n_1 n_2^2 n_3^2},$$

where

$$\begin{aligned} M^{(1)}_{n_1,n_2,n_3} &= c_{n_1,n_2}\left(\frac{\partial^4 f}{\partial x^2 \partial y^2}(\cdot,\cdot,0)\right) + (-1)^{n_3+1} c_{n_1,n_2}\left(\frac{\partial^4 f}{\partial x^2 \partial y^2}(\cdot,\cdot,1)\right), \\ M^{(2)}_{n_1,n_2,n-3} &= c_{n_1,n_3}\left(\frac{\partial^4 f}{\partial x^2 \partial z^2}(\cdot,0,\cdot)\right) + (-1)^{n_2+1} c_{n_1,n_3}\left(\frac{\partial^4 f}{\partial x^2 \partial z^2}(\cdot,1,\cdot)\right), \\ M^{(3)}_{n_1,n_2,n_3} &= c_{n_2,n_3}\left(\frac{\partial^4 f}{\partial y^2 \partial z^2}(0,\cdot,\cdot)\right) + (-1)^{n_1+1} c_{n_2,n_3}\left(\frac{\partial^4 f}{\partial y^2 \partial z^2}(1,\cdot,\cdot)\right); \end{aligned}$$

(iv)

$$c_{n_1,n_2,n_3}(T) = \frac{c_{n_1,n_2,n_3}\left(\frac{\partial^6 f}{\partial x^2 \partial y^2 z^2}\right)}{\pi^6 n_1^2 n_2^2 n_3^2}.$$

From this and Proposition 6.1, we get the following theorem.

Theorem 6.2. Suppose that $f \in W^{(2,2,2)}([0,1]^3)$, i.e., $\frac{\partial^6 f}{\partial x^2 \partial y^2 \partial z^2}(x,y,z) \in C([0,1]^3)$. Then

$$c_{n_1,n_2,n_3}(f) = \frac{8}{\pi^3 n_1 n_2 n_3}\left(U_{n_1,n_2,n_3} + o\left(\frac{1}{n_1}\right) + o\left(\frac{1}{n_2}\right) + o\left(\frac{1}{n_3}\right)\right),$$

where U_{n_1,n_2,n_3} is stated in (6.5).

Let $n_i = 2p_i + q_i$ $(i = 1,2,3)$. Then

$$c_{2p_1+q_1,2p_2+q_2,2p_3+q_3}(f) = \frac{1}{\pi^3 p_1 p_2 p_3}\left(U_{2p_1+q_1,2p_2+q_2,2p_3+q_3} + o\left(\frac{1}{p_1}\right) + o\left(\frac{1}{p_2}\right) + o\left(\frac{1}{p_3}\right)\right),$$

where U_{n_1,n_2,n_3} is stated in (6.5). It is clear from (6.5) that $U_{2p_1+q_1,2p_2+q_2,2p_3+q_3} = U_{q_1,q_2,q_3}$. So

$$c^2_{2p_1+q_1,2p_2+q_2,2p_3+q_3}(f) = \frac{1}{\pi^2 p_1^2 p_2^2 p_3^2}\left(U_{q_1,q_2,q_3} + o\left(\frac{1}{p_1}\right) + o\left(\frac{1}{p_2}\right) + o\left(\frac{1}{p_3}\right)\right)$$

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and

$$\sum_{(q_1,q_2,q_3)\in\{0,1\}^3} c^2_{2p_1+q_1,2p_2+q_2,2p_3+q_3}(f) = \frac{1}{\pi^2 p_1^2 p_2^2 p_3^2}\left(\sum_{(q_1,q_2,q_3)\in\{0,1\}^3} U^2_{q_1,q_2,q_3} + o\left(\frac{1}{p_1}\right) + o\left(\frac{1}{p_2}\right) + o\left(\frac{1}{p_3}\right)\right).$$

By (6.5),

$$\begin{aligned} U_{q_1,q_2,q_3} &= f(0,0,0) + (-1)^{q_2+1} f(0,1,0) + (-1)^{q_2+q_3} f(0,1,1) \\ &+ (-1)^{q_3+1} f(0,0,1) + (-1)^{q_1+1} f(1,0,0) + (-1)^{q_1+q_2} f(1,1,0) \\ &+ (-1)^{q_1+q_2+q_3+1} f(1,1,1) + (-1)^{q_1+q_3} f(1,0,1). \end{aligned}$$

A direct computation shows that

$$\begin{aligned} \sum_{(q_1,q_2,q_3)\in\{0,1\}^3} U_{q_1,q_2,q_3} &= 8(f^2(0,0,0) + f^2(0,1,0) + f^2(0,1,1) + f^2(0,0,1) \\ &+ f^2(1,0,0) + f^2(1,1,0) + f^2(1,1,1) + f^2(1,0,1)) \\ &= 8 \sum_{\lambda\in\{0,1\}^3} f^2(\lambda). \end{aligned}$$

Therefore,

$$\sum_{(q_1,q_2,q_3)\in\{0,1\}^3} c^2_{2p_1+q_1,2p_2+q_2,2p_3+q_3}(f) = \frac{1}{\pi^6 p_1^2 p_2^2 p_3^2}\left(8 \sum_{\lambda\in\{0,1\}^3} f^2(\lambda) + o\left(\frac{1}{p_1}\right) + o\left(\frac{1}{p_2}\right) + o\left(\frac{1}{p_3}\right)\right).$$

From this, we deduce the following proposition.

Proposition 6.3. Let $f \in W^{(2,2,2)}([0,1]^3)$. Then its Fourier sine coefficients

$$c_{n_1,n_2,n_3}(f) = o\left(\frac{1}{n_1 n_2 n_3}\right)\left(o\left(\frac{1}{n_1}\right) + o\left(\frac{1}{n_2}\right) + o\left(\frac{1}{n_3}\right)\right)$$

if and only if $f(\lambda) = 0$ for all $\lambda \in \{0,1\}^3$.

Suppose that $f \in W^{(2,2,2)}([0,1]^3)$. Then the hyperbolic cross truncation of its Fourier sine series:

$$\begin{aligned} & S_N^{(h)}(f;x,y,z) \\ = & \sum_{\substack{1\le n_1,n_2,n_3\le N-1 \\ 1\le n_1 n_2 n_3\le N-1}} c_{n_1,n_2,n_3}(f)\, \sin(\pi n_1 x)\sin(\pi n_2 y)\sin(\pi n_3 z) \\ = & \sum_{\substack{1\le p_1,p_2,p_3\le \left[\frac{N-1}{2}\right] \\ 1\le p_1 p_2 p_3\le \left[\frac{N-1}{8}\right]}} \sum_{(q_1,q_2,q_3)\in\{0,1\}^3} c_{2p_1+q_1,2p_2+q_2,2p_3+q_3}(f) \sin\pi(2p_1+q_1)x\, \sin\pi(2p_2+q_2)y\, \sin\pi(2p_3+q_3)z. \end{aligned}$$

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By the Parseval identity and (6.3), it follows that

$$8 \parallel f - S_N^{(h)}(f) \parallel_2^2$$

$$= \left(\sum_{p_1,p_2,p_3=1}^{\infty} - \sum_{\substack{1 \le p_1,p_2,p_3 \le \left[\frac{N-1}{2}\right] \\ 1 \le p_1p_2p_3 \le \left[\frac{N-1}{8}\right]}} \right) \sum_{(q_1,q_2,q_3)\in\{0,1\}^3} c^2_{2p_1+q_1,2p_2+q_2,2p_3+q_3}(f)$$

$$= \tfrac{1}{\pi^6} \left(\sum_{p_1,p_2,p_3=1}^{\infty} - \sum_{\substack{1 \le p_1,p_2,p_3 \le \left[\frac{N-1}{2}\right] \\ 1 \le p_1p_2p_3 \le \left[\frac{N-1}{8}\right]}} \tfrac{8}{p_1^2p_2^2p_3^2} \left(\sum_{\lambda\in\{0,1\}^3} f^2(\lambda) + o(1) \right) \right)$$

$$= \tfrac{1}{\pi^6} \sum_{p_1=1}^{\left[\frac{N-1}{2}\right]} \sum_{p_2=1}^{\left[\frac{N}{4p_1}\right]} \sum_{p_3=\left[\frac{N}{2p_1p_2}\right]}^{\infty} \left(\sum_{\lambda\in\{0,1\}^3} f^2(\lambda) + o(1) \right).$$

Notice that

$$\sum_{p_3=\left[\frac{N}{8p_1p_2}\right]}^{\infty} \tfrac{1}{p_1^2p_2^2p_3^2}$$

$$= \tfrac{1}{p_1^2p_2^2} \int_{\frac{N}{8p_1p_2}}^{\infty} \tfrac{\mathrm{d}t}{t^2} + O\left(\tfrac{1}{N^2}\right) = \tfrac{8}{p_1p_2N} + O\left(\tfrac{1}{N^2}\right)$$

and

$$\sum_{p_2=1}^{\left[\frac{N}{4p_1}\right]} \sum_{p_3=\left[\frac{N}{8p_1p_2}\right]}^{\infty} \tfrac{1}{p_1^2p_2^2p_3^2}$$

$$= \sum_{p_2=1}^{\left[\frac{N}{4p_1}\right]} \tfrac{8}{p_1p_2N} + O\left(\tfrac{1}{Np_1}\right) = \tfrac{8}{p_1N} \int_1^{\frac{N}{4p_1}} \tfrac{\mathrm{d}t}{t} + O\left(\tfrac{1}{Np_1}\right)$$

$$= \tfrac{8}{Np_1}(\log N - \log p_1) + O\left(\tfrac{1}{Np_1}\right).$$

Since

$$\sum_{p_1=1}^{\left[\frac{N-1}{2}\right]} \frac{1}{p_1} = \log N + O(1)$$

and

$$\sum_{p_1=1}^{\left[\frac{N-1}{2}\right]} \frac{\log p_1}{p_1} = \int_1^{\frac{N}{2}} \frac{\log t}{t}\,\mathrm{d}t + O(1) = \frac{1}{2}\int_1^{\frac{N}{2}} \mathrm{d}\log^2 t + O(1) = \frac{1}{2}\log^2 N + O(1),$$

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we have

$$\sum_{p_1=1}^{\left[\frac{N-1}{2}\right]} \sum_{p_2=1}^{\left[\frac{N}{4p_1}\right]} \sum_{p_3=\left[\frac{N}{8p_1p_2}\right]}^{\infty} \frac{1}{p_1^2p_2^2p_3^2}$$

$$= \frac{8\log N}{N} \sum_{p_1=1}^{\left[\frac{N-1}{2}\right]} \frac{1}{p_1} - \frac{8}{N} \sum_{p_1=1}^{\left[\frac{N-1}{2}\right]} \frac{\log p_1}{p_1} + O\left(\frac{\log N}{N}\right)$$

$$= \frac{8\log^2 N}{N} - \frac{4\log^2 N}{N} + O\left(\frac{\log N}{N}\right) = \frac{4\log^2 N}{N} + O\left(\frac{\log N}{N}\right).$$

Finally, we have

$$\| f - S_N^{(h)}(f) \|_2^2 = \frac{4}{\pi^6} \left(\sum_{\lambda \in \{0,1\}^3} f^2(\lambda) \right) \frac{\log^2 N}{N}(1 + o(1)) \qquad (N \to \infty).$$

For stochastic processes, we have the corresponding result.

Theorem 6.4. Suppose that f is a stochastic process and $f \in W^{(2,2,2)}([0,1]^3)$. Then

$$E\left[\| f - S_N^{(h)}(f) \|_2^2\right] = \frac{4}{\pi^6} \left(\sum_{\lambda \in \{0,1\}^3} E[f^2(\lambda)] \right) \frac{\log^2 N}{N}(1 + o(1)) \qquad (N \to \infty).$$

For a three-variate function f on $[0,1]^3$, in its decomposition formula (6.1)-(6.4), we expand univariate functions $f_1(x,0,0), ..., f_1(1,y,1)$, bivariate functions $f_2(x,y,0), ..., f_2(x,y,1)$, and three-variate function $T(x,y,z)$ into Fourier sine series, we get the Fourier sine series with polynomial factors. We again define the corresponding hyperbolic cross truncations as follows:

$$(\widetilde{S}_N^{(h)} f)(x,y,z) = P(x,y,z) + Q_N(x,y,z) + R_N(x,y,z) + T_N(x,y,z), \tag{6.6}$$

where $P(x,y,z)$ is stated in (6.2), $Q_N(x,y,z)$ is obtained by replacing eight univariate functions by their Nth partial sums in (6.3), $R_N(x,y,z)$ is obtained by replacing four bivariate functions by their Nth hyperbolic cross truncations in (6.4), and $T_N^{(h)}$ is the Nth hyperbolic cross truncation of $T(x,y,z)$.

Theorem 6.5. Let $f \in W^{(2,2,2)}([0,1]^3)$. Then hyperbolic cross truncations of the Fourier sine series of f with polynomial factors satisfy

$$\| f - \widetilde{S}_N^{(h)}(f) \|_2^2 = O\left(\frac{\log^2 N}{N^3}\right),$$

where $\widetilde{S}_N^{(h)}(f)$ is defined in (6.6).

The number of Fourier sine coefficients in $\widetilde{S}_N^{(h)}(f)$ satisfy $N_c \sim N \log^2 N$. From this and (6.7), we have

$$\| f - S_N^{(h)}(f) \|_2^2 = O\left(\frac{\log^8 N_c}{N_c^3}\right).$$

Therefore, we can use fewest Fourier sine coefficients to reconstruct f. For stochastic processes, the corresponding result is

$$E\left[\| f - S_N^{(h)}(f) \|_2^2\right] = O\left(\frac{\log^8 N_c}{N_c^3}\right).$$

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A new relaxation method for mathematical programs with nonlinear complementarity constraints *

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Abstract. In this paper, mathematical programs with nonlinear complementarity constraints (MPCC) are investigated and a new relaxed method is proposed. Firstly, based on Mangasarian complementarity function, MPCC is relaxed. The relaxed problem is a parametrized nonlinear programming. Secondly, it is proved that the sequence of stationary points of the relaxed problems converges to M-stationary point of MPCC under some mild assumptions; further, it is shown that the stationary point is strong for MPCC if some additional conditions are satisfied. Thirdly, we analyze the existence of the Lagrange multipliers for the relaxed problem. We show that Guignard constraint qualification holds for the relaxed problem under MPCC-linear independence constraint qualifications, and then obtain the existence theorem of the Lagrange multipliers.

Key words. Nonlinear complementarity constraints; Mathematical programs; Relaxed method; Constraint qualifications; Stationary points; Global convergence

AMS subject classification 90C, 49M.

1. Introduction

In this paper, we consider the following MPCC :

$$\begin{array}{ll} \min & f(z) \\ \text{s.t.} & g_i(z) \leq 0, \quad i \in I = \{1, \cdots, m\}, \\ & h_i(z) = 0, \quad i \in I_e = \{1, \cdots, m_e\}, \\ & G_i(z) \geq 0, \quad i \in I_c = \{1, \cdots, m_c\}, \\ & H_i(z) \geq 0, \quad i \in I_c, \\ & G_i(z) H_i(z) \leq 0, \quad i \in I_c, \end{array} \tag{1.1}$$

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where $f,\ g_i,\ h_i,\ G_i,\ H_i:\ \mathbb{R}^n \to \mathbb{R}$ are all continuously differentiable. The MPCC (1.1) has many applications in game theory, traffic transportation, engineering design and so on. The interested reader is referred to the monograph [1] for more details.

As we know, the MPCC (1.1) is a highly difficult nonlinear program since the standard Mangasarian-Fromovitz constraint qualification (MFCQ) is violated at any feasible point (see [2]). This implies that the well-developed approaches for the standard nonlinear programs typically have severe difficulties if they are directly used to solve the MPCC (1.1). So MPCC-tailed algorithms have to be studied.

During last decade, several kinds of efficient methods for the MPCC (1.1) have been developed, such as relaxation (or regularization) ([4–8]), smoothing ([1, 9–17]), interior point method ([1, 18–20]) and penalization ([21]). In this paper, our focus is on relaxation method. The basic idea of the relaxation method is to relax the complicated complementarity constraints

$$G_i(z) \geq 0,\ H_i(z) \geq 0,\ G_i(z)H_i(z) \leq 0, \quad i \in I_c$$

in a suitable way. The interested reader is referred to the recent review paper on relaxation method [5] for more knowledge.

Kadrani et al. proposed a relaxation scheme in [8] as follows:

$$\begin{array}{ll} \min & f(z) \\ \text{s.t.} & g_i(z) \leq 0, \quad i \in I, \\ & h_i(z) = 0, \quad i \in I_e, \\ & G_i(z) \geq -t, \quad i \in I_c, \\ & H_i(z) \geq -t, \quad i \in I_c, \\ & (G_i(z) - t)\,(H_i(z) - t) \leq 0, \quad i \in I_c, \end{array} \tag{1.2}$$

where t is a nonnegative parameter. It is shown that any accumulation point of the stationary point sequence of (1.2) converges to an M-stationary point of MPCC (1.1) when $t \to 0$ under the MPCC-linear independence constraint qualification (MPCC-LICQ) condition and some mild conditions. They also showed that existence of KKT multipliers for the relaxed problem (1.2) under the MPCC-LICQ assumption. Figure 1, however, shows that there exist two disadvantages: (1) the feasible region of the relaxed problem (1.2) is almost disconnected. Therefore, one has to meet severe difficulties when solving (1.2) by means of a standard NLP algorithm; (2) the feasible region of the MPCC (1.1) is not included in that of the relaxed problem (1.2), regardless of the choice of $t > 0$.

In order to overcome the above drawbacks, Kanzow et al. recently proposed a new relaxation

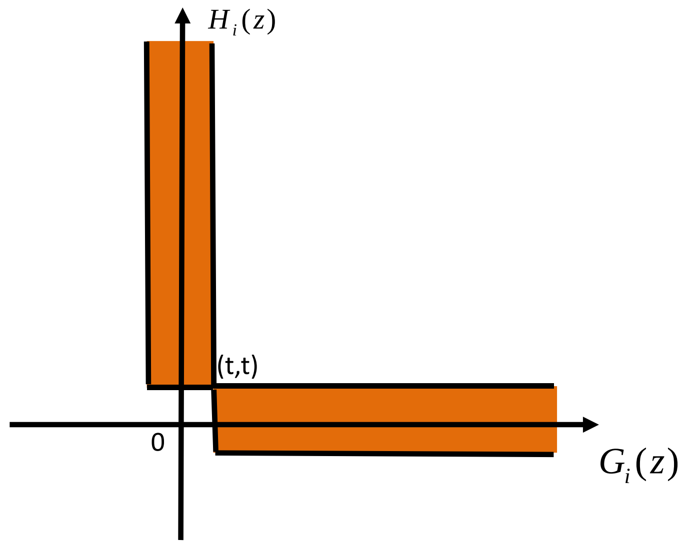

Figure 1: Geometric interpretation of relaxation in [8]

scheme in [5] as follows:

$$\begin{aligned} \min \quad & f(z) \\ \text{s.t.} \quad & g_i(z) \le 0, \;\; i \in I, \\ & h_i(z) = 0, \;\; i \in I_e, \\ & G_i(z) \ge -t, \;\; i \in I_c, \\ & H_i(z) \ge -t, \;\; i \in I_c, \\ & \Psi(z;t) = (\Psi_i(z;t), i \in I_c) \le 0, \end{aligned} \tag{1.3}$$

where $t \ge 0$ is a parameter, $\Psi_i(z;t) = \varphi(G_i(z) - t, H_i(z) - t)$, the complementarity function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\varphi(x,y) = \begin{cases} xy, & x + y \ge 0, \\ -\frac{1}{2}(x^2 + y^2), & x + y < 0. \end{cases}$$

The geometric interpretation of the relaxation scheme (1.3) is given in Figure 2. It is shown that any accumulation point of the stationary point sequence of (1.3) converges to an M-stationary point of MPCC (1.1) when $t \to 0$ under much weaker MPCC-constant positive linear dependence (MPCC-CPLD) condition and some mild conditions. And they also showed the existence of the Lagrange multipliers for the relaxed problem (1.3) under the MPCC-LICQ assumption.

It is worth noting that the feasible region of the original problem (1.1) is part of the boundary of that of the relaxed problem (1.3). Consequently, some additional stricter conditions is required for the search directions when solving the relaxed problem (1.3) by a standard NLP algorithm.

In this paper, motivated from the ideas in [5, 8] and based on the Mangasarian complementarity

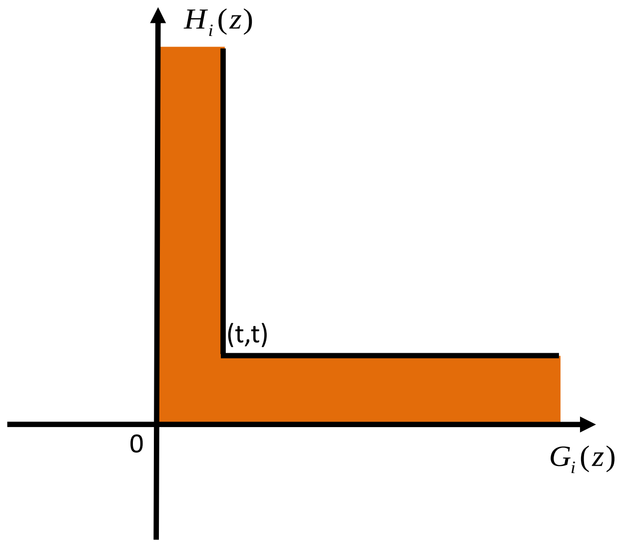

Figure 2: Geometric interpretation of relaxation in [5]

function ([25]) defined by

$$\phi(a,b) = \rho(a) + \rho(b) - \rho(|a-b|) \tag{1.4}$$

with $\rho: \ \mathbb{R} \to \mathbb{R}$ being given by

$$\rho(\tau) = \begin{cases} \tau^2, & \text{if } \ \tau \geq 0 \\ -\tau^2, & \text{if } \ \tau < 0, \end{cases}$$

we propose a new relaxation scheme:

$$\begin{array}{ll} \min & f(z) \\ \text{s.t.} & g_i(z) \leq 0, \ i \in I, \\ & h_i(z) = 0, \ i \in I_e, \\ & G_i(z) \geq -t, \ i \in I_c, \\ & H_i(z) \geq -t, \ i \in I_c, \\ & \Phi_i(z;t) \leq 0, \ i \in I_c, \end{array} \tag{1.5}$$

where t is a nonnegative parameter and

$$\Phi_i(z;t) = \phi(G_i(z) - t, H_i(z) - t). \tag{1.6}$$

The geometric interpretation of the relaxation scheme (1.5) is given in Figure 3.

We show that any accumulation point of the stationary point sequence of (1.5) converges to an M-stationary point of MPCC (1.1) when $t \to 0$ under much weaker MPCC-CPLD condition and some mild conditions, and converges to a strongly stationary point of MPCC (1.1) under additional conditions. We also show that the standard Guignard constraint qualification (GCQ) holds at every

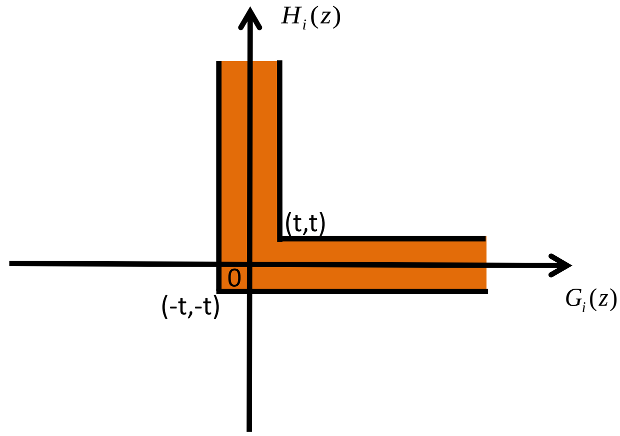

Figure 3: Geometric interpretation of the proposed relaxation scheme

feasible point of the relaxed problem (1.5) and the existence of Lagrange multipliers of the relaxed problem (1.5) is verified under some mild conditions.

The rest of the paper is organized as follows. Some definitions of different stationary points and constraint qualifications and preliminary results about MPCC are restated in Section 2. Section 3 contains the analysis and proof of the convergent results. The existence of Lagrange multipliers for the relaxed problem is analyzed and verified in Section 4 and some concluding remarks are given in the final section.

2. Preliminaries

As we know, except for Guignard CQ, all standard constraint qualifications are far too restrictive for MPCCs ([22]). Some MPCC-tailed CQs are introduced in the past. Furthermore, due to the fact that most standard CQs are likely to be violated at a local minimum of an MPCC, the KKT conditions can not be considered as the optimality conditions. Hence, several weaker stationarity notions have been proposed. For convenience and completeness, in this section we briefly restate some concepts and results about the MPCC (1.1) which are needed in the sequel analysis. The reader is also referred to [5, 22–24].

Let S be the feasible set of the MPCC (1.1). For any $z^* \in S$, define different index sets for the MPCC (1.1) as follows:

$$\begin{aligned} I_{0+}(z^*) &= \{i \in I_c \mid G_i(z^*) = 0,\ H_i(z^*) > 0\}, \\ I_{00}(z^*) &= \{i \in I_c \mid G_i(z^*) = 0,\ H_i(z^*) = 0\}, \\ I_{+0}(z^*) &= \{i \in I_c \mid G_i(z^*) > 0,\ H_i(z^*) = 0\}. \end{aligned} \tag{2.1}$$

Definition 2.1 [5, 23] Let z^* be a feasible point of the MPCC (1.1). Then z^* is said to be
(1) weakly stationary for the MPCC (1.1), if there exist multipliers $(\alpha^*, \beta^*, \gamma^*, \delta^*) \in \mathbb{R}^m \times \mathbb{R}^{m_e} \times$

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$\mathbb{R}^{m_c} \times \mathbb{R}^{m_c}$ such that

$$\begin{aligned} &\nabla f(z^*) + \sum_{i\in I} \alpha_i^* \nabla g_i(z^*) + \sum_{i\in I_e} \beta_i^* \nabla h_i(z^*) - \sum_{i\in I_c} \gamma_i^* \nabla G_i(z^*) - \sum_{i\in I_c} \delta_i^* \nabla H_i(z^*) = 0, \\ &\alpha_i^* \geq 0, \quad \alpha_i^* g_i(z^*) = 0 \ (i \in I), \qquad \gamma_i^* = 0 \ (i \in I_{+0}(z^*)), \quad \delta_i^* = 0 \ (i \in I_{0+}(z^*)); \end{aligned} \tag{2.2}$$

(2) C-stationarity, if it is weakly stationarity and $\gamma_i^* \delta_i^* \geq 0, \ \forall \ i \in I_{00}(z^*)$;

(3) M-stationary, if it is weakly stationarity and $\gamma_i^* > 0, \ \delta_i^* > 0$ or $\gamma_i^* \delta_i^* = 0, \ \forall \ i \in I_{00}(z^*)$;

(4) strongly stationary, if it is weakly stationarity and $\gamma_i^*, \ \delta_i^* \geq 0, \ \forall \ i \in I_{00}(z^*)$.

Obviously, we know that strong stationarity implies M-stationarity, M-stationarity implies C-stationarity and C-stationarity implies weak stationarity. Moreover, it is shown in [22] that strong stationarity is equivalent to the standard KKT conditions of an MPCC. However, a counterexample given in [23] indicates that strong stationarity may not hold at a global minimum, even for very simple MPCCs.

Definition 2.2 [5, 23] Let z^* be a feasible point of the MPCC (1.1). Then

(1) MPCC-LICQ is said to hold at z^* if the gradients

$$\begin{aligned} &\{\nabla g_i(z^*) \mid i \in I_g(z^*)\} \cup \{\nabla h_i(z^*) \mid i \in I_e\} \cup \{\nabla G_i(z^*) \mid i \in I_{00}(z^*) \cup I_{0+}(z^*)\} \\ &\cup \{\nabla H_i(z^*) \mid i \in I_{00}(z^*) \cup I_{+0}(z^*)\} \end{aligned}$$

are linearly independent.

(2) MPCC-CPLD is said to hold at z^* if, for any subsets $I_1 \subseteq I_g(z^*), \ I_2 \subseteq I_e, \ I_3 \subseteq I_{00}(z^*) \cup I_{0+}(z^*), \ I_4 \subseteq I_{00}(z^*) \cup I_{+0}(z^*)$ such that the gradients

$$\{\nabla g_i(z^*) \mid i \in I_1\} \cup \{\{\nabla h_i(z^*) \mid i \in I_2\} \cup \{\nabla G_i(z^*) \mid i \in I_3\} \cup \{\nabla H_i(z^*) \mid i \in I_4\}\}$$

are positive-linearly dependent, there exists a neighborhood $N(z^*)$ of z^* such that the gradients

$$\{\nabla g_i(z) \mid i \in I_1\} \cup \{\nabla h_i(z) \mid i \in I_2\} \cup \{\nabla G_i(z) \mid i \in I_3\} \cup \{\nabla H_i(z) \mid i \in I_4\}$$

are linearly dependent for all $z \in N(z^*)$.

It follows from [24] that MPCC-LICQ implies MPCC-CPLD.

3. Convergence results

In this section, we analyze the convergence behavior of the relaxed problem (1.5) as $t \to 0$. For convenience, denote by $R_{MPCC}(t)$ (1.5) the relaxed problem (1.5), and define the following index sets for $R_{MPCC}(t)$ (1.5):

$$\begin{aligned} &I_g(z) = \{i \in I \mid g_i(z) = 0\}, \quad I_G(z;t) = \{i \in I_c \mid G_i(z) = -t\}, \\ &I_H(z;t) = \{i \in I_c \mid H_i(z) = -t\}, \quad I_\Phi(z;t) = \{i \in I_c \mid \Phi_i(z;t) = 0\}, \\ &I_\Phi^{0+}(z;t) = \{i \in I_\Phi(z;t) \mid G_i(z) - t = 0, \ H_i(z) - t > 0\}, \\ &I_\Phi^{00}(z;t) = \{i \in I_\Phi(z;t) \mid G_i(z) - t = 0, \ H_i(z) - t = 0\}, \\ &I_\Phi^{+0}(z;t) = \{i \in I_\Phi(z;t) \mid G_i(z) - t > 0, \ H_i(z) - t = 0\}, \\ &supp(c) = \{i \mid c_i \neq 0, \ i = 1, \ldots, l. \ c = (c_i) \in \mathbb{R}^l\}. \end{aligned}$$

Obviously, we have $I_\Phi^{0+}(z;t)\cap I_\Phi^{00}(z;t)\cap I_\Phi^{+0}(z;t)=\emptyset,\ \ I_\Phi^{0+}(z;t)\cup I_\Phi^{00}(z;t)\cup I_\Phi^{+0}(z;t)=I_\Phi(z;t)$.

By elementary computation and analysis, we can obtain the important properties of the complementarity function ϕ given in (1.4), which play a key role in the subsequently analysis.

Lemma 3.1 [25] (1) $\phi(a,b)=0$ if and only if $a\geq 0,\quad b\geq 0,\quad ab=0$.

(2) ϕ is continuously differentiable, and its gradient is

$$\nabla\phi(a,b)=\begin{cases}\begin{pmatrix}-4a+2b\\-4b+2a\end{pmatrix}, & \text{if } a<0 \text{ and } b<0,\\ \begin{pmatrix}-4a+2b\\2a\end{pmatrix}, & \text{if } a<0 \text{ and } b\geq 0,\\ \begin{pmatrix}2b\\-4b+2a\end{pmatrix}, & \text{if } a\geq 0 \text{ and } b<0,\\ \begin{pmatrix}2b\\2a\end{pmatrix}, & \text{if } a\geq 0 \text{ and } b\geq 0.\end{cases}$$

(3) The following inequality holds:

$$\phi(a,b)\begin{cases}>0, & \text{if } a>0 \text{ and } b>0,\\ <0, & \text{if } a<0 \text{ or } b<0.\end{cases}\tag{3.1}$$

According to the definition (1.6) of Φ_i, one obtains the expressions of $\Phi_i(z;t)$ and the gradient of $\Phi_i(z;t)$, respectively :

$$\begin{aligned}&\Phi_i(z;t)=\phi(G_i(z)-t,H_i(z)-t)\\ &=\begin{cases}-2(G_i(z)-t)^2-2(H_i(z)-t)^2+2(G_i(z)-t)(H_i(z)-t), & G_i(z)-t<0 \text{ and } H_i(z)-t<0,\\ -2(G_i(z)-t)^2+2(G_i(z)-t)(H_i(z)-t), & G_i(z)-t<0 \text{ and } H_i(z)-t\geq 0,\\ -2(H_i(z)-t)^2+2(G_i(z)-t)(H_i(z)-t), & G_i(z)-t\geq 0 \text{ and } H_i(z)-t<0,\\ 2(G_i(z)-t)(H_i(z)-t), & G_i(z)-t\geq 0 \text{ and } H_i(z)-t\geq 0,\end{cases}\end{aligned}\tag{3.2}$$

$$\begin{aligned}&\nabla\Phi_i(z;t)\\ &=\begin{cases}(2H_i(z)-4G_i(z)+2t)\nabla G_i(z)+(2G_i(z)-4H_i(z)+2t)\nabla H_i(z), & G_i(z)-t<0 \text{ and } H_i(z)-t<0,\\ (2H_i(z)-4G_i(z)+2t)\nabla G_i(z)+2(G_i(z)-t)\nabla H_i(z), & G_i(z)-t<0 \text{ and } H_i(z)-t\geq 0,\\ 2(H_i(z)-t)\nabla G_i(z)+(2G_i(z)-4H_i(z)+2t)\nabla H_i(z), & G_i(z)-t\geq 0 \text{ and } H_i(z)-t<0,\\ 2(H_i(z)-t)\nabla G_i(z)+2(G_i(z)-t)\nabla H_i(z), & G_i(z)-t\geq 0 \text{ and } H_i(z)-t\geq 0,\end{cases}\end{aligned}\tag{3.3}$$

where the parameter $t\geq 0$.

Let $S(t)$ be the feasible set of $R_{MPCC}(t)$ (1.5). Then the following result is true.

Lemma 3.2 (1) $S(0)=S$; (2) $S(t_1)\subseteq S(t_2),\ \ 0\leq t_1\leq t_2$; (3) $S=\bigcap\limits_{t>0}S(t)$.

Next, we establish the convergence theorem of the proposed relaxation method.

Theorem 3.1 Suppose that $\{t_k\} \downarrow 0$, $(z^k, \alpha^k, \beta^k, \gamma^k, \delta^k, \nu^k)$ is a KKT pair of $R_{MPCC}(t_k)$ (1.5) for all k, z^* is an accumulation point of the sequence $\{z^k\}$, and MPCC-CPLD holds at z^*. Then the following statements hold:
(1) z^* is M-stationary for the MPCC (1.1);
(2) If $\{z^k\}$ additionally satisfies $I_\Phi^{0+}(z^k;t_k) = I_\Phi^{+0}(z^k;t_k) = \emptyset$, then z^* is strongly stationary for the MPCC (1.1).

Proof. (1) Note that $z^k \to z^*$, $t_k \to 0$ and the continuity of g_i, h_i, G_i, H_i, we obtain that z^* is feasible for MPCC (1.1) and the following inclusion relations are true:

$$\begin{array}{c} I_g(z^k) \subseteq I_g(z^*), \\ I_G(z^k;t_k) \cup I_\Phi^{00}(z^k;t_k) \cup I_\Phi^{0+}(z^k;t_k) \subseteq I_{00}(z^*) \cup I_{0+}(z^*), \\ I_H(z^k;t_k) \cup I_\Phi^{00}(z^k;t_k) \cup I_\Phi^{+0}(z^k;t_k) \subseteq I_{00}(z^*) \cup I_{+0}(z^*). \end{array} \tag{3.4}$$

Since $(z^k, \alpha^k, \beta^k, \gamma^k, \delta^k, \nu^k)$ is a KKT pair of $R_{MPCC}(t_k)$ (1.5), we have

$$\begin{aligned} 0 = \ & \nabla f(z^k) + \sum_{i\in I} \alpha_i^k \nabla g_i(z^k) + \sum_{i\in I_e} \beta_i^k \nabla h_i(z^k) - \sum_{i\in I_c} \gamma_i^k \nabla G_i(z^k) - \sum_{i\in I_c} \delta_i^k \nabla H_i(z^k) \\ & + \sum_{i\in I_c} \nu_i^k \nabla \Phi_i(z^k;t_k), \end{aligned} \tag{3.5}$$

$$\begin{array}{ll} \alpha_i^k \begin{cases} \geq 0, & i \in I_g(z^k), \\ = 0, & i \notin I_g(z^k); \end{cases} & \gamma_i^k \begin{cases} \geq 0, & i \in I_G(z^k;t_k), \\ = 0, & i \notin I_G(z^k;t_k); \end{cases} \\ \delta_i^k \begin{cases} \geq 0, & i \in I_H(z^k;t_k), \\ = 0, & i \notin I_H(z^k;t_k); \end{cases} & \nu_i^k \begin{cases} \geq 0, & i \in I_\Phi(z^k;t_k), \\ = 0, & i \notin I_\Phi(z^k;t_k). \end{cases} \end{array} \tag{3.6}$$

From (3.3), one has

$$\nabla \Phi_i(z^k;t_k) = \begin{cases} 2(H_i(z^k) - t_k)\nabla G_i(z^k), & i \in I_\Phi^{0+}(z^k;t_k), \\ 2(G_i(z^k) - t_k)\nabla H_i(z^k), & i \in I_\Phi^{+0}(z^k;t_k), \\ 0, & i \in I_\Phi^{00}(z^k;t_k). \end{cases}$$

Define $\nu^{G,k} = (\nu_i^{G,k}, i \in I_c)$ and $\nu^{H,k} = (\nu_i^{H,k}, i \in I_c)$ with

$$\nu_i^{G,k} = \begin{cases} 2\nu_i^k(H_i(z^k) - t_k), & \text{if } i \in I_\Phi^{0+}(z^k;t_k), \\ 0, & \text{otherwise;} \end{cases} \qquad \nu_i^{H,k} = \begin{cases} 2\nu_i^k(G_i(z^k) - t_k), & \text{if } i \in I_\Phi^{+0}(z^k;t_k), \\ 0, & \text{otherwise.} \end{cases}$$

Note that $I_\Phi(z^k;t_k) = I_\Phi^{0+}(z^k;t_k) \cup I_\Phi^{00}(z^k;t_k) \cup I_\Phi^{+0}(z^k;t_k)$, (3.5) can be rewritten as follows:

$$\begin{aligned} 0 = \ & \nabla f(z^k) + \sum_{i\in I} \alpha_i^k \nabla g_i(z^k) + \sum_{i\in I_e} \beta_i^k \nabla h_i(z^k) - \sum_{i\in I_c} \gamma_i^k \nabla G_i(z^k) - \sum_{i\in I_c} \delta_i^k \nabla H_i(z^k) \\ & + \sum_{i\in I_c} \nu_i^{G,k} \nabla G_i(z^k) + \sum_{i\in I_c} \nu_i^{H,k} \nabla H_i(z^k). \end{aligned} \tag{3.7}$$

Note that the multipliers $v_i^{G,k}$ and $\delta_i^{H,k}$ are nonnegative, too, according to [4, Lemma A.1], we suppose, without loss of generality, the gradients corresponding to nonzero multipliers, that is,

$$\begin{aligned} & \{\nabla g_i(z^k) \mid i \in supp(\alpha^k)\} \cup \{\nabla h_i(z^k) \mid i \in supp(\beta^k)\} \cup \{\nabla G_i(z^k) \mid i \in supp(\gamma^k) \cup supp(\nu^{G,k})\} \\ & \cup \{\nabla H_i(z^k) \mid i \in supp(\delta^k) \cup supp(\nu^{H,k})\}, \end{aligned} \tag{3.8}$$

are linearly independent.

In what follows, we show that the sequence $\{(\alpha^k, \beta^k, \gamma^k, \delta^k, \nu^{G,k}, \nu^{H,k})\}$ is bounded. Suppose, by contradiction, that the conclusion is not true. Then there is a vector $(\alpha, \beta, \gamma, \delta, \nu^G, \nu^H)$ and a subset $K \subseteq \{1, 2, \dots\}$ such that

$$\frac{(\alpha^k, \beta^k, \gamma^k, \delta^k, \nu^{G,k}, \nu^{H,k})}{\|(\alpha^k, \beta^k, \gamma^k, \delta^k, \nu^{G,k}, \nu^{H,k})\|} \quad \xrightarrow{K} \quad (\alpha, \beta, \gamma, \delta, \nu^G, \nu^H) \neq 0.$$

Dividing by $\|(\alpha^k, \beta^k, \gamma^k, \delta^k, \nu^{G,k}, \nu^{H,k})\|$ in (3.7) and passing to the limit, we obtain

$$\begin{aligned} 0 = \; & \sum_{i\in I} \alpha_i \nabla g_i(z^*) + \sum_{i\in I_e} \beta_i \nabla h_i(z^*) - \sum_{i\in I_c} \gamma_i \nabla G_i(z^*) - \sum_{i\in I_c} \delta_i \nabla H_i(z^*) \\ & + \sum_{i\in I_c} \nu_i^G \nabla G_i(z^*) + \sum_{i\in I_c} \nu_i^H \nabla H_i(z^*), \end{aligned}$$

which implies the gradients

$$\begin{aligned} & \{\nabla g_i(z^*) \mid i \in supp(\alpha)\} \cup \{\nabla h_i(z^*) \mid i \in supp(\beta)\} \cup \{\nabla G_i(z^*) \mid i \in supp(\gamma) \cup supp(\nu^G)\} \\ & \cup \{\nabla H_i(z^*) \mid i \in supp(\delta) \cup supp(\nu^H)\} \end{aligned}$$

are positive-linearly dependent.

Since MPCC-CPLD holds at z^*, there exists a neighbourhood $U(z^*)$ of z^* such that $\forall\, z \in U(z^*)$, the gradients

$$\begin{aligned} & \{\nabla g_i(z) \mid i \in supp(\alpha)\} \cup \{\nabla h_i(z) \mid i \in supp(\beta)\} \cup \{\nabla G_i(z) \mid i \in supp(\gamma) \cup supp(\nu^G)\} \\ & \cup \{\nabla H_i(z) \mid i \in supp(\delta) \cup supp(\nu^H)\} \end{aligned}$$

are linearly dependent. This contradicts the linear independence in (3.8) since $supp(\alpha, \beta, \gamma, \delta, \nu^G, \nu^H) \subseteq supp(\alpha^k, \beta^k, \gamma^k, \delta^k, \nu^{G,k}, \nu^{H,k})$ for k sufficiently large. Therefore, $\{(\alpha^k, \beta^k, \gamma^k, \delta^k, \nu^{G,k}, \nu^{H,k})\}$ is bounded.

We suppose, without loss of generality, that $\{(\alpha^k, \beta^k, \gamma^k, \delta^k, \nu^{G,k}, \nu^{H,k})\}$ converges to $(\alpha^*, \beta^*, \gamma^*, \delta^*, \nu^{G,*}, \nu^{H,*})$. Since $I_G(z^k; t_k) \cap I_\Phi^{0+}(z^k; t_k) = \emptyset,\; I_H(z^k; t_k) \cap I_\Phi^{+0}(z^k; t_k) = \emptyset$, we define

$$\widetilde{\gamma}_i = \begin{cases} \gamma_i^*, & \text{if } i \in supp(\gamma^*), \\ -\nu_i^{G,*}, & \text{if } i \in supp(\nu^{G,*}), \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad \widetilde{\delta}_i = \begin{cases} \delta_i^*, & \text{if } i \in supp(\delta^*), \\ -\nu_i^{H,*}, & \text{if } i \in supp(\nu^{H,*}), \\ 0, & \text{otherwise.} \end{cases} \tag{3.9}$$

By passing to the limit in (3.7), we have

$$0 = \nabla f(z^*) + \sum_{i\in I} \alpha_i^* \nabla g_i(z^*) + \sum_{i\in I_e} \beta_i^* \nabla h_i(z^*) - \sum_{i\in I_c} \widetilde{\gamma}_i \nabla G_i(z^*) - \sum_{i\in I_c} \widetilde{\delta}_i \nabla H_i(z^*), \tag{3.10}$$

where $\alpha_i^* \geq 0,\; \alpha_i^* g_i(z^*) = 0,\; i \in I$. And it follows for k sufficiently large that

$$\begin{aligned} supp(\alpha^*) &\subseteq I_g(z^k) \subseteq I_g(z^*), \\ supp(\widetilde{\gamma}) = supp(\gamma^*) \cup supp(\nu^{G,*}) &\subseteq I_G(z^k; t_k) \cup I_\Phi^{0+}(z^k; t_k) \subseteq I_{00}(z^*) \cup I_{0+}(z^*), \\ supp(\widetilde{\delta}) = supp(\delta^*) \cup supp(\nu^{H,*}) &\subseteq I_H(z^k; t_k) \cup I_\Phi^{+0}(z^k; t_k) \subseteq I_{00}(z^*) \cup I_{+0}(z^*). \end{aligned} \tag{3.11}$$

From (3.11), one has $\widetilde{\gamma}_i = 0,\ i \in I_{+0}(z^*);\quad \widetilde{\delta}_i = 0,\ i \in I_{0+}(z^*)$, together with (3.10), we can conclude that z^* is weakly stationary for MPCC (1.1).

In what follows, we prove z^* is M-stationary, i.e., either $\widetilde{\gamma}_i > 0$, $\widetilde{\delta}_i > 0$ or $\widetilde{\gamma}_i\widetilde{\delta}_i = 0$, for all $i \in I_{00}(z^*)$. Suppose, by contradiction, that there is an $i \in I_{00}(z^*)$ with $\widetilde{\gamma}_i < 0$ and $\widetilde{\delta}_i \neq 0$ (the case $\widetilde{\gamma}_i \neq 0$ and $\widetilde{\delta}_i < 0$ can be proven in a similar way). According to (3.9) and (3.11), one has $i \in supp(\nu^{G,*}) \subseteq I_\Phi^{0+}(z^k;t_k)$ for k sufficiently large. Note that $I_\Phi^{0+}(z^k;t_k) \cap (I_H(z^k;t_k) \cup I_\Phi^{+0}(z^k;t_k)) = \emptyset$, it follows from (3.9) that $\widetilde{\delta}_i = 0$, which yields a contradiction. Hence, z^* is an M-stationary point.

(2) In order to show z^* is a strongly stationary point of the MPCC (1.1), based on the result (1), it is sufficient to show that $\widetilde{\gamma}_i \geq 0,\ \forall i \in I_{00}(z^*);\ \widetilde{\delta}_i \geq 0,\ \forall i \in I_{00}(z^*)$.

By (3.11), the equality (3.10) can be rewirtten as

$$0 = \nabla f(z^*) + \sum_{i \in supp(\alpha^*)} \alpha_i^* \nabla g_i(z^*) + \sum_{i \in I_e} \beta_i^* \nabla h_i(z^*) - \sum_{i \in supp(\widetilde{\gamma})} \widetilde{\gamma}_i \nabla G_i(z^*) - \sum_{i \in supp(\widetilde{\delta})} \widetilde{\delta}_i \nabla H_i(z^*). \quad (3.12)$$

In view of $I_\Phi^{0+}(z^k;t_k) = \emptyset$, one gets from (3.9)

$$\widetilde{\gamma}_i = \begin{cases} \gamma_i^*, & i \in supp(\gamma^*), \\ 0, & \text{else}. \end{cases} \quad (3.13)$$

For $\forall\ i \in I_{00}(z^*)$, if $i \in supp(\gamma^*)$, then one obtains from (3.11) that $\widetilde{\gamma} = \gamma_i^* > 0$; otherwise, $\widetilde{\gamma} = \gamma_i^* = 0$. This indicates $\widetilde{\gamma}_i \geq 0$ for all $i \in I_{00}(z^*)$.

Similarly, one can show $\widetilde{\delta}_i \geq 0$ for all $i \in I_{00}(z^*)$.

Thus, z^* is a strongly stationary point of the MPCC (1.1). □

4. Existence of multipliers

In the convergent theorem, i.e., Theorem 3.1, we assume that there exists a KKT point for $R_{MPCC}(t_k)$ (1.5). Whether does a KKT point for $R_{MPCC}(t_k)$ (1.5) exist or not, or what conditions can ensure the existence of KKT point ? In order to answer these questions, we will further discuss the existence of Lagrange multipliers of $R_{MPCC}(t)$ (1.5) in this section.

Let $\widetilde{z}$ be feasible for $R_{MPCC}(t)$ (1.5) and J be an arbitrary subset of $I_\Phi^{00}(\widetilde{z};t)$, define an auxiliary program ($AP(t,J)$ for short) as follows:

$$\begin{array}{ll} \min & f(z) \\ \text{s.t.} & g_i(z) \leq 0, \quad i \in I, \\ & h_i(z) = 0, \quad i \in I_e, \\ & G_i(z) \geq -t,\ H_i(z) \geq -t, G_i(z) \leq t,\ i \in J, \\ & G_i(z) \geq -t,\ H_i(z) \geq -t, H_i(z) \leq t,\ i \in \overline{J}, \\ & G_i(z) \geq -t,\ H_i(z) \geq -t, \Phi_i(z;t) \leq 0,\ i \notin I_\Phi^{00}(\widetilde{z};t), \end{array} \quad (4.1)$$

where $\overline{J}$ means the complement of J in $I_\Phi^{00}(\widetilde{z};t)$.

It is obvious that $\widetilde{z}$ is feasible for $AP(t,J)$. Denote by $S(t,J)$ the feasible set of $AP(t,J)$ (4.1). It is not difficult to obtain the relation of feasible sets between $AP(t,J)$ (4.1) and $R_{MPCC}(t)$ (1.5).

Lemma 4.1 Let J be an arbitrary subset of $I_\Phi^{00}(\widetilde{z};t)$ and $t \geq 0$. Then $S(t,J) \subseteq S(t)$.

Lemma 4.2 For any $t \geq 0$ and any feasible point $\widetilde{z}$ of $R_{MPCC}(t)$ (1.5), the following equality holds true:

$$\mathcal{T}_{S(t)}(\widetilde{z}) = \bigcup_{J \subseteq I_\Phi^{00}(\widetilde{z};t)} \mathcal{T}_{S(t,J)}(\widetilde{z}),$$

where $\mathcal{T}_{S(t)}$ and $\mathcal{T}_{S(t,J)}(\widetilde{z})$ are the tangent cones of $R_{MPCC}(t)$ (1.5) and $AP(t,J)$ (4.1) at $\widetilde{z}$, respectively.

Proof. For any $d \in \mathcal{T}_{S(t)}(\widetilde{z})$, the definition of tangent cone tells us that there exists a sequence $\{z^k\} \subseteq S(t)$, $z^k \to \widetilde{z}$, and a sequence $\{\tau_k\} \downarrow 0$ such that $d = \lim\limits_{k\to\infty} \frac{z^k - \widetilde{z}}{\tau_k}$.

In the following, we show that $d \in \bigcup\limits_{J \subseteq I_\Phi^{00}(\widetilde{z};t)} \mathcal{T}_{S(t,J)}(\widetilde{z})$. Note that $z^k \in S(t)$, one has

$$\begin{aligned} &g_i(z^k) \leq 0,\ i \in I, \quad h_i(z^k) = 0,\ i \in I_e, \\ &G_i(z^k) \geq -t,\ H_i(z^k) \geq -t,\ \Phi_i(z^k;t) \leq 0,\ i \in I_c. \end{aligned}$$

Hence, one has $\Phi_i(z^k;t) \leq 0$ for any $i \in I_c$.

If $i \in I_\Phi^{00}(\widetilde{z};t)$, there are six cases for $\Phi_i(z^k;t) \leq 0$ as follows:

$$\begin{aligned} &G_i(z^k) - t < 0,\ H_i(z^k) - t < 0; \\ &G_i(z^k) - t < 0,\ H_i(z^k) - t \geq 0; \\ &G_i(z^k) - t \geq 0,\ H_i(z^k) - t < 0; \\ &G_i(z^k) - t > 0,\ H_i(z^k) - t = 0; \\ &G_i(z^k) - t = 0,\ H_i(z^k) - t = 0; \\ &G_i(z^k) - t = 0,\ H_i(z^k) - t > 0. \end{aligned}$$

Thus, there exists an infinity subset $K \subseteq \{1,2,\ldots\}$ such that $G_i(z^k) - t \leq 0,\ \forall\ k \in K$. Let $J = \{i \in I_\Phi^{00}(\widetilde{z};t) \mid G_i(z^k) - t \leq 0,\ \forall\ k \in K\}$, $\overline{J} = I_\Phi^{00}(\widetilde{z};t) \backslash J$, then one gets $\{z^k\} \subseteq S(t,J)$. This implies $d \in \bigcup\limits_{J \subseteq I_\Phi^{00}(\widetilde{z};t)} \mathcal{T}_{S(t,J)}(\widetilde{z})$. Therefore, we have $\mathcal{T}_{S(t)}(\widetilde{z}) \subseteq \bigcup\limits_{J \subseteq I_\Phi^{00}(\widetilde{z};t)} \mathcal{T}_{S(t,J)}(\widetilde{z})$.

Conversely, for any $d \in \bigcup\limits_{J \subseteq I_\Phi^{00}(\widetilde{z};t)} \mathcal{T}_{S(t,J)}(\widetilde{z})$, there exists a subset $J \subseteq I_\Phi^{00}(\widetilde{z};t)$ such that $d \in \mathcal{T}_{S(t,J)}(\widetilde{z})$. Accordingly, there exists a sequence $\{z^k\} \subseteq S(t,J)$, $z^k \to \widetilde{z}$ and a sequence $\{\tau_k\} \downarrow 0$ such that $d = \lim\limits_{k\to\infty} \frac{z^k - \widetilde{z}}{\tau_k}$.

By Lemma 4.1, one has $\{z^k\} \subseteq S(t)$, so $d \in \mathcal{T}_{S(t)}(\widetilde{z})$. Thus one obtains

$$\bigcup_{J \subseteq I_\Phi^{00}(\widetilde{z};t)} \mathcal{T}_{S(t,J)}(\widetilde{z}) \subseteq \mathcal{T}_{S(t)}(\widetilde{z}).$$

Hence, the result is true. The proof is finished. $\square$

For the sake of convenience, we now give a conclusion in [26], which is used in the proof of our Theorem 4.1.

Lemma 4.3 Suppose that $C_1,\ C_2 \subseteq \mathbb{R}^n$ are cones defined by

$$C_1 = \{p \in \mathbb{R}^n \mid x_i^T p \leq 0,\ i \in I;\ y_i^T p = 0,\ i \in I_e\},$$
$$C_2 = \{q \in \mathbb{R}^n \mid q = \sum_{i\in I} \lambda_i x_i + \sum_{i \in I_e} \mu_i y_i,\ \lambda_i \geq 0,\ \forall\ i \in I\}.$$

Then $C_1 = C_2^\circ$ and $C_2 = C_1^\circ$, where C_1° and C_2° are the polar cones of C_1 and C_2, respectively.

The following theorem shows that standard Guignard CQ holds for $R_{MPCC}(t)$ (1.5) only under MPCC-LICQ assumption.

Theorem 4.1 Let z^* be feasible for MPCC (1.1) such that MPCC-LICQ holds at z^*. Then there exists a $\bar{t} > 0$ and a neighborhood $U(z^*)$ of z^* such that standard GCQ holds for $R_{MPCC}(t)$ (1.5) for $\forall\ t \in (0, \bar{t}]$ and $\forall\ z \in U(z^*) \cap S(t)$.

Proof. Since MPCC-LICQ holds at z^*, the gradients

$$\begin{aligned}&\{\nabla g_i(z) \mid i \in I_g(z^*)\} \cup \{\nabla h_i(z) \mid i \in I_e\} \cup \{\nabla G_i(z) \mid i \in I_{0+}(z^*) \cup I_{00}(z^*)\}\\ &\cup\{\nabla H_i(z) \mid i \in I_{+0}(z^*) \cup I_{00}(z^*)\}\end{aligned} \tag{4.2}$$

are linearly independent at z^*. Since g_i, h_i, G_i and H_i are continuously differentiable, the gradients (4.2) remain linearly independent in some neighborhood of z^*. Hence, there exists a $\bar{t} > 0$ and sufficiently small neighborhood $U(z^*)$ of z^* such that for all $t \in (0, \bar{t}]$ and all $z \in U(z^*) \cap S(t)$, the gradients (4.2) are linearly independent at z, and the following inclusions hold from (3.4)

$$\begin{aligned}&I_g(z) \subseteq I_g(z^*), \quad I_G(z;t) \subseteq I_{00}(z^*) \cup I_{0+}(z^*), \quad I_H(z;t) \subseteq I_{00}(z^*) \cup I_{+0}(z^*),\\ &I_\Phi^{00}(z;t) \cup I_\Phi^{0+}(z;t) \subseteq I_{00}(z^*) \cup I_{0+}(z^*), \quad I_\Phi^{00}(z;t) \cup I_\Phi^{+0}(z;t) \subseteq I_{00}(z^*) \cup I_{+0}(z^*).\end{aligned} \tag{4.3}$$

For any $t \in (0, \bar{t}]$ and $\tilde{z} \in U(z^*) \cap S(t)$, we have $\tilde{z} \in S(t, J)$ for any $J \subseteq I_\Phi^{00}(\tilde{z}; t)$, and the active gradients of $AP(t, J)$ (4.1) are

$$\begin{aligned}&\{\nabla g_i(\tilde{z}) \mid i \in I_g(\tilde{z})\} \cup \{\nabla h_i(\tilde{z}) \mid i \in I_e\} \cup \{\nabla G_i(\tilde{z}) \mid i \in I_G(\tilde{z};t) \cup I_\Phi^{0+}(\tilde{z};t) \cup J\}\\ &\cup\{\nabla H_i(\tilde{z}) \mid i \in I_H(\tilde{z};t) \cup I_\Phi^{+0}(\tilde{z};t) \cup \overline{J}\}.\end{aligned}$$

Thus, the standard LICQ for $AP(t, J)$ (4.1) holds at $\tilde{z}$. Since LICQ implies ACQ, we have

$$\mathcal{T}_{S(t,J)}(\tilde{z}) = \mathcal{L}_{S(t,J)}(\tilde{z}), \quad \forall\ J \subseteq I_\Phi^{00}(\tilde{z};t),$$

which together with Lemma 4.1 yields

$$\mathcal{T}_{S(t)}(\tilde{z}) = \bigcup_{J \subseteq I_\Phi^{00}(\tilde{z};t)} \mathcal{T}_{S(t,J)}(\tilde{z}) = \bigcup_{J \subseteq I_\Phi^{00}(\tilde{z};t)} \mathcal{L}_{S(t,J)}(\tilde{z}).$$

From Theorem 3.1.9 in [26], we obtain

$$\mathcal{T}_{S(t)}(\tilde{z})^\circ = \bigcap_{J \subseteq I_\Phi^{00}(\tilde{z};t)} \mathcal{L}_{S(t,J)}(\tilde{z})^\circ. \tag{4.4}$$

To prove that GCQ for $R_{MPCC}(t)$ (1.5) holds at $\widetilde{z}$, we need to show that $\mathcal{L}_{S(t)}(\widetilde{z})^\circ = \mathcal{T}_{S(t)}(\widetilde{z})^\circ$. Note that $\mathcal{L}_{S(t)}(\widetilde{z})^\circ \subseteq \mathcal{T}_{S(t)}(\widetilde{z})^\circ$ since $\mathcal{T}_{S(t)}(\widetilde{z}) \subseteq \mathcal{L}_{S(t)}(\widetilde{z})$, we only prove the inclusion

$$\mathcal{T}_{S(t)}(\widetilde{z})^\circ \subseteq \mathcal{L}_{S(t)}(\widetilde{z})^\circ.$$

The linearized tangent cone of $AP(t,J)$ (4.1) at $\widetilde{z}$ is given by

$$\begin{aligned}\mathcal{L}_{S(t,J)}(\widetilde{z}) = \quad & \{p \in \mathbb{R}^n \mid \nabla g_i(\widetilde{z})^T p \leq 0,\ i \in I_g(\widetilde{z}),\\ & \qquad \nabla h_i(\widetilde{z})^T p = 0,\ i \in I_e,\\ & \qquad \nabla G_i(\widetilde{z})^T p \geq 0,\ i \in I_G(\widetilde{z};t),\\ & \qquad \nabla H_i(\widetilde{z})^T p \geq 0,\ i \in I_H(\widetilde{z};t),\\ & \qquad \nabla G_i(\widetilde{z})^T p \leq 0,\ i \in I_\Phi^{0+}(\widetilde{z};t) \cup J,\\ & \qquad \nabla H_i(\widetilde{z})^T p \leq 0,\ i \in I_\Phi^{+0}(\widetilde{z};t) \cup \overline{J}\},\end{aligned}$$

so it follows from Lemma 4.2 that

$$\begin{aligned}\mathcal{L}_{S(t,J)}(\widetilde{z})^\circ = \Bigg\{ q \in \mathbb{R}^n \mid q = & \sum_{i\in I_g(\widetilde{z})} \alpha_i \nabla g_i(\widetilde{z}) + \sum_{i\in I_e} \beta_i \nabla h_i(\widetilde{z}) - \sum_{i\in I_G(\widetilde{z};t)} \gamma_i \nabla G_i(\widetilde{z})\\ & - \sum_{i\in I_H(\widetilde{z};t)} \delta_i \nabla H_i(\widetilde{z}) + \sum_{i\in I_\Phi^{0+}(\widetilde{z};t)\cup J} \nu_i \nabla G_i(\widetilde{z})\\ & + \sum_{i\in I_\Phi^{+0}(\widetilde{z};t)\cup \overline{J}} \sigma_i \nabla H_i(\widetilde{z}),\ \alpha,\ \gamma,\ \delta,\ \nu,\ \sigma \geq 0 \Bigg\}.\end{aligned} \tag{4.5}$$

Now for $q \in \mathcal{T}_{S(t)}(\widetilde{z})^\circ$, one obtains from (4.4) that $q \in \mathcal{L}_{S(t,J)}(\widetilde{z})^\circ$ for any $J \subseteq I_\Phi^{00}(\widetilde{z};t)$. So it follows from (4.5) that

$$\begin{aligned} q = & \sum_{i\in I_g(\widetilde{z})} \alpha_i \nabla g_i(\widetilde{z}) + \sum_{i\in I_e} \beta_i \nabla h_i(\widetilde{z}) - \sum_{i\in I_G(\widetilde{z};t)} \gamma_i \nabla G_i(\widetilde{z}) - \sum_{i\in I_H(\widetilde{z};t)} \delta_i \nabla H_i(\widetilde{z})\\ & + \sum_{i\in I_\Phi^{0+}(\widetilde{z};t)\cup J} \nu_i \nabla G_i(\widetilde{z}) + \sum_{i\in I_\Phi^{+0}(\widetilde{z};t)\cup \overline{J}} \sigma_i \nabla H_i(\widetilde{z}),\end{aligned} \tag{4.6}$$

where $\alpha_i,\ \gamma_i,\ \delta_i,\ \nu_i,\ \sigma_i \geq 0$.

On the other hand, in view of $\overline{J} \subseteq I_\Phi^{00}(\widetilde{z};t)$, we have from (4.4) that $q \in \mathcal{L}_{S(t,\overline{J})}(\widetilde{z})^\circ$, thus it follows that

$$\begin{aligned} q \ = & \sum_{i\in I_g(\widetilde{z})} \overline{\alpha}_i \nabla g_i(\widetilde{z}) + \sum_{i\in I_e} \overline{\beta}_i \nabla h_i(\widetilde{z}) - \sum_{i\in I_G(\widetilde{z};t)} \overline{\gamma}_i \nabla G_i(\widetilde{z}) - \sum_{i\in I_H(\widetilde{z};t)} \overline{\delta}_i \nabla H_i(\widetilde{z})\\ & + \sum_{i\in I_\Phi^{0+}(\widetilde{z};t)\cup \overline{J}} \overline{\nu_i} \nabla G_i(\widetilde{z}) + \sum_{i\in I_\Phi^{+0}(\widetilde{z};t)\cup J} \overline{\sigma}_i \nabla H_i(\widetilde{z}),\end{aligned} \tag{4.7}$$

where $\overline{\alpha}_i,\ \overline{\gamma}_i,\ \overline{\delta}_i,\ \overline{\nu}_i,\ \overline{\sigma}_i \geq 0$.

Note that the gradients

$$\begin{aligned}&\{\nabla g_i(\widetilde{z}) \mid i \in I_g(\widetilde{z})\} \cup \{\nabla h_i(\widetilde{z}) \mid i \in I_e\} \cup \{\nabla G_i(\widetilde{z}) \mid i \in I_G(\widetilde{z};t) \cup I_\Phi^{0+}(\widetilde{z};t) \cup I_\Phi^{00}(\widetilde{z};t)\}\\ &\cup \{\nabla H_i(\widetilde{z}) \mid i \in I_H(\widetilde{z};t) \cup I_\Phi^{+0}(\widetilde{z};t) \cup I_\Phi^{00}(\widetilde{z};t)\}\end{aligned}$$

are linearly independent, hence, the corresponding coefficients in (4.6) and (4.7) must be equal. In particular, we obtain

$$\nu_i = 0,\ i \in J; \qquad \sigma_i = 0,\ i \in \overline{J}.$$

Further, we obtain

$$\begin{aligned} q = & \sum_{i\in I_g(\widetilde{z})} \alpha_i \nabla g_i(\widetilde{z}) + \sum_{i\in I_e} \beta_i \nabla h_i(\widetilde{z}) - \sum_{i\in I_G(\widetilde{z};t)} \gamma_i \nabla G_i(\widetilde{z}) - \sum_{i\in I_H(\widetilde{z};t)} \delta_i \nabla H_i(\widetilde{z}) \\ & + \sum_{i\in I_\Phi^{0+}(\widetilde{z};t)} \nu_i \nabla G_i(\widetilde{z}) + \sum_{i\in I_\Phi^{+0}(\widetilde{z};t)} \sigma_i \nabla H_i(\widetilde{z}). \end{aligned} \tag{4.8}$$

Note that the linearized cone of $R_{MPCC}(t)$ (1.5) is given as follows:

$$\begin{aligned} \mathcal{L}_{S(t)}(\widetilde{z}) = \ & \{p \in \mathbb{R}^n \mid \nabla g_i(\widetilde{z})^T p \le 0,\ i \in I_g(\widetilde{z}), \\ & \nabla h_i(\widetilde{z})^T p = 0,\ i \in I_e, \\ & \nabla G_i(\widetilde{z})^T p \ge 0,\ i \in I_G(\widetilde{z};t), \\ & \nabla H_i(\widetilde{z})^T p \ge 0,\ i \in I_H(\widetilde{z};t), \\ & \nabla \Phi_i(\widetilde{z};t)^T p \le 0,\ i \in I_\Phi(\widetilde{z};t)\}. \end{aligned}$$

In view of $\nabla\Phi_i(\widetilde{z};t) = 0,\ i \in I_\Phi^{00}(\widetilde{z};t)$ and $I_\Phi(\widetilde{z};t) = I_\Phi^{0+}(\widetilde{z};t) \cup I_\Phi^{00}(\widetilde{z};t) \cup I_\Phi^{+0}(\widetilde{z};t)$, $I_\Phi^{0+}(\widetilde{z};t) \cap I_\Phi^{00}(\widetilde{z};t) \cap I_\Phi^{+0}(\widetilde{z};t) = \emptyset$, the representation above can be rewritten as follows:

$$\begin{aligned} \mathcal{L}_{S(t)}(\widetilde{z}) = \ & \{p \in \mathbb{R}^n \mid \nabla g_i(\widetilde{z})^T p \le 0,\ i \in I_g(\widetilde{z}), \\ & \nabla h_i(\widetilde{z})^T p = 0,\ i \in I_e, \\ & \nabla G_i(\widetilde{z})^T p \ge 0,\ i \in I_G(\widetilde{z};t), \\ & \nabla H_i(\widetilde{z})^T p \ge 0,\ i \in I_H(\widetilde{z};t), \\ & \nabla G_i(\widetilde{z})^T p \le 0,\ i \in I_\Phi^{0+}(\widetilde{z};t), \\ & \nabla H_i(\widetilde{z})^T p \le 0,\ i \in I_\Phi^{+0}(\widetilde{z};t)\}. \end{aligned}$$

By Lemma 4.3 and (4.8), one obtains $q \in \mathcal{L}_{S(t)}(\widetilde{z})^\circ$. The arbitrariness of q implies $\mathcal{T}_{S(t)}(\widetilde{z})^\circ \subseteq \mathcal{L}_{S(t)}(\widetilde{z})^\circ$. The proof is finished. □

The following result shows that stronger CQ for $R_{MPCC}(t)$ (1.5) holds at all points where $I_\Phi^{00}(z;t) = \emptyset$ holds.

Theorem 4.2 Let z^* be feasible for the MPCC (1.1) such that MPCC-CPLD (MPCC-LICQ) holds at z^*. Then there exists a $\bar{t} > 0$ and a neighborhood $U(z^*)$ of z^* such that the following statement holds for all $t \in (0, \bar{t}]$: If $z \in U(z^*) \cap S(t)$ with $I_\Phi^{00}(z;t) = \emptyset$, then standard CPLD (LICQ) for $R_{MPCC}(t)$ (1.5) holds at z.

Proof. We first prove the conclusion for MPCC-CPLD. Suppose, by contradiction, that there were a sequence $\{t^k\} \downarrow 0$ and $z^k \to z^*$ with z^k feasible for $R_{MPCC}(t^k)$ (1.5), and $I_\Phi^{00}(z^k;t_k) = \emptyset$ for all $k \in \{1, 2, \ldots\}$ such that standard CPLD is not satisfied in z^k for all $k \in \{1, 2, \ldots\}$. Thus there exist index subsets

$$I_1^k \subseteq I_g(z^k),\ I_2^k \subseteq I_e,\ I_3^k \subseteq I_G(z^k;t_k),\ I_4^k \subseteq I_H(z^k;t_k),\ I_5^k \subseteq I_\Phi^{0+}(z^k;t_k),\ I_6^k \subseteq I_\Phi^{+0}(z^k;t_k)$$

such that the gradients

$$\big\{\{\nabla g_i(z) \mid i \in I_1^k\} \cup \{-\nabla G_i(z) \mid i \in I_3^k\} \cup \{-\nabla H_i(z) \mid i \in I_4^k\} \cup \{(H_i(z) - t_k)\nabla G_i(z) \mid i \in I_5^k\}$$
$$\cup\{(G_i(z) - t_k)\nabla H_i(z) \mid i \in I_6^k\}\big\} \cup \{\nabla h_i(z) \mid i \in I_2^k\}$$

are positive-linearly dependent at z^k, but linearly independent at points arbitrary close to z^k. Since $I_g(z^k)$, I_e, $I_G(z^k; t_k)$, $I_H(z^k; t_k)$, $I_\Phi^{0+}(z^k; t_k)$, $I_\Phi^{+0}(z^k; t_k)$ are all finite sets, without loss of generality, we may assume $I_i^k \equiv I_i$ $(i = 1, 2, \ldots 6)$. Note that $I_g(z^k) \subseteq I_g(z^*)$ for all k sufficiently large, thus $I_1 \subseteq I_g(z^*)$. Similarly, we obtain $I_3 \cup I_5 \subseteq I_{00}(z^*) \cup I_{0+}(z^*)$, $I_4 \cup I_6 \subseteq I_{00}(z^*) \cup I_{+0}(z^*)$. Positive-linearly dependence at z^k implies positive-linearly dependence of the gradients

$$\{\nabla g_i(z^k) \mid i \in I_1\} \cup \Big\{\{\nabla h_i(z^k) \mid i \in I_2\} \cup \{\nabla G_i(z^k) \mid i \in I_3 \cup I_5\} \cup \{\nabla H_i(z^k) \mid i \in I_4 \cup I_6\}\Big\}. \tag{4.9}$$

Because the standard CPLD is not satisfied, there exists a sequence $\{y^k\} \to z^*$ such that the gradients (4.9) are linearly independent at y^k. If the gradients (4.9) were positive-linearly independent at z^*, then from Theorem 2.2 in [27] we know that these gradients are also positive-linearly independent at any point close to z^*. This is a contradiction. If the gradients (4.9) were positive-linearly dependent at z^*, MPCC-CPLD would imply that they remain linearly dependent in some neighborhood of z^*, which contradicts the statement the gradients (4.9) are linearly independent at y^k. Hence, CPLD holds at z.

Next we prove the conclusion for MPCC-LICQ. For all $z \in U(z^*) \cap S(t)$ and $t \in (0, \bar{t})$ sufficiently small, we have the following relations:

$$\begin{aligned}
&I_g(z) \subseteq I_g(z^*),\\
&I_G(z;t) \cup I_\Phi^{00}(z;t) \cup I_\Phi^{0+}(z;t) \subseteq I_{00}(z^*) \cup I_{0+}(z^*),\\
&I_H(z;t) \cup I_\Phi^{00}(z;t) \cup I_\Phi^{+0}(z;t) \subseteq I_{00}(z^*) \cup I_{+0}(z^*),\\
&I_G(z;t) \cap \big(I_\Phi^{00}(z;t) \cup I_\Phi^{0+}(z;t)\big) = \emptyset,\\
&I_H(z;t) \cap \big(I_\Phi^{00}(z;t) \cup I_\Phi^{+0}(z;t)\big) = \emptyset.
\end{aligned} \tag{4.10}$$

In view of MPCC-LICQ assumption and (4.10), for any $z \in U(z^*)$, the gradients

$$\begin{aligned}
&\{\nabla g_i(z) \mid i \in I_g(z)\} \cup \{\nabla h_i(z) \mid i \in I_e\} \cup \{\nabla G_i(z) \mid i \in I_G(z;t) \cup I_\Phi^{0+}(z;t)\}\\
&\cup\{\nabla H_i(z) \mid i \in I_H(z;t) \cup I_\Phi^{+0}(z;t)\}
\end{aligned} \tag{4.11}$$

are linearly independent.

The active gradients of $R_{MPCC}(t)$ (1.5) at feasible point $z \in U(z^*)$ are

$$\begin{aligned}
&\nabla g_i(z),\ i \in I_g(z),\\
&\nabla h_i(z),\ i \in I_e,\\
&\nabla G_i(z),\ i \in I_G(z;t),\\
&\nabla H_i(z),\ i \in I_H(z;t),\\
&\nabla \Phi_i(z;t) = \begin{cases} 2(H_i(z) - t)\nabla G_i(z), & i \in I_\Phi^{0+}(z;t),\\ 2(G_i(z) - t)\nabla H_i(z), & i \in I_\Phi^{+0}(z;t).\end{cases}
\end{aligned} \tag{4.12}$$

From (4.11), we know that the following equality

$$\begin{aligned}
& \sum_{i\in I_g(z)} \alpha_i \nabla g_i(z) + \sum_{i\in I_e} \beta_i \nabla h_i(z) - \sum_{i\in I_G(z;t)} \gamma_i \nabla G_i(z) - \sum_{i\in I_H(z;t)} \delta_i \nabla H_i(z) + \sum_{i\in I_\Phi(z;t)} \nu_i \nabla \Phi_i(z;t) \\
= & \sum_{i\in I_g(z)} \alpha_i \nabla g_i(z) + \sum_{i\in I_e} \beta_i \nabla h_i(z) - \sum_{i\in I_G(z;t)} \gamma_i \nabla G_i(z) - \sum_{i\in I_H(z;t)} \delta_i \nabla H_i(z) \\
& + \sum_{i\in I_\Phi^{0+}(z;t)} \nu_i [2(H_i(z)-t)] \nabla G_i(z;t) + \sum_{i\in I_\Phi^{+0}(z;t)} \nu_i [2(G_i(z)-t)] \nabla H_i(z;t) \\
= & \; 0
\end{aligned} \tag{4.13}$$

implies that

$$\begin{aligned}
& \alpha_i = 0,\ i\in I_g(z);\ \beta_i = 0,\ i \in I_e;\ \gamma_i = 0,\ i\in I_G(z;t);\ \delta_i = 0,\ i\in I_H(z;t); \\
& \nu_i[2(H_i(z)-t)] = 0,\ i\in I_\Phi^{0+}(z;t);\quad \nu_i[2(G_i(z)-t)] = 0,\ i\in i\in I_\Phi^{+0}(z;t).
\end{aligned}$$

Note that $H_i(z) - t > 0$ for $i \in I_\Phi^{0+}(z;t)$, so $\nu_i = 0$ for $i \in I_\Phi^{0+}(z;t)$.

Similarly, we have $\nu_i = 0$ for $i \in I_\Phi^{+0}(z;t)$.

Summing up the above discussion, we can conclude that standard LICQ holds at $z \in U(z^*) \cap S(t)$ for $R_{MPCC}(t)$ (1.5). □

The following result shows that the existence of Lagrange multipliers in a local minimum of $R_{MPCC}(t)$ (1.5) can be guaranteed, which is a direct consequence of Theorem 4.1.

Theorem 4.3 Let z^* be feasible for MPCC (1.1) such that MPCC-LICQ holds at z^*. Then there is a $\bar{t} > 0$ and a neighborhood $U(z^*)$ of z^* such that for all $t \in (0, \bar{t}]$: If $z \in U(z^*)$ is a local minimizer of feasible point for $R_{MPCC}(t)$ (1.5), then there exists Lagrange multipliers such that z together with Lagrange multiplier vector w is a KKT point of $R_{MPCC}(t)$ (1.5).

5. Concluding remarks

In this paper, based on Mangasarian complementarity function, a new relaxed method for mathematical program with complementarity constraints is proposed. Under MPCC-CPLD, any limit point of a sequence of stationary points of a sequence of relaxed problems is M-stationary for MPCC (1.1), and it is strongly stationary under additional conditions which is easily to be checked. Moreover, we further analyze the existence of the Lagrange multipliers for relaxed problems. The existence of the Lagrange multipliers can be guaranteed under MPCC-LICQ.

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Some fixed point results of generalized Lipschitz mappings on cone b-metric spaces over Banach algebras

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Abstract: In this paper, we introduce the concept of cone b-metric space over Banach algebra and obtain some fixed point theorems for generalized Lipschitz mappings in such settings without the assumption of normality. Moveover, we obtain some periodic properties of the fixed points. In addition, we give two examples to illustrate our assertions and show our results are never equivalent with the counterparts in b-metric versions.

MSC: 47H10; 54H25

Keywords: cone b-metric space over Banach algebra, generalized Lipschitz condition, P property, c-sequence

1 Introduction

In 2007, Huang and Zhang [1] defined cone metric spaces with a different view in respect to previous works. They substituted a normed space instead of the real line, but went further, defining convergent and Cauchy sequences in terms of interior points of the underlying cone. Moreover, they obtained some fixed point theorems in cone metric spaces. Afterwards, some scholars focused on the investigation of fixed point theorems in such spaces. According to incomplete statistics, more than six hundred papers dealing with cone metric spaces have been published so far (see [9]). But now it is not popular since

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some researchers constructed several mappings from cone metric spaces to metric spaces, and found some fixed point results in cone metric spaces could been directly obtained from the corresponding metric versions (see [3-10]). This makes it become meaningless to study fixed point theorems in cone metric spaces. However, the current situation changed, since, very recently, Liu and Xu [18] introduced cone metric space over Banach algebra and defined generalized Lipschitz mapping where the contractive coefficient is vector instead of usual real constant. They proved the existence of fixed points in such settings under the conditions that the underlying cones are normal cones. Furthermore, they gave an example to explain that the fixed point theorems in cone metric spaces over Banach algebras are not equivalent to those in metric spaces. Subsequently, Xu and Radenović [20] omitted the normality of cones by using c-sequences. Starting with the similar approach of [18], several papers have appeared (see [20-25]).

The main purpose of this article is to introduce a concept called cone b-metric space over Banach algebra, which is a great improvement since it expands the concept of cone metric space over Banach algebra. We present some fixed point theorems in such frameworks without the assumption of normal cones. Moreover, we obtain the P properties of the mappings. Further, by two examples, we support our results and establish the non-equivalence of fixed point results between cone b-metric spaces over Banach algebras and b-metric spaces.

We need the following definitions and results, consistent with [18], in the sequel.

Let $\mathbb{A}$ be a real Banach algebra, $\|\cdot\|$ be its norm and θ be its zero element. A nonempty closed subset K of $\mathbb{A}$ is called a cone if $K+K \subset K$, $K^2 = K \cap K \subset K$, $K \cap (-K) = \{\theta\}$ and $\lambda K \subset K$ for all $\lambda \geq 0$. We denote int$K$ as the interior of K. If int$K \neq \emptyset$, then K is said to be a solid cone. Define a partial ordering $\preceq$ with respect to K by $u \preceq v$ iff $v - u \in K$. Write $u \prec v$ iff $v - u \in K$ and $u \neq v$. Define $u \ll v$ iff $v - u \in \text{int}K$. The cone K is called normal if there is a real number $M > 0$ such that for all $u, v \in \mathbb{A}$, $\theta \preceq u \preceq v$ implies $\|u\| \leq M\|v\|$. The least positive number satisfying above is called the normal constant of K.

In the sequel, unless otherwise specified, we always suppose that $\mathbb{A}$ is a real Banach algebra with a unit e, K is a solid cone in $\mathbb{A}$, and $\preceq$, $\prec$ and $\ll$ are partial orderings with respect to K.

Definition 1.1([18]) Let X be a nonempty set and $\mathbb{A}$ be a Banach algebra. Suppose

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that a mapping $d : X \times X \to \mathbb{A}$ satisfies for all $x, y, z \in X$,

(c1) $\theta \preceq d(x, y)$ and $d(x, y) = \theta$ iff $x = y$;

(c2) $d(x, y) = d(y, x)$;

(c3) $d(x, z) \preceq d(x, y) + d(y, z)$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space over Banach algebra.

Inspired by Definition 1.1 and [12, Definition 2.1], we introduce the notion of cone b-metric space over Banach algebra.

Definition 1.2 Let X be a nonempty set, $s \geq 1$ be a constant and $\mathbb{A}$ be a Banach algebra. Suppose that a mapping $d : X \times X \to \mathbb{A}$ satisfies for all $x, y, z \in X$,

(d1) $\theta \preceq d(x, y)$ and $d(x, y) = \theta$ iff $x = y$;

(d2) $d(x, y) = d(y, x)$;

(d3) $d(x, z) \preceq s[d(x, y) + d(y, z)]$.

Then d is called a cone b-metric on X, and (X, d) is called a cone b-metric space over Banach algebra.

Remark 1.3 A cone metric space over Banach algebra must be a cone b-metric space over Banach algebra. Conversely, it is not true. As a result, the concept of cone b-metric space over Banach algebra greatly generalizes the concept of cone metric space over Banach algebra.

We shall give some examples in an attempt to illustrate that it is an interesting increase from cone b-metric space over Banach algebra to cone metric space over Banach algebra, since there exist a lot of cone b-metric spaces over Banach algebras which are not cone metric spaces over Banach algebras.

Example 1.4 Let $\mathbb{A} = C[0, 1]$ be the usual Banach space with the supremum norm. Define multiplication in the usual way: $(xy)(t) = x(t)y(t)$. Then $\mathbb{A}$ is a Banach algebra with a unit 1. Put $K = \{x \in \mathbb{A} : x(t) \geq 0, t \in [0, 1]\}$ and $X = \mathbb{R}$. Define a mapping $d : X \times X \to \mathbb{A}$ by $d(x, y)(t) = |x - y|^p e^t$ for all $x, y \in X$, where $p > 1$ is a constant. This makes (X, d) into a cone b-metric space over Banach algebra with the coefficient $s = 2^{p-1}$, but it is not a cone metric space over Banach algebra.

Example 1.5 Let $X = l^p = \{x = (x_n)_{n\geq 1} : \sum\limits_{n=1}^{\infty} |x_n|^p < \infty\}$ $(0 < p < 1)$. Let $d : X \times X \to \mathbb{R}^+$,

$$d(x, y) = \Big(\sum_{n=1}^{\infty} |x_n - y_n|^p \Big)^{\frac{1}{p}},$$

where $x = (x_n)_{n\geq 1},\ y = (y_n)_{n\geq 1} \in l^p$. Then (X, d) is a b-metric space (see [11]). Put $\mathbb{A} = l^1 = \{a = (a_n)_{n\geq 1} : \sum\limits_{n=1}^{\infty} |a_n| < \infty\}$ with convolution as multiplication:

$$ab = (a_n)_{n\geq 1}(b_n)_{n\geq 1} = \Big(\sum_{i+j=n} a_i b_j \Big)_{n\geq 1}.$$

Then $\mathbb{A}$ is a Banach algebra with a unit $e = (1, 0, 0, \ldots)$. Let $K = \{a = (a_n)_{n\geq 1} \in \mathbb{A} : a_n \geq 0,$ for all $n \geq 1\}$, which is a normal cone in $\mathbb{A}$. Defining a mapping $\tilde{d} : X \times X \to \mathbb{A}$ by $\tilde{d}(x, y) = (\frac{d(x,y)}{2^n})_{n\geq 1}$, we conclude that $(X, \tilde{d})$ is a cone b-metric space over Banach algebra with the coefficient $s = 2^{\frac{1}{p}-1} > 1$, but it is not a cone metric space over Banach algebra.

Definition 1.6 Let (X, d) be a cone b-metric space over Banach algebra $\mathbb{A}$ and $\{x_n\}$ a sequence in X. We say that

(i) $\{x_n\}$ is a convergent sequence if, for every $c \in \mathbb{A}$ with $\theta \ll c$, there is an N such that $d(x_n, x) \ll c$ for all $n \geq N$. Ones write it by $x_n \to x$ $(n \to \infty)$;

(ii) $\{x_n\}$ is a Cauchy sequence if, for every $c \in \mathbb{A}$ with $\theta \ll c$, there is an N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$;

(iii) (X, d) is a complete cone b-metric space if every Cauchy sequence in X is convergent.

Definition 1.7([17]) Let K be a solid cone in a Banach space $\mathbb{A}$. A sequence $\{u_n\} \subset K$ is said to be a c-sequence if for each $c \gg \theta$ there exists a natural number N such that $u_n \ll c$ for all $n > N$.

Lemma 1.8([20]) Let K be a solid cone in a Banach algebra $\mathbb{A}$, $\{u_n\}$ and $\{v_n\}$ be two c-sequences in K. If $\alpha, \beta \in K$ are two arbitrarily given vectors, then $\{\alpha u_n + \beta v_n\}$ is a c-sequence.

Proof It is evident that $\{u_n + v_n\}$ is a c-sequence (see [17]). We only show that $\{\alpha u_n\}$ is a c-sequence. Indeed, without loss of generality, put $\theta \prec \alpha$. For any $c \gg \theta$, there is a

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$\delta > 0$ such that

$$U(c,\delta) = \{u \in \mathbb{A} : \|u - c\| < \delta\} \subset K.$$

Set $c_0 \gg \theta$ and $\|c_0\| < \frac{\delta}{\|\alpha\|}$. Notice that

$$\|(c - \alpha c_0) - c\| = \|\alpha c_0\| \leq \|\alpha\|\|c_0\| < \delta \Rightarrow c - \alpha c_0 \in U(c,\delta) \subset K,$$

which implies that $c - \alpha c_0 \in \text{int}K$, i.e., $\alpha c_0 \ll c$. Since $\{u_n\}$ is a c-sequence, then there exists N such that $u_n \ll c_0$ for all $n > N$, thus $\alpha u_n \ll c$ $(n > N)$.

Lemma 1.9([19]) Let $\mathbb{A}$ be a Banach algebra with a unit e, then the spectral radius $\rho(u)$ of $u \in \mathbb{A}$ holds

$$\rho(u) = \lim_{n\to\infty} \|u^n\|^{\frac{1}{n}} = \inf \|u^n\|^{\frac{1}{n}}.$$

Further, $e - u$ is invertible and $(e-u)^{-1} = \sum\limits_{i=0}^{\infty} u^i$ provided that $\rho(u) < 1$.

Lemma 1.10([19]) Let $\mathbb{A}$ be a Banach algebra with a unit e and $u, v \in \mathbb{A}$. If u commutes with v, then

$$\rho(u+v) \leq \rho(u) + \rho(v), \qquad \rho(uv) \leq \rho(u)\rho(v).$$

Lemma 1.11([20]) Let $\mathbb{A}$ be a Banach algebra with a unit e and let k be a vector in $\mathbb{A}$. If $\rho(k) < 1$, then

$$\rho((e-k)^{-1}) < \frac{1}{1-\rho(k)}.$$

The following properties are often used (in particular when dealing with cone b-metric spaces over Banach algebras in which the cones need not be normal)(see [2], [20]).

(p1) If $\theta \preceq u \ll c$ for each $c \in \text{int}K$, then $u = \theta$.

(p2) If $u \preceq v$ and $v \ll w$, then $u \ll w$.

(p3) If $u \in K$ and $\rho(u) < 1$, then $\|u^n\| \to 0$ $(n \to \infty)$.

(p4) If $u \preceq ku$, where $u, k \in K$ and $\rho(k) < 1$, then $u = \theta$.

(p5) If $c \in \text{int}K$ and $u_n \to \theta$ $(n \to \infty)$, then there exists N such that, for all $n > N$, one has $u_n \ll c$.

2 Main results

In this section, by deleting the assumption of normality of cones, we shall prove some fixed point theorems of generalized Lipschitz mappings in the setting of cone b-metric s-

paces over Banach algebras. We also obtain the P properties of the mappings. Otherwise, we present two examples to verify our results. Our examples indicate that cone b-metric spaces over Banach algebras are never equivalent to b-metric spaces in terms of the existence of the fixed points of the mappings involved.

Theorem 2.1 Let (X, d) be a complete cone b-metric space over Banach algebra $\mathbb{A}$ with the coefficient $s \geq 1$. Let K be a solid not necessarily normal cone of $\mathbb{A}$. Suppose $T : X \to X$ is a mapping and suppose that there exists $k \in K$ such that, for all $x, y \in X$, at least one of the following generalized Lipschitz conditions holds:

(i) $d(Tx, Ty) \preceq kd(x, y)$ and $\rho(k) < \frac{1}{s}$;

(ii) $d(Tx, Ty) \preceq k(d(Tx, x) + d(Ty, y))$ and $\rho(k) < \frac{1}{1+s}$;

(iii) $d(Tx, Ty) \preceq k(d(Tx, y) + d(Ty, x))$ and $\rho(k) < \frac{1}{s+s^2}$.

Then T has a unique fixed point in X.

Proof Fix $x_0 \in X$ and set $x_1 = Tx_0$ and $x_{n+1} = Tx_n = T^{n+1}x_0$. Then for all three cases (i)-(iii), we shall prove that

$$d(x_{n+1}, x_n) \preceq \lambda d(x_n, x_{n-1}), \tag{2.1}$$

where $\lambda \in K$ and $\rho(\lambda) < \frac{1}{s}$.

For the case (i), it ensures us that

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \preceq kd(x_n, x_{n-1}).$$

Let $\lambda = k$, (2.1) is clear.

For the case (ii), it is easy to see that

$$\begin{aligned} d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) &\preceq k(d(Tx_n, x_n) + d(Tx_{n-1}, x_{n-1})) \\ &= k(d(x_{n+1}, x_n) + d(x_n, x_{n-1})). \end{aligned} \tag{2.2}$$

As a consequence of $\rho(k) < \frac{1}{1+s} < 1$, it follows from Lemma 1.9 that $e - k$ is invertible. Hence by (2.2), we deduce that

$$d(x_{n+1}, x_n) \preceq (e - k)^{-1}kd(x_n, x_{n-1}).$$

By Lemma 1.10 and Lemma 1.11, we speculate that

$$\rho((e - k)^{-1}k) \leq \frac{\rho(k)}{1 - \rho(k)} < \frac{\frac{1}{1+s}}{1 - \frac{1}{1+s}} = \frac{1}{s}. \tag{2.3}$$

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Put $\lambda = (e-k)^{-1}k$, (2.1) is obvious.

For the case (iii), we have

$$\begin{aligned} d(x_{n+1},x_n) &= d(Tx_n,Tx_{n-1}) \preceq k(d(Tx_n,x_{n-1})+d(x_n,Tx_{n-1})) \\ &= k(d(x_{n+1},x_{n-1})+d(x_n,x_n)) = kd(x_{n+1},x_{n-1}) \\ &\preceq sk(d(x_{n+1},x_n)+d(x_n,x_{n-1})). \end{aligned} \tag{2.4}$$

On account of $\rho(k) < \frac{1}{s}$, it follows from Lemma 1.9 that $e-sk$ is invertible, then by (2.4), one has

$$d(x_{n+1},x_n) \preceq (e-sk)^{-1}skd(x_n,x_{n-1}).$$

Take advantage of Lemma 1.10 and Lemma 1.11, it establishes that

$$\begin{aligned} &\rho((e-sk)^{-1}sk) \leq \rho((e-sk)^{-1})\rho(sk) \\ \leq\ &\frac{\rho(sk)}{1-\rho(sk)} = \frac{s\rho(k)}{1-s\rho(k)} < \frac{\frac{s}{s+s^2}}{1-\frac{s}{s+s^2}} = \frac{1}{s}. \end{aligned} \tag{2.5}$$

Choose $\lambda = (e-sk)^{-1}sk$, (2.1) is valid.

Making full use of (2.1), we get

$$d(x_{n+1},x_n) \preceq \lambda d(x_n,x_{n-1}) \preceq \lambda^2 d(x_{n-1},x_{n-2}) \preceq \cdots \preceq \lambda^n d(x_1,x_0).$$

Note that $\rho(\lambda) < \frac{1}{s}$ implies $e-s\lambda$ is invertible and

$$(e-s\lambda)^{-1} = \sum_{i=0}^{\infty}(s\lambda)^i.$$

Hence, for any $m \geq 1,\ p \geq 1$ and $\lambda \in K$ with $\rho(\lambda) < \frac{1}{s}$, we have that

$$\begin{aligned} d(x_m,x_{m+p}) &\preceq s[d(x_m,x_{m+1})+d(x_{m+1},x_{m+p})] \\ &\preceq sd(x_m,x_{m+1})+s^2[d(x_{m+1},x_{m+2})+d(x_{m+2},x_{m+p})] \\ &\preceq sd(x_m,x_{m+1})+s^2d(x_{m+1},x_{m+2})+s^3d(x_{m+2},x_{m+3}) \\ &\quad +\cdots+s^{p-1}d(x_{m+p-2},x_{m+p-1})+s^{p-1}d(x_{m+p-1},x_{m+p}) \\ &\preceq s\lambda^m d(x_1,x_0)+s^2\lambda^{m+1}d(x_1,x_0)+s^3\lambda^{m+2}d(x_1,x_0) \\ &\quad +\cdots+s^{p-1}\lambda^{m+p-2}d(x_1,x_0)+s^p\lambda^{m+p-1}d(x_1,x_0) \\ &= s\lambda^m[e+s\lambda+s^2\lambda^2+\cdots+(s\lambda)^{p-1}]d(x_1,x_0) \\ &\preceq s\lambda^m(e-s\lambda)^{-1}d(x_1,x_0). \end{aligned}$$

Since $\rho(\lambda) < \frac{1}{s} \leq 1$ implies that $\|\lambda^m\| \to 0$ $(m \to \infty)$, further, $\{\lambda^m\}$ is a c-sequence. Thus we derive from Lemma 1.8 that $\{s\lambda^m(e-s\lambda)^{-1}d(x_1,x_0)\}$ is a c-sequence. This means $\{x_n\}$ is a Cauchy sequence in X. Since (X,d) is complete, there exists $x^* \in X$ such that $x_n \to x^*$ $(n \to \infty)$. Next, we shall show x^* is the fixed point. In order to complete it, we consider three cases as follows.

For the case (i), one has

$$d(x^*,Tx^*) \preceq s[d(x_{n+1},x^*) + d(Tx_n,Tx^*)] \preceq s[d(x_{n+1},x^*) + kd(x_n,x^*)].$$

In view of $x_n \to x^*$ $(n \to \infty)$, it follows that $\{d(x_{n+1},x^*)\}$ and $\{d(x_n,x^*)\}$ are c-sequences. So by Lemma 1.8 that $\{s[d(x_{n+1},x^*) + kd(x_n,x^*)]\}$ is also a c-sequence. We obtain $Tx^* = x^*$.

For the case (ii), it is not hard to verify that

$$\begin{aligned} d(x^*,Tx^*) &\preceq s[d(x_{n+1},x^*) + d(Tx_n,Tx^*)] \\ &\preceq sd(x_{n+1},x^*) + sk[d(x_n,x_{n+1}) + d(x^*,Tx^*)]. \end{aligned} \tag{2.6}$$

Note that $e - sk$ is invertible, then (2.6) implies that

$$d(x^*,Tx^*) \preceq s(e-sk)^{-1}[d(x_{n+1},x^*) + kd(x_n,x_{n+1})].$$

Because $\{x_n\}$ is a Cauchy and convergent sequence, it means $\{d(x_{n+1},x^*)\}$ and $\{d(x_n,x_{n+1})\}$ are c-sequences. Hence by Lemma 1.8 that $\{s(e-sk)^{-1}[d(x_{n+1},x^*)+kd(x_n,x_{n+1})]\}$ is also a c-sequence. We have $Tx^* = x^*$.

For the case (iii), it is evident that

$$\begin{aligned} d(x^*,Tx^*) &\preceq s[d(x_{n+1},x^*) + d(Tx_n,Tx^*)] \\ &\preceq sd(x_{n+1},x^*) + sk[d(x_n,Tx^*) + d(x^*,x_{n+1})] \\ &\preceq sd(x_{n+1},x^*) + s^2k[d(x_n,x^*) + d(x^*,Tx^*)] + skd(x^*,x_{n+1}). \end{aligned} \tag{2.7}$$

Now that $\rho(k) < \frac{1}{s+s^2} < \frac{1}{s^2}$ determines that $e - s^2k$ is invertible, then (2.7) leads to

$$d(x^*,Tx^*) \preceq s(e-s^2k)^{-1}[(e+k)d(x_{n+1},x^*) + skd(x_n,x^*)].$$

Since $\{d(x_n,x^*)\}$ is a c-sequence, then by Lemma 1.8, $\{s(e-s^2k)^{-1}[(e+k)d(x_{n+1},x^*) + skd(x_n,x^*)]\}$ is also a c-sequence. Accordingly, $Tx^* = x^*$.

Finally, we shall prove the fixed point is unique. To this end, we suppose for absurd that there exists another fixed point y^*. We need to show it for three cases.

For the case (i), it may be verified that

$$d(x^*,y^*) = d(Tx^*,Ty^*) \preceq kd(x^*,y^*).$$

Consequently, $y^* = x^*$.

For the case (ii), it is valid that

$$d(x^*,y^*) = d(Tx^*,Ty^*) \preceq k[d(x^*,Tx^*) + d(y^*,Ty^*)] = \theta.$$

That is, $y^* = x^*$.

For the case (iii), we arrive at

$$d(x^*,y^*) = d(Tx^*,Ty^*) \preceq k[d(x^*,Ty^*) + d(y^*,Tx^*)] = 2kd(x^*,y^*).$$

Because $\rho(k) < \frac{1}{s+s^2} \leq \frac{1}{2}$ leads to $\rho(2k) < 1$, we get $y^* = x^*$. Finally, the claim holds.

Remark 2.2 Theorem 2.1 generalizes [20, Theorem 3.1-3.3]. Indeed, take $s = 1$ in Theorem 2.1. Otherwise, Theorem 2.1 also generalizes [27, Corollary 3.8] from b-metric (or metric type) space to cone b-metric (or cone metric type) space over Banach algebra.

It is well-known that if x^* is a fixed point of a mapping T, then x^* is also a fixed point of T^n for each $n \in \mathbb{N}$. But the converse is not true. If a mapping T holds $F(T) = F(T^n)$ for each $n \in \mathbb{N}$, where $F(T)$ denotes the set of all fixed points of T, then ones call T has a P property (see [28-30]). The following results are generalizations of the corresponding results in metric and cone metric spaces (see [28-30]). It will be obtained also without using normality of cones.

Theorem 2.3 Let (X,d) be a cone b-metric space over Banach algebra $\mathbb{A}$ with the coefficient $s \geq 1$. Let K be a solid not necessarily normal cone of $\mathbb{A}$. Suppose $T : X \to X$ is a mapping such that $F(T) \neq \emptyset$ and that

$$d(Tx,T^2x) \preceq \mu d(x,Tx) \tag{2.8}$$

for all $x \in X$, where $\mu \in K$ is a generalized Lipschitz constant with $\rho(\mu) < 1$. Then T has a P property.

Proof We shall always assume that $n > 1$, since the statement for $n = 1$ is trivial. Let $z \in F(T^n)$. By the assumption, it is clear that

$$\begin{aligned} d(z, Tz) &= d(TT^{n-1}z, T^2T^{n-1}z) \preceq \mu d(T^{n-1}z, T^n z) = \mu d(TT^{n-2}z, T^2T^{n-2}z) \\ &\preceq \mu^2 d(T^{n-2}z, T^{n-1}z) \preceq \cdots \preceq \mu^n d(z, Tz). \end{aligned}$$

By virtue of $\rho(\mu) < 1$, it follows that $\|\mu^n\| \to 0$ $(n \to \infty)$. Accordingly, $\{\mu^n d(z, Tz)\}$ is a c-sequence. Then $d(z, Tz) = \theta$, i.e., $Tz = z$.

Theorem 2.4 Let (X, d) be a complete cone b-metric space over Banach algebra $\mathbb{A}$ with the coefficient $s \geq 1$. Let K be a solid not necessarily normal cone of $\mathbb{A}$. Suppose $T : X \to X$ is a mapping and suppose that there exists $k \in K$ such that, for all $x, y \in X$, at least one of the following generalized Lipschitz conditions holds:

(i) $d(Tx, Ty) \preceq kd(x, y)$ and $\rho(k) < \frac{1}{s}$;

(ii) $d(Tx, Ty) \preceq k(d(Tx, x) + d(Ty, y))$ and $\rho(k) < \frac{1}{1+s}$;

(iii) $d(Tx, Ty) \preceq k(d(Tx, y) + d(Ty, x))$ and $\rho(k) < \frac{1}{s+s^2}$.

Then T has a P property.

Proof Making full use of Theorem 2.1, we claim T has a unique fixed point. In order to utilize Theorem 2.3, we have to show (2.8). To this end, we divide it into three cases.

For the case (i), it follows that

$$d(Tx, T^2x) = d(Tx, TTx) \preceq kd(x, Tx).$$

Let $\mu = k$, (2.8) is valid.

For the case (ii), we have

$$d(Tx, T^2x) = d(Tx, TTx) \preceq k(d(x, Tx) + d(Tx, T^2x)),$$

which establishes that

$$d(Tx, T^2x) \preceq (e - k)^{-1}kd(x, Tx).$$

Owing to (2.3), $\rho((e - k)^{-1}k) < \frac{1}{s} \leq 1$, then let $\mu = (e - k)^{-1}k$, (2.8) is trivial.

For the case (iii), we have

$$\begin{aligned} d(Tx, T^2x) &= d(Tx, TTx) \preceq k(d(x, T^2x) + d(Tx, Tx)) \\ &= kd(x, T^2x) \preceq sk(d(x, Tx) + d(Tx, T^2x)), \end{aligned}$$

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which means that

$$d(Tx, T^2x) \preceq (e - sk)^{-1}skd(x, Tx).$$

In view of (2.5), $\rho((e - sk)^{-1}sk) < \frac{1}{s} \leq 1$, then let $\mu = (e - sk)^{-1}sk$, (2.8) is trivial.

Theorem 2.5 Let (X, d) be a complete cone b-metric space over Banach algebra $\mathbb{A}$ with the coefficient $s \geq 1$. Let K be a solid not necessarily normal cone of $\mathbb{A}$. Suppose $T : X \to X$ is a mapping and there exists $k \in K$ and $\rho(k) < \frac{1}{s}$ such that, for all $x, y \in X$, the following generalized Lipschitz condition holds:

$$d(Tx, Ty) \preceq k \cdot u(x, y),$$

where

$$u(x, y) \in \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty)}{2s}, \frac{d(y, Tx)}{2s} \right\}.$$

Then T has a unique fixed point in X. Moreover, T has a P property.

Proof If $u = d(x, y)$, then by Theorem 2.1(i) and Theorem 2.4(i), the proof is valid. We shall consider the other cases.

Fix $x_0 \in X$ and set $x_1 = Tx_0$ and $x_{n+1} = Tx_n = T^{n+1}x_0$. Then we have that

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \preceq k \cdot u(x_n, x_{n-1}),$$

where

$$\begin{aligned} u(x_n, x_{n-1}) &\in \left\{ d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1}), \frac{d(x_n, Tx_{n-1})}{2s}, \frac{d(x_{n-1}, Tx_n)}{2s} \right\} \\ &= \left\{ d(x_n, x_{n+1}), d(x_{n-1}, x_n), \theta, \frac{d(x_{n-1}, x_{n+1})}{2s} \right\} \end{aligned}$$

If $d(x_{n+1}, x_n) \preceq kd(x_n, x_{n+1})$ or $d(x_{n+1}, x_n) \preceq \theta$, then for all $n \in \mathbb{N}$, $d(x_{n+1}, x_n) = \theta$. That is, $Tx_n = x_{n+1} = x_n$ for all $n \in \mathbb{N}$, thus x_n is the fixed point. If $d(x_{n+1}, x_n) \preceq kd(x_n, x_{n-1})$, i.e., (2.1) holds if $\lambda = k$. If $d(x_{n+1}, x_n) \preceq k \cdot \frac{d(x_{n-1}, x_{n+1})}{2s}$, then

$$d(x_{n+1}, x_n) \preceq k \cdot \frac{d(x_{n-1}, x_{n+1})}{2s} \preceq k \cdot \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}. \tag{2.9}$$

Since $\rho(k) < \frac{1}{s}$ implies that $2e - k$ is invertible, then (2.9) leads to

$$d(x_{n+1}, x_n) \preceq (2e - k)^{-1}kd(x_n, x_{n-1}).$$

Note that

$$\begin{aligned}\rho((2e-k)^{-1}k) &= \frac{1}{2}\rho\Big((e-\frac{k}{2})^{-1}k\Big) \leq \frac{1}{2}\cdot\frac{\rho(k)}{1-\rho(\frac{k}{2})}\\ &= \frac{\rho(k)}{2-\rho(k)} < \frac{\frac{1}{s}}{2-\frac{1}{s}} = \frac{1}{2s-1} \leq \frac{1}{s}.\end{aligned}$$

Take $\lambda = (2e-k)^{-1}k$, hence (2.1) holds. Therefore, following an argument similar to that given in Theorem 2.1, we obtain that there exists $x^* \in X$ such that $x_n \to x^*$ $(n \to \infty)$.

In the following, we shall divide two cases to prove that x^* is the fixed point.

For the case that $u(x,y) = d(y,Ty)$, we have

$$\begin{aligned}d(x^*,Tx^*) &\preceq s[d(x^*,Tx_n) + d(Tx_n,Tx^*)]\\ &\preceq s[d(x^*,x_{n+1}) + kd(x^*,Tx^*)],\end{aligned}$$

which follows that

$$d(x^*,Tx^*) \preceq s(e-sk)^{-1}d(x^*,x_{n+1}).$$

Because $\{d(x_{n+1},x^*)\}$ is a c-sequence, then $x^* = Tx^*$.

For the case that $u(x,y) = \frac{d(y,Tx)}{2s}$, we arrive at

$$\begin{aligned}d(x^*,Tx^*) &\preceq s[d(x^*,Tx_n) + d(Tx_n,Tx^*)]\\ &\preceq s\Big[d(x^*,x_{n+1}) + k\cdot\frac{d(x^*,x_{n+1})}{2s}\Big]\\ &= \Big(se + \frac{1}{2}k\Big)d(x^*,x_{n+1}).\end{aligned}$$

Now that $\{d(x_{n+1},x^*)\}$ is a c-sequence, then $x^* = Tx^*$.

Next, we shall prove that the fixed point is unique. Assume there exists another fixed point y^*, then

$$d(x^*,y^*) = d(Tx^*,Ty^*) \preceq k\cdot u(x^*,y^*),$$

where

$$u(x^*,y^*) \in \Big\{d(x^*,Tx^*), d(y^*,Ty^*), \frac{d(x^*,Ty^*)}{2s}, \frac{d(y^*,Tx^*)}{2s}\Big\} = \Big\{\theta, \frac{d(x^*,y^*)}{2s}\Big\}.$$

It is not hard to verify that $x^* = y^*$.

Finally, we shall prove T has a P property. In order to end this, we have to show (2.8). We divide it into four cases.

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For the case that $u(x,y) = d(x,Tx)$, from

$$d(Tx,T^2x) = d(Tx,TTx) \preceq kd(x,Tx),$$

we have (2.8).

For the case that $u(x,y) = d(y,Ty)$, we get

$$d(Tx,T^2x) = d(Tx,TTx) \preceq kd(Tx,T^2x),$$

which means that $d(Tx,T^2x) = \theta$. Hence (2.8) is clear.

For the case that $u(x,y) = \frac{d(x,Ty)}{2s}$, we obtain

$$d(Tx,T^2x) = d(Tx,TTx) \preceq k \cdot \frac{d(x,T^2x)}{2s} \preceq \frac{k}{2}[d(x,Tx) + d(Tx,T^2x)],$$

which implies that $d(Tx,T^2x) \preceq (2e-k)^{-1}kd(x,Tx)$. So (2.8) is obvious.

For the case that $u(x,y) = \frac{d(y,Tx)}{2s}$, we obtain

$$d(Tx,T^2x) = d(Tx,TTx) \preceq k \cdot \frac{d(Tx,Tx)}{2s} = \theta,$$

which establishes that $d(Tx,T^2x) = \theta$. Thus (2.8) is obvious.

Therefore, by using Theorem 2.3, T has a P property.

In the following, we shall furnish two nontrivial examples to support our main results.

Example 2.6(the case of a non-normal cone) Let $\mathbb{A} = C^1_{\mathbb{R}}[0,1]$ and $\|u\| = \|u\|_\infty + \|u'\|_\infty$ be its norm. Define usual pointwise multiplication as its multiplication. Clearly, $\mathbb{A}$ is a Banach algebra with a unit $e = 1$. Put $K = \{u \in \mathbb{A} : u = u(t) \geq 0, t \in [0,1]\}$. Then K is a non-normal cone (see [2]). Set $X = \{a,b,c\}$ and define a mapping $d : X \times X \to \mathbb{A}$ by $d(a,b)(t) = d(b,a)(t) = e^t$, $d(b,c)(t) = d(c,b)(t) = 2e^t$, $d(a,c)(t) = d(c,a)(t) = 4e^t$, $d(a,a)(t) = d(b,b)(t) = d(c,c)(t) = 0$. One claims that (X,d) is a cone b-metric space over Banach algebra $\mathbb{A}$ with the coefficient $s = \frac{4}{3}$. But it is not a cone metric space over Banach algebra since it does not satisfy the triangle inequality. Now let a mapping $T : X \to X$ be $Ta = Tb = b$, $Tc = a$ and let $k \in K$ with $k(t) = \frac{1}{6}t + \frac{1}{2}$. It is not hard to verify that all conditions of Theorem 2.1 in the case of (i) hold. Therefore, $x^* = b$ is the unique fixed point. Clearly, T has a P property.

Example 2.7(the case of a normal cone) Let $\mathbb{A} = \mathbb{R}^2$ with the norm $\|(u_1, u_2)\| = |u_1| + |u_2|$ and the multiplication by

$$uv = (u_1, u_2)(v_1, v_2) = (u_1v_1, u_1v_2 + u_2v_1).$$

Put $K = \{u = (u_1, u_2) \in \mathbb{A} : u_1, u_2 \geq 0\}$. It is easy to see that K is a normal cone and $\mathbb{A}$ is a Banach algebra with a unit $e = (1, 0)$. Let $X = [0, 0.55] \times [-2, 2]$ and for all $x = (x_1, x_2), y = (y_1, y_2) \in X$, $d(x, y) = (|x_1 - y_1|^2, |x_2 - y_2|^2)$. We demonstrate that (X, d) is a complete cone b-metric space over Banach algebra $\mathbb{A}$ with the coefficient $s = 2$.

Define a mapping $T : X \to X$ as

$$Tx = T(x_1, x_2) = \Big(\frac{1}{2}\big(\cos\frac{x_1}{2} - |x_1 - \frac{1}{2}|\big), \arctan(2 + |x_2|) + \ln(x_1 + 1)\Big).$$

By using mean value theorem of differentials, we have

$$\begin{aligned}
d(Tx, Ty) &= d(T(x_1, x_2), T(y_1, y_2)) \\
&= \Big(\Big|\frac{1}{2}\big(\cos\frac{x_1}{2} - \cos\frac{y_1}{2} - |x_1 - \frac{1}{2}| + |y_1 - \frac{1}{2}|\big)\Big|^2, \\
&\qquad |\arctan(2 + |x_2|) - \arctan(2 + |y_2|) + \ln(x_1 + 1) - \ln(y_1 + 1)|^2\Big) \\
&\preceq \Big(\Big(|\frac{x_1 + y_1}{4}||\frac{x_1 - y_1}{4}| + \frac{1}{2}|x_1 - y_1|\Big)^2, \Big(\frac{1}{5}|x_2 - y_2| + |x_1 - y_1|\Big)^2\Big) \\
&\preceq \Big(\Big(\frac{|x_1 + y_1|}{16} + \frac{1}{2}\Big)^2|x_1 - y_1|^2, 2\Big(\frac{1}{25}|x_2 - y_2|^2 + |x_1 - y_1|^2\Big)\Big) \\
&\preceq \Big(\frac{1}{3}|x_1 - y_1|^2, \frac{2}{25}|x_2 - y_2|^2 + 2|x_1 - y_1|^2\Big) \\
&\preceq \Big(\frac{1}{3}, 2\Big)\big(|x_1 - y_1|^2, |x_2 - y_2|^2\big) \\
&= \Big(\frac{1}{3}, 2\Big)d(x, y)
\end{aligned}$$

for all $x, y \in X$. Denote $k = (\frac{1}{3}, 2)$. Careful calculations show that all conditions of Theorem 2.1 in the case of (i) hold. Thus T has a unique fixed point in X.

It is well-known that some results concerning fixed points and other results, in case of cone spaces with non-normal cones, cannot be provided by reducing to metric spaces (see [2]). In other words, if the underlying cones are non-normal, then some fixed point results in cone spaces are not equivalent to those of metric spaces. Otherwise, [3-10] appeal to the equivalence if the cones are normal cones. However, next, we shall claim our fixed point results in cone b-cone metric spaces over Banach algebras are never equivalent to the

counterparts in b-metric spaces even if the cones are normal cones. For this, we consider Example 2.7. Put

$$d_\xi(x,y) = \xi_e \circ d(x,y) = \inf\{r \in \mathbb{R} : d(x,y) \preceq re\}, \quad x, y \in X,$$

where $e = (e_1, e_2) \in \mathrm{int}K$, $\xi_e(y) = \inf\{r \in \mathbb{R} : y \in re - K\}$ $(y \in \mathbb{A})$. Then by Theorem 2.1 of [8], d_ξ is a b-metric. We shall prove our conclusions are not equivalent to the well-known Banach contraction principle in b-metric space. Indeed, taking $x' = (\frac{1}{2}, 0)$, $y' = (0,0)$ and $e = (1, \frac{1}{2})$, we have

$$\begin{aligned} d_\xi(Tx', Ty') &= \inf\Big\{r \in \mathbb{R} : \Big(\frac{1}{4}\Big(\cos\frac{1}{4} - \frac{1}{2}\Big)^2, \Big(\ln\frac{3}{2}\Big)^2\Big) \preceq r\Big(1, \frac{1}{2}\Big)\Big\} \\ &= \max\Big\{\Big(\frac{1}{4}\Big(\cos\frac{1}{4} - \frac{1}{2}\Big)^2, 2\Big(\ln\frac{3}{2}\Big)^2\Big)\Big\} = 2\Big(\ln\frac{3}{2}\Big)^2 \\ &\geq \frac{1}{4} = d_\xi(x', y'), \end{aligned}$$

which implies that there does not exist $\lambda \in [0,1)$ such that

$$d_\xi(Tx, Ty) \leq \lambda d_\xi(x,y)$$

for all $x, y \in X$. Thus it does not satisfy the contractive condition of Banach contraction principle in b-metric space. That is to say, the proof of [8, Theorem 2.6] will be unreasonable if under the setting of cone b-cone metric space over Banach algebra.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions

Both authors contribute equally and significantly in writing this paper. Both authors read and approve the final manuscript.

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SOME IDENTITIES OF BELL POLYNOMIALS

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Abstract. In this paper, we study some properties of Bell polynomials which are represented by the the linear combination of special polynomials. By using those properties, we give some new identities of Bell polynomials associated with special numbers and polynomials.

1. Introduction

The stirling number of the first kind is defined as

$$(x)_n = x(x-1)\cdots(x-n-1) = \sum_{l=0}^{n} s_1(n,l)x^l,\ (n \geq 0) \tag{1}$$

and the stirling number of the second kind is given by

$$x^n = \sum_{l=0}^{n} s_2(n,l)x^l,\ (n \geq 0) \qquad \text{(see [10, 13, 17])}. \tag{2}$$

It is known that the Bell polynomials are defined by the generating function to be

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!} \qquad \text{(see [4, 6, 16, 17, 18])}. \tag{3}$$

When $x = 1, Bel_n = Bel_n(1)$ are the Bell numbers. Note that $Bel_n(0) = \delta_{0,n}$, $(n \geq 0)$.

As is well known, the Bernoulli polynomials are defined by the generating function to be

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \qquad \text{(see [1, 7])}, \tag{4}$$

and the Euler polynomials are given by the generating function to be

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \qquad \text{(see [1 − 18])}. \tag{5}$$

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The Cauchy polynomials are given by

$$\frac{t}{log(t+1)}(1+t)^x = \sum_{n=0}^{\infty} C_n(x)\frac{t^n}{n!} \qquad \text{(see [6, 11])}, \tag{6}$$

and the Daehee polynomials are defined by the generating function to be

$$\frac{log(t+1)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!} \qquad \text{(see [9])}. \tag{7}$$

Finally, we introduce the Changhee polynomials which are given by the generating function to be

$$\frac{2}{t+2}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!} \qquad \text{(see [10])}. \tag{8}$$

Recently, several authors have studied these several special polynomials(see [1-18]).

In this paper, we study some properties of Bell polynomials which are represented by the the linear combination of special polynomials. By using those properties, we give some new identities of Bell polynomials associated with special numbers and polynomials.

2. Some identities of Bell polynomials

From (3), we easily derive the following equation:

$$Bel_n(x) = e^{-x}\sum_{l=0}^{\infty}\frac{x^l}{l!}l^n. \tag{9}$$

By replacing t by e^{e^t-1} in (4), we get

$$\begin{aligned} \frac{e^t-1}{e^{e^t-1}-1}e^{x(e^t-1)} &= \sum_{m=0}^{\infty} B_m(x)\frac{1}{m!}(e^t-1)^m \\ &= \sum_{m=0}^{\infty} B_m(x)\frac{m!}{m!}\sum_{n=m}^{\infty} s_2(n,m)\frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}\left\{\sum_{m=0}^{n} B_m(x)s_2(n,m)\right\}\frac{t^m}{m!}, \end{aligned} \tag{10}$$

and

$$\begin{aligned} \frac{e^t-1}{e^{e^t-1}-1}e^{x(e^t-1)} &= \left(\sum_{m=0}^{\infty} B_m\frac{1}{m!}(e^t-1)^m\right)\left(\sum_{l=0}^{\infty} Bel_l(x)\frac{t^l}{l!}\right) \\ &= \left(\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k} B_m s_2(k,m)\right)\frac{t^k}{k!}\right)\left(\sum_{l=0}^{\infty} Bel_l(x)\frac{t^l}{l!}\right) \\ &= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}\sum_{m=0}^{k} B_m s_2(k,m)Bel_{n-k}(x)\right\}\frac{t^n}{n!}. \end{aligned} \tag{11}$$

Therefore, by (10) and (11), we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} B_m(x)s_2(n,m) = \sum_{k=0}^{n}\binom{n}{k}\sum_{m=0}^{k} B_m s_2(k,m)Bel_{n-k}(x). \tag{12}$$

Let us take $e^t - 1$ instead of t in (5). Then we have

$$\begin{aligned}\frac{2}{e^{e^t-1}+1}e^{x(e^t-1)} &= \sum_{m=0}^{\infty} E_m(x)\frac{1}{m!}(e^t-1)^m \\ &= \sum_{m=0}^{\infty} E_m(x)\sum_{n=m}^{\infty} s_2(n,m)\frac{t^n}{n!} \\ &= \sum_{n=0}^{i} nfty\left(\sum_{m=0}^{n} s_2(n,m)E_m(x)\right)\frac{t^n}{n!}, \end{aligned} \tag{13}$$

and

$$\begin{aligned}\frac{e^{e^t-1}-1}{e^t-1}e^{x(e^t-1)} &= \left(\sum_{m=0}^{\infty} E_m\frac{(e^t-1)^m}{m!}\right)\left(\sum_{l=0} Bel_l(x)\frac{t^l}{l!}\right) \\ &= \left(\sum_{m=0}^{\infty} E_m\sum_{k=m}^{\infty} s_2(k,m)\frac{t^k}{k!}\right)\left(\sum_{l=0} Bel_l(x)\frac{t^l}{l!}\right) \\ &= \left(\sum_{k=0}^{\infty}\sum_{m=0}^{k} E_m s_2(k,m)\frac{t^k}{k!}\right)\left(\sum_{l=0} Bel_l(x)\frac{t^l}{l!}\right) \\ &= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{m}\binom{n}{k}\sum_{m=0}^{k} E_m s_2(k,m)Bel_{n-k}(x)\right\}\frac{t^n}{n!}. \end{aligned} \tag{14}$$

Therefore, by (13) and (14), we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} s_2(n,m)E_m(x) = \sum_{k=0}^{m}\binom{n}{k}\sum_{m=0}^{k} E_m s_2(k,m)Bel_{n-k}(x). \tag{15}$$

From (6), by replacing t by $e^{e^t-1} - 1$, we get

$$\begin{aligned}\frac{e^{e^t-1}-1}{e^t-1}e^{x(e^t-1)} &= \sum_{n=0}^{\infty} C_n(x)\frac{1}{n!}\left(e^{e^t-1}-1)^n\right) \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} C_n(x)s_2(m,n)\right)\frac{t^m}{m!} \end{aligned} \tag{16}$$

and

$$\begin{aligned}\frac{e^{e^t-1}-1}{e^t-1}e^{x(e^t-1)} &= \frac{e^{(x+1)(e^t-1)} - e^{x(e^t-1)}}{e^t-1} \\ &= \frac{1}{t}\left(\frac{t}{e^t-1}\right)\left(\sum_{m=1}^{\infty}\left\{Bel_m(x+1) - Bel_m(x)\right\}\frac{t^m}{m!}\right) \\ &= \left(\sum_{k=0}^{\infty} B_k\frac{t^k}{k!}\right)\left(\sum_{m=0}^{\infty}\left\{\frac{Bel_{m+1}(x+1) - Bel_{m+1}(x)}{m+1}\right\}\frac{t^m}{m!}\right) \end{aligned}$$

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$$= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\left(\frac{Bel_{m+1}(x+1)-Bel_{m+1}(x)}{m+1}\right)B_{n-m}\binom{n}{m}\right\}\frac{t^m}{m!}. \quad (17)$$

Therefore, by (16) and (17), we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} C_m(x)s_2(n,m) = \sum_{m=0}^{n}\binom{n}{m}B_{n-m}\left(\frac{Bel_{m+1}(x+1)-Bel_{m+1}(x)}{m+1}\right). \quad (18)$$

Let us take $e^{e^t-1}-1$ instead of t in (7). Then we have

$$\begin{aligned}\frac{e^t-1}{e^{e^t-1}-1}e^{x(e^t-1)} &= \sum_{n=0}^{\infty} D_n(x)\frac{1}{n!}(e^{e^t-1}-1)^n \\ &= \sum_{n=0}^{\infty} D_n(x)\frac{1}{n}n!\sum_{m=n}^{\infty} s_2(m,n)\frac{(e^t-1)^m}{m!} \\ &= \sum_{k=0}^{\infty}\left\{\sum_{m=0}^{k}\sum_{n=0}^{m} D_n(x)s_2(m,n)s_2(k,m)\right\}\frac{k!}{k!} \quad (19)\end{aligned}$$

and

$$\begin{aligned}\frac{e^t-1}{e^{e^t-1}-1}e^{x(e^t-1)} &= \left(\sum_{m=0}^{\infty} B_m\frac{1}{m!}(e^t-1)^m\right)\left(\sum_{l=0}^{\infty} Bel_l(x)\frac{t^l}{l!}\right) \\ &= \left(\sum_{m=0}^{\infty} B_m(x)\sum_{k=m}^{\infty} s_2(k,m)\frac{t^k}{k!}\right)\left(\sum_{l=0}^{\infty} Bel_l(x)\frac{t^l}{l!}\right) \\ &= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}\sum_{m=0}^{k} B_m s_2(k,m)Bel_{n-k}(x)\right\}\frac{t^n}{n!}. \quad (20)\end{aligned}$$

Therefore, by (19) and (20), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$\sum_{k=0}^{n}\sum_{m=0}^{k} D_m(x)s_2(k,m)s_2(n,k) = \sum_{k=0}^{n}\binom{n}{k}\sum_{m=0}^{k} B_m s_2(k,m)Bel_{n-k}(x). \quad (21)$$

By replacing t by $e^{e^t-1}-1$, we get

$$\begin{aligned}\frac{2}{e^{e^t-1}+1}e^{x(e^t-1)} &= \sum_{n=0}^{\infty} Ch_n(x)\frac{(e^{e^t-1}-1)^n}{n!} \\ &= \sum_{n=0}^{\infty} Ch_n(x)\sum_{m=n}^{\infty} s_2(m,n)\frac{(e^t-1)^m}{m!} \\ &= \sum_{m=0}^{\infty}\left\{\sum_{n=0}^{m} Ch_n(x)s_2(m,n)\right)\frac{1}{m!}(e^t-1)^m \\ &= \sum_{m=0}^{\infty}\left\{\sum_{n=0}^{m} Ch_n(x)s_2(m,n)\right)\sum_{k=m}^{\infty} s_2(k,m)\frac{t^k}{k!}\end{aligned}$$

$$= \sum_{k=0}^{\infty}\left\{\sum_{m=0}^{k}\sum_{n=0}^{m} Ch_n(x)s_2(m,n)s_2(k,m)\right)\frac{t^k}{k!} \tag{22}$$

and

$$\begin{aligned}\frac{2}{e^{e^t-1}+1}e^{x(e^t-1)} &= \left(\sum_{n=0}^{\infty} E_n\frac{(e^t-1)^n}{n!}\right)\left(\sum_{l=0}^{\infty} Bel_l(x)\frac{t^l}{l!}\right)\\ &= \left(\sum_{n=0}^{\infty} E_n(x)\frac{1}{n!}n!\sum_{k=n}^{\infty} s_2(k,n)\frac{t^k}{k!}\right)\left(\sum_{l=0}^{\infty} Bel_l(x)\frac{t^l}{l!}\right)\\ &= \left(\sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} E_n s_2(m,n)\right)\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty} Bel_l(x)\frac{t^l}{l!}\right)\\ &= \sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}\binom{k}{m}\sum_{n=0}^{m} E_n s_2(m,n)Bel_{k-m}(x)\right)\frac{t^k}{k!}. \end{aligned} \tag{23}$$

Therefore, by (22) and (23), we obtain the following theorem.

Theorem 2.5. *For $k \geq 0$, we have*

$$\sum_{m=0}^{k}\binom{k}{m}\sum_{n=0}^{m} E_n s_2(m,n)Bel_{k-m}(x) = \sum_{m=0}^{k}\sum_{n=0}^{m} Ch_n(x)s_2(m,n)s_2(k,m). \tag{24}$$

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On a type of rough intuitionistic fuzzy sets and its topological structure

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The rough intuitionistic fuzzy set theory is an extension of the theory of rough fuzzy sets. For further studying the theories and applications of rough intuitionistic fuzzy sets, in this paper, we propose a type of rough intuitionistic fuzzy sets and investigate its topological structure. It is proved that an intuitionistic fuzzy topology is induced by a binary relation in a crisp approximation space, and a preorder is generated by a family of intuitionistic fuzzy sets. Moreover, there exists a one-to-one correspondence between the set of all intuitionistic fuzzy topologies having property $(*)$ and the set of all preorders. That is to say, there exists a one-to-one correspondence between the set of all intuitionistic fuzzy topological spaces having property $(*)$ and the set of all crisp approximation spaces whose relations are preorders.

Keywords: approximation operator; preorder; rough intuitionistic fuzzy set; intuitionistic fuzzy topology.

1 Introduction

Rough set theory was proposed by Pawlak [14] to deal with imprecision, vagueness, and uncertainty in data analysis. In classical Pawlak rough set theory, the lower and upper approximation operators are two important basic concepts. The equivalence (indiscernibility) relations or partitions are the simplest formulation of the lower and upper approximation operators. However, the requirement of the equivalence relation in Pawlak rough set model seems to be a very restrictive condition that may limit the application domain of the rough set model. To solve this problem, many authors have generalized the notion of approximation operators by using more general binary relations [4, 20, 21, 26, 27] or by employing coverings [2, 3, 28, 33]. Moreover,

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rough sets can also be generalized into the fuzzy environment and the results are called rough fuzzy sets and fuzzy rough sets [8, 9, 10, 12, 13, 15, 16, 19, 22, 23, 24, 30].

As an extension of the theory of fuzzy sets, the theory of intuitionistic fuzzy (IF, for short) sets is originated by Atanassov [1]. A fuzzy set gives a degree of which element belongs to a set, but an IF set gives both a membership degree and a nonmembership degree. Obviously, an IF set is more objective than a fuzzy set to describe the vagueness of data or information. Many authors generalized the concepts and operations in fuzzy set theory into IF set theory, to enrich the theory of IF sets and enlarge the application of IF sets. Therefore, the combination of IF set theory and rough set theory is an interesting research issue over the years [5, 7, 17, 18, 29, 31]. The rough IF sets are indeed natural generalizations of rough fuzzy sets and will be applied in decision analysis.

Topology is a mathematical tool to study information systems and rough sets. It is important to discuss topological structures of rough sets. Many authors investigated topological structures of rough sets in the fuzzy environment [11, 32, 25]. Zhou et al. presented a one-to-one correspondence between the set of all IF reflexive and transitive approximation spaces and the set of all IF rough topological spaces [32]. Lin and Wang discussed the topological properties of IF rough sets [11]. Xu and Wu investigated topological structures of a type rough IF sets [25].

This paper is devoted to the discussion of a type of rough IF sets and its topological structure. Firstly, in a crisp approximation space, an IF topology is generated by the relation, whose interior and closure operators are IF lower and upper approximation operators respectively. Then, a preorder is induced by a family of IF sets. Moreover, there exists a one-to-one correspondence between the set of all intuitionistic fuzzy topological spaces having property $(*)$ and the set of all crisp approximation spaces whose relations are preorders.

2 Basic Concepts and properties

In this section, we introduce the basic concepts about binary relation, intuitionistic fuzzy set and intuitionistic fuzzy topological space.

Throughout this paper, U will be a nonempty set called the universe of discourse. The class of all subsets (intuitionistic fuzzy subsets, respectively) of U will be denoted by $\mathcal{P}(U)$ (by $\mathcal{IF}(U)$, respectively).

Definition 1. Let U be a set, $U \times U$ the product set of U and U. Any subset R of $U \times U$ is called a binary relation on U. For any $(x, y) \in U \times U$, if $(x, y) \in R$, we say x has relation R with y, and denote this relationship as xRy. For any $x \in U$, we call the set $\{y \in U | xRy\}$ the successor neighborhood of x in R and denote it as $R_s(x)$, and the set $\{y \in U | yRx\}$ the predecessor neighborhood of x in R and denote it as $R_p(x)$. Let R be a relation on U.

(Reflexive relation) If for any $x \in U$, xRx, we say R is reflexive. In another word, if for any $x \in U$, $x \in R_s(x)$, R is reflexive.

(Transitive relation) If for any $x, y, z \in U$, xRy and $yRz \Rightarrow xRz$, we say R is transitive. In another word, if for any $x, y \in U$, $y \in R_s(x) \Rightarrow R_s(y) \subseteq R_s(x)$, R is transitive.

(Preorder) A binary relation R is referred to as a preorder if R is reflexive and transitive.

Definition 2 [1]. Let U be a non-empty set. An intuitionistic fuzzy set A in U is an object

having the form

$A = \{< x, \mu_A(x), \gamma_A(x) > | x \in U\}$,

where $\mu_A : U \to [0,1]$ and $\gamma_A : U \to [0,1]$ satisfy $0 \le \mu_A(x) + \gamma_A(x) \le 1$ for all $x \in U$, and $\mu_A(x)$ and $\gamma_A(x)$ are, respectively, called the degree of membership and the degree of nonmembership of the element $x \in U$ to A.

Obviously, a fuzzy set $A = \{< x, \mu_A(x) > | x \in U\}$, can be identified with the IF set of the form $A = \{< x, \mu_A(x), 1 - \mu_A(x) > | x \in U\}$. Thus, an IF set is indeed an extension of a fuzzy set. We introduce some basic operations on $\mathcal{IF}(U)$ in the following definition.

Definition 3 [1]. Let $A, B \in \mathcal{IF}(U)$ and $\{A_j | j \in J\} \subseteq \mathcal{IF}(U)$, where J is an index set. Define the operations as follows:

$A \subseteq B \Leftrightarrow \mu_A(x) \le \mu_B(x)$ and $\gamma_A(x) \ge \gamma_B(x)$ for all $x \in U$,
$A \supseteq B \Leftrightarrow B \supseteq A$,
$A = B \Leftrightarrow A \subseteq B$ and $B \supseteq A$,
$A \cap B = \{< x, \mu_A(x) \wedge \mu_B(x), \gamma_A(x) \vee \gamma_B(x) > | x \in U\}$,
$A \cup B = \{< x, \mu_A(x) \vee \mu_B(x), \gamma_A(x) \wedge \gamma_B(x) > | x \in U\}$,
$A^c = \{< x, \gamma_A(x), \mu_A(x) > | x \in U\}$,
$\cap A_j = \{< x, \wedge \mu_{A_j}(x), \vee \gamma_{A_j}(x) > | x \in U\}$,
$\cup A_j = \{< x, \vee \mu_{A_j}(x), \wedge \gamma_{A_j}(x) > | x \in U\}$,
$0_\sim = \{< x, 0, 1 > | x \in U\}$, $1_\sim = \{< x, 1, 0 > | x \in U\}$.

Definition 4 [6]. An IF topology τ on a nonempty set U is a family of IF sets in U satisfying the following axioms:

(T_1) $0_\sim, 1_\sim \in \tau$,
(T_2) $G_1 \cap G_2 \in \tau$ for all $G_1, G_2 \in \tau$,
(T_3) $\cup_{j \in J} G_j \in \tau$ for an arbitrary family $\{G_j | j \in J\} \subseteq \tau$.

In this case the pair (U, τ) is called an IF topological space and each IF set $G \in \tau$ is known as an IF open set in U, and the complement G^c of an IF open set G in (U, τ) is called an IF closed set in U. For any $A \in \mathcal{IF}(U)$, the IF interior and IF closure of A are, respectively, defined as follows:

$int(A) = \cup\{G | G \in \tau,\ G \subseteq A\}$,
$cl(A) = \cap\{K | K^c \in \tau,\ A \subseteq K\}$,

where int and cl are, respectively, called the IF interior operator and the IF closure operator of τ.

It can be shown that $cl(A)$ is an IF closed set and $int(A)$ is an IF open set in U, A is an IF open set in U if and only if $int(A) = A$, and A is an IF closed set in U if and only if $cl(A) = A$. Some properties of IF interior operator and IF closure operator are presented as

Proposition 1. *Let (U, τ) be an IF topological space and $A, B \in \mathcal{IF}(U)$. Then the following*

3

properties hold:

(1) $cl(A^c) = (int(A))^c$, $int(A^c) = (cl(A))^c$,
(2) $int(A) \subseteq A \subseteq cl(A)$,
(3) $int(A \cap B) = int(A) \cap int(B)$, $cl(A \cup B) = cl(A) \cup cl(B)$,
(4) $int(int(A)) = int(A)$, $cl(cl(A)) = cl(A)$,
(5) $int(1_\sim) = 1_\sim$, $cl(0_\sim) = 0_\sim$.

Conversely, it is easy to verify that if an IF operator $i : \mathcal{IF}(U) \to \mathcal{IF}(U)$ $(c : \mathcal{IF}(U) \to \mathcal{IF}(U)$, *respectively*) *satisfies the following properties: for any* $A, B \in \mathcal{IF}(U)$,

(1) $i(1_\sim) = 1_\sim$, $(c(0_\sim) = 0_\sim,)$
(2) $i(A) \subseteq A$, $(A \subseteq c(A),)$
(3) $i(A \cap B) = i(A) \cap i(B)$, $(c(A \cup B) = c(A) \cup c(B),)$
(4) $i(i(A)) = i(A)$, $(c(c(A)) = c(A),)$

then $\{A|i(A) = A,\ A \in \mathcal{IF}(U)\}$ $(\{A^c|c(A) = A,\ A \in \mathcal{IF}(U)\}$, *respectively*) *is an IF topology on* U *and denoted by* $\tau(i)$ $(\tau(c)$, *respectively*).

3 The one-to-one correspondence between IF approximation operators and IF topological spaces

Firstly, we introduce the definition of IF approximation operators.

Definition 5. Let R be a binary relation on U. Then (U, R) is called a crisp approximation space. Define a family of IF sets as follows:

$$\mathcal{A}(R) = \{A \in \mathcal{IF}(U)|\forall (x, y) \in R, \mu_A(x) \le \mu_A(y), \gamma_A(x) \ge \gamma_A(y)\}.$$

Then a pair of rough IF approximation operators are defined by

$$\underline{R}(X) = \cup\{A|A \subseteq X, A \in \mathcal{A}(R)\},$$
$$\overline{R}(X) = \cap\{A|X \subseteq A, A \in \mathcal{A}(R)\}.$$

Since Dubois and Prade proposed rough fuzzy set [8], much authors have discussed properties of rough fuzzy set [9, 23, 24]. At the same time, the definitions of rough fuzzy set in [9, 23, 24] were extended to rough IF set [17, 18, 25, 31]. It is easy to verify that the definition of the rough IF in Definition 5 is different from that in [17, 18, 25, 31].

It is easy to get properties of rough IF approximation operators: $\forall A, B \in \mathcal{IF}(U)$,

(1) $\underline{R}(1_\sim) = 1_\sim$, $\overline{R}(0_\sim) = 0_\sim$;
(2) $\underline{R}(A) = (\overline{R}(A^c))^c$, $\overline{R}(A) = (\underline{R}(A^c))^c$;
(3) $\underline{R}(A) \subseteq A \subseteq \overline{R}(A)$;
(4) $A \subseteq B \Rightarrow \underline{R}(A) \subseteq \underline{R}(B), \overline{R}(A) \subseteq \overline{R}(B)$.

3.1 From a crisp approximation space to an intuitionistic fuzzy topological space

In this subsection, we will present more properties of $\mathcal{A}(R)$, $\underline{R}$ and $\overline{R}$.

Proposition 2. *Let R be a binary relation on U. Then, for any $\mathcal{B} \subseteq \mathcal{A}(R)$, $\cup\mathcal{B}, \cap\mathcal{B} \in \mathcal{A}(R)$.*

Proof. For any $(x, y) \in R$ and $B \in \mathcal{B}$, we get $\mu_B(x) \leq \mu_B(y)$ and $\gamma_B(x) \geq \gamma_B(y)$. Thus $\mu_{\cup\mathcal{B}}(x) \leq \mu_{\cup\mathcal{B}}(y)$ and $\gamma_{\cup\mathcal{B}}(x) \geq \gamma_{\cup\mathcal{B}}(y)$, $\mu_{\cap\mathcal{B}}(x) \leq \mu_{\cap\mathcal{B}}(y)$ and $\gamma_{\cap\mathcal{B}}(x) \geq \gamma_{\cap\mathcal{B}}(y)$. So $\cup\mathcal{B}, \cap\mathcal{B} \in \mathcal{A}(R)$.

Corollary 1. *Let R be a binary relation on U. Then*

(1) $\mathcal{A}(R)$ is an IF topology,

(2) $\underline{R}$ and $\overline{R}$ are, respectively, the IF interior operator and the IF closure operator of $\mathcal{A}(R)$.

Proof. (1) It is clear that $0_\sim, 1_\sim \in \mathcal{A}(R)$. Thus, according to Definition 4 and Proposition 2, $\mathcal{A}(R)$ is an IF topology.

(2) By (1) and Definition 4, we can get this proposition.

From Corollary 1, we know that $\mathcal{A}(R)$ is a IF topology if R is an arbitrarily relation, and $\underline{R}$ and $\overline{R}$ are, respectively, the IF interior and closure operators of $\mathcal{A}(R)$. Hence $\underline{R}(A)\cap\underline{R}(B) = \underline{R}(A\cap B)$ and $\overline{R}(A) \cup \overline{R}(B) = \overline{R}(A \cup B)$ for all $A, B \in \mathcal{IF}(U)$. To get more properties of $\mathcal{A}(R)$, we suppose R is a preorder in the following.

Proposition 3. *Let R be a preorder on U. Then, for any $x, y \in U$, xRy if and only if $\mu_A(x) \leq \mu_A(y)$ and $\gamma_A(x) \geq \gamma_A(y)$ for all $A \in \mathcal{A}(R)$.*

Proof. "$\Rightarrow$". If xRy, by the definition of $\mathcal{A}(R)$, it is easy to obtain that $\mu_A(x) \leq \mu_A(y)$ and $\gamma_A(x) \geq \gamma_A(y)$ for all $A \in \mathcal{A}(R)$.

"$\Leftarrow$". Suppose $(x, y) \notin R$, then define an IF set B as follows: for any $u \in U$,

$$\mu_B(u) = \begin{cases} 0, & u \in R_p(y); \\ 1, & u \notin R_p(y), \end{cases} \quad \gamma_B(u) = 1 - \mu_B(u).$$

For any $(u_1, u_2) \in R$, if $u_2 \notin R_p(y)$, then $\mu_B(u_2) = 1$. So $\mu_B(u_1) \leq \mu_B(u_2)$. If $u_2 \in R_p(y)$, hence $(u_2, y) \in R$. Since R is transitive and $(u_1, u_2) \in R$, we have $(u_1, y) \in R$. Thus $u_1 \in R_p(y)$, $\mu_B(u_1) = 0$, which implies $\mu_B(u_1) \leq \mu_B(u_2)$. So we can conclude that $\mu_B(u_1) \leq \mu_B(u_2)$. Then $\gamma_B(u_1) = 1 - \mu_B(u_1) \geq 1 - \mu_B(u_2) = \gamma_B(u_2)$. Consequently, $B \in \mathcal{A}(R)$.

Since R is reflexive, we obtain $y \in R_p(y)$, so $\mu_B(y) = 0$. By $(x, y) \notin R$, $x \notin R_p(y)$, thus $\mu_B(x) = 1$.

In conclusion, there exists $B \in \mathcal{A}(R)$ such that $\mu_B(x) > \mu_B(y)$, which contradicts the assumption of this theorem.

Proposition 4. *Let R be a preorder on U. Then, for any $x \in U$ and $a, b \in [0, 1]$ with $a + b \leq 1$, there exists an $A \in \mathcal{A}(R)$ such that for any $z \in U$,*

$$\mu_A(z) = \begin{cases} a, & z \in R_s(x); \\ 0, & z \notin R_s(x), \end{cases} \qquad \gamma_A(z) = \begin{cases} b, & z \in R_s(x); \\ 1, & z \notin R_s(x). \end{cases}$$

Proof. We only prove $A \in \mathcal{A}(R)$. In fact, for any $(u,v) \in R$, if $u \in R_s(x)$, we have $v \in R_s(x)$ since R is transitive. R is reflexive, so $\mu_A(x) = \mu_A(u) = \mu_A(v) = a$ and $\gamma_A(x) = \gamma_A(u) = \gamma_A(v) = b$. If $u \notin R_s(x)$, then $\mu_A(u) = 0$ and $\gamma_A(u) = 1$. Hence $\mu_A(u) \leq \mu_A(v)$ and $\gamma_A(u) \geq \gamma_A(v)$. In conclusion, $\mu_A(u) \leq \mu_A(v)$ and $\gamma_A(u) \geq \gamma_A(v)$. Therefore, $A \in \mathcal{A}(R)$.

3.2 From an intuitionistic fuzzy topological space to a crisp approximation space

Definition 6. Let $\mathcal{A} \subseteq IF(U)$, then define a binary relation from $\mathcal{A}$ as follows:

$$R(\mathcal{A})= \{(x,y) \in U \times U | \forall A \in \mathcal{A},\ \mu_A(x) \leq \mu_A(y), \gamma_A(x) \geq \gamma_A(y)\}.$$

In Definition 6, a binary relation is induced from a family of intuitionistic fuzzy sets. Proposition 5 below gives the properties of $R(\mathcal{A})$.

Proposition 5. *Let $\mathcal{A} \subseteq IF(U)$, then $R(\mathcal{A})$ is a preorder.*

Proof. For any $x \in U$ and $A \in \mathcal{A}$, $\mu_A(x) = \mu_A(x)$ and $\gamma_A(x) = \gamma_A(x)$, then $(x,x) \in R(\mathcal{A})$. We obtain that $R(\mathcal{A})$ is reflexive.

For any $x,y,z \in U$, if $(x,y) \in R(\mathcal{A})$ and $(y,z) \in R(\mathcal{A})$, then for any $A \in \mathcal{A}$, $\mu_A(x) \leq \mu_A(y)$ and $\gamma_A(x) \geq \gamma_A(y)$, $\mu_A(y) \leq \mu_A(z)$ and $\gamma_A(y) \geq \gamma_A(z)$. So $\mu_A(x) \leq \mu_A(z)$ and $\gamma_A(x) \geq \gamma_A(z)$. Hence $(x,z) \in R(\mathcal{A})$. It follows that $R(\mathcal{A})$ is transitive.

By Proposition 5, we can induce a preorder from a family of IF sets. We first convert a preorder R into a family of IF sets $\mathcal{A}(R)$, then convert the family of IF sets $\mathcal{A}(R)$ into a preorder $R(\mathcal{A}(R))$, and consider the relationship between R and $R(\mathcal{A}(R))$.

Theorem 1. *Let R be a preorder on U. Then $R = R(\mathcal{A}(R))$.*

Proof. For any $(x,y) \in R$, by the definition of $\mathcal{A}(R)$, $\mu_A(x) \leq \mu_A(y)$ and $\gamma_A(x) \geq \gamma_A(y)$ for all $A \in \mathcal{A}(R)$. According to Definition 6, $(x,y) \in R(\mathcal{A}(R))$, so $R \subseteq R(\mathcal{A}(R))$. Conversely, for any $(x,y) \in R(\mathcal{A}(R))$, $\mu_A(x) \leq \mu_A(y)$ and $\gamma_A(x) \geq \gamma_A(y)$ for all $A \in \mathcal{A}(R)$. From Proposition 3, $(x,y) \in R$, so $R(\mathcal{A}(R)) \subseteq R$. Therefore, we obtain $R = R(\mathcal{A}(R))$.

If we first convert a family of IF sets $\mathcal{A}$ into a preorder $R(\mathcal{A})$, then change the preorder $R(\mathcal{A})$ into the family of IF sets $\mathcal{A}(R(\mathcal{A}))$, $\mathcal{A} = \mathcal{A}(R(\mathcal{A}))$?

Proposition 6. *Let $\mathcal{A} \subseteq \mathcal{IF}(U)$, then $\mathcal{A} \subseteq \mathcal{A}(R(\mathcal{A}))$.*

Proof. For any $A \in \mathcal{A}$, by the definition of $R(\mathcal{A})$, $\mu_A(x) \leq \mu_A(y)$ and $\gamma_A(x) \geq \gamma_A(y)$ for all $(x,y) \in R(\mathcal{A})$. According to Definition 5, we have $A \in \mathcal{A}(R(\mathcal{A}))$. So $\mathcal{A} \subseteq \mathcal{A}(R(\mathcal{A}))$.

Generally, $\mathcal{A}(R(\mathcal{A}))$ is not equal to $\mathcal{A}$.

Example 1. Let $U = \{a,b\}$ and $\mathcal{A}= \{A\}$, where $A = \{< a, \frac{1}{3}, \frac{1}{2} >, < b, \frac{1}{2}, \frac{1}{3} >\}$. Then $R(\mathcal{A})= \{(a,a),(a,b),(b,b)\}$. Thus, we have $B = \{< a, \frac{1}{2}, \frac{1}{2} >, < b, \frac{1}{2}, \frac{1}{2} >\} \in \mathcal{A}(R(\mathcal{A}))$ and $B \notin \mathcal{A}$. So $\mathcal{A}(R(\mathcal{A})) \neq \mathcal{A}$.

In order to give a sufficient and necessary condition for $\mathcal{A}(R(\mathcal{A})) = \mathcal{A}$, we propose two properties for a family of IF sets $\mathcal{A}$.

Property $(*)$: for any $x \in U$ and $a, b \in [0,1]$ with $a + b \leq 1$, there exists an $A \in \mathcal{A}$ such that for any $z \in U$,

$$\mu_A(z) = \begin{cases} a, & z \in R(\mathcal{A})_s(x); \\ 0, & z \notin R(\mathcal{A})_s(x), \end{cases} \quad \gamma_A(z) = \begin{cases} b, & z \in R(\mathcal{A})_s(x); \\ 1, & z \notin R(\mathcal{A})_s(x). \end{cases}$$

Property $(**)$: for any $\mathcal{B} \subseteq \mathcal{A}$, $\cup \mathcal{B} \in \mathcal{A}$.

Theorem 2. *Let $\mathcal{A} \subseteq \mathcal{IF}(U)$, then $\mathcal{A} = \mathcal{A}(R(\mathcal{A}))$ if and only if $\mathcal{A}$ has properties $(*)$ and $(**)$.*

Proof. "$\Rightarrow$". From Proposition 5, $R(\mathcal{A})$ is a preorder. By Proposition 4, for any $x \in U$ and $a, b \in [0,1]$ with $a + b \leq 1$, there exists an $A \in \mathcal{A}(R(\mathcal{A}))$ such that for any $z \in U$,

$$\mu_A(z) = \begin{cases} a, & z \in R(\mathcal{A})_s(x); \\ 0, & z \notin R(\mathcal{A})_s(x), \end{cases} \quad \gamma_A(z) = \begin{cases} b, & z \in R(\mathcal{A})_s(x); \\ 1, & z \notin R(\mathcal{A})_s(x). \end{cases}$$

Since $\mathcal{A} = \mathcal{A}(R(\mathcal{A}))$, $\mathcal{A}$ satisfies property $(*)$ by Proposition 2. According to Proposition 4, $\mathcal{A}(R(\mathcal{A}))$ satisfies property $(**)$, which implies that $\mathcal{A}$ has property $(**)$.

"$\Leftarrow$". By Proposition 6, $\mathcal{A} \subseteq \mathcal{A}(R(\mathcal{A}))$. Now we prove $\mathcal{A}(R(\mathcal{A})) \subseteq \mathcal{A}$. Let $A \in \mathcal{A}(R(\mathcal{A}))$, then for any $(u, v) \in R(\mathcal{A})$, $\mu_A(u) \leq \mu_A(v)$ and $\gamma_A(u) \geq \gamma_A(v)$. Since $\mathcal{A}$ satisfies property $(*)$, for any $x \in U$, there is $B_x \in \mathcal{A}$ such that for any $z \in U$,

$$\mu_{B_x}(z) = \begin{cases} \mu_A(x), & z \in R(\mathcal{A})_s(x); \\ 0, & z \notin R(\mathcal{A})_s(x), \end{cases} \quad \gamma_{B_x}(z) = \begin{cases} \gamma_A(x), & z \in R(\mathcal{A})_s(x); \\ 1, & z \notin R(\mathcal{A})_s(x). \end{cases}$$

Then $A = \bigcup_{x \in U} B_x$. In fact, for any $y \in U$, since $R(\mathcal{A})$ is reflexive, We have $y \in R(\mathcal{A})_s(y)$. So $\mu_{B_y}(y) = \mu_A(y)$ and $\gamma_{B_y}(y) = \gamma_A(y)$. Hence

$$\mu_A(y) \leq \vee_{x \in U} \mu_{B_x}(y) = \mu_{(\bigcup_{x \in U} B_x)}(y), \; \gamma_A(y) \geq \wedge_{x \in U} \gamma_{B_x}(y) = \gamma_{(\bigcup_{x \in U} B_x)}(y).$$

Conversely, for any $x \in U$, if $y \notin R(\mathcal{A})_s(x)$, then $\mu_{B_x}(y) = 0$ and $\gamma_{B_x}(y) = 1$. If $y \in R(\mathcal{A})_s(x)$, then $\mu_{B_x}(y) = \mu_A(x) \leq \mu_A(y)$ and $\gamma_{B_x}(y) = \gamma_A(x) \geq \gamma_A(y)$. So

$$\mu_{\bigcup_{x \in U} B_x}(y) = \vee_{x \in U} \mu_{B_x}(y) \leq \mu_A(y), \; \gamma_{(\bigcup_{x \in U} B_x)}(y) = \wedge_{x \in U} \gamma_{B_x}(y) \geq \gamma_A(y).$$

Therefore, by property $(**)$, $A = \bigcup_{x \in U} B_x \in \mathcal{A}$, which implies $\mathcal{A}(R(\mathcal{A})) \subseteq \mathcal{A}$.

From Theorem 2, $\mathcal{A}$ having properties $(*)$ and $(**)$ is a sufficiency and necessary condition for $\mathcal{A} = \mathcal{A}(R(\mathcal{A}))$. By Corollary 1 and Theorem 2, it is easy to obtain

Corollary 2. *Let $\mathcal{A} \subseteq \mathcal{IF}(U)$, then $\mathcal{A} = \mathcal{A}(R(\mathcal{A}))$ if and only if $\mathcal{A}$ is an IF topology satisfying property $(*)$.*

Denote the set of all preorders on U as $\widetilde{R}$, and denote the family of all IF topologies on U having property $(*)$ as $\widetilde{A}$. Combining Theorem 1, Corollaries 1 and 2, we have

Theorem 2. *Let U be a non-empty set. Then there exists a one-to-one correspondence between $\widetilde{R}$ and $\widetilde{A}$.*

Proof. Define a mapping $f : \widetilde{R} \to \widetilde{A}$ by $f(R) = \mathcal{A}(R)$. And define a mapping $g : \widetilde{A} \to \widetilde{R}$ by $g(\mathcal{A}) = R(\mathcal{A})$. For any $R \in \widetilde{R}$, by Theorem 1 and Corollary 1, we get $g \circ f(R) = g(\mathcal{A}(R)) = R(\mathcal{A}(R)) = R$. For any $\mathcal{A} \in \widetilde{A}$, according to Proposition 5 and Corollary 2, $f \circ g(\mathcal{A}) = f(R(\mathcal{A})) = \mathcal{A}(R(\mathcal{A})) = \mathcal{A}$. Then there exists a one-to-one correspondence between $\widetilde{R}$ and $\widetilde{A}$.

By Theorem 2, there exists a one-to-one correspondence between crisp approximation spaces whose relations are preorders and IF topological spaces having property $(*)$.

4 Conclusion

In this paper, an IF topology has been induced in a crisp approximation space, whose IF interior and closure operators are IF lower and upper approximation operators respectively. Conversely, a preorder has been generated by a family of IF sets. The important contribution of this paper is that we establish a one-to-one correspondence between the set of all intuitionistic fuzzy topological spaces having property $(*)$ and the set of all crisp approximation spaces whose relations are preorders. In our future work, we will discuss relationships between this type of rough IF sets and other types of rough IF sets, and explore connections between rough IF sets and covering-based rough sets.

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Modelling by Shepard-type curves and surfaces

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Dedicated to the memory of Prof. D.D. Stancu

Abstract

First parametric curves of Shepard-type are studied, which overcome some of the original Shepard operator's drawbacks, have some advantages with respect to the Bézier case and are optimal in some sense. Bounds for the deviation and approximation results for Shepard-type operators faster converging than the original one are proved. As an application a weighted progressive iterative approximation technique interesting in CAGD and an extension to tensor product surfaces case are given.

Key-words: Shepard-type operators; deviation; weighted progressive iterative approximation; tensor product surfaces.

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1 Introduction

Shepard-type operators are rational operators of interpolatory type widely used in classical Approximation Theory and in scatter data interpolation problems and they allow approximation results not possible by polynomials. However they suffer of drawback of flat spots phenomenon, which makes them unsuitable for CAGD.

The purpose of the present paper is to study a new class of Shepard-type curves $S_{n,\lambda}$ overcoming the above drawback (Section 2). The paper is organized as follows. In Section 3 Theorem 1 gives an estimation of the maximal distance between $S_{n,\lambda}[P]$ and the control polygon P in terms of the maximal absolute first order difference of the control points. In Section 4 we construct a sequence of Shepard-type operators based on $S_{n,\lambda}$ converging to the global Shepard-type interpolating operator and in Theorems 2, 3, 4, 5 and 6 we give convergence results and approximation error estimates. The results are applied in CAGD to study the weighted progressive iterative approximation (WPIA in short) technique in Theorems 7 and 8. The key idea is to iteratively change the control points of the active curve to deform towards the target shape represented by the point data. So by adjusting the control points of $S_{n,\lambda}$ curves and by using a weight, the WPIA process generates sequences of curves converging to the global Shepard-type interpolating curve at the original control points. Moreover an optimal value of the weight giving the fastest convergence rate is shown in Theorem 7. Based on such format, data points can be adaptively fit. Finally in Section 5 the results are extended to tensor product surfaces. The proofs of main results are in Section 6. The demonstration techniques are based on direct estimates of $S_{n,\lambda}$ operator and preliminary Lemmas on the eigenstructure of $S_{n,\lambda}$ operator interesting in themselves.

Near-interpolating curves $S_{n,\lambda}$ have some advantages with respect to Bézier curves: the parameter λ can be used as a shape control tool to draw a pencil of curves and choose the desired shape; $S_{n,\lambda}$ curves have pseudo-local control property against the global behaviour of Bézier curves; the deviation between $S_{n,\lambda}$ function and its control polygon is smaller than for the Bézier case (see Section 3); the corresponding weighted progressive iterative approximation process has faster rate of convergence than for Bézier case (Section 4); these advantages extend to the surfaces case (Section 5). Numerical experiments are also shown, verifying our theoretical analysis.

2 Near-interpolating curves

Let $A_n(t) = [A_{n,0}(t), A_{n,1}(t), \ldots, A_{n,n}(t)]^T$, where

$$A_{n,i}(t) = \frac{1/\left((t-t_i)^s + \lambda\right)}{\sum_{i=0}^n 1/\left((t-t_i)^s + \lambda\right)}, \tag{1}$$

for $0 \le i \le n$, $n \in \mathbb{N}$, $t \in [0,1]$, $t_i = i/n$, $i = 0, \ldots, n$, s even > 2 and $0 < n^s\lambda \le 1/(6\zeta(s))$, with ζ being the zeta Riemann function.

In the following Lemma 5 we will show that $A_{n,i}(t)$, $0 \le i \le n$, form a basis generating a subspace S of rational functions of degree (sn, sn), with

$$0 \le A_{n,i}(t) \le 1,\ i = 0, \dots, n,\ \sum_{i=0}^{n} A_{n,i}(t) = 1. \tag{2}$$

Hence in the following the functions $A_{n,i}$, $i = 0, \dots, n$, are called blending functions. Given the blending functions $A_{n,i}(t)$ defined by (1) and a control polygon $P = [P_0, P_1, \dots, P_n]^T$, $P_i \in \mathbb{R}^d$, $i = 0, \dots, n$, $d \ge 2$, introduce the near-interpolating parametric Shepard-type curve $S_{n,\lambda}[P, t]$ defined by

$$S_{n,\lambda}[P, t] = \sum_{i=0}^{n} A_{n,i}(t) P_i = A_n(t) P. \tag{3}$$

Hence by (1)–(3) it is easy to check that $S_{n,\lambda}[P, t]$:

- is a rational curve of degree (sn, sn);
- it reproduces points;
- it is symmetric (i.e., $S_{n,\lambda}[P, 1-t] = S_{n,\lambda}[\tilde{P}, t]$, $\tilde{P} = [P_n, \dots, P_0]$, $\forall t \in [0,1]$);
- it is smooth;
- it is nondegenerate;
- it lies in the convex hull of the control polygon P;
- it satisfies the pseudo-local control property (indeed each function $A_{n,j}(t)$, $0 \le j \le n$, attains its maximum value close to 1 at $t = t_j$ and is very small for $|t - t_j| > 1/n$, in other words the point P_j influences strongly the shape of the curve in a neighborhood of $t = t_j$);
- it interpolates at the control points, as λ tends to 0 (see the following remark on Balazs-Shepard operator);
- it satisfies the degree elevation-type formula

$$\overline{S}_{n+1,\lambda}[P \cup \overline{P}, t] = \frac{S_{n,\lambda}[P, t] D_n(t)}{\overline{D}_{n+1}(t)} + \frac{\overline{P} / \left((t - \bar{t})^s + \lambda\right)}{\overline{D}_{n+1}(t)},\ \bar{t} \ne t_k,\ k = 0, \dots, n,$$

with

$$\overline{S}_{n+1,\lambda}[P \cup \overline{P}, t] = \frac{\sum_{k=0}^{n} P_i / \left((t - t_i)^s + \lambda\right) + \overline{P} / \left((t - \bar{t})^s + \lambda\right)}{\sum_{k=0}^{n} 1 / \left((t - t_i)^s + \lambda\right) + 1 / \left((t - \bar{t})^s + \lambda\right)},$$

$$D_n(t) = \sum_{k=0}^{n} \frac{1}{(t - t_i)^s + \lambda},$$

$$\overline{D}_{n+1}(t) = \sum_{k=0}^{n} \frac{1}{(t - t_i)^s + \lambda} + \frac{1}{(t - \bar{t})^s + \lambda}.$$

By the above remarks $S_{n,\lambda}[P, t]$ can be considered a parametric curve approximating the control polygon P.

In the nonparametric case, i.e. $P_i = (t_i, f(t_i))$, with f a continuous function on $[0,1]$, operators similar to (3) were studied to approximate surface data [1] or noisy values [8]. When λ tends to 0, then $S_{n,\lambda}$ tends to the well-known Balazs-Shepard operator [14]

$$S_n(f,t) = \frac{\sum_{i=0}^{n} f(t_i)/(t-t_i)^s}{\sum_{i=0}^{n} 1/(t-t_i)^s}, \ s \text{ even } \geq 2,$$

interpolating f at t_i, $i = 0, \ldots, n$. Such operators are extensively used in applicative problems involving scattered data interpolation and they have been subject of several papers proving approximation results not possible by polynomials [2, 5, 6, 7].

It is easy to see that the presence of parameter λ in (3) makes the structure of $S_{n,\lambda}$ not far from the simple S_n and analogous convergence results and error estimates can be proved as in [5, 7, 12].

For example if $\| \ \|$ denotes the usual supremum norm on $[0,1]$ and $\omega(f)$ the modulus of continuity of f, then working as in [6, 7, 12]

$$\|f - S_{n,\lambda}(f)\| \leq \text{const } \omega\left(f; \frac{1}{n}\right). \tag{4}$$

On the other hand the choice $0 < \lambda n^s \leq 1/(6\zeta(s))$, makes $S_{n,\lambda}[P,t]$ a curve near-interpolating the control polygon, overcoming the flat spots drawback affecting the original Shepard operator. If $\lambda \to \infty$, then $S_{n,\lambda}[P,t]$ tends to the arithmetic mean of P_i, $i = 0, \ldots, n$.

Here for the sake of simplicity and in analogy to the Bernstein-Bézier case we assumed that the knots are uniformly spaced.

3 Bounds for the deviation

In this Section we give an answer to the question if near-interpolating curves are good curves, in the sense that they do not deviate too much from the data points polygon. This problem is interesting in many CAGD applications, like intersection testing, creating tolerance envelopes, rendering or design (see, e.g., [11, 13]). The following theorem estimates the maximum distance between $S_{n,\lambda}[P,t]$ and its control polygon in terms of the maximal absolute first order difference of the control points of P. The implication of this result is to give a finer localization for the function than by standard convex-hull or mini-max bound. To this end following [11, 13] we view the polygon $P = [P_0, P_1, \ldots, P_n]$, $P_i \in \mathbb{R}$, $i = 0, \ldots, n$, as the piecewise linear function $p(t)$ given by

$$p(t) = P_i + n(t - t_i)(P_{i+1} - P_i), \quad t_i \leq t \leq t_{i+1}, \ i = 0, \ldots, n-1.$$

Then let $e(t) = e_n(t) = |S_{n,\lambda}[P,t] - p(t)|$, with $S_{n,\lambda}[P,t]$ the univariate near-interpolating Shepard-type function and $\Delta P = \max_{0 \leq i \leq n-1} |P_{i+1} - P_i|$. We have

Theorem 1 *For every* $t \in [0,1]$

$$e(t) \leq \Delta P C_{n,\lambda}(t), \tag{5}$$

where

$$C_{n,\lambda}(t) = a_j(t) + a_j^-(t) + a_j^+(t),$$

when $t \in [t_j, t_{j+1}]$ *for some* $0 \leq j \leq n-1$, *and*

$$a_j(t) := \frac{1}{D} \begin{cases} \left| \frac{n(t-t_j)}{(t-t_j)^s+\lambda} + \frac{n(t-t_j)-1}{(t_{j+1}-t)^s+\lambda} \right|, & \text{if } |t-t_j| \leq \frac{1}{2n}, \\ \left| \frac{n(t_{j+1}-t)}{(t_{j+1}-t)^s+\lambda} + \frac{n(t_{j+1}-t)-1}{(t_j-t)^s+\lambda} \right|, & \text{if } |t_{j+1}-t| \leq \frac{1}{2n}, \end{cases}$$

$$a_j^-(t) := \frac{1}{D} \begin{cases} \sum_{k=0}^{j-1} \frac{j-k+n(t-t_j)}{(t-t_k)^s+\lambda}, & \text{if } |t-t_j| \leq \frac{1}{2n}, \\ \sum_{i=0}^{j-1} \frac{j+1-i-n(t_{j+1}-t)}{(t-t_i)^s+\lambda}, & \text{if } |t_{j+1}-t| \leq \frac{1}{2n}, \end{cases}$$

$$a_j^+(t) := \frac{1}{D} \begin{cases} \sum_{k=j+2}^{n} \frac{k-j-n(t-t_j)}{(t-t_k)^s+\lambda}, & \text{if } |t-t_j| \leq \frac{1}{2n}, \\ \sum_{i=j+2}^{n} \frac{i-j-1+n(t_{j+1}-t)}{(t-t_i)^s+\lambda}, & \text{if } |t_{j+1}-t| \leq \frac{1}{2n}, \end{cases}$$

with

$$D := D_n(t) := \sum_{i=0}^{n} \frac{1}{(t-t_i)^s + \lambda}.$$

Remarks. Pointwise estimate (5) corresponds to the fact that for λ vanishing $e(t)$ goes to 0 at t_i, $i = 0, \ldots, n$, as expected since $S_{n,\lambda}[P]$ approaches original interpolating Shepard function. Compare [13, p. 582] for an analogous result for Bernstein-Bézier polynomials.

By Theorem 1, if n and λ are fixed, we find a domain where $S_{n,\lambda}[P,t]$ lies. When n is fixed and λ goes to 0, $C_{n,\lambda}(t)$ tends to $C_{n,0}(t)$, where $C_{n,0}(t)$ corresponds to the function bounding the deviation for the original Shepard function. By numerical tests we found $C_{n,0}(t) \leq 0.28$, $t \in [0,1]$, $n \leq 50$. On the other hand the choice of λ very close to 0 is to be avoided, because the corresponding $S_{n,\lambda}$ function is very close to the original Shepard function giving a very bad approximation near the control points because of the flat spots phenomenon.

Then we have implemented the following procedure in a Matlab environment: given $n+1$ control points, we draw $S_{n,\lambda}$ functions for various λ, we choose that value $\overline{\lambda}$ giving a satisfactory shape and by Theorem 1 we draw a region where the corresponding function $S_{n,\overline{\lambda}}$ lies.

The following examples refer to the cases presented in [11, p. 629], where the deviation of a Bernstein-Bézier polynomial from its control polygon was studied. Here the control polygon P is represented by a continuous line, the near-interpolating Shepard-type function by a dashed line, the bounding region is shaded. We have chosen $s = 4$ (indeed also an even $s > 4$ does not give real improvements, on the other hand the choice $s = 2$ gives worse results, as expected from the theoretical results (cfr. [7])). From our experiments it follows that by a proper choice of λ the bound for the deviation of $S_{n,\lambda}$ functions is smaller than for the Bézier case (cfr. [11, Theorem 4.2, p. 621]).

3.1 Example 1

Here $P_0 = 0$, $P_1 = 1$, $P_2 = 1$, $P_3 = 0$. The choice $\lambda = 2 \times 10^{-3}$ gives a satisfactory shape for the corresponding near-interpolating function and by the estimate in Theorem 1 we get the corresponding bounding region clipped with the min-max bound (see Fig. 1). We have $\max_{t\in[0,1]} C_{n,\lambda}(t) = 0.2319$, $\Delta P = 1$; so by (5) $\max_{t\in[0,1]} e(t) \leq 0.232$.

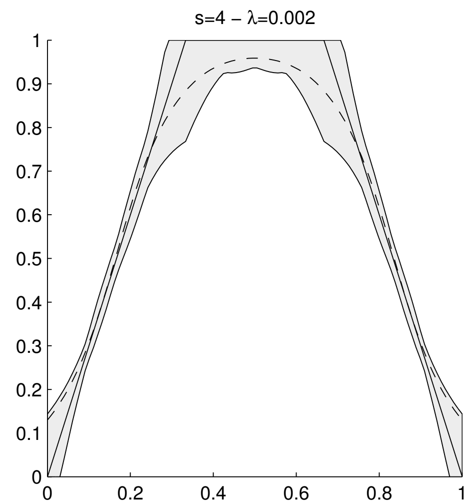

Figure 1: Deviation for Example 1.

3.2 Example 2

Here $P_0 = 0$, $P_1 = 1$, $P_2 = -1$, $P_3 = 0$. We have chosen $\lambda = 1.5 \times 10^{-3}$ and the corresponding bounding region clipped with the min-max bound is drawn in Fig. 2. We have $\max_{t\in[0,1]} C_{n,\lambda}(t) = 0.1893$, $\Delta P = 2$; so by (5) $\max_{t\in[0,1]} e(t) \leq 0.379$.

3.3 Example 3

Here $P_0 = 0$, $P_1 = 1$, $P_2 = 2$, $P_3 = 3$, $P_4 = 2$, $P_5 = 1$. We have chosen $\lambda = 3 \times 10^{-4}$ and the corresponding bounding region clipped with the min-max bound is drawn in Fig. 3. Here $\max_{t\in[0,1]} C_{n,\lambda}(t) = 0.274$, $\Delta P = 1$; so by (5) $\max_{t\in[0,1]} e(t) \leq 0.274$.

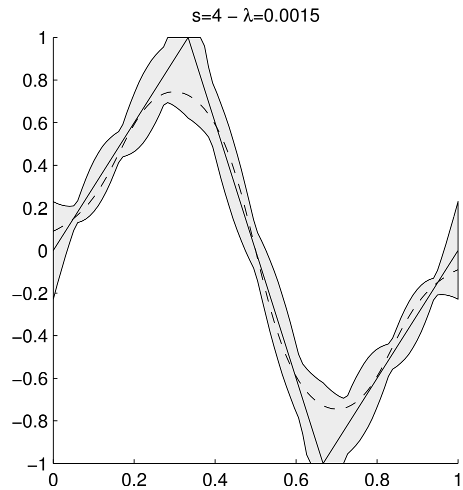

Figure 2: Deviation for Example 2.

4 Weighted progressive iterative approximation

In this Section we study the WPIA property of $S_{n,\lambda}$ curves defined in Section 2. Consider the nonparametric case, i.e.,

$$S_{n,\lambda}(f;t) = \sum_{i=0}^{n} A_{i,n}(t) f(t_i) = A_n(t)\overline{f},$$

with $f \in C([0,1])$ and $\overline{f} = [f(t_0), f(t_1), \ldots, f(t_n)]^T$. From (2) it follows that $S_{n,\lambda}$ preserves constants.

Introduce the global interpolating Shepard-type operator defined by

$$G_{n,\lambda}(f;t) = \sum_{i=0}^{n} A_{n,i}(t) f_i^G = A_n(t)\overline{f}^G, \quad \overline{f}^G = \left[f_0^G, f_1^G, \ldots, f_n^G\right]^T, \qquad (6)$$

with

$$G_{n,\lambda}(f;t_i) = f(t_i), \; i = 0, \ldots, n. \qquad (7)$$

In other words the values f_i^G, $i = 0, \ldots, n$, are determined by solving the linear system (7), guaranteeing for the Shepard-type operator $G_{n,\lambda}$ the interpolation condition at the given values $f(t_i)$ and overcoming the flat spot phenomenon of the original interpolating Shepard operator.

If $n^s\lambda = o(1)$, then $G_{n,\lambda}$ tends to the original Shepard operator and f_i^G tend to $f(t_i)$, $i = 0, \ldots, n$.

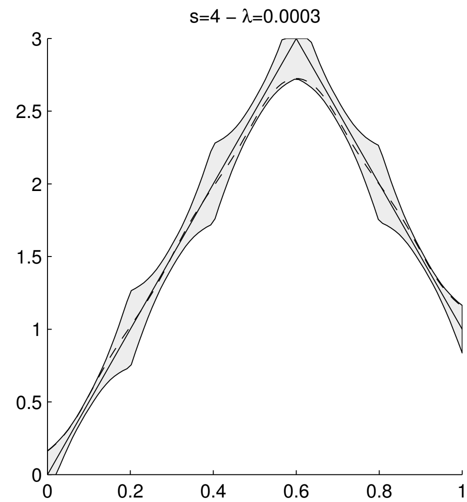

Figure 3: Deviation for Example 3.

We remark that the system (7) can be written as $B\overline{f}^G = \overline{f}$, where

$$B = \begin{pmatrix} A_{n,0}(t_0) & A_{n,1}(t_0) & \cdots & A_{n,n}(t_0) \\ A_{n,0}(t_1) & A_{n,1}(t_1) & \cdots & A_{n,n}(t_1) \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,0}(t_n) & A_{n,1}(t_n) & \cdots & A_{n,n}(t_n) \end{pmatrix} \tag{8}$$

is the collocation matrix of the basis $A_{n,i}$, $i = 0, \ldots, n$.

We observe that B is a symmetric, centrosymmetric and positive stochastic matrix.

Remark. If $n^s\lambda = o(1)$, then $G_{n,\lambda}$ tends to the original Shepard operator, therefore B tends to the identity matrix, consequently all the eigenvalues of B tend to 1.

We denote by $M^k = M \ldots M$ k times the k-th iterate of M operator.

It is easy to deduce from (4) that for any fixed m

$$\left\| f - S^m_{n,\lambda}(f) \right\| \leq \text{const}\ \omega\left(f; \frac{1}{n}\right).$$

Moreover if $\lambda^{(n)}_{n-1} < 1$ denotes the second largest eigenvalue of B, then

Theorem 2 *For any $m \in \mathbb{N}$ and $t \in [0,1]$,*

$$\left| f(t_0) - S^m_{n,\lambda}(f;t) \right| \leq \|f\| \left(\lambda^{(n)}_{n-1}\right)^{m-1}.$$

Hence

$$\lim_m S^m_{n,\lambda}(f;t) = f(t_0). \tag{9}$$

Remark. It is well-known that an analogous result to (9) holds for Bernstein-type operators (see, e.g., [3]).

From (9) we deduce that we have to smartly combine the iterates of $S_{n,\lambda}$ operator to improve the approximation by such operator.

To this aim for every $f \in C([0,1])$ we construct the sequence of rational functions of degree (sn, sn) $\left\{\tilde{S}_m(f;t)\right\}_{m=0}^{\infty}$ defined by

$$\tilde{S}_m(f;t) = A_n(t)\left[I + w\sum_{i=1}^{m}(I - wB)^{i-1}(I - B)\right]\overline{f}, \tag{10}$$

with $0 < w$ a fixed parameter that can take as any possible value as long as it can guarantee the convergence of the above rational functions (see later). It is easy to see that

$$\begin{aligned}\tilde{S}_0(f;t) =& S_{n,\lambda}(f;t)\\ \tilde{S}_1(f;t) =& S_{n,\lambda}(f;t) + wS_{n,\lambda}(f - S_{n,\lambda}(f);t)\\ =&(1+w)S_{n,\lambda}(f;t) - wS^2_{n,\lambda}(f;t)\end{aligned}$$

and

$$\tilde{S}_2(f;t) = (1+2w)S_{n,\lambda}(f;t) - w(2+w)S^2_{n,\lambda}(f;t) + w^2S^3_{n,\lambda}(f;t).$$

Note that in general $\tilde{S}_m$ is not a positive operator.

In particular if $w = 1$, then

$$\tilde{S}_m(f;t) = A_n(t)\left[\sum_{i=0}^{m}(I - B)^i\right]\overline{f}. \tag{11}$$

Cfr. [10, 15] for an analogous operator based on Bernstein polynomials.

We note that since the matrix B is symmetric and centrosymmetric, the number of operations to compute the quantity inside brackets in (10) is reduced by 1/4, that is $(m-1)n^3/2 + O(n^2)$.

In the sequel C will denote a positive constant which may assume different values even in the same formula.

The following Theorem 3 gives a motivation for the construction of $\tilde{S}_m$, i.e. it shows that $\tilde{S}_m(f)$ approximates f at the knots t_i, $i = 0,\ldots,n$, better than original $S_{n,\lambda}(f)$, in other words the loss of positivity is compensated by a better degree of approximation at the knots. Indeed if $\lambda_0^{(n)}$ denotes the smallest eigenvalue of B, then

Theorem 3 *Let $\rho(I - wB) < 1$. Then for any $f \in C([0,1])$ and for any fixed $m > 0$*

$$\left|f(t_i) - \tilde{S}_m(f;t_i)\right| < 2\left\|f\right\|\left(1 - w\lambda_0^{(n)})^m(1 - \lambda_0^{(n)}\right),\ i = 0,\ldots,n. \tag{12}$$

In particular if $w = 1$, then

$$\left\|f(t_i) - \tilde{S}_m(f;t_i)\right\| < 2\left\|f\right\|\left(1 - \lambda_0^{(n)}\right)^{m+1}\ \ i = 0,\ldots,n. \tag{13}$$

Now we examine the behaviour of $\tilde{S}_m$ for n fixed and $m \to \infty$.

Theorem 4 *Let $\rho(I - wB) < 1$. If $f \in C([0,1])$, then for every $t \in [0,1]$ and n*

$$\tilde{S}_\infty(f;t) := \lim_{m\to\infty} \tilde{S}_m(f;t) = G_{n,\lambda}(f;t) = A_n(t)\overline{f}^G = A_n(t)B^{-1}\overline{f}. \tag{14}$$

In particular if $w < 2$, then (14) holds true for every $t \in [0,1]$ and n.

Moreover

$$\left\| f - \tilde{S}_\infty(f) \right\| \le C\omega\left(f; \frac{1}{n}\right). \tag{15}$$

Furthermore if $f \in C^{n+1}([0,1])$

$$\left| f(t) - \tilde{S}_\infty(f;t) \right| \le \frac{|(t-t_0)(t-t_1)\cdots(t-t_n)|}{(n-1)!} M, \tag{16}$$

with

$$M = \max_{0\le t\le 1} \left| f^{(n+1)}(t) - \tilde{S}_\infty^{(n+1)}(f;t) \right|.$$

Theorem 4 says that by the sequence (10) we can reach the global interpolating operator (6) without solving the linear system (7). In other words the sequence $\{\tilde{S}_m\}_m$ continuously links $S_{n,\lambda}$ operator to $G_{n,\lambda}$ operator.

Compare with [15] for an analogous result for Bernstein operator.

Moreover we give an estimate of approximation error of $\tilde{S}_\infty$ by $\tilde{S}_m$.

Theorem 5 *Let $\rho(I - wB) < 1$. For any $f \in C([0,1])$*

$$\left\| \tilde{S}_\infty(f) - \tilde{S}_m(f) \right\| < 2\|f\| \left(1 - w\lambda_0^{(n)}\right)^m \left(1 - \lambda_0^{(n)}\right). \tag{17}$$

The fastest rate is attained when $w = 2/\left(1 + \lambda_0^{(n)}\right)$, therefore

$$\left\| \tilde{S}_\infty(f) - \tilde{S}_m(f) \right\| < 2\|f\| \left(\frac{1 - \lambda_0^{(n)}}{1 + \lambda_0^{(n)}}\right)^m \left(1 - \lambda_0^{(n)}\right). \tag{18}$$

In addition

Theorem 6 *Let $w = 1$. Then*

$$\left\| \tilde{S}_\infty(f) - \tilde{S}_m(f) \right\| < \|f\| \frac{1}{2^m}. \tag{19}$$

If $n^s\lambda = o(1)$, then

$$\left\| \tilde{S}_\infty(f) - \tilde{S}_m(f) \right\| = o(1)^{m+1}. \tag{20}$$

Remark. From Theorem 6 we deduce that the rate of convergence of $\tilde{S}_m$ to $\tilde{S}_\infty$ is faster than in the analogous Bernstein case [15].

The above results find application in CAGD to construct sequences of curves based on $S_{n,\lambda}$ operator converging to the global interpolating Shepard-type curve based on $G_{n,\lambda}$ operator. Let us see in detail the WPIA process.

Given the control polygon $P = [P_0, \dots, P_n]^T$ and the basis $A_{n,i}(t)$, $i = 0, \dots, n$, defined by (1), we can generate the initial curve

$$\gamma_w^0(t) = \sum_{i=0}^{n} A_{n,i}(t) P_i^0 = S_{n,\lambda}[P, t],$$

with $P_i^0 = P_i$, $i = 0, \dots, n$. Then we calculate the remaining curves of the sequence $\gamma_w^{k+1}(t)$, for $k \geq 0$ as follows

$$\gamma_w^{k+1}(t) = \sum_{i=0}^{n} P_i^{k+1} A_{n,i}(t), \tag{21}$$

with

$$P_i^{k+1} = P_i^k + w\overline{\Delta}_i^k,$$

and $\overline{\Delta}_i^k$ the adjusting vectors given by

$$\overline{\Delta}_i^k = P_i - \gamma_w^k(t_i), \; i = 0, 1, \dots, n, \tag{22}$$

in other words we multiply all the adjusting vectors by a common weight w. Then the iterative process can be written in matrix form as follows:

$$\begin{aligned} \left[\overline{\Delta}_0^k, \overline{\Delta}_1^k, \dots, \overline{\Delta}_n^k\right]^T &= (I - wB)\left[\overline{\Delta}_0^{k-1}, \overline{\Delta}_1^{k-1}, \dots, \overline{\Delta}_n^{k-1}\right]^T \\ &= (I - wB)^k \left[\overline{\Delta}_0^0, \overline{\Delta}_1^0, \dots, \overline{\Delta}_n^0\right]^T. \end{aligned} \tag{23}$$

The weight w in (23) can be taken as any possible value, as long as it can guarantee the convergence of the above iterative process.

Remark. In the not-parametric case curves γ_w^k correspond to the rational functions defined by (10).

Now we show how to determine the value of w to obtain the fastest convergence rate. We say that γ_w^0 curve satisfies the WPIA property iff $\lim_k \gamma_w^k(t_i) = P_i$, $i = 0, \dots, n$.

We have

Theorem 7 *If $\rho(I - wB) < 1$, curve γ_w^0 satisfies the WPIA property. In particular if $w < 2$, curve γ_w^0 satisfies WPIA property. Moreover the WPIA process has the fastest convergence rate when*

$$w = \frac{2}{1 + \lambda_0^{(n)}} \tag{24}$$

and in such case

$$\rho(I - wB) = \frac{1 - \lambda_0^{(n)}}{1 + \lambda_0^{(n)}}.$$

Remarks. WPIA property makes possible to construct a sequence of control polygons converging to the control polygon of an interpolating curve of Shepard-type. Moreover the parameter k can be used as shape parameter in order to model different shapes, obtaining as extreme cases the Shepard-type curve and the global interpolating Shepard-type curve. By choosing an optimal value of the weight w, Theorem 7 shows that the weighted PIA shares the progressive iterative approximation property and has the fastest convergence rate. As remarked before the rate is faster than for the Bézier case (cfr. [9]).

We observe that here firstly the convergence results for WPIA property are deduced from the analogous approximation results in the nonparametric case (cfr. [9]).

If $n^s\lambda = o(1)$, from the remark to (8) it follows that the value of w in (24) approaches 1 from above. Therefore we call the WPIA process for $w = 1$ (see following PIA technique) "quasi" optimal.

If $w = 1$, then the corresponding progressive iterative approximation process (called PIA in short) is given by

$$\gamma^0(t) = \sum_{i=0}^{n} A_{n,i}(t)P_i^0, \quad \gamma^{k+1}(t) = \sum_{i=0}^{n} P_i^{k+1} A_{n,i}(t), \quad k \geq 0, \tag{25}$$

where $P_i^0 = P_i$ for all $i = 0, \dots, n$, and $\tilde{\Delta}_i^k$ are the adjusting vectors given by

$$P_i^{k+1} = P_i^k + \tilde{\Delta}_i^k, \ \tilde{\Delta}_i^k = P_i - \gamma^k(t_i), \ i = 0, 1, \dots, n.$$

Then the iterative process can be written in matrix form as follows:

$$\begin{aligned}\left[\tilde{\Delta}_0^k, \tilde{\Delta}_1^k, \dots, \tilde{\Delta}_n^k\right]^T &= (I - B)\left[\tilde{\Delta}_0^{k-1}, \tilde{\Delta}_1^{k-1}, \dots, \tilde{\Delta}_n^{k-1}\right]^T \\ &= (I - B)^k \left[\tilde{\Delta}_0^0, \tilde{\Delta}_1^0, \dots, \tilde{\Delta}_n^0\right]^T.\end{aligned}$$

Remark. Curves γ^k in the not-parametric case correspond to rational functions in (11).

We say that γ^0 curve satisfies the PIA property iff $\lim_k \gamma^k(t_i) = P_i$, $i = 0, \dots, n$.

We have

Theorem 8 *Curve γ^0 satisfies the PIA property. Moreover under the assumptions of Theorem 6, the rate of convergence is faster than in Bézier case.*

Remarks. Based on PIA format, we can design an adaptive fitting method to fit data points, by adjusting the control points corresponding to these data points, if fitting precision is above a predefined threshold.

We observe that the above PIA and weighted PIA processes can be interpreted in terms of classical iterative methods for linear systems; indeed PIA and weighted PIA iterations correspond to classical Richardson and classical modified Richardson method, respectively (compare [4] for an analogous revisitation for Bézier curves).

4.1 Example

Consider a helix of radius 5 given by (cfr. [9])

$$(x(t), y(t), z(t)) = (5\cos t, 5\sin t, t),\ \ t \in [0, 6\pi].$$

A sequence of 19 control points is sampled from the helix as

$$(x(s_i), y(s_i), z(s_i))\,,\ \ s_i = i\frac{\pi}{3},\ \ i = 0, 1, \ldots, 18. \tag{26}$$

Starting with these control points we fit the helix by two sequences of curves generated by the WPIA process defined by (21), (22) and (24) with $s = 4$, $\lambda = 4 \times 10^{-6}$ and $w \simeq 1.57$, and by the PIA process defined by (25) with the same value of $\lambda = 4\ \times 10^{-6}$, respectively. Figures 4 and 5 show the starting, the second and the fourth curves of such sequences for WPIA and PIA, respectively, and star symbol denotes the control points given in (26). Note that the results in Fig. 5 are very similar to Fig. 4, as expected from the remark to (24). The fitting errors of the above WPIA and PIA processes (the maximal Euclidean norm of the corresponding adjusting vectors of such curves) are shown in Fig. 6 for the first 40 iterations. Figure 6 shows that the WPIA process reaches a faster rate of convergence than PIA, as expected from Theorem 7.

5 Modelling by Shepard-type surfaces

Letting $P_{i,j} \in \mathbb{R}^3$, $i = 0, \ldots, m$, $j = 0, \ldots, n$, $m, n \in \mathbb{N}$, be the vertices of the control net P, introduce tensor product near-interpolating surface $S_{m,n,\lambda,\mu}[P, u, v]$ defined by

$$\begin{aligned} S_{m,n,\lambda,\mu}[P, u, v] &= \sum_{i=0}^{m}\sum_{j=0}^{n} P_{i,j} A_{m,i}(u) A_{n,j}(v) \\ &= A_m^T(u) P A_n(v) \end{aligned} \tag{27}$$

with $0 < m^s\lambda < 1/\left(2\zeta(s)\right)$, $0 < n^s\mu < 1/\left(2\zeta(s)\right)$, $(u, v) \in [0, 1]^2$, $t_i = t_{i,m} = i/m$, $y_j = y_{j,n} = j/n$, $P = (P_{i,j})_{\substack{i=0,\ldots,m \\ j=0,\ldots,n}}$.

Since

$$\sum_{i=0}^{m}\sum_{j=0}^{n} A_{m,i}(n) A_{n,j}(v) = 1,\ \ 0 \le A_{m,i}(u) A_{n,j}(v) \le 1,$$

it follows from (27) that $S_{m,n,\lambda,\mu}[P, u, v]$ is a rational surface of degree (sm, sm) with respect to u and (sn, sn) with respect to v, lying in the convex hull of the control net P.

If λ and μ tend to 0, then $S_{m,n,\lambda,\mu}[P, t_{i,m}, t_{j,n}] \to P_{i,j}$, $0 \le i \le m$, $0 \le j \le n$, hence analogously to the curve case, $S_{m,n,\lambda,\mu}[P]$ may be considered a near-interpolating surface, overcoming the flat spots drawback occurring for original

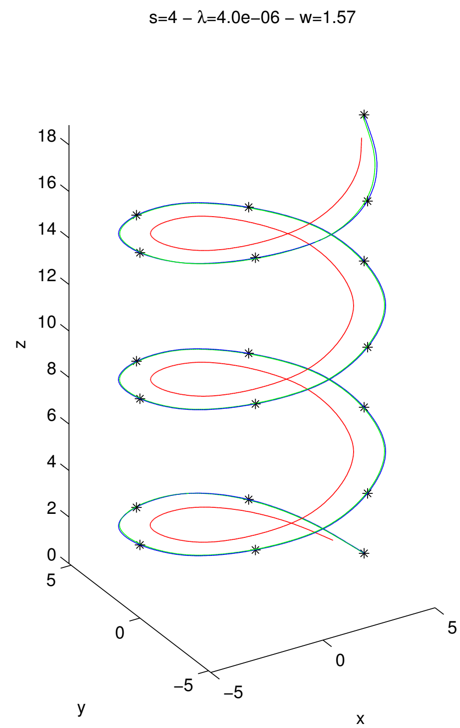

Figure 4: Weighted progressive iterative approximation at the starting, second and fourth iteration

Shepard surfaces. As in the curve case, here the parameters λ and μ may be used as shape control tools to model the form of an object.

In Fig. 7 we use $S_{m,n,\lambda,\mu}$ with $s = 4$ and $\lambda = \mu = 10^{-6}$ to model the shape of the Vesuvius, a volcano near Napoli (Italy), based on 650×380 control vertices.

The results of Section 3 readily generalize to the tensor product setting. Indeed, if $h(u, v)$ denotes the piecewise bilinear function corresponding to the control net P and

$$D(u, v) := S_{m,n,\mu,\lambda}[P, u, v] - h(u, v)$$

denotes the deviation of the Shepard surface $S_{m,n,\lambda,\mu}$ from h, then we can follow [13, Section 3, p. 584] and by Theorem 1 we can bound $D(u, v)$ by directional first forward differences. We omit details.

Also the WPIA process of Section 4 can be easily extended to the tensor

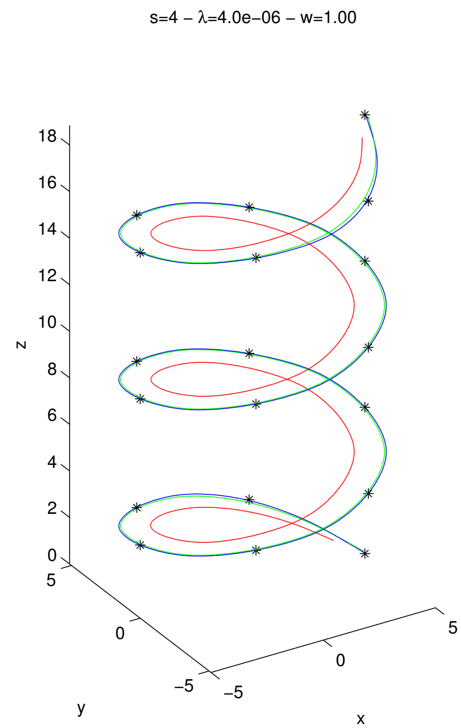

Figure 5: Progressive iterative approximation at the starting, second and fourth iteration

product surfaces. We can generate the initial surface

$$S^0(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} P^0_{ij} A_{n,i}(x) A_{m,j}(y)$$

with $P^0_{ij} = P_{i,j}$ for all $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n$. Then the remaining surfaces of the sequence $S^{k+1}(x,y)$ for $k \geq 0$ can be calculated as follows

$$S^{k+1}(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} P^{k+1}_{i,j} A_{m,i}(x) A_{n,j}(y),$$

where

$$P^{k+1}_{i,j} = P^k_{i,j} + w\Delta^k_{i,j}, \Delta_{i,j} = P_{i,j} - S^k(t_i, y_j), i = 0, \ldots, m, j = 0, \ldots, n.$$

The iterative process can be written in matrix form

$$\Delta^k = (I - wB)\Delta^{k-1} = (I - wB)^k \Delta^0,$$

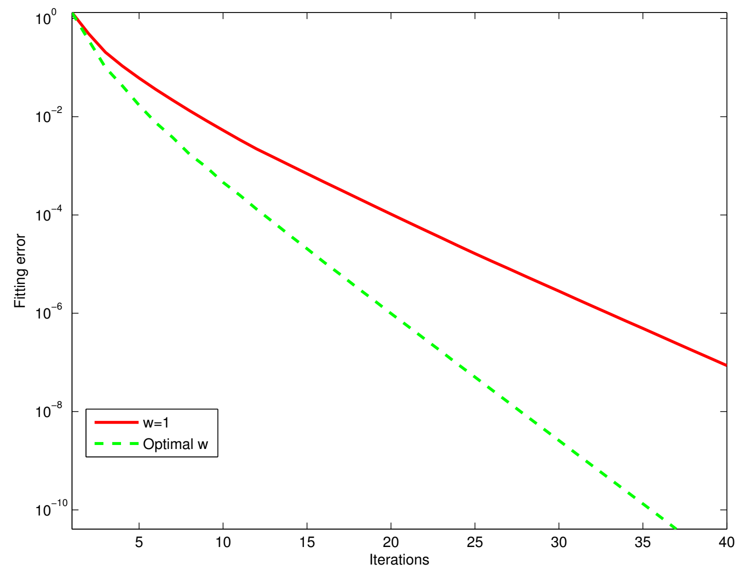

Figure 6: Fitting error vs. iterations

where

$$\Delta^j = \left[\Delta^j_{0,0}, \Delta^j_{0,1}, \ldots, \Delta^j_{0,n}, \Delta^j_{1,0}, \Delta^j_{1,1}, \ldots, \Delta^j_{1,n}, \ldots, \Delta^j_{m,0}, \Delta^j_{m,1}, \ldots \Delta^j_{m,n},\right],\ j = 0, \ldots, k,$$

I is the identity matrix of order $(m+1)(n+1)$ and $B = B_1 \otimes B_2$ is the Kronecker product of two collocation matrices

$$B_1 = A_{m,j}(t_i)^{i=0,\ldots,m}_{j=0,\ldots,m},\ B_2 = A_{n,j}(y_i)^{i=0,\ldots,n}_{j=0,\ldots,n}.$$

Thus we get the surface sequence $S^k(x,y)$, $k = 0, \ldots$. If $\lim_{k\to\infty} S^k(x_i, y_j) = P_{i,j}$, $i = 0, \ldots, m$, $j = 0, \ldots, n$ then we say that the initial surface S^0 has the weighted progressive iteration approximation (WPIA in short) property.

We have

Theorem 9 *The surface S^0 has WPIA property if $\rho(I - wB) < 1$. Moreover the weighted PIA approximation has the fastest convergence rate when*

$$w = \frac{2}{1 + \lambda_m(B_1)\mu_n(B_2)},$$

where $\lambda_m(B_1)$ and $\mu_n(B_2)$ are the smallest eigenvalues of B_1 and B_2, respectively, and in such case

$$\rho(I - wB) = \frac{1 - \lambda_m(B_1)\mu_n(B_2)}{1 + \lambda_m(B_1)\mu_n(B_2)}.$$

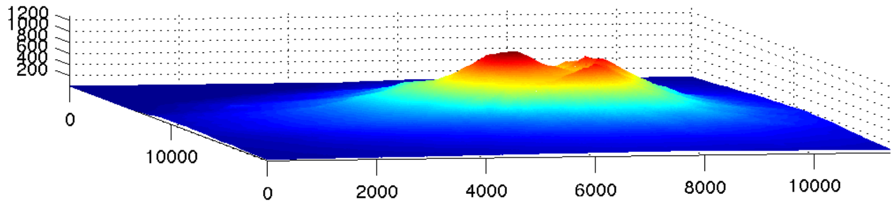

Figure 7: Modelled shape of Vesuvius volcano.

Remarks. Obviously if $w = 1$ then from Theorem 9 we prove that S^0 has the PIA property (cfr. Theorem 8). Finally, as remarked after Theorem 6, the convergence rate is faster than for the Bézier surfaces case [9].

6 Proofs of main results

6.1 Proof of Theorem 1

Let $t \in [t_j, t_{j+1}]$, for some $0 \le j \le n-1$.
Case 1. $|t - t_j| = \min_{i=0,\dots,n-1} |t - t_i| \le 1/(2n)$.
It follows that

$$\begin{aligned} e(t) &= \left| P_j + n(t-t_j)(P_{j+1} - P_j) - \frac{1}{D} \sum_{i=0}^{n} \frac{P_i}{(t-t_i)^s + \lambda} \right| \\ &\le \frac{1}{D} \left| \frac{n(t-t_j)(P_{j+1}-P_j)}{(t-t_j)^s + \lambda} + \frac{P_j + n(t-t_j)(P_{j+1}-P_j) - P_{j+1}}{(t-t_{j+1})^s + \lambda} \right| \\ &\quad + \frac{1}{D} \left\{ \sum_{i=0}^{j-1} + \sum_{i=j+2}^{n} \right\} \frac{|P_j + n(t-t_j)(P_{j+1}-P_j) - P_i|}{(t-t_i)^s + \lambda} \\ &:= A_j + A_j^- + A_j^+. \end{aligned}$$

Clearly

$$A_j \le \frac{1}{D} |P_{j+1} - P_j| \left| \frac{n(t-t_j)}{(t-t_j)^s + \lambda} + \frac{n(t-t_j) - 1}{(t_{j+1} - t)^s + \lambda} \right|.$$

Moreover we can prove that

$$A_j^- \le \frac{1}{D} \Delta P \sum_{k=0}^{j-1} \frac{|j - k + n(t-t_j)|}{(t-t_k)^s + \lambda}.$$

and similarly

$$A_j^+ \leq \frac{1}{D}\Delta P \sum_{k=j+2}^{n} \frac{|k-j-n(t-t_j)|}{(t-t_k)^s+\lambda}.$$

Case 2. $|t_{j+1}-t| \leq 1/(2n)$.
Since $p(t) = P_{j+1} + n(t_{j+1}-t)(P_j - P_{j+1})$

$$\begin{aligned}
e(t) &= \left| P_{j+1} + n(t_{j+1}-t)(P_j-P_{j+1}) - \frac{1}{D}\sum_{i=0}^{n}\frac{P_i}{(t-t_i)^s+\lambda}\right| \\
&\leq \frac{1}{D}\left| n\frac{(t_{j+1}-t)(P_j-P_{j+1})}{(t-t_{j+1})^s+\lambda} + \frac{(n(t_{j+1}-t)-1)\,(P_j-P_{j+1})}{(t_j-t)^s+\lambda}\right| \\
&\quad + \frac{1}{D}\left\{\sum_{i=0}^{j-1} + \sum_{i=j+2}^{n}\right\}\left|\frac{P_{j+1}+n(t_{j+1}-t)(P_j-P_{j+1})-P_i}{(t-t_i)^s+\lambda}\right| \\
&:= B_j + B_j^- + B_j^+.
\end{aligned}$$

Again, clearly

$$B_j \leq \frac{1}{D}\left|P_{j+1}-P_j\right| \left|\frac{n(t_{j+1}-t)}{(t-t_{j+1})^s+\lambda} + \frac{n(t_{j+1}-t)-1}{(t_j-t)^s+\lambda}\right|.$$

Moreover

$$B_j^- \leq \frac{1}{D}\Delta P \sum_{i=0}^{j-1} \frac{j+1-i-n(t_{j+1}-t)}{(t-t_i)^s+\lambda},$$

and

$$B_j^+ \leq \frac{1}{D}\Delta P \sum_{i=j+2}^{n} \frac{i-j-1+n(t_{j+1}-t)}{(t-t_i)^s+\lambda}.$$

And the statement follows. □

The proofs of results of Section 4 are based on some preliminary lemmas interesting in themselves.

In the following denote by $\lambda_i^{(n)} = \lambda_i$, $i = 0, \ldots, n$, the $n+1$ eigenvalues of B, which are sorted in increasing order, i.e., $\lambda_0^{(n)} \leq \lambda_1^{(n)} \leq \cdots \leq \lambda_n^{(n)}$, and denote by $\rho(B)$ its spectral radius, which is the maximal absolute value of its eigenvalues, i.e., $\rho(B) = \max\left\{\left|\lambda_0^{(n)}\right|, \left|\lambda_n^{(n)}\right|\right\}$.

Lemma 1 *For $i = 0, \ldots, n$, we have $1/2 < \lambda_i^{(n)} \leq 1$.*

Proof. We observe that B is a symmetric positive stochastic matrix. If $n^s\lambda \leq 1/(6\zeta(s))$, then by Gerschgorin's theorem and from a well-known result for such matrices $\lambda_n^{(n)} = 1$ and $1/2 < \lambda_i^{(n)} < 1$, $i = 0, \ldots, n-1$. And the assertion follows. □

Remark. Lemma 1 implies that $||I-B|| = \rho(I-B) < 1$, where $\rho(I-B)$ is the spectral radius of the matrix $I-B$, with I the identity matrix.

If $\hat{f}_k(t)$ is the eigenfunction of $S_{n,\lambda}$ corresponding to $\lambda_k^{(n)}$, then

$$S_{n,\lambda}(\hat{f}_k; t) = \lambda_k^{(n)} \hat{f}_k(t). \tag{28}$$

Therefore

Lemma 2 *The eigenvalues of B coincide with the eigenvalues of $S_{n,\lambda}$.*

Proof. In the representation (28) we set $t = 0, 1/n, 2/n, \ldots, 1$ and obtain

$$B\hat{\overline{f}}_k = \lambda_k^{(n)} I \hat{\overline{f}}_k,$$

where $\hat{\overline{f}}_k = \left[\hat{f}_k(0), \hat{f}_k(1/n), \ldots, \hat{f}_k(1)\right]^T$. The last equation implies that

$$\det\left(B - \lambda_k^{(n)} I\right) = 0,$$

i.e., $\lambda_k^{(n)}$ is an eigenvalue of B. $\square$

The next statement is well-known from the theory of linear algebra (cfr. [15]).

Lemma 3 *If T is a square matrix with $\rho(T) < 1$, then the matrix $I - T$ is invertible and we have*

$$(I - T)^{-1} = I + T + T^2 + \cdots.$$

If we set $T = I - B$ in the last formula we obtain

Lemma 4 *The matrix B is invertible and we have*

$$B^{-1} = I + (I - B) + (I - B)^2 + \cdots = \sum_{k=0}^{\infty} (I - B)^k.$$

Lemma 4 shows that the matrix B is nonsingular and gives a useful representation of the inverse matrix B^{-1}.

Lemma 5 *The functions $A_{n,i}$, $i = 0, \ldots, n$, form a basis generating a subspace of rational functions of degree (sn, sn).*

Proof. Assume that $A_{n,i}$ is linearly dependent. Consequently there exist $\alpha_0, \alpha_1, \ldots, \alpha_n$, at least one of which different from 0, such that

$$h_n(x) := \alpha_0 A_{n,0}(x) + \alpha_1 A_{n,1}(x) + \ldots + \alpha_n A_{n,n}(x) = 0$$

for all $x \in [0, 1]$. In particular $h_n(x) = 0$ at $x = 0, 1/n, 2/n, \ldots, 1$. This implies that the rows of B are linearly dependent. But this is impossible because by Lemma 4 we know that the matrix B is invertible, hence its rows are linearly independent. So $A_{n,i}$ are linearly independent. $\square$

We end our study on B with the following observation.

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Lemma 6 *All rows of B and B^{-1}sum to 1.*

Proof. We know that B is stochastic. On the other hand from (6)-(8) if $\overline{f} = [1, \ldots, 1]^T$, then $\overline{f}^G = [1, \ldots, 1]^T$, which implies the rows of B^{-1} sum to 1. □

Lemma 7 *Let $\rho(I - wB) < 1$. Then the sequence of rational operators $\left\{\tilde{S}_m\right\}_{m=0}^{\infty}$ uniformly tends to its limiting operator $\tilde{S}_\infty$ on $[0, 1]$, as $m \to \infty$ and*

$$\tilde{S}_\infty(f; t) = A_n(t) B^{-1} \overline{f}. \tag{29}$$

Proof. The representation (10) implies

$$\tilde{S}_\infty(f; t) - \tilde{S}_m(f; t) = wA_n(t) \left[\sum_{i=m+1}^{\infty} (I - wB)^{i-1} (I - B) \right] \overline{f}. \tag{30}$$

From the assumption $\rho(I - wB) < 1$ by Lemma 3 we know that the power series $I + (I - wB) + (I - wB)^2 \ldots$ is convergent and this implies that the matrix

$$\sum_{i=m+1}^{\infty} (I - wB)^{i-1} \to 0,$$

as $n \to \infty$. From (10) and Lemma 3 we deduce

$$\tilde{S}_\infty(f; t) = A_n(t)(I + w(I - B)w^{-1}B^{-1})\overline{f},$$

that is (29). □

6.2 Proof of Theorem 2

Letting

$$f(t_0) = \bar{I}\bar{f}, \ \bar{I} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & 0 \end{pmatrix},$$

it results $\forall t \in [0, 1]$

$$\left| f(t_0) - S^m_{n,\lambda}(f; t) \right| \leq A_n(t) \left[\bar{I} - B^{m-1} \right] \bar{f}.$$

On the other hand

$$\left\| A_n(t) \left[\bar{I} - B^{m-1} \right] \right\| \leq \left\| \bar{I} - B^{m-1} \right\| = \rho \left(\bar{I} - B^{m-1} \right) \leq \left(\lambda^{(n)}_{n-1} \right)^{m-1}.$$

Since by Lemma 1 $\lambda^{(n)}_{n-1} < 1$, the assertion follows. □

6.3 Proof of Theorem 3

From (29) and (30) and Lemmas 1 and 3

$$\begin{aligned}
\left|\tilde{S}_\infty(f;t_i) - \tilde{S}_m(f;t_i)\right| &= \left|f(t_i) - \tilde{S}_m(f;t_i)\right| \\
&\le w \left\| B(I-B) \sum_{i=m+1}^{\infty} (I-wB)^{i-1} \right\| \|f\| \\
&= w \left\| B(I-B)(I-wB)^m \sum_{i=0}^{\infty} (I-wB)^i \right\| \|f\| \\
&= w \left\| B(I-B)(I-wB)^m w^{-1} B^{-1} \right\| \|f\| \\
&\le ||f|| \left(1-\lambda_0^{(n)}\right) \left(1-w\lambda_0^{(n)}\right)^m \left(\lambda_0^{(n)}\right)^{-1} \\
&< 2||f|| \left(1-\lambda_0^{(n)}\right) \left(1-w\lambda_0^{(n)}\right)^m .
\end{aligned}$$

In particular if $w = 1$, we deduce (13). □

6.4 Proof of Theorem 4

For a given function f let us denote by $\overline{S}_{n,\lambda}(f)$ the vector

$$\overline{S}_{n,\lambda}(f) = \left[S_{n,\lambda}(f;0), S_{n,\lambda}\left(f;\frac{1}{n}\right), \dots, S_{n,\lambda}(f;1) \right]^T .$$

Obviously

$$\overline{S}_{n,\lambda}(f) = B\overline{f}.$$

In a similar way if we denote by

$$\overline{S}_\infty(f) = \left[\tilde{S}_\infty(f;0), \tilde{S}_\infty\left(f;\frac{1}{n}\right), \dots, \tilde{S}_\infty(f;1) \right]^T ,$$

then (29) implies

$$\overline{S}_\infty(f) = BB^{-1}\overline{f} = \overline{f},$$

that is $\tilde{S}_\infty(f)$ is the rational function of type $S_{n,\lambda}$ interpolating f at the knots $0, 1/n, 2/n, \dots, 1$, i.e., $\tilde{S}_\infty = G_{n,\lambda}$. Finally if $0 < w < 2$, then $\rho(I - wB) = (1 - w\lambda_0^{(n)}) < 1$ and we can follow the first part of the proof.

Now we prove (15). From (4) and Lemmas 1, 6 and 7

$$\left|f(t) - \tilde{S}_\infty(f;t)\right| \le |f(t) - S_{n,\lambda}(f;t)| + \left|S_{n,\lambda}(f;t) - \tilde{S}_\infty(f;t)\right|$$

$$\le C\omega\left(f;\frac{1}{n}\right) + A_n(t)B^{-1}\left|B\overline{f} - \overline{f}\right| \le C\omega\left(f;\frac{1}{n}\right).$$

Working as in the proof of the well-known pointwise estimate of Lagrange interpolating polynomial, we get (16). The proof of Theorem 4 is completed. □

6.5 Proof of Theorem 5

From (30) and Lemma 1 working as in the proof of Theorem 3 we have

$$\begin{aligned}\left\|\tilde{S}_\infty(f)-\tilde{S}_m(f)\right\| &\le w\,\|f\|\,\|A_n\|\left\|(I-B)\sum_{i=m+1}^{\infty}(I-wB)^{i-1}\right\|\\ &\le\ \|f\|\left(1-\lambda_0^{(n)}\right)\left(1-w\lambda_0^{(n)}\right)^m\left(\lambda_0^{(n)}\right)^{-1},\end{aligned}$$

that is (17). Working as in [9] the assertion follows. □

6.6 Proof of Theorem 6

We know that

$$A_{n,j}(t_j)=1-\sum_{i\neq j}A_{n,i}(t_j),\ \ j=0,1,\ldots,n,$$

hence by Gerschgorin's theorem for some $j=0,1,\ldots,n$,

$$\begin{aligned}1-\lambda_0^{(n)} &\le 2\sum_{i\neq j}A_{n,i}(t_j)\\ &=2\frac{\sum_{i\neq j}n^s/\left((i-j)^s+n^s\lambda\right)}{1/\lambda+\sum_{i\neq j}n^s/\left((i-j)^s+n^s\lambda\right)}\\ &=2n^s\lambda\frac{\sum_{i\neq j}1/\left((i-j)^s+n^s\lambda\right)}{1+n^s\lambda\sum_{i\neq j}1/\left((i-j)^s+n^s\lambda\right)}\end{aligned}\tag{31}$$

for some j.

If $n^s\lambda=o(1)$, then by (31) one gets $1-\lambda_0^{(n)}=o(1)$ and by Theorem 5 (20) follows. Since $1-\lambda_0^{(n)}<1/2$ (see Lemma 1) by Theorem 5 (19) follows. □

6.7 Proof of Theorem 7

WPIA property follows from the remark to (23) and Theorem 3. From Theorem 5 we deduce the assertion. □

6.8 Proof of Theorem 8

From Theorem 7 and the remark to Lemma 1 we immediately deduce the PIA property for γ^0 curves. By Theorem 6 the assertion follows. □

6.9 Proof of Theorem 9

We can work as in [9]. □

7 Conclusions

Near-interpolating parametric curves of Shepard-type are introduced and studied. Such curves, overcoming the flat spots drawback of original Shepard curves, present some advantages with respect to Bézier curves, like the shape control parameter λ, the pseudo-local control property, a smaller deviation from the control polygon, a faster rate of convergence of weighted progressive iterative approximation process and higher flexibility to model objects. Theorem 1 gives a bound of the maximal distance between such curves and their control polygon in terms of the maximal absolute first order difference of the control points. Such a bound gives a strong localization of the curves, useful in some CAGD problems.

New operators of Shepard-type faster converging than in the analogous Bernstein-Bézier case are studied and approximation results are established in Theorems 2–6. The results are applied to get a weighted progressive iterative approximation algorithm in (21)–(22) and Theorems 7–8 to reach the global interpolating Shepard-type curve without solving a linear system. Such technique gives a straightward and intuitive algorithm to generate sequences of curves with finer and finer precision for data point fitting, of interest in Computer-Aided-Modelling.

Analogously tensor product near-interpolation surfaces of Shepard-type are introduced and similar results are presented to model surfaces. Further research is necessary to address questions on the optimal choice in some sense of the parameter λ, on the optimal choice for the knots instead of equidistant nodes, on sharp bound in the estimate of the deviation and on surfaces extension to triangular patches.

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Hesitant fuzzy set theory applied to BCK/BCI-algebras

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Abstract. The notions of hesitant fuzzy subalgebras and hesitant fuzzy ideals of BCK/BCI-algebras are introduced, and related properties are investigated. Characterizations of hesitant fuzzy subalgebras and hesitant fuzzy ideals of BCK/BCI-algebras are discussed. Given a special set, conditions for this set to be a hesitant fuzzy ideal are provided.

1. **Introduction**

The notions of Atanassov's intuitionistic fuzzy sets, type 2 fuzzy sets and fuzzy multisets etc. are a generalization of fuzzy sets. As another generalization of fuzzy sets, Torra [6] introduced the notion of hesitant fuzzy sets which are a very useful to express peoples hesitancy in daily life. The hesitant fuzzy set is a very useful tool to deal with uncertainty, which can be accurately and perfectly described in terms of the opinions of decision makers. Xu and Xia [11] proposed a variety of distance measures for hesitant fuzzy sets, based on which the corresponding similarity measures can be obtained. They investigated the connections of the aforementioned distance measures and further develop a number of hesitant ordered weighted distance measures and hesitant ordered weighted similarity measures. Also, hesitant fuzzy set theory is used in decision making problem etc. (see [5, 8, 9, 10, 12]), and is applied to residuated lattices and MTL-algebras (see [2, 4]).

In this paper, we introduce the notions of hesitant fuzzy subalgebras and hesitant fuzzy ideals of BCK/BCI-algebras, and investigate their relations and properties. We consider characterizations of hesitant fuzzy subalgebras and hesitant fuzzy ideals of BCK/BCI-algebras. Given a special set, we provide conditions for this set to be a hesitant fuzzy ideal.

2. **Preliminaries**

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X;*,0)$ of type $(2,0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x,y,z\in X)\ (((x*y)*(x*z))*(z*y)=0)$,

(II) $(\forall x,y\in X)\ ((x*(x*y))*y=0)$,

(III) $(\forall x\in X)\ (x*x=0)$,

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(IV) $(\forall x, y \in X)$ $(x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)$ $(0 * x = 0)$,

then X is called a *BCK-algebra*. A BCK-algebra X is said to be *positive implicative* if it satisfies:

$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * (y * z)). \tag{2.1}$$

A BCK-algebra X is said to be *implicative* if it satisfies:

$$(\forall x, y \in X)\,(x = x * (y * x)). \tag{2.2}$$

Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x), \tag{2.3}$$

$$(\forall x, y, z \in X)\,(x \leq y \;\Rightarrow\; x * z \leq y * z,\; z * y \leq z * x), \tag{2.4}$$

$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y), \tag{2.5}$$

$$(\forall x, y, z \in X)\,((x * z) * (y * z) \leq x * y) \tag{2.6}$$

where $x \leq y$ if and only if $x * y = 0$. A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset A of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

(b1) $0 \in A$.

(b2) $(\forall x \in X)(\forall y \in A)(x * y \in A \;\Rightarrow\; x \in A)$.

We refer the reader to the books [1, 3] for further information regarding BCK/BCI-algebras.

3. Hesitant fuzzy subalgebras/ideals

Definition 3.1 ([6, 7]). Let E be a reference set. A *hesitant fuzzy set* on E is defined in terms of a function that when applied to E returns a subset of $[0, 1]$, which can be viewed as the following mathematical representation:

$$H_E := \{(e, h_E(e)) \mid e \in E\}$$

where $h_E : E \to \mathscr{P}([0, 1])$.

In what follows, we take a BCK/BCI-algebra X as a reference set unless otherwise specified.

Definition 3.2. Given a non-empty subset A of X, a hesitant fuzzy set

$$H_X := \{(x, h_X(x)) \mid x \in X\}$$

on X satisfying the following condition:

$$h_X(x) = \emptyset \text{ for all } x \notin A \tag{3.1}$$

is called a *hesitant fuzzy set related to A* (briefly, *A-hesitant fuzzy set*) on X, and is represented by $H_A := \{(x, h_A(x)) \mid x \in X\}$, where h_A is a mapping from X to $\mathscr{P}([0, 1])$ with $h_A(x) = \emptyset$ for all $x \notin A$.

Definition 3.3. Given a non-empty subset (subalgebra as much as possible) A of X, let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy set on X. Then $H_A := \{(x, h_A(x)) \mid x \in X\}$ is called a *hesitant fuzzy subalgebra of X related to A* (briefly, *A-hesitant fuzzy subalgebra* of X) if it satisfies the following condition:

$$(\forall x, y \in A)\,(h_A(x * y) \supseteq h_A(x) \cap h_A(y))\,. \tag{3.2}$$

An A-hesitant fuzzy subalgebra of X with $A = X$ is called a *hesitant fuzzy subalgebra* of X.

Example 3.4. Let $X = \{0, 1, a, b, c\}$ be a BCI-algebra with the following Cayley table:

$*$	0	1	a	b	c
0	0	0	c	b	a
1	1	0	c	b	a
a	a	a	0	c	b
b	b	b	a	0	c
c	c	c	b	a	0

For a subalgebra $A = \{0, a, b, c\}$ of X, let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy set on X defined by

$$h_A(x) = \begin{cases} [0,1] & \text{if } o(x) = 1, \\ \{k \in [0,1] \mid k \leq \frac{1}{r}\} & \text{if } o(x) = r \neq 1 \end{cases}$$

where $o(x) = \min\{n \in \mathbb{N} \mid 0 * x^n = 0\}$. Then

$$H_A = \left\{(0, [0,1]), (1, \emptyset), (a, [0, \tfrac{1}{4}]), (b, [0, \tfrac{1}{2}]), (c, [0, \tfrac{1}{4}])\right\}$$

and it is an A-hesitant fuzzy subalgebra of X.

Theorem 3.5. *Let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy set on X where A is a non-empty subset (subalgebra as much as possible) of X. Then the following are equivalent:*

(1) *$H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy subalgebra of X.*
(2) *For any $\varepsilon \in \mathscr{P}([0,1])$, the set $H_A(\varepsilon) := \{x \in X \mid h_A(x) \supseteq \varepsilon\}$ is a subalgebra of X whenever it is non-empty.*

The set $H_A(\varepsilon) := \{x \in X \mid h_A(x) \supseteq \varepsilon\}$ is called a *hesitant fuzzy ε-level set* of $H_A := \{(x, h_A(x)) \mid x \in X\}$.

Proof. Assume that $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy subalgebra of X. Let $\varepsilon \in \mathscr{P}([0,1])$ be such that $H_A(\varepsilon) := \{x \in X \mid h_A(x) \supseteq \varepsilon\} \neq \emptyset$. If $x, y \in H_A(\varepsilon)$, then $h_A(x) \supseteq \varepsilon$ and $h_A(y) \supseteq \varepsilon$. It follows from (3.2) that

$$h_A(x * y) \supseteq h_A(x) \cap h_A(y) \supseteq \varepsilon$$

and so that $x * y \in H_A(\varepsilon)$. Therefore $H_A(\varepsilon) := \{x \in X \mid h_A(x) \supseteq \varepsilon\}$ is a subalgebra of X for all $\varepsilon \in \mathscr{P}([0,1])$ whenever it is non-empty.

Conversely, suppose that the non-empty hesitant fuzzy ε-level set of $H_A := \{(x, h_A(x)) \mid x \in X\}$ is a subalgebra of X for all $\varepsilon \in \mathscr{P}([0,1])$. For any $x, y \in A$, let $h(x) = \varepsilon_x$ and $h(y) = \varepsilon_y$. Take

$\varepsilon = \varepsilon_x \cap \varepsilon_y$. Then $x, y \in H_A(\varepsilon)$, and so $x * y \in H_A(\varepsilon)$. Hence $h_A(x * y) \supseteq \varepsilon = \varepsilon_x \cap \varepsilon_y = h(x) \cap h(y)$. Therefore $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy subalgebra of X. □

Definition 3.6. Given a non-empty subset (subalgebra as much as possible) A of X, an A-hesitant fuzzy set $H_A := \{(x, h_A(x)) \mid x \in X\}$ on X is called a *hesitant fuzzy ideal of X related to A* (briefly, *A-hesitant fuzzy ideal* of X) if it satisfies:

$$(\forall x, y \in A)\left(h_A(x * y) \cap h_A(y) \subseteq h_A(x) \subseteq h_A(0)\right). \tag{3.3}$$

An A-hesitant fuzzy ideal of X with $A = X$ is called a *hesitant fuzzy ideal* of X.

Example 3.7. (1) Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

For a subalgebra $A = \{0, a, b\}$ of X, let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_A = \left\{(0, [\tfrac{1}{4}, \tfrac{3}{4}]), (a, (\tfrac{1}{4}, \tfrac{1}{2})), (b, [\tfrac{1}{2}, \tfrac{3}{4})), (c, \emptyset)\right\}.$$

Then $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy ideal of X.
(2) Let $X = \{0, 1, a, b, c\}$ be a BCI-algebra with the following Cayley table:

$*$	0	1	a	b	c
0	0	0	a	b	c
1	1	0	a	b	c
a	a	a	0	c	b
b	b	b	c	0	a
c	c	c	b	a	0

Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_X = \{(0, [0, 1)), (1, [0.2, 0.7]), (a, (0.2, 0.3]), (b, \{0.4, 0.5\}), (c, [0.6, 0.7))\}.$$

It is routine to verify that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy ideal of X.

Theorem 3.8. *Let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy set on X where A is a non-empty subset (subalgebra as much as possible) of X. Then the following are equivalent:*

1. *$H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy ideal of X.*
2. *The non-empty hesitant fuzzy ε-level set of $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an ideal of X for all $\varepsilon \in \mathscr{P}([0, 1])$.*

Proof. Suppose that $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy ideal of X. Let $x, y \in X$ and $\varepsilon \in \mathscr{P}([0,1])$ be such that $x * y \in H_A(\varepsilon)$ and $y \in H_A(\varepsilon)$. Then $h_A(x*y) \supseteq \varepsilon$ and $h_A(y) \supseteq \varepsilon$. It follows from (3.3) that

$$h_A(0) \supseteq h_A(x) \supseteq h_A(x*y) \cap h_A(y) \supseteq \varepsilon.$$

Hence $0 \in H_A(\varepsilon)$ and $x \in H_A(\varepsilon)$, and therefore $H_A(\varepsilon)$ is an ideal of X.

Conversely, assume that the second condition is valid. For any $x \in X$, let $h_A(x) = \varepsilon_x$. Then $x \in H_A(\varepsilon_x)$. Since $H_A(\varepsilon_x)$ is an ideal of X, we have $0 \in H_A(\varepsilon_x)$ and so $h_A(x) = \varepsilon_x \subseteq h_A(0)$. For any $x, y \in A$, let $h_A(x*y) = \varepsilon_{x*y}$ and $h_A(y) = \varepsilon_y$. If we take $\varepsilon = \varepsilon_{x*y} \cap \varepsilon_y$, then $x * y \in H_A(\varepsilon)$ and $y \in H_A(\varepsilon)$ which imply that $x \in H_A(\varepsilon)$ Thus $h_A(x) \supseteq \varepsilon = \varepsilon_{x*y} \cap \varepsilon_y = h_A(x*y) \cap h_A(y)$. Therefore $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy ideal of X. □

Proposition 3.9. *Let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy ideal of X where A is a subalgebra of X. Then the following assertions are valid.*

(1) $(\forall x, y \in A)\,(x \leq y \;\Rightarrow\; h_A(x) \supseteq h_A(y))$.
(2) $(\forall x, y, z \in A)\,(x * y \leq z \;\Rightarrow\; h_A(x) \supseteq h_A(y) \cap h_A(z))$.

Proof. (1) Let $x, y \in A$ be such that $x \leq y$. Then $0 = x * y \in A$, and so

$$h_A(y) = h_A(0) \cap h_A(y) = h_A(x*y) \cap h_A(y) \subseteq h_A(x)$$

by (3.3).
(2) Let $x, y, z \in A$ be such that $x * y \leq z$. Then

$$h_A(z) = h_A(0) \cap h_A(z) = h_A((x*y)*z) \cap h_A(z) \subseteq h_A(x*y).$$

It follows that $h_A(y) \cap h_A(z) \subseteq h_A(x*y) \cap h_A(y) \subseteq h_A(x)$. □

Corollary 3.10. *Every hesitant fuzzy ideal $H_X := \{(x, h_X(x)) \mid x \in X\}$ of X satisfies the following assertions.*

(1) $h_X(y) \subseteq h_X(x)$ *for all* $x, y \in X$ *with* $x \leq y$.
(2) $h_X(y) \cap h_X(z) \subseteq h_X(x)$ *for all* $x, y, z \in X$ *with* $x * y \leq z$.

The following corollary is easily proved by induction.

Corollary 3.11. *Given a subalgebra A of X, every A-hesitant fuzzy ideal $H_A := \{(x, h_A(x)) \mid x \in X\}$ of X satisfies the following condition:*

$$(\cdots(x * a_1) * \cdots) * a_n = 0 \;\Rightarrow\; \bigcap_{k=1}^{n} h_A(a_k) \subseteq h_A(x) \tag{3.4}$$

for all $x, a_1, a_2, \cdots, a_n \in X$.

Proposition 3.12. *Let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy ideal of X where A is a subalgebra of X. Then the following assertions are equivalent.*

(1) $(\forall x, y \in A)\,(h_A((x*y)*y) \subseteq h_A(x*y))$.

(2) $(\forall x, y, z \in A)\,(h_A((x*y)*z) \subseteq h_A((x*z)*(y*z)))$.

Proof. Assume that (1) is valid and let $x, y, z \in A$. Since

$$((x*(y*z))*z)*z = ((x*z)*(y*z))*z \leq (x*y)*z,$$

it follows from Proposition 3.9(1), (1) and (2.5) that

$$\begin{aligned} h_A((x*y)*z) &\subseteq h_A(((x*(y*z))*z)*z) \\ &\subseteq h_A((x*(y*z))*z) \\ &= h_A((x*z)*(y*z)). \end{aligned}$$

Conversely, suppose that (2) holds. If we put $z := y$ in (2), then

$$h_A((x*y)*y) \subseteq h_A((x*y)*(y*y)) = h_A((x*y)*0) = h_A(x*y)$$

which proves (1). □

Theorem 3.13. *For a subalgebra A of a BCK-algebra X, every A-hesitant fuzzy ideal is an A-hesitant fuzzy subalgebra.*

Proof. Let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy ideal of X. Then

$$\begin{aligned} h_A(x*y) &\supseteq h_A((x*y)*x) \cap h_A(x) = h_A((x*x)*y) \cap h_A(x) \\ &= h_A(0*y) \cap h_A(x) = h_A(0) \cap h_A(x) \supseteq h_A(x) \cap h_A(y) \end{aligned}$$

by (3.3), (2.5), (III) and (V). Hence $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy subalgebra of X. □

The following example shows that the converse of Theorem 3.13 is not true in general.

Example 3.14. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	c	c	a	0

Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_X = \left\{(0,[0,1]), (a,(0,\tfrac{1}{4})), (b,(0,\tfrac{1}{2})), (c,[\tfrac{1}{4},\tfrac{3}{4})), (d,(\tfrac{6}{8},\tfrac{7}{8}))\right\}.$$

Then $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy subalgebra of X, but not a hesitant fuzzy ideal of X since $h_X(d*b) \cap h_X(b) = [\frac{1}{4}, \frac{1}{2}) \nsubseteq (\frac{6}{8}, \frac{7}{8}) = h_X(d)$.

In a BCI-algebra X, Theorem 3.13 is not true in general as seen in the following example.

Example 3.15. Let $(Y, *, 0)$ be a BCI-algebra and $(\mathbb{Z}, +, 0)$ an additive group of integers. Let $(\mathbb{Z}, -, 0)$ be the adjoint BCI-algebra of $(\mathbb{Z}, +, 0)$ and let $X := Y \times \mathbb{Z}$. Then $(X, \otimes, (0, 0))$ is a BCI-algebra where the operation $\otimes$ is given by

$$(\forall (x, m), (y, n) \in X)\,((x, m) \otimes (y, n) = (x * y, m - n))\,.$$

For a subset $A := Y \times \mathbb{N}_0$ of X where $\mathbb{N}_0$ is the set of nonnegative integers, let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X in which h_X is given as follows:

$$h_X : X \to \mathscr{P}([0, 1]),\ x \mapsto \begin{cases} [\frac{1}{2}, 1] & \text{if } x \in A, \\ [\frac{1}{3}, 1] & \text{otherwise.} \end{cases}$$

Then $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy ideal of X. But it is not a hesitant fuzzy subalgebra of X since

$$h_X((0, 0) \otimes (0, 1)) = h_X(0, -1) = [\tfrac{1}{3}, 1] \not\supseteq [\tfrac{1}{2}, 1] = h_X(0, 0) \cap h_X(0, 1).$$

Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X. For any $a, b \in X$ and $n \in \mathbb{N}$, let

$$h_X[b; a^n] := \{x \in X \mid h_X((x * b) * a^n) = h_X(0)\}$$

where $(x * b) * a^n = ((\cdots((x * b) * a) * a) * \cdots) * a$ in which a appears n-times. Obviously, $a, b, 0 \in h_X[b; a^n]$.

Proposition 3.16. *Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X such that $h_X(x) \subseteq h_X(0)$ and $h_X(x * y) = h_X(x) \cup h_X(y)$ for all $x, y \in X$. For any $a, b \in X$ and $n \in \mathbb{N}$, if $x \in h_X[b; a^n]$ then $x * y \in h_X[b; a^n]$ for all $y \in X$.*

Proof. Let $x \in h_X[b; a^n]$. Then $h_X((x * b) * a^n) = h_X(0)$, and thus

$$\begin{aligned} h_X(((x * y) * b) * a^n) &= h_X(((x * b) * y) * a^n) \\ &= h_X(((x * b) * a^n) * y) \\ &= h_X((x * b) * a^n) \cup h_X(y) \\ &= h_X(0) \cup h_X(y) = h_X(0) \end{aligned}$$

for all $y \in X$. Hence $x * y \in h_X[b; a^n]$ for all $y \in X$. □

Proposition 3.17. *Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on a BCK-algebra X. If an element $a \in X$ satisfies:*

$$(\forall x \in X)\,(x \leq a)\,, \tag{3.5}$$

then $h_X[b; a^n] = X = h_X[a; b^n]$ for all $b \in X$ and $n \in \mathbb{N}$.

Proof. Let $b, x \in X$ and $n \in \mathbb{N}$. Then

$$\begin{aligned} h_X((x * b) * a^n) &= h_X(((x * b) * a) * a^{n-1}) \\ &= h_X(((x * a) * b) * a^{n-1}) \\ &= h_X((0 * b) * a^{n-1}) \\ &= h_X(0) \end{aligned}$$

by (2.5), (3.5) and (V), and so $x \in h_X[b; a^n]$, which shows that $h_X[b; a^n] = X$. Similarly $h_X[a; b^n] = X$. □

Corollary 3.18. *If $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy set on a bounded BCK-algebra X, then $h_X[b; u^n] = X = h_X[u; b^n]$ for all $b \in X$ and $n \in \mathbb{N}$ where u is the unit of X.*

Proposition 3.19. *Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X such that*

$$(\forall x, y \in X)\,(x \leq y \;\Rightarrow\; h_X(x) \supseteq h_X(y))\,. \tag{3.6}$$

If $b \leq c$ in X, then $h_X[b; a^n] \subseteq h_X[c; a^n]$ for all $a \in X$ and $n \in \mathbb{N}$.

Proof. Let $b, c \in X$ be such that $b \leq c$. For any $a \in X$ and $n \in \mathbb{N}$, if $x \in h_X[b; a^n]$ then

$$\begin{aligned} h_X(0) &= h_X((x * b) * a^n) = h_X((x * a^n) * b) \\ &\subseteq h_X((x * a^n) * c) = h_X((x * c) * a^n) \end{aligned}$$

by (2.4) and (3.6), and so $h_X((x * c) * a^n) = h_X(0)$. Thus $x \in h_X[c; a^n]$, and therefore $h_X[b; a^n] \subseteq h_X[c; a^n]$ for all $a \in X$ and $n \in \mathbb{N}$. □

Corollary 3.20. *If $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy ideal of X, then $h_X[b; a^n] \subseteq h_X[c; a^n]$ for all $n \in \mathbb{N}$ and $a, b, c \in X$ with $b \leq c$.*

The following example shows that there exists a hesitant fuzzy set $H_X := \{(x, h_X(x)) \mid x \in X\}$ on X such that the set $h_X[b; a^n]$ is not an ideal of X for some $a, b \in X$ and $n \in \mathbb{N}$.

Example 3.21. Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_X = \left\{(0, [0, \tfrac{1}{2}]), (a, [0, \tfrac{1}{3}]), (b, [0, \tfrac{1}{3}]), (c, [0, \tfrac{1}{3}])\right\}.$$

Then $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy set (moreover, hesitant fuzzy ideal) on X and $h_X[a; c] = \{x \in X \mid h_X((x * a) * c) = h_X(0)\} = \{0, a, c\}$ which is not an ideal of X since $b * a = a \in h_X[a; c]$ but $b \notin h_X[a; c]$.

We now consider conditions for a set $h_X[b; a^n]$ to be an ideal of X.

Theorem 3.22. *Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on a positive implicative BCK-algebra X in which h_X is injective. Then $h_X[b; a^n]$ is an ideal of X for all $a, b \in X$ and $n \in \mathbb{N}$.*

Proof. Let $a, b, x, y \in X$ and $n \in \mathbb{N}$ be such that $x * y \in h_X[b; a^n]$ and $y \in h_X[b; a^n]$. Then $h_X((y * b) * a^n) = h_X(0)$, which implies that $(y * b) * a^n = 0$ since h_X is injective. Hence

$$\begin{aligned}
h_X(0) &= h_X(((x * y) * b) * a^n) \\
&= h_X((((x * y) * b) * a) * a^{n-1}) \\
&= h_X((((x * b) * (y * b)) * a) * a^{n-1}) \\
&= h_X(((((x * b) * a) * ((y * b) * a)) * a) * a^{n-2}) \\
&= \cdots \\
&= h_X(((x * b) * a^n) * ((y * b) * a^n)) \\
&= h_X(((x * b) * a^n) * 0) \\
&= h_X((x * b) * a^n),
\end{aligned}$$

which shows that $x \in h_X[b; a^n]$. Therefore $h_X[b; a^n]$ is an ideal of X for all $a, b \in X$ and $n \in \mathbb{N}$. □

Theorem 3.23. *Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on a positive implicative BCK-algebra X such that*

$$(\forall x, y \in X)\,(h_X(x) \subseteq h_X(0),\ h_X(x * y) = h_X(x) \cap h_X(y)). \tag{3.7}$$

Then $h_X[b; a^n]$ is an ideal of X for all $a, b \in X$ and $n \in \mathbb{N}$.

Proof. Let $a, b, x, y \in X$ and $n \in \mathbb{N}$ be such that $x * y \in h_X[b; a^n]$ and $y \in h_X[b; a^n]$. Then $h_X((y * b) * a^n) = h_X(0)$, which implies from (3.7) that

$$\begin{aligned}
h_X(0) &= h_X(((x * y) * b) * a^n) \\
&= h_X((((x * y) * b) * a) * a^{n-1}) \\
&= h_X((((x * b) * (y * b)) * a) * a^{n-1}) \\
&= h_X(((((x * b) * a) * ((y * b) * a)) * a) * a^{n-2}) \\
&= \cdots \\
&= h_X(((x * b) * a^n) * ((y * b) * a^n)) \\
&= h_X((x * b) * a^n) \cap h_X((y * b) * a^n) \\
&= h_X((x * b) * a^n) \cap h_X(0) \\
&= h_X((x * b) * a^n).
\end{aligned}$$

Hence $x \in h_X[b; a^n]$, and therefore $h_X[b; a^n]$ is an ideal of X for all $a, b \in X$ and $n \in \mathbb{N}$. □

Since every implicative BCK-algebra is positive implicative, we have the following corollary.

Corollary 3.24. *Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on an implicative BCK-algebra X in which the condition (3.7) is valid or h_X is injective. Then $h_X[b; a^n]$ is an ideal of X for all $a, b \in X$ and $n \in \mathbb{N}$.*

Corollary 3.25. *Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on a BCK-algebra X, where X satisfies any one of the following conditions:*

$$(\forall x, y \in X)\,(x * y = (x * y) * y)\,, \tag{3.8}$$

$$(\forall x, y \in X)\,((x * (x * y)) * (y * x) = x * (x * (y * (y * x))))\,, \tag{3.9}$$

$$(\forall x, y \in X)\,(x * y = (x * y) * (x * (x * y)))\,, \tag{3.10}$$

$$(\forall x, y \in X)\,(x * (x * y) = (x * (x * y)) * (x * y))\,, \tag{3.11}$$

$$(\forall x, y \in X)\,((x * (x * y)) * (y * x) = (y * (y * x)) * (x * y))\,. \tag{3.12}$$

If $H_X := \{(x, h_X(x)) \mid x \in X\}$ satisfies the condition (3.7) or h_X is injective, then $h_X[b; a^n]$ is an ideal of X for all $a, b \in X$ and $n \in \mathbb{N}$.

Proposition 3.26. *Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X in which h_X is injective. If J is an ideal of X, then the following assertion is valid.*

$$(\forall a, b \in J)\,(\forall n \in \mathbb{N})\,(h_X[b; a^n] \subseteq J)\,. \tag{3.13}$$

Proof. For any $a, b \in J$ and $n \in \mathbb{N}$, let $x \in h_X[b; a^n]$. Then

$$h_X(((x * b) * a^{n-1}) * a) = h_X((x * b) * a^n) = h_X(0)$$

and so $((x * b) * a^{n-1}) * a = 0 \in J$ because h_X is injective. Since J is an ideal of X, it follows from (b2) that $(x * b) * a^{n-1} \in J$. Continuing this process, we have $x * b \in J$ and thus $x \in J$. Therefore $h_X[b; a^n] \subseteq J$ for all $a, b \in J$ and $n \in \mathbb{N}$. □

Theorem 3.27. *Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on a BCK-algebra X. For any subset J of X, if the condition (3.13) holds, then J is an ideal of X.*

Proof. Suppose that the condition (3.13) is valid. Not that $0 \in h_X[b; a^n] \subseteq J$. Let $x, y \in X$ be such that $x * y \in J$ and $y \in J$. Taking $b := x * y$ implies that

$$\begin{aligned} h_X((x * b) * y^n) &= h_X((x * (x * y)) * y^n) \\ &= h_X(((x * (x * y)) * y) * y^{n-1}) \\ &= h_X(((x * y) * (x * y)) * y^{n-1}) \\ &= h_X(0 * y^{n-1}) = h_X(0), \end{aligned}$$

and so $x \in h_X[b; y^n] \subseteq J$ with $b = x * y$. Therefore J is an ideal of X. □

Theorem 3.28. *If $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy ideal of X, then the set*

$$H_a := \{x \in X \mid h_X(a) \subseteq h_X(x)\}$$

is an ideal of X for all $a \in X$.

Proof. Let $x, y \in X$ be such that $x * y \in H_a$ and $y \in H_a$. Then $h_X(a) \subseteq h_X(x * y)$ and $h_X(a) \subseteq h_X(y)$. It follows from (3.3) that

$$h_X(a) \subseteq h_X(x * y) \cap h_X(y) \subseteq h_X(x) \subseteq h_X(0)$$

and so that $0 \in H_a$ and $x \in H_a$. Therefore H_a is an ideal of X for all $a \in X$. □

Theorem 3.29. *Let $a \in X$ and let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X. Then*

(1) *If H_a is an ideal of X, then $H_X := \{(x, h_X(x)) \mid x \in X\}$ satisfies:*

$$(\forall x, y \in X)\,(h_X(a) \subseteq h_X(x * y) \cap h_X(y) \;\Rightarrow\; h_X(a) \subseteq h_X(x))\,. \tag{3.14}$$

(2) *If $H_X := \{(x, h_X(x)) \mid x \in X\}$ satisfies the condition (3.14) and $h_X(x) \subseteq h_X(0)$ for all $x \in X$, then H_a is an ideal of X.*

Proof. (1) Assume that H_a is an ideal of X and let $x, y \in X$ be such that $h_X(a) \subseteq h_X(x*y) \cap h_X(y)$. Then $x * y \in H_a$ and $y \in H_a$, which imply that $x \in H_a$, that is, $h_X(a) \subseteq h_X(x)$.
(2) Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X in which the condition (3.14) holds and $h_X(x) \subseteq h_X(0)$ for all $x \in X$. Then $0 \in H_a$. Let $x, y \in X$ be such that $x * y \in H_a$ and $y \in H_a$. Then $h_X(a) \subseteq h_X(x * y)$ and $h_X(a) \subseteq h_X(y)$, and so $h_X(a) \subseteq h_X(x * y) \cap h_X(y)$. It follows from (3.14) that $h_X(a) \subseteq h_X(x)$, that is, $x \in H_a$. Therefore H_a is an ideal of X. □

4. Conclusions

We have introduced the notions of hesitant fuzzy subalgerbas and hesitant fuzzy ideals of BCK/BCI-algebras, and have investigated their relations and properties. We have considered characterizations of hesitant fuzzy subalgerbas and hesitant fuzzy ideals of BCK/BCI-algebras. Given a special set, we have provided conditions for this set to be a hesitant fuzzy ideal. Future research will focus on applying the notions/contents to other types of ideals in BCK/BCI-algebras and related algebraic structures.

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Fractional q-integrodifference equations and inclusions with nonlocal fractional q-integral conditions

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Abstract

In this paper, we study a class of fractional q-integrodifference equations with nonlocal fractional q-integral boundary conditions which have different quantum numbers. Some new existence and uniqueness results are obtained by using fixed point theorems. Both cases the single- and multi-valued are considered. Some examples illustrating our results are also presented.

Keywords: fractional q-difference equation; boundary value problem; existence; fixed point theorems
2010 Mathematics Subject Classifications: 34A08; 34B18; 39A13.

1 Introduction

In this paper, we will study the existence and uniqueness of solutions of a class of fractional q-integrodifference equations with nonlocal fractional q-integral conditions which have different quantum numbers. In the first part, we deal with the following nonlocal fractional q-integral boundary value problem of nonlinear fractional q-integrodifference equation

$$\begin{cases} {}^cD_q^{\alpha}x(t) = f(t, x(t), I_z^{\delta}x(t)), \quad t \in (0,T), \\ x(\zeta) = g(x), \qquad \lambda I_p^{\beta}x(\eta) = I_r^{\gamma}x(\xi), \quad 0 < \zeta < \eta < \xi < T, \end{cases} \tag{1.1}$$

where $0 < p, q, r, z < 1$, $1 < \alpha \le 2$, $\beta, \gamma, \delta > 0$, $\lambda \in \mathbb{R}$ are given constants, D_q^{α} is the fractional q-derivative of Caputo type of order α, I_{ϕ}^{ψ} is the fractional ϕ-integral of order ψ with $\phi = p, r, z$ and $\psi = \beta, \gamma, \delta$, $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : C([0,T], \mathbb{R}) \to \mathbb{R}$ are continuous functions.

The study of q-difference equations, initiated by Jackson [20, 21], Carmichael [12], Mason [24] and Adams [1] in the first quarter of 20th century, has been developed over the years, for instance, see [14, 17, 22]. In recent years, the topic has attracted the attention of several researchers and a variety of new results can be found in the papers [3], [4], [5], [6], [7], [8], [13], [15], [16], [18], [19], [23].

The case $\zeta = 0, g = 0$ was studied recently in [2], where existence and uniqueness results are proved by applying Banach's contraction principle, Krasnoselskii's fixed point theorem and Leray-Schauder's Nonlinear Alternative.

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Nonlocal conditions were initiated by Bitsadze [9]. As remarked by Byszewski [10, 11], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena. For example, $g(x)$ may be given by $g(x) = \sum_{i=1}^{p} c_i x(t_i)$ where $c_i, i = 1, \ldots, p$, are given constants and $0 < t_1 < \ldots < t_p \leq T$.

In Section 3 we give some sufficient conditions for the existence and uniqueness of solutions and for the existence of at least one solution of problem (1.1). The first result is based on Banach's contraction principle and the second on a fixed point theorem due to D. O'Regan [25]. Concrete examples are also provided to illustrate the possible applications of the established analytical results.

In the second part we consider the multi-valued analogue of problem (1.1) given by

$$\begin{cases} {}^cD_q^\alpha x(t) \in F(t, x(t), I_z^\delta x(t)), \quad t \in (0,T), \\ x(\zeta) = g(x), \qquad \lambda I_p^\beta x(\eta) = I_r^\gamma x(\xi), \quad 0 < \zeta < \eta < \xi < T, \end{cases} \tag{1.2}$$

where $F : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalue map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subset of $\mathbb{R}$.

We give an existence result for the problem (1.2) in the case when the right hand side is convex valued by using the Nonlinear Alternative for contractive maps.

Note that there are four different quantum numbers and the boundary condition of (1.1) does not contain the value of unknown function x at the right-side of boundary point $t = T$. *One may interpret the q-integral boundary condition in (1.1) as the q-integrals with different quantum numbers are related through a real number λ.*

The paper is organized as follows: In Section 2, for the convenience of the reader, we cite some definitions and fundamental results on q-calculus as well as the fractional q-calculus. An auxiliary lemma, needed in the proofs of our main results is presented in Section 3. In Section 4 we prove our main results for single-valued case and in Section 5 we prove our main results for multi-valued case.

2 Preliminaries

To make this paper self-contained, below we recall some known facts on fractional q-calculus. The presentation here can be found in, for example, [7], [16], [26].

For $q \in (0,1)$, define

$$[a]_q = \frac{1-q^a}{1-q}, \quad a \in \mathbb{R}. \tag{2.1}$$

The q-analogue of the power function $(1-b)^k$ with $k \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ is

$$(1-b)^{(0)} = 1, \quad (1-b)^{(k)} = \prod_{i=0}^{k-1} (1 - bq^i), \quad k \in \mathbb{N}, \quad b \in \mathbb{R}. \tag{2.2}$$

More generally, if $\gamma \in \mathbb{R}$, then

$$(1-b)^{(\gamma)} = \prod_{i=0}^{\infty} \frac{1-bq^i}{1-bq^{\gamma+i}}. \tag{2.3}$$

We use the notation $0^{(\gamma)} = 0$ for $\gamma > 0$. The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}. \tag{2.4}$$

Obviously, $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$.

The q-derivative of a function h is defined by

$$(D_q h)(x) = \frac{h(x) - h(qx)}{(1-q)x} \quad \text{for} \quad x \neq 0 \quad \text{and} \quad (D_q h)(0) = \lim_{x \to 0} (D_q h)(x), \tag{2.5}$$

and q-derivatives of higher order are given by

$$(D_q^0 h)(x) = h(x) \quad \text{and} \quad (D_q^k h)(x) = D_q(D_q^{k-1} h)(x), \quad k \in \mathbb{N}. \tag{2.6}$$

The q-integral of a function h defined on the interval $[0,b]$ is given by

$$(I_q h)(x) = \int_0^x h(s)d_q s = x(1-q)\sum_{i=0}^{\infty} h(xq^i)q^i, \quad x \in [0,b]. \tag{2.7}$$

If $a \in [0,b]$ and h is defined in the interval $[0,b]$, then its integral from a to b is defined by

$$\int_a^b h(s)d_q s = \int_0^b h(s)d_q s - \int_0^a h(s)d_q s. \tag{2.8}$$

Similar to derivatives, an operator I_q^k is given by

$$(I_q^0 h)(x) = h(x) \ \text{ and } \ (I_q^k h)(x) = I_q(I_q^{k-1}h)(x), \quad k \in \mathbb{N}. \tag{2.9}$$

The fundamental theorem of calculus applies to these operators D_q and I_q, i.e.,

$$(D_q I_q h)(x) = h(x), \tag{2.10}$$

and if h is continuous at $x = 0$, then

$$(I_q D_q h)(x) = h(x) - h(0). \tag{2.11}$$

Definition 2.1 *Let $\nu \geq 0$ and h be a function defined on $[0,T]$. The fractional q-integral of Riemann-Liouville type is given by $(I_q^0 h)(x) = h(x)$ and*

$$(I_q^\nu h)(x) = \frac{1}{\Gamma_q(\nu)}\int_0^x (x-qs)^{(\nu-1)}h(s)d_q s, \quad \nu > 0, \quad x \in [0,T]. \tag{2.12}$$

Definition 2.2 *The fractional q-derivative of the Riemann-Liouville type of order $\nu \geq 0$ is defined by $(D_q^0 h)(x) = h(x)$ and*

$$(D_q^\nu h)(x) = (D_q^{[\nu]} I_q^{[\nu]-\nu} h)(x), \quad \nu > 0, \tag{2.13}$$

where $[\nu]$ is the smallest integer greater than or equal to ν.

Definition 2.3 *The fractional q-derivative of the Caputo type of order $\alpha \geq 0$ is defined by*

$$({}^c D_q^\alpha h)(x) = (I_q^{[\alpha]-\alpha} D_q^{[\alpha]} h)(x), \quad \alpha > 0, \tag{2.14}$$

provided that $D_q^{[\alpha]}h(x)$ exists on $[0,T]$.

Lemma 2.4 *[27] Let $\alpha, \beta \geq 0$ and f be a function defined in $[0,T]$. Then, the following formulas hold:*

(1) $(I_q^\beta I_q^\alpha f)(x) = (I_q^{\alpha+\beta} f)(x)$,

(2) $(D_q^\alpha I_q^\alpha f)(x) = f(x)$.

Lemma 2.5 *[28] Let $\alpha > 0$ and ν be a positive integer. Then, the following equality holds:*

$$(I_q^\alpha D_q^\nu f)(x) = (D_q^\nu I_q^\alpha f)(x) - \sum_{k=0}^{\nu-1} \frac{x^{\alpha-\nu+k}}{\Gamma_q(\alpha+k-\nu+1)}(D_q^k f)(0). \tag{2.15}$$

Lemma 2.6 *[29] Let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$ and $a < x$. Then, the following formula is valid*

$$(I_q^{\alpha\, c} D_q^\alpha f)(x) = f(x) - \sum_{k=0}^{[\alpha]-1} \frac{(D_q^k f)(a)}{\Gamma_q(k+1)}(x-a)^{(k)}. \tag{2.16}$$

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Definition 2.7 *For any $m, n > 0$,*

$$B_q(m,n) = \int_0^1 u^{(m-1)}(1-qu)^{(n-1)} d_q u \tag{2.17}$$

is called the q-beta function.

The expression of q-beta function in terms of the q-gamma function can be written as

$$B_q(m,n) = \frac{\Gamma_q(m)\Gamma_q(n)}{\Gamma_q(m+n)}.$$

Lemma 2.8 *[2] Let $\alpha, \beta, \gamma > 0$ and $0 < p, q, r < 1$. Then we have*

$$\begin{aligned}\int_0^\eta \int_0^x \int_0^y (\eta - px)^{(\alpha-1)}(x-qy)^{(\beta-1)}(y-rz)^{(\gamma-1)} d_r z d_q y d_p x \\ = \frac{1}{[\gamma]_r} B_p(\alpha, \beta+\gamma+1) B_q(\beta, \gamma+1)\eta^{\alpha+\beta+\gamma}.\end{aligned} \tag{2.18}$$

3 An auxiliary lemma

Lemma 3.1 *Let $\beta, \gamma > 0$, $\lambda \in \mathbb{R}$ and $0 < p, q, r < 1$. Then, for $y \in C([0,T], \mathbb{R})$, the unique solution of boundary value problem*

$$^cD_q^\alpha x(t) = y(t), \quad t \in (0,T), \quad 1 < \alpha \le 2, \tag{3.1}$$

subject to the nonlocal fractional condition

$$x(\zeta) = g(x), \qquad \lambda I_p^\beta x(\eta) = I_r^\gamma x(\xi), \quad 0 < \zeta < \eta < \xi < T, \tag{3.2}$$

is given by

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s) d_q s \\ &\quad - \frac{t-\zeta}{D\Gamma_q(\alpha)} \left\{ \frac{1}{\Gamma_r(\gamma)} \int_0^\xi \int_0^s (\xi - rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)} y(u) d_q u d_r s \right. \\ &\quad \left. - \frac{\lambda}{\Gamma_p(\beta)} \int_0^\eta \int_0^s (\eta - ps)^{(\beta-1)}(s-qu)^{(\alpha-1)} y(u) d_q u d_p s \right\} \\ &\quad - \frac{t\Psi - \Omega}{D\Gamma_q(\alpha)} \int_0^\zeta (\zeta - qs)^{\alpha-1} y(s) d_q s + \frac{t\Psi - \Omega}{D} g(x), \end{aligned} \tag{3.3}$$

where

$$D = \zeta\Psi - \Omega \neq 0, \tag{3.4}$$

and

$$\Omega = \frac{\lambda\eta^{\beta+1}}{\Gamma_p(\beta+2)} - \frac{\xi^{\gamma+1}}{\Gamma_r(\gamma+2)}, \qquad \Psi = \frac{\lambda\eta^\beta}{\Gamma_p(\beta+1)} - \frac{\xi^\gamma}{\Gamma_r(\gamma+1)}. \tag{3.5}$$

Proof. From $1 < \alpha \le 2$, we let $n = 2$. Using the Definition 2.3 and Lemma 2.4, the equation (3.1) can be expressed as

$$(I_q^\alpha I_q^{[\alpha]-\alpha} D_q^{[\alpha]} x)(t) = (I_q^\alpha y)(t).$$

From Lemma 2.6, we have

$$x(t) = c_1 t + c_2 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s) d_q s \tag{3.6}$$

for some constants $c_1, c_2 \in \mathbb{R}$. It follows from the first condition of (3.2) that

$$c_1 \zeta + c_2 = g(x) - \int_0^\zeta \frac{(\zeta-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s) d_q s. \tag{3.7}$$

Applying the Riemann-Liouville fractional p-integral of order $\beta > 0$ for (3.6) we have

$$\begin{aligned} I_p^\beta x(t) &= \int_0^t \frac{(t-ps)^{(\beta-1)}}{\Gamma_p(\beta)} \left(c_1 s + c_2 + \int_0^s \frac{(s-qu)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(u) d_q u \right) d_p s \\ &= \frac{c_1}{\Gamma_p(\beta)} \int_0^t (t-ps)^{(\beta-1)} s d_p s + \frac{c_2}{\Gamma_p(\beta)} \int_0^t (t-ps)^{(\beta-1)} d_p s \\ &\quad + \frac{1}{\Gamma_p(\beta)\Gamma_q(\alpha)} \int_0^t \int_0^s (t-ps)^{(\beta-1)} (s-qu)^{(\alpha-1)} y(u) d_q u d_p s \\ &= c_1 \frac{t^{\beta+1}}{\Gamma_p(\beta+2)} + c_2 \frac{t^\beta}{\Gamma_p(\beta+1)} \\ &\quad + \frac{1}{\Gamma_p(\beta)\Gamma_q(\alpha)} \int_0^t \int_0^s (t-ps)^{(\beta-1)} (s-qu)^{(\alpha-1)} y(u) d_q u d_p s, \end{aligned}$$

since

$$\begin{aligned} \int_0^t (t-ps)^{(\beta-1)}\, s\, d_p s &= (1-q)t \sum_{n=0}^\infty q^n (t-qtq^n)^{(\beta-1)} t q^n \\ &= (1-q) t^{\beta+1} \sum_{n=0}^\infty q^n (1-qq^n)^{(\beta-1)} q^n \\ &= t^{\beta+1} \int_0^1 (1-qs)^{(\beta-1)} s d_p s = t^{\beta+1} B_p(\beta, 2) = t^{\beta+1} \frac{\Gamma_p(\beta)}{\Gamma_p(\beta+2)}, \end{aligned}$$

with $\Gamma_p(2) = 1$.

In particular, we have

$$\begin{aligned} I_p^\beta x(\eta) &= c_1 \frac{\eta^{\beta+1}}{\Gamma_p(\beta+2)} + c_2 \frac{\eta^\beta}{\Gamma_p(\beta+1)} \\ &\quad + \frac{1}{\Gamma_p(\beta)\Gamma_q(\alpha)} \int_0^\eta \int_0^s (\eta-ps)^{(\beta-1)} (s-qu)^{(\alpha-1)} y(u) d_q u d_p s. \end{aligned} \tag{3.8}$$

Using the Riemann-Liouville fractional r-integral of order $\gamma > 0$ and repeating the above process, we get

$$\begin{aligned} I_r^\gamma x(\xi) &= c_1 \frac{\xi^{\gamma+1}}{\Gamma_r(\gamma+2)} + c_2 \frac{\xi^\gamma}{\Gamma_r(\gamma+1)} \\ &\quad + \frac{1}{\Gamma_r(\gamma)\Gamma_q(\alpha)} \int_0^\xi \int_0^s (\xi-rs)^{(\gamma-1)} (s-qu)^{(\alpha-1)} y(u) d_q u d_r s. \end{aligned} \tag{3.9}$$

The second nonlocal condition of (3.2) implies

$$c_1 \Omega + c_2 \Psi = \frac{1}{\Gamma_r(\gamma)\Gamma_q(\alpha)} \int_0^\xi \int_0^s (\xi-rs)^{(\gamma-1)} (s-qu)^{(\alpha-1)} y(u) d_q u d_r s$$

$$-\frac{\lambda}{\Gamma_p(\beta)\Gamma_q(\alpha)}\int_0^{\eta}\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}y(u)d_qud_ps. \tag{3.10}$$

By solving the system of equations (3.7), (3.10) we find

$$\begin{aligned} c_1 &= \frac{\Psi}{D}\left(g(x)-\int_0^{\zeta}\frac{(\zeta-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}y(s)d_qs\right)\\ &\quad -\frac{1}{D\Gamma_r(\gamma)\Gamma_q(\alpha)}\int_0^{\xi}\int_0^s(\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}y(u)d_qud_rs\\ &\quad +\frac{\lambda}{D\Gamma_p(\beta)\Gamma_q(\alpha)}\int_0^{\eta}\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}y(u)d_qud_ps, \end{aligned}$$

and

$$\begin{aligned} c_2 &= -\frac{\Omega}{D}\left(g(x)-\int_0^{\zeta}\frac{(\zeta-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}y(s)d_qs\right)\\ &\quad +\frac{\zeta}{D\Gamma_r(\gamma)\Gamma_q(\alpha)}\int_0^{\xi}\int_0^s(\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}y(u)d_qud_rs\\ &\quad -\frac{\lambda\zeta}{D\Gamma_p(\beta)\Gamma_q(\alpha)}\int_0^{\eta}\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}y(u)d_qud_ps. \end{aligned}$$

Substituting the values of c_1 and c_2 in (3.6), we get the desired result in (3.3). □

4 Existence results for single-valued problem (1.1)

In this section, we denote by $\mathcal{C}=C([0,T],\mathbb{R})$ the Banach space of all continuous functions from $[0,T]$ to $\mathbb{R}$ endowed with the norm defined by $\|x\|=\sup_{t\in[0,T]}|x(t)|$. In view of Lemma 3.1, we define an operator $\mathcal{P}:\mathcal{C}\to\mathcal{C}$ by

$$\begin{aligned} (\mathcal{P}x)(t) &= \int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s,x(s),I_z^{\delta}x(s))d_qs\\ &\quad -\frac{t-\zeta}{D\Gamma_q(\alpha)}\left\{\frac{1}{\Gamma_r(\gamma)}\int_0^{\xi}\int_0^s(\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}f(u,x(u),I_z^{\delta}x(u))d_qud_rs\right.\\ &\quad \left.-\frac{\lambda}{\Gamma_p(\beta)}\int_0^{\eta}\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}f(u,x(u),I_z^{\delta}x(u))d_qud_ps\right\}\\ &\quad -\frac{t\Psi-\Omega}{D\Gamma_q(\alpha)}\int_0^{\zeta}(\zeta-qs)^{\alpha-1}f(s,x(s),I_z^{\delta}x(s))d_qs+\frac{t\Psi-\Omega}{D}g(x), \end{aligned} \tag{4.1}$$

where $D\neq 0$ is defined by (3.4) and Ω,Ψ are defined by (3.5). It should be noticed that problem (1.1) has solutions if and only if the operator $\mathcal{P}$ has fixed points.

Let us define $\mathcal{P}_{1,2}:\mathcal{C}\to\mathcal{C}$ by

$$\begin{aligned} (\mathcal{P}_1x)(t) &= \int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s,x(s),I_z^{\delta}x(s))d_qs\\ &\quad -\frac{t-\zeta}{D\Gamma_q(\alpha)}\left\{\frac{1}{\Gamma_r(\gamma)}\int_0^{\xi}\int_0^s(\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}f(u,x(u),I_z^{\delta}x(u))d_qud_rs\right.\\ &\quad \left.-\frac{\lambda}{\Gamma_p(\beta)}\int_0^{\eta}\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}f(u,x(u),I_z^{\delta}x(u))d_qud_ps\right\}\\ &\quad -\frac{t\Psi-\Omega}{D\Gamma_q(\alpha)}\int_0^{\zeta}(\zeta-qs)^{\alpha-1}f(s,x(s),I_z^{\delta}x(s))d_qs,\qquad t\in[0,T], \end{aligned} \tag{4.2}$$

and

$$(\mathcal{P}_2x)(t) = \frac{t\Psi - \Omega}{D}g(x), \qquad t \in [0,T]. \tag{4.3}$$

Clearly

$$(\mathcal{P}x)(t) = (\mathcal{P}_1x)(t) + (\mathcal{P}_2x)(t), \qquad t \in [0,T]. \tag{4.4}$$

For convenience we introduce the notations:

$$\begin{aligned} p_0 : \quad &= \frac{T^\alpha}{\Gamma_q(\alpha+1)} + \frac{T+\zeta}{|D|\Gamma_r(\gamma)}\,\frac{\xi^{\alpha+\gamma}B_r(\gamma,\alpha+1)}{\Gamma_q(\alpha+1)} \\ &+\frac{|\lambda|(T+\zeta)}{|D|\Gamma_p(\beta)}\,\frac{\eta^{\alpha+\beta}B_p(\beta,\alpha+1)}{\Gamma_q(\alpha+1)} + \frac{T|\Psi|+|\Omega|}{D\Gamma_q(\alpha)}\,\frac{\zeta^\alpha}{\Gamma_q(\alpha+1)}, \end{aligned} \tag{4.5}$$

$$\begin{aligned} q_0 : \quad &= \frac{T^{\alpha+\delta}B_q(\alpha,\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)} + \frac{T+\zeta}{|D|\Gamma_r(\gamma)}\,\frac{\xi^{\alpha+\gamma+\delta}B_q(\alpha,\delta+1)B_r(\gamma,\alpha+\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)} \\ &+\frac{|\lambda|(T+\zeta)}{|D|\Gamma_p(\beta)}\,\frac{\eta^{\alpha+\beta+\delta}B_q(\alpha,\delta+1)B_p(\beta,\alpha+\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)} \\ &+\frac{T|\Psi|+|\Omega|}{D\Gamma_q(\alpha)}\,\frac{\zeta^{\alpha+\delta}B_q(\alpha,\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}, \end{aligned} \tag{4.6}$$

and

$$k_0 := \frac{T|\Psi|+|\Omega|}{|D|}. \tag{4.7}$$

Theorem 4.1 *Let $f : [0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Assume that*

(A_1) $|f(t,w_1,w_2)-f(t,\bar{w}_1,\bar{w}_2)| \le L_1|w_1-\bar{w}_1|+L_2|w_2-\bar{w}_2|, \forall t\in[0,T],\ L_1>0,\ L_2>0,\ w_1,\bar{w}_1,w_2,\bar{w}_2\in\mathbb{R}$;

(A_2) $g : C([0,T],\mathbb{R})\to\mathbb{R}$ *is a continuous function satisfying the condition:*

$$|g(u)-g(v)| \le \ell\|u-v\|,\ \ \ell < k_0^{-1}, \quad \forall\ u,v\in C([0,T],\mathbb{R}),\ \ \ell>0;$$

(A_3) $\kappa := L_1p_0 + L_2q_0 + \ell\, k_0 < 1$.

Then the boundary value problem (1.1) *has a unique solution.*

Proof. For $x,y\in\mathcal{C}$ and for each $t\in[0,T]$, from the definition of $\mathcal{P}$, assumptions (A_1), (A_2) and Lemma 2.8, we obtain

$$\begin{aligned} |(\mathcal{P}x)(t)-(\mathcal{P}y)(t)| \quad \le \quad & \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,x(s),I_z^\delta x(s))-f(s,y(s),I_z^\delta y(s))|d_qs \\ &+\frac{T+\zeta}{|D|\Gamma_q(\alpha)}\Bigg\{\frac{1}{\Gamma_r(\gamma)}\int_0^\xi\int_0^s(\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}\times \\ &\times|f(u,x(u),I_z^\delta x(u))-f(u,y(u),I_z^\delta y(u))|d_qud_ps \\ &+\frac{|\lambda|}{\Gamma_p(\beta)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}\times \\ &\times|f(u,x(u),I_z^\delta x(u))-f(u,y(u),I_z^\delta y(u))|d_qud_ps\Bigg\} \\ &+\frac{T|\Psi|+|\Omega|}{D\Gamma_q(\alpha)}\int_0^\zeta(\zeta-qs)^{\alpha-1}|f(s,x(s),I_z^\delta x(s))-f(s,y(s),I_z^\delta y(s))|d_qs \\ &+\frac{T|\Psi|+|\Omega|}{|D|}|g(x)-g(y)| \end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left(L_1\|x-y\|+L_2\|x-y\|\int_0^s \frac{(s-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qs \\
&\quad +\frac{T+\zeta}{|D|\Gamma_r(\gamma)\Gamma_q(\alpha)}\int_0^\xi\int_0^s (\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)} \\
&\quad \times\left(L_1\|x-y\|+L_2\|x-y\|\int_0^u \frac{(u-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qud_rs \\
&\quad +\frac{|\lambda|(T+\zeta)}{|D|\Gamma_p(\beta)\Gamma_q(\alpha)}\int_0^\eta\int_0^s (\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)} \\
&\quad \times\left(L_1\|x-y\|+L_2\|x-y\|\int_0^u \frac{(u-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qud_ps \\
&\quad +\frac{T|\Psi|+|\Omega|}{D\Gamma_q(\alpha)}\int_0^\zeta (\zeta-qs)^{\alpha-1}\left(L_1\|x-y\|+L_2\|x-y\|\int_0^s \frac{(s-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qs \\
&\quad +\frac{T|\Psi|+|\Omega|}{|D|}|g(x)-g(y)| \\
&\leq \|x-y\|\Bigg\{\frac{T^\alpha}{\Gamma_q(\alpha+1)}L_1+\frac{T^{\alpha+\delta}B_q(\alpha,\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}L_2 \\
&\quad +\frac{T+\zeta}{|D|\Gamma_r(\gamma)}\left[\frac{\xi^{\alpha+\gamma}B_r(\gamma,\alpha+1)L_1}{\Gamma_q(\alpha+1)}+\frac{\xi^{\alpha+\gamma+\delta}B_q(\alpha,\delta+1)B_r(\gamma,\alpha+\delta+1)L_2}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}\right] \\
&\quad +\frac{|\lambda|(T+\zeta)}{|D|\Gamma_p(\beta)}\left[\frac{\eta^{\alpha+\beta}B_p(\beta,\alpha+1)L_1}{\Gamma_q(\alpha+1)}+\frac{\eta^{\alpha+\beta+\delta}B_q(\alpha,\delta+1)B_p(\beta,\alpha+\delta+1)L_2}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}\right] \\
&\quad +\frac{T|\Psi|+|\Omega|}{|D|\Gamma_q(\alpha)}\left[\frac{\zeta^\alpha}{\Gamma_q(\alpha+1)}L_1+\frac{\zeta^{\alpha+\delta}B_q(\alpha,\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}L_2\right]\Bigg\}+\frac{T|\Psi|+|\Omega|}{|D|}\ell\|x-y\| \\
&= (L_1p_0+L_2q_0+k_0\ell)\|x-y\|.
\end{aligned}
$$

Hence

$$\|(\mathcal{P}x)-(\mathcal{P}y)\|\leq\kappa\|x-y\|.$$

As $\kappa<1$ by (A_3), the operator $\mathcal{P}$ is a contraction map from the Banach space $\mathcal{C}$ into itself. Hence the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

Example 4.2 *Consider the following nonlocal problem of q-integrodifference equation*

$$
\begin{cases}
{}^cD_{\frac{1}{2}}^{\frac{3}{2}}x(t)=\dfrac{e^{-3t}}{2(t+\sqrt{5})^2}\cdot\dfrac{|x|}{1+|x|}+\dfrac{1}{2}I_{\frac{3}{8}}^{\frac{5}{2}}x(t)+\dfrac{3}{2}, \quad t\in(0,1/2), \\
x\left(\dfrac{1}{8}\right)=\dfrac{1}{15}x\left(\dfrac{3}{8}\right)+\dfrac{2}{3}, \quad \dfrac{2}{5}I_{\frac{2}{5}}^{\frac{7}{3}}x\left(\dfrac{1}{4}\right)=I_{\frac{2}{3}}^{\frac{5}{4}}x\left(\dfrac{1}{3}\right).
\end{cases}
\tag{4.8}
$$

Here, $\alpha=3/2$, $q=1/2$, $\delta=5/2$, $z=3/8$, $T=1/2$, $\zeta=1/8$, $\lambda=2/5$, $\beta=7/3$, $p=2/5$, $\eta=1/4$, $\gamma=5/4$, $r=2/3$, $\xi=1/3$, $g(x)=(1/15)x+(2/3)$ and $f(t,x,I_z^\delta x)=(e^{-3t}|x|)/(2(t+\sqrt{5})^2(1+|x|))+(1/2)I_{3/8}^{5/2}x+(3/2)$. By using the Maple program, we find that $\Omega=-0.04119212$, $\Psi=-0.22035718$, $D=0.01364747$, $p_0=1.57981377$, $q_0=0.02586708$ and $k_0=11.09148475$.

As $|f(t,w_1,w_2)-f(t,\bar{w}_1,\bar{w}_2)|\leq(1/10)|w_1-\bar{w}_1|+(1/2)|w_2-\bar{w}_2|$ and $|g(x)-g(y)|\leq(1/15)|x-y|$, therefore, (A_1) and (A_2) are satisfied with $L_1=1/10$, $L_2=1/2$ and $\ell=1/15<1/11.09148475=k_0^{-1}$, respectively. Hence $\kappa=L_1p_0+L_2q_0+\ell k_0=0.91034723<1$. By the conclusion of Theorem 4.1, the nonlocal problem (4.8) has a unique solution on $[0,1/2]$.

Our next existence result relies on a fixed point theorem due to O'Regan in [25].

Lemma 4.3 *Denote by U an open set in a closed, convex set C of a Banach space E. Assume $0 \in U$. Also assume that $F(\bar{U})$ is bounded and that $F : \bar{U} \to C$ is given by $F = F_1 + F_2$, in which $F_1 : \bar{U} \to E$ is continuous and completely continuous and $F_2 : \bar{U} \to E$ is a nonlinear contraction (i.e., there exists a nonnegative nondecreasing function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(z) < z$ for $z > 0$, such that $\|F_2(x) - F_2(y)\| \le \phi(\|x - y\|)$ for all $x, y \in \bar{U}$). Then, either*

(C1) F has a fixed point $u \in \bar{U}$; or

(C2) there exist a point $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda F(u)$, where $\bar{U}$ and ∂U, respectively, represent the closure and boundary of U.

Let

$$\Omega_r = \{x \in C([0, T], \mathbb{R}) : \|x\| < r\},$$

and denote the maximum number by

$$M_r = \max\{|f(t, x, y)| : (t, x, y) \in [0, T] \times [-r, r] \times [-r, r]\}.$$

Theorem 4.4 *Let $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that (A_2) holds. In addition we assume that*

(A_4) $g(0) = 0$;

(A_5) *there exist a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ and a function $p \in C([0, T], \mathbb{R}^+)$ such that*

$$|f(t, w_1, w_2)| \le p(t)\psi(|w_1|) + |w_2| \quad for\ each\ (t, w_1, w_2) \in [0, T] \times \mathbb{R}^2;$$

(A_6) $\sup_{r\in(0,\infty)} \dfrac{r}{p_0\psi(r)\|p\|} > \dfrac{1}{1 - k_0\ell - q_0}$, *where p_0, q_0 and k_0 are defined in (4.5), (4.6) and (4.7), respectively, and $k_0\ell + q_0 < 1$.*

Then the boundary value problem (1.1) has at least one solution on $[0, T]$.

Proof. By the assumption (A_6), there exists a number $r_0 > 0$ such that

$$\frac{r_0}{p_0\psi(r_0)\|p\|} > \frac{1}{1 - k_0\ell - q_0}. \tag{4.9}$$

We shall show that the operators $\mathcal{P}_1$ and $\mathcal{P}_2$ defined by (4.2) and (4.3), respectively, satisfy all the conditions of Lemma 4.3.

Step 1. *The operator $\mathcal{P}_1$ is continuous and completely continuous.* We first show that $\mathcal{P}_1(\bar{\Omega}_{r_0})$ is bounded. For any $x \in \bar{\Omega}_{r_0}$, we have

$$\begin{aligned}
\|\mathcal{P}_1 x\| \le\ & \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s, x(s), I_z^\delta x(s))| d_q s \\
& + \frac{T + \zeta}{|D|\Gamma_r(\gamma)\Gamma_q(\alpha)} \int_0^\xi \int_0^s (\xi - rs)^{(\gamma-1)} (s - qu)^{(\alpha-1)} |f(u, x(u), I_z^\delta x(u))| d_q u d_r s \\
& + \frac{|\lambda|(T + \zeta)}{|D|\Gamma_p(\beta)\Gamma_q(\alpha)} \int_0^\eta \int_0^s (\eta - ps)^{(\beta-1)} (s - qu)^{(\alpha-1)} |f(u, x(u), I_z^\delta x(u))| d_q u d_p s \\
& + \frac{T|\Psi| + |\Omega|}{D\Gamma_q(\alpha)} \int_0^\zeta (\zeta - qs)^{\alpha-1} |f(s, x(s), I_z^\delta x(s))| d_q s \\
\le\ & \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left(p(s)\psi(\|x\|) + \|x\| \int_0^s \frac{(s - zv)^{(\delta-1)}}{\Gamma_z(\delta)} d_z v \right) d_q s \\
& + \frac{T + \zeta}{|D|\Gamma_r(\gamma)\Gamma_q(\alpha)} \int_0^\xi \int_0^s (\xi - rs)^{(\gamma-1)} (s - qu)^{(\alpha-1)}
\end{aligned}$$

$$
\begin{aligned}
& \times\left(p(u)\psi(\|x\|)+\|x\|\int_0^u\frac{(u-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qud_rs\\
&+\frac{|\lambda|(T+\zeta)}{|D|\Gamma_p(\beta)\Gamma_q(\alpha)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}\\
&\times\left(p(u)\psi(\|x\|)+\|x\|\int_0^u\frac{(u-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qud_ps\\
&+\frac{T|\Psi|+|\Omega|}{D\Gamma_q(\alpha)}\int_0^\zeta(\zeta-qs)^{\alpha-1}\left(p(s)\psi(\|x\|)+\|x\|\int_0^s\frac{(s-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qs\\
\leq\;& \frac{\|p\|\psi(r_0)T^\alpha}{\Gamma_q(\alpha+1)}+\frac{r_0T^{\alpha+\delta}B_q(\alpha,\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}\\
& \frac{T+\zeta}{|D|\Gamma_r(\gamma)}\left(\frac{\xi^{\alpha+\gamma}B_r(\gamma,\alpha+1)\|p\|\psi(r_0)}{\Gamma_q(\alpha+1)}+\frac{r_0\xi^{\alpha+\gamma+\delta}B_q(\alpha,\delta+1)B_r(\gamma,\alpha+\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}\right)\\
& \frac{|\lambda|(T+\zeta)}{|D|\Gamma_p(\beta)}\left(\frac{\eta^{\alpha+\beta}B_p(\beta,\alpha+1)\|p\|\psi(r_0)}{\Gamma_q(\alpha+1)}+\frac{r_0\eta^{\alpha+\beta+\delta}B_q(\alpha,\delta+1)B_p(\beta,\alpha+\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}\right)\\
&+\frac{T|\Psi|+|\Omega|}{D\Gamma_q(\alpha)}\left(\frac{\|p\|\psi(r_0)\zeta^\alpha}{\Gamma_q(\alpha+1)}+\frac{r_0\zeta^{\alpha+\delta}B_q(\alpha,\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}\right)\\
=\;& \|p\|\psi(r_0)p_0+r_0q_0:=G.
\end{aligned}
$$

Thus the operator $\mathcal{P}_1(\bar{\Omega}_{r_0})$ is uniformly bounded. For any $t_1,t_2\in[0,T],t_1<t_2$, we have

$$
\begin{aligned}
& |(\mathcal{P}_1x)(t_2)-(\mathcal{P}_1x)(t_1)|\\
\leq\;& \left|\int_0^{t_2}\frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,x(s),I_z^\delta x(s))|d_qs\right.\\
&\left.-\int_0^{t_1}\frac{(t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,x(s),I_z^\delta x(s))|d_qs\right|\\
&+\frac{|t_2-t_1|}{|D|\Gamma_r(\gamma)\Gamma_q(\alpha)}\int_0^\xi\int_0^s(\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}|f(u,x(u),I_z^\delta x(u))|d_qud_rs\\
&+\frac{|\lambda||t_2-t_1|}{|D|\Gamma_p(\beta)\Gamma_q(\alpha)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}|f(u,x(u),I_z^\delta x(u))|d_qud_ps\\
&+\frac{|\Psi||t_2-t_1|}{|D|\Gamma_q(\alpha)}\int_0^\zeta(\zeta-qs)^{\alpha-1}|f(s,x(s),I_z^\delta x(s))|d_qs\\
\leq\;& \int_0^{t_2}\frac{|(t_2-qs)^{(\alpha-1)}-(t_1-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)}\left(p(s)\psi(\|x\|)+\|x\|\int_0^s\frac{(s-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qs\\
&+\int_{t_1}^{t_2}\frac{(t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left(p(s)\psi(\|x\|)+\|x\|\int_0^s\frac{(s-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qs\\
&+\frac{|t_2-t_1|}{|D|\Gamma_r(\gamma)\Gamma_q(\alpha)}\int_0^\xi\int_0^s(\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}\\
&\times\left(p(u)\psi(\|x\|)+\|x\|\int_0^u\frac{(u-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qud_rs\\
&+\frac{|\lambda||t_2-t_1|}{|D|\Gamma_p(\beta)\Gamma_q(\alpha)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}\\
&\times\left(p(u)\psi(\|x\|)+\|x\|\int_0^u\frac{(u-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qud_ps\\
&+\frac{|\Psi||t_2-t_1|}{|D|\Gamma_q(\alpha)}\left(p(s)\psi(\|x\|)+\|x\|\int_0^s\frac{(s-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qs,
\end{aligned}
$$

which is independent of x and tends to zero as $t_2-t_1\to0$. Thus, $\mathcal{P}_1$ is equicontinuous. Hence, by the

Arzelá-Ascoli Theorem, $\mathcal{P}_1(\bar{\Omega}_{r_0})$ is a relatively compact set. Now, let $x_n, y_n \subset \bar{\Omega}_{r_0}$ with $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$. Then the limits $\|x_n(t) - x(t)\| \to 0$ and $\|y_n(t) - y(t)\| \to 0$ are uniformly valid on $[0, T]$. From the uniform continuity of $f(t, x, y)$ on the compact set $[0, T] \times [-r_0, r_0] \times [-r_0, r_0]$, it follows that $\|f(t, x_n(t), y_n(t)) - f(t, x(t), y(t))\| \to 0$ is uniformly valid on $[0, T]$. Hence $\|\mathcal{P}_1 x_n - \mathcal{P}_1 x\| \to 0$ as $n \to \infty$ which proves the continuity of $\mathcal{P}_1$. This completes the proof of Step 1.

Step 2. *The operator $\mathcal{P}_2 : \bar{\Omega}_{r_0} \to C([0, T], \mathbb{R})$ is contractive.* This is a consequence of (A_2). Indeed, for $x, y \in C([0, T], \mathbb{R})$, we have

$$\begin{aligned} |\mathcal{P}_2 x(t) - \mathcal{P}_2 y(t)| &= \left|\frac{t\Psi - \Omega}{D}\right| |g(x) - g(y)| \\ &\leq \frac{T|\Psi| + |\Omega|}{|D|} |g(x) - g(y)|, \\ &\leq k_0 \ell \|x - y\|, \end{aligned}$$

which, on taking supremum over $t \in [0, T]$, yields

$$\|\mathcal{P}_2 x - \mathcal{P}_2 y\| \leq L_0 \|x - y\|, \quad L_0 = k_0 \ell < 1.$$

This shows that $\mathcal{P}_2$ is a contraction as $L_0 < 1$.

Step 3. *The set $\mathcal{P}(\bar{\Omega}_{r_0})$ is bounded.* The assumptions (A_2) and (A_4) imply that

$$\|\mathcal{P}_2(x)\| \leq k_0 \ell r_0,$$

for any $x \in \bar{\Omega}_{r_0}$. This, with the boundedness of the set $\mathcal{P}_1(\bar{\Omega}_{r_0})$ implies that the set $\mathcal{P}(\bar{\Omega}_{r_0})$ is bounded.

Step 4. *Finally, it will be shown that the case (C2) in Lemma 4.3 does not hold.* On the contrary, we suppose that (C2) holds. Then, we have that there exist $\theta \in (0, 1)$ and $x \in \partial\Omega_{r_0}$ such that $x = \theta \mathcal{P} x$. So, we have $\|x\| = r_0$ and

$$\begin{aligned} x(t) &= \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, x(s), I_z^\delta x(s)) d_q s \\ &-\frac{t - \zeta}{D\Gamma_q(\alpha)} \left\{ \frac{1}{\Gamma_r(\gamma)} \int_0^\xi \int_0^s (\xi - rs)^{(\gamma-1)} (s - qu)^{(\alpha-1)} f(u, x(u), I_z^\delta x(u)) d_q u d_r s \right. \\ &\left. -\frac{\lambda}{\Gamma_p(\beta)} \int_0^\eta \int_0^s (\eta - ps)^{(\beta-1)} (s - qu)^{(\alpha-1)} f(u, x(u), I_z^\delta x(u)) d_q u d_p s \right\} \\ &-\frac{t\Psi - \Omega}{D\Gamma_q(\alpha)} \int_0^\zeta (\zeta - qs)^{\alpha-1} f(s, x(s), I_z^\delta x(s)) d_q s + \frac{t\Psi - \Omega}{D} g(x), \quad t \in [0, T]. \end{aligned}$$

Using the assumptions $(A_4) - (A_6)$, we get

$$r_0 \leq p_0 \psi(r_0) \|p\| + r_0 q_0 + k_0 \ell r_0.$$

Thus, we get a contradiction:

$$\frac{r_0}{p_0 \psi(r_0) \|p\|} \leq \frac{1}{1 - k_0 \ell - q_0}.$$

Thus the operators $\mathcal{P}_1$ and $\mathcal{P}_2$ satisfy all the conditions of Lemma 4.3. Hence, the operator $\mathcal{P}$ has at least one fixed point $x \in \bar{\Omega}_{r_0}$, which is the solution of the problem (1.1). This completes the proof. □

Example 4.5 *Consider the following nonlocal problem of q-integrodifference equation*

$$\begin{cases} {}^c D_{\frac{1}{5}}^{\frac{7}{4}} x(t) = \dfrac{t^2 + 1}{35} \left(|x| + \dfrac{|x| + 1}{|x| + 2} \right) + I_{\frac{2}{7}}^{\frac{3}{4}} x, \quad t \in (0, 1/2), \\ x\left(\dfrac{1}{7}\right) = \dfrac{1}{12} \sin\left(x\left(\dfrac{1}{4}\right)\right), \quad \dfrac{1}{10} I_{\frac{1}{6}}^{\frac{4}{3}} x\left(\dfrac{3}{10}\right) = I_{\frac{1}{2}}^{\frac{4}{7}} x\left(\dfrac{2}{5}\right). \end{cases} \tag{4.10}$$

Here, $\alpha = 7/4$, $q = 1/5$, $\delta = 3/4$, $z = 2/7$, $T = 1/2$, $\zeta = 1/7$, $\lambda = 1/10$, $\beta = 4/3$, $p = 1/6$, $\eta = 3/10$, $\gamma = 4/7$, $r = 1/2$, $\xi = 2/5$, $g(x) = (1/12)\sin x$ and $f(t,x,I_z^\delta x) = ((t^2+1)/35)(|x|+((|x|+1)/(|x|+2)))+I_{2/7}^{3/4}x$. By using the Maple program, we find that $\Omega = -0.18824514$, $\Psi = -0.62150021$, $D = 0.09945940$, $p_0 = 0.92752573$, $q_0 = 0.37709650$ and $k_0 = 5.01707462$.

As $|g(x) - g(y)| \le (1/12)|x-y|$ with $\ell = (1/12) < (1/5.01707462) = k_0^{-1}$ and $g(0) = 0$, therefore, (A_2) and (A_4) are satisfied, respectively. Since $|f(t,w_1,w_2)| = |((t^2+1)/35)(|w_1| + ((|w_1|+1)/(|w_1|+2)))+w_2| \le ((t^2+1)/35)(w_1^2+3|w_1|+1)+|w_2|$, we choose $p(t) = (t^2+1)/35$ and $\psi(|w_1|) = w_1^2+3|w_1|+1$. We can show that

$$\sup_{r\in(0,\infty)} \frac{r}{p_0\psi(r)\|p\|} = 6.03756836 > 4.88247997 = \frac{1}{1-k_0\ell-q_0}.$$

Therefore, by Theorem 4.4, the boundary value problem (4.10) has at least one solution on $[0,1/2]$.

5 Existence results for multi-valued problem (1.2)

Let us recall some basic definitions on multi-valued maps [30, 31].

For a normed space $(X, \|\cdot\|)$, let $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$. A multi-valued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in P_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in P_b(X)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : [0;1] \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

Definition 5.1 *A function $x \in AC^1([0,T],\mathbb{R})$ is a solution of the problem (1.2) if $x(\zeta) = g(x)$, $\lambda I_p^\beta x(\eta) = I_r^\gamma x(\xi)$, and there exists a function $f \in L^1([0,T],\mathbb{R})$ such that $f(t) \in F(t,x(t),I_z^\delta x(t))$ a.e. on $[0,T]$ and*

$$\begin{aligned} x(t) \;&= \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s)d_qs \\ &-\frac{t-\zeta}{D\Gamma_q(\alpha)}\left\{\frac{1}{\Gamma_r(\gamma)}\int_0^\xi\int_0^s (\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}f(u)d_qud_rs\right. \\ &\left.-\frac{\lambda}{\Gamma_p(\beta)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}f(u)d_qud_ps\right\} \\ &-\frac{t\Psi-\Omega}{D\Gamma_q(\alpha)}\int_0^\zeta(\zeta-qs)^{\alpha-1}f(s)d_qs+\frac{t\Psi-\Omega}{D}g(x). \end{aligned} \tag{5.1}$$

Here $AC^1([0,T],\mathbb{R})$ will denote the space of functions $x : [0,T] \to \mathbb{R}$ that are absolutely continuous and whose first derivative is absolutely continuous.

Definition 5.2 *A multivalued map $F : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if*

(i) $t \longmapsto F(t,x,y)$ is measurable for each $x, y \in \mathbb{R}$;

(ii) $(x,y) \longmapsto F(t,x,y)$ is upper semicontinuous for almost all $t \in [0,T]$;

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\alpha > 0$, there exists $\varphi_\alpha \in L^1([0,T],\mathbb{R}^+)$ such that

$$\|F(t,x,y)\| = \sup\{|v| : v \in F(t,x,y)\} \le \varphi_\alpha(t)$$

for all $\|x\|, \|y\| \le \alpha$ and for a.e. $t \in [0,T]$.

For each $x, y \in C([0,T],\mathbb{R})$, define the set of selections of F by

$$S_{F,x,y} := \{v \in L^1([0,T],\mathbb{R}) : v(t) \in F(t,x(t),y(t)) \text{ for a.e. } t \in [0,T]\}.$$

The following lemma will be used in the sequel.

Lemma 5.3 *([32]) Let X be a Banach space. Let $F : [0,T] \times \mathbb{R}^2 \to \mathcal{P}_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0,T],X,X)$ to $C([0,T],X,X)$. Then the operator*

$$\Theta \circ S_F : C([0,T],X,X) \to \mathcal{P}_{cp,c}(C([0,T],X,X)), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x,y})$$

is a closed graph operator in $C([0,T],X,X) \times C([0,T],X,X)$.

To prove our main result in this section, we use the following form of the Nonlinear Alternative for contractive maps [33, Corollary 3.8].

Theorem 5.4 *Let X be a Banach space, and D a bounded neighborhood of $0 \in X$. Let $Z_1 : X \to \mathcal{P}_{cp,c}(X)$ and $Z_2 : \bar{D} \to \mathcal{P}_{cp,c}(X)$ two multi-valued operators satisfying*

(a) Z_1 is contraction, and

(b) Z_2 is u.s.c and compact.

Then, if $G = Z_1 + Z_2$, either

(i) G has a fixed point in $\bar{D}$ or

(ii) there is a point $u \in \partial D$ and $\lambda \in (0,1)$ with $u \in \lambda G(u)$.

Theorem 5.5 *Assume that (A_2) holds. In addition we suppose that:*

(H_1) $F : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory multivalued map;

(H_2) there exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in C([0,T],\mathbb{R}^+)$ such that

$$\|F(t,x,y)\|_{\mathcal{P}} := \sup\{|v| : v \in F(t,x,y)\} \le p(t)\psi(\|x\|) + |y|$$

for each $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}$;

(H_3) there exists a number $M > 0$ such that

$$\frac{(1 - k_0\ell - q_0)M}{\psi(M)p_0\|p\|} > 1, \quad k_0\ell + q_0 < 1, \tag{5.2}$$

where p_0, q_0, k_0 are defined in (4.5), (4.6) and (4.7) respectively.

Then the boundary value problem (1.2) has at least one solution on $[0,T]$.

Proof. To transform the problem (1.2) to a fixed point, we introduce an operator $\mathcal{N}: C([0,T],\mathbb{R}) \longrightarrow \mathcal{P}(C([0,T],\mathbb{R}))$ defined by

$$\mathcal{N}(x) = \left\{ \begin{array}{l} h \in C([0,T],\mathbb{R}): \\ h(t) = \left\{ \begin{array}{l} \displaystyle\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s) d_q s \\ \displaystyle -\frac{t-\zeta}{D\Gamma_q(\alpha)} \left\{ \frac{1}{\Gamma_r(\gamma)} \int_0^\xi \int_0^s (\xi - rs)^{(\gamma-1)} (s-qu)^{(\alpha-1)} f(u) d_q u d_r s \right. \\ \displaystyle \left. -\frac{\lambda}{\Gamma_p(\beta)} \int_0^\eta \int_0^s (\eta - ps)^{(\beta-1)} (s-qu)^{(\alpha-1)} f(u) d_q u d_p s \right\} \\ \displaystyle -\frac{t\Psi - \Omega}{D\Gamma_q(\alpha)} \int_0^\zeta (\zeta - qs)^{\alpha-1} f(s) d_q s + \frac{t\Psi-\Omega}{D} g(x), \end{array} \right. \end{array} \right\}$$

for $f \in S_{F,x}$.

Now, we define two operators $\mathcal{A}: C([0,T],\mathbb{R}) \longrightarrow C([0,T],\mathbb{R})$ by

$$\mathcal{A}x(t) = \frac{t\Psi - \Omega}{D} g(x), \tag{5.3}$$

and a multi-valued operator $\mathcal{B}: C([0,T],\mathbb{R}) \longrightarrow \mathcal{P}(C([0,T],\mathbb{R}))$ by

$$\mathcal{B}(x) = \left\{ \begin{array}{l} h \in C([0,T],\mathbb{R}): \\ h(t) = \left\{ \begin{array}{l} \displaystyle\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s) d_q s \\ \displaystyle -\frac{t-\zeta}{D\Gamma_q(\alpha)} \left\{ \frac{1}{\Gamma_r(\gamma)} \int_0^\xi \int_0^s (\xi - rs)^{(\gamma-1)} (s-qu)^{(\alpha-1)} f(u) d_q u d_r s \right. \\ \displaystyle \left. -\frac{\lambda}{\Gamma_p(\beta)} \int_0^\eta \int_0^s (\eta - ps)^{(\beta-1)} (s-qu)^{(\alpha-1)} f(u) d_q u d_p s \right\} \\ \displaystyle -\frac{t\Psi - \Omega}{D\Gamma_q(\alpha)} \int_0^\zeta (\zeta - qs)^{\alpha-1} f(s) d_q s. \end{array} \right. \end{array} \right\} \tag{5.4}$$

Observe that $\mathcal{N} = \mathcal{A} + \mathcal{B}$. We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 5.4 on $[0,T]$. The proof consists of several steps and claims.

Step 1: We show that $\mathcal{A}$ *is a contraction on* $C([0,T],\mathbb{R})$. The proof is similar to the one for the operator $\mathcal{Q}_2$ in Step 2 of Theorem 4.4.

Step 2: $\mathcal{B}$ *is compact and convex valued and it is completely continuous.* This will be established in several claims.

Claim I: $\mathcal{B}$ *maps bounded sets into bounded sets in* $C([0,T],\mathbb{R})$. Let $B_R = \{x \in C([0,T],\mathbb{R}) : \|x\| \le R\}$ be a bounded set in $C([0,T],\mathbb{R})$. Then, for each $h \in \mathcal{B}(x), x \in B_\rho$, there exists $f \in S_{F,x}$ such that

$$\begin{aligned} h(t) &= \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s) d_q s - \frac{t-\zeta}{D\Gamma_q(\alpha)} \left\{ \frac{1}{\Gamma_r(\gamma)} \int_0^\xi \int_0^s (\xi - rs)^{(\gamma-1)} (s-qu)^{(\alpha-1)} f(u) d_q u d_r s \right. \\ &\quad \left. -\frac{\lambda}{\Gamma_p(\beta)} \int_0^\eta \int_0^s (\eta - ps)^{(\beta-1)} (s-qu)^{(\alpha-1)} f(u) d_q u d_p s \right\} - \frac{t\Psi - \Omega}{D\Gamma_q(\alpha)} \int_0^\zeta (\zeta - qs)^{\alpha-1} f(s) d_q s. \end{aligned}$$

Then, for $t \in [0,T]$, we have

$$\begin{aligned} |h(t)| &\le \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s)| d_q s \\ &\quad + \frac{T+\zeta}{|D|\Gamma_q(\alpha)} \left\{ \frac{1}{\Gamma_r(\gamma)} \int_0^\xi \int_0^s (\xi - rs)^{(\gamma-1)} (s-qu)^{(\alpha-1)} |f(u)| d_q u d_r s \right. \end{aligned}$$

$$
\begin{aligned}
&+\frac{|\lambda|(T+\zeta)}{\Gamma_p(\beta)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}|f(u)|d_qud_ps\Bigg\}\\
&+\frac{T|\Psi|+|\Omega|}{|D|\Gamma_q(\alpha)}\int_0^\zeta(\zeta-qs)^{\alpha-1}|f(s)|d_qs\\
\leq\;& \frac{\|p\|\psi(R)T^\alpha}{\Gamma_q(\alpha+1)}+\frac{RT^{\alpha+\delta}B_q(\alpha,\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}\\
&+\frac{T+\zeta}{|D|\Gamma_r(\gamma)}\left(\frac{\xi^{\alpha+\gamma}B_r(\gamma,\alpha+1)\|p\|\psi(R)}{\Gamma_q(\alpha+1)}+\frac{R\xi^{\alpha+\gamma+\delta}B_q(\alpha,\delta+1)B_r(\gamma,\alpha+\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}\right)\\
&+\frac{|\lambda|(T+\zeta)}{|D|\Gamma_p(\beta)}\left(\frac{\eta^{\alpha+\beta}B_p(\beta,\alpha+1)\|p\|\psi(R)}{\Gamma_q(\alpha+1)}+\frac{R\eta^{\alpha+\beta+\delta}B_q(\alpha,\delta+1)B_p(\beta,\alpha+\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}\right)\\
&+\frac{T|\Psi|+|\Omega|}{|D|\Gamma_q(\alpha)}\left(\frac{\|p\|\psi(R)\zeta^\alpha}{\Gamma_q(\alpha+1)}+\frac{R\zeta^{\alpha+\delta}B_q(\alpha,\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}\right).
\end{aligned}
$$

Thus,

$$\|h\|\leq\psi(R)p_0\|p\|+Rq_0.$$

Claim II: $\mathcal{B}$ *maps bounded sets into equi-continuous sets.* Let $t_1,t_2\in[0,T]$ with $t_1<t_2$ and $x\in B_R$. Then, for each $h\in\mathcal{B}(x)$, we obtain

$$
\begin{aligned}
|h(t_2)-h(t_1)|\;\leq\;&\left|\int_0^{t_2}\frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s)d_qs-\int_0^{t_1}\frac{(t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s)d_qs\right|\\
&+\frac{|t_2-t_1|}{|D|\Gamma_r(\gamma)\Gamma_q(\alpha)}\int_0^\xi\int_0^s(\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}|f(u)|d_qud_rs\\
&+\frac{|\lambda||t_2-t_1|}{|D|\Gamma_p(\beta)\Gamma_q(\alpha)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}|f(u)|d_qud_ps\\
&+\frac{|\Psi||t_2-t_1|}{|D|\Gamma_q(\alpha)}\int_0^\zeta(\zeta-qs)^{\alpha-1}|f(s)|d_qs\\
\leq\;&\int_0^{t_2}\frac{|(t_2-qs)^{(\alpha-1)}-(t_1-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)}\left(p(s)\psi(\|x\|)+\|x\|\int_0^s\frac{(s-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qs\\
&+\int_{t_1}^{t_2}\frac{(t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left(p(s)\psi(\|x\|)+\|x\|\int_0^s\frac{(s-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qs\\
&+\frac{|t_2-t_1|}{|D|\Gamma_r(\gamma)\Gamma_q(\alpha)}\int_0^\xi\int_0^s(\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}\\
&\times\left(p(u)\psi(\|x\|)+\|x\|\int_0^u\frac{(u-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qud_rs\\
&+\frac{|\lambda||t_2-t_1|}{|D|\Gamma_p(\beta)\Gamma_q(\alpha)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}\\
&\times\left(p(u)\psi(\|x\|)+\|x\|\int_0^u\frac{(u-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qud_ps\\
&+\frac{|\Psi||t_2-t_1|}{|D|\Gamma_q(\alpha)}\int_0^\zeta(\zeta-qs)^{\alpha-1}\left(p(s)\psi(\|x\|)+\|x\|\int_0^s\frac{(s-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right)d_qs.
\end{aligned}
$$

Obviously the right hand side of the above inequality tends to zero independently of $x\in B_\rho$ as $t_2-t_1\to 0$. Therefore it follows by the Ascoli-Arzelá theorem that $\mathcal{B}:C([0,T],\mathbb{R})\to\mathcal{P}(C([0,T],\mathbb{R}))$ is completely continuous.

Claim III: $\mathcal{B}$ *has a closed graph.* Let $x_n\to x_*, h_n\in\mathcal{B}(x_n)$ and $h_n\to h_*$. Then we need to show that

$h_* \in \mathcal{B}(x_*)$. Associated with $h_n \in \mathcal{B}(x_n)$, there exists $f_n \in S_{F,x_n}$ such that for each $t \in [0,T]$,

$$\begin{aligned} h_n(t) &= \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f_n(s)d_qs - \frac{t-\zeta}{D\Gamma_q(\alpha)}\left\{\frac{1}{\Gamma_r(\gamma)}\int_0^\xi\int_0^s (\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}f_n(u)d_qud_rs\right. \\ &\left. -\frac{\lambda}{\Gamma_p(\beta)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}f_n(u)d_qud_ps\right\} - \frac{t\Psi-\Omega}{D\Gamma_q(\alpha)}\int_0^\zeta(\zeta-qs)^{\alpha-1}f_n(s)d_qs. \end{aligned}$$

Thus it suffices to show that there exists $f_* \in S_{F,x_*}$ such that for each $t \in [0,T]$,

$$\begin{aligned} h_*(t) &= \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f_*(s)d_qs - \frac{t-\zeta}{D\Gamma_q(\alpha)}\left\{\frac{1}{\Gamma_r(\gamma)}\int_0^\xi\int_0^s (\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}f_*(u)d_qud_rs\right. \\ &\left. -\frac{\lambda}{\Gamma_p(\beta)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}f_*(u)d_qud_ps\right\} - \frac{t\Psi-\Omega}{D\Gamma_q(\alpha)}\int_0^\zeta(\zeta-qs)^{\alpha-1}f_*(s)d_qs. \end{aligned}$$

Let us consider the linear operator $\Theta : L^1([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ given by

$$\begin{aligned} f \mapsto \Theta(f)(t) &= \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s)d_qs - \frac{t-\zeta}{D\Gamma_q(\alpha)}\left\{\frac{1}{\Gamma_r(\gamma)}\int_0^\xi\int_0^s (\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}f(u)d_qud_rs\right. \\ &\left. -\frac{\lambda}{\Gamma_p(\beta)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}f(u)d_qud_ps\right\} - \frac{t\Psi-\Omega}{D\Gamma_q(\alpha)}\int_0^\zeta(\zeta-qs)^{\alpha-1}f(s)d_qs. \end{aligned}$$

Observe that

$$\begin{aligned} \|h_n(t)-h_*(t)\| &\le \left\| \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}(f_n(s)-f_*(s))d_qs \right. \\ &\quad -\frac{t-\zeta}{D\Gamma_q(\alpha)}\left\{\frac{1}{\Gamma_r(\gamma)}\int_0^\xi\int_0^s(\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}(f_n(u)-f_*(u))d_qud_rs\right. \\ &\quad \left. -\frac{\lambda}{\Gamma_p(\beta)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}(f_n(u)-f_*(u))d_qud_ps\right\} \\ &\quad \left. -\frac{t\Psi-\Omega}{D\Gamma_q(\alpha)}\int_0^\zeta(\zeta-qs)^{\alpha-1}(f_n(s)-f_*(s))d_qs\right\| \to 0, \end{aligned}$$

as $n \to \infty$. Thus, it follows by Lemma 5.3 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$\begin{aligned} h_*(t) &= \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f_*(s)d_qs - \frac{t-\zeta}{D\Gamma_q(\alpha)}\left\{\frac{1}{\Gamma_r(\gamma)}\int_0^\xi\int_0^s (\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}f_*(u)d_qud_rs\right. \\ &\left. -\frac{\lambda}{\Gamma_p(\beta)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}f_*(u)d_qud_ps\right\} - \frac{t\Psi-\Omega}{D\Gamma_q(\alpha)}\int_0^\zeta(\zeta-qs)^{\alpha-1}f_*(s)d_qs \end{aligned}$$

for some $f_* \in S_{F,x_*}$. Hence $\mathcal{B}$ has a closed graph (and therefore has closed values). In consequence, the operator $\mathcal{B}$ is compact valued.

Thus the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 5.4 and hence its conclusion implies either condition (i) or condition (ii) holds. We show that the conclusion (ii) is not possible. If $x \in \theta\mathcal{A}(x) + \theta\mathcal{B}(x)$ for $\theta \in (0,1)$, then there exists $f \in S_{F,x}$ such that

$$x(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s)d_qs - \frac{t-\zeta}{D\Gamma_q(\alpha)}\left\{\frac{1}{\Gamma_r(\gamma)}\int_0^\xi\int_0^s (\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)}f(u)d_qud_rs\right.$$

$$-\frac{\lambda}{\Gamma_p(\beta)}\int_0^\eta\int_0^s(\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)}f(u)d_qud_ps\Bigg\}$$
$$-\frac{t\Psi-\Omega}{D\Gamma_q(\alpha)}\int_0^\zeta(\zeta-qs)^{\alpha-1}f(s)d_qs+\frac{t\Psi-\Omega}{D}g(x),\quad t\in[0,T].$$

Following the method for proof of Claim I, we can obtain

$$\|x\|\le\psi(\|x\|)p_0\|p\|+q_0\|x\|+k_0\ell\|x\|. \tag{5.5}$$

If condition (ii) of Theorem 5.4 holds, then there exists $\theta\in(0,1)$ and $x\in\partial B_r$ with $x=\theta\mathcal{N}(x)$. Then, x is a solution of (1.2) with $\|x\|=M$. Now, by the inequality (5.5), we get

$$\frac{(1-k_0\ell-q_0)M}{\psi(M)p_0\|p\|}\le 1$$

which contradicts (5.2). Hence, $\mathcal{N}$ has a fixed point in $[0,T]$ by Theorem 5.4, and consequently the problem (1.2) has a solution. This completes the proof. □

Example 5.6 *Consider the following nonlocal problem of q-integrodifference inclusion*

$$\begin{cases} {}^cD_{\frac{2}{9}}^{\frac{5}{3}}x(t)\in F\left(t,x,I_{\frac{3}{7}}^{\frac{5}{6}}x\right), \quad t\in(0,1/2),\\ x\left(\frac{1}{6}\right)=\dfrac{|x(1/16)|}{60(1+|x(1/16)|)},\quad \dfrac{2}{3}I_{\frac{1}{3}}^{\frac{7}{4}}x\left(\frac{1}{5}\right)=I_{\frac{3}{5}}^{\frac{4}{3}}x\left(\frac{3}{8}\right),\end{cases} \tag{5.6}$$

where $F:[0,1/2]\times\mathbb{R}^2\to\mathcal{P}(\mathbb{R})$ *is a multivalued map given by*

$$x\to F(t,x,I_{\frac{3}{7}}^{\frac{5}{6}}x)=\left[\frac{t|x|(1+\cos^2 4x)}{12(1+|x|)}+I_{\frac{3}{7}}^{\frac{5}{6}}x,\frac{(t+1)(|x|+1)e^{-3x^2}}{16(1+\sin^2 2x)}+I_{\frac{3}{7}}^{\frac{5}{6}}x\right].$$

Here, $\alpha=5/3$, $q=2/9$, $\delta=5/6$, $z=3/7$, $T=1/2$, $\zeta=1/6$, $\lambda=2/3$, $\beta=7/4$, $p=1/3$, $\eta=1/5$, $\gamma=4/3$, $r=3/5$, $\xi=3/8$, $g(x)=(1/60)(|x|/(1+|x|))$. By using the Maple program, we find that $\Omega=-0.04690826$, $\Psi=-0.20641547$, $D=0.01250568$, $p_0=1.95003166$, $q_0=0.58637355$ and $k_0=12.00382078$.

As $|g(x)-g(y)|\le(1/60)|x-y|$, therefore, (A_2) is satisfied with $\ell=(1/60)<(1/12.00382078)=k_0^{-1}$. For $f\in F$ and $x,y\in\mathbb{R}$, we have

$$|f|\le\max\left(\frac{t|x|(1+\cos^2 4x)}{12(1+|x|)}+y,\frac{(t+1)(|x|+1)e^{-3x^2}}{16(1+\sin^2 2x)}+y\right)\le\frac{t+1}{16}(|x|+1)+|y|.$$

Thus

$$\|F(t,x,y)\|_{\mathcal{P}}:=\sup\{|v|\ :\ v\in F(t,x,y)\}\le p(t)\psi(\|x\|)+|y|,\quad x,y\in\mathbb{R},$$

with $p(t)=(t+1)/16$ and $\psi(\|x\|)=\|x\|+1$. By computing directly, we found that there exists a constant $M>5.94574011$ such that

$$\frac{(1-k_0\ell-q_0)M}{\psi(M)p_0\|p\|}>1.$$

Clearly, all the conditions of Theorem 5.5 are satisfied. Hence, the nonlocal boundary value problem (5.6) has at least one solution on $[0,1/2]$.

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On the Solvability of a System of Multi-Point Second Order Boundary Value Problem

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Abstract

In this paper, by using Leray-Schauder fixed point theorem, we obtain existence of at least one symmetric solution for a multi-point second order boundary value problem. We give an example to demonstrate our main result.

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1 Introduction

The study of multi-point boundary value problems for linear second-order ordinary differential equations were initiated by Il'in and Moiseev [7]. Motivated by the study of Il'in and Moiseev [7], Gupta [4] studied nonlinear three-point boundary value problems for nonlinear ordinary differential equations. Since then, the more general nonlinear multi-point boundary value problems have been studied by many authors. We refer the reader to [5, 8, 10, 13, 16].

On the other hand, the existence of symmetric positive solutions of second order boundary value problems have been studied by some authors, see [1, 6]. Most of the study of symmetric positive solution is limited to Dirichlet boundary value problem, Sturm-Liouville boundary value problem and Neumann boundary value problem. However, there is not so much work on symmetric positive solutions for second-order m-point boundary value problems see [2, 9, 12].

Young and Tisdell [14], studied the following singular boundary value problem (BVP)

$$\left\{ \frac{1}{p}(py^{'})^{'} = qf(t,y), \quad t \in (0,T), \right.$$

coupled with various forms of the following boundary conditions:

$$\left\{ \begin{array}{l} -\alpha y(0) + \beta \lim\limits_{t\to 0^+} p(t)y^{'}(t) = c, \\ \gamma y(T) + \delta \lim\limits_{t\to T^-} p(t)y^{'}(t) = d. \end{array} \right.$$

They established the existence of solutions to second-order singular boundary value problems by using Leray-Schauder fixed point theorem.

We notice that a type of symmetric problem has received much attention, for instance,[5, 8, 15], and the references therein. Jiang et al. [8] studied the following a singular system

$$\left\{ \begin{array}{l} -x^{''}(t) = a_1(t)f(t,x(t),y(t)), \quad t \in (0,1), \\ -y^{''}(t) = a_2(t)g(t,x(t),y(t)), \quad t \in (0,1), \\ x(0) = \sum\limits_{i=1}^{m} \alpha_i y(\xi_i), \ x(1) = \sum\limits_{i=1}^{m} \alpha_i y(\widetilde{\xi}_i), \\ y(0) = \sum\limits_{i=1}^{m} \beta_i x(\eta_i), \ y(1) = \sum\limits_{i=1}^{m} \beta_i x(\widetilde{\eta}_i). \end{array} \right.$$

By using a fixed point theorem in a cone, they obtained at least one or two symmetric positive solutions.

Motivated by this results mentioned above, in this paper, we consider the following second order multi-point boundary value problem

$$\begin{cases} (p(t)u^{'}(t))^{'} = f(t, u(t), u^{'}(t)), \quad t \in (a, b), \\ u(a) = \sum\limits_{i=1}^{m-2} \alpha_i p(\eta_i) u^{'}(\eta_i), u(b) = \sum\limits_{i=1}^{m-2} \alpha_i p(\xi_i) u^{'}(\xi_i). \end{cases} \tag{1.1}$$

Throughout this paper we assume that following conditions hold:

$(C1)$ $f \in \mathcal{C}([a, b] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ and $f(t, 0, 0) \not\equiv 0$ for $t \in [a, b]$, $p \in \mathcal{C}'([a, b], \mathbb{R})$ with $p > 0$ on (a, b) and $\int_a^b \frac{ds}{p(s)} < +\infty$ and f is symmetric function on $[a, b]$ such that $f(b + a - t, u, v) = f(t, u, v)$, and p antisymmetric function on $[a, b]$ such that $p(b + a - t) = -p(t)$ and f is an even function in v, i.e., $f(t, u, v) = f(t, u, -v)$.

$(C2)$ $\alpha, \beta \in [0, \infty)$, $\alpha_i \in [0, \infty)$, $\xi_i, \eta_i \in (a, b)$ such that $\xi_i = b + a - \eta_i$ for $i \in \{1, 2, ..., m - 2\}$.

By using Leray-Schauder fixed point theorem, we get the existence of symmetric solution for the BVP (1.1).

This paper contains three sections besides the Introduction. In Section 2, we present some necessary preliminaries that will be used to prove our main result. In Section 3, we obtain the existence of at least one symmetric solution for the BVP (1.1). Finally we give an example to illustrate our result in Section 4.

To the best of our knowledge, there is no earlier literature studying this problem. This paper attempts to fill part of this gap in the literatures.

As for notation, if y, $z \in \mathbb{R}$, then $\langle y, z\rangle$ denotes their usual inner product and $\|z\|$ denotes the Euclidean norm of z. We adopt the standart norm for elements u of $\mathcal{C}^{'}([a, b], \mathbb{R}^n)$, namely

$$||u(t)||_0 := \max\{\max_{t\in[a,b]} \|u(t)\|, \max_{t\in[a,b]} \|u^{'}(t)\|\}.$$

For all $t \in (a, b)$, we have

$$\begin{aligned} \langle u(t), p(t)u'(t)\rangle^{'} &= \langle u(t), (p(t)u'(t))^{'}\rangle + \langle u^{'}(t), p(t)u'(t)\rangle \\ &= \langle u(t), f(t, u(t), u'(t))\rangle + p(t)\|u^{'}(t)\|^2. \end{aligned}$$

The above identity will be needed in the proof of our main result and our technique is based on a priori bound. We refer the reader to the papers [11, 14] and the references therein.

2 Preliminaries

In this section, we will employ several lemmas to prove the main result in this paper. When $n = 1$, (1.1) reduces to the scaler equation.

Lemma 2.1 *Suppose the condition* $D = -\int_a^b \frac{1}{p(s)} ds \neq 0$ *hold. Then, for* $h \in \mathcal{C}([a, b], \mathbb{R}^n)$, *and symmetric on* $[a, b]$, *the BVP*

$$\begin{cases} (p(t)u^{'}(t))^{'} = h(t), \quad t \in (a, b), \\ u(a) = \sum\limits_{i=1}^{m-2} \alpha_i p(\eta_i) u^{'}(\eta_i), u(b) = \sum\limits_{i=1}^{m-2} \alpha_i p(\xi_i) u^{'}(\xi_i), \end{cases} \tag{2.1}$$

has a unique solution u

$$u(t) = \int_a^b G(t, s)h(s)ds + \sum_{i=1}^{m-2} \alpha_i \int_a^b {G_t}^{[1]}(\eta_i, s)h(s)ds,$$

where ${G_t}^{[1]}(t, s) = p(t){G_t}^{'}(t, s)$ *and* $G(t, s)$ *be the Green's function for* (2.1) *is given by*

$$G(t, s) = \frac{1}{D} \begin{cases} \theta(t)\varphi(s), & a \le t \le s \le b, \\ \theta(s)\varphi(t), & a \le s \le t \le b, \end{cases} \tag{2.2}$$

2

where $\theta(t)$ and $\varphi(t)$ are given by

$$\theta(t) = \int_a^t \frac{1}{p(\tau)} d\tau, \tag{2.3}$$

$$\varphi(t) = \int_t^b \frac{1}{p(\tau)} d\tau, \tag{2.4}$$

respectively.

Proof. $u(t) = \int_a^b G(t,s)h(s)ds + \sum_{i=1}^{m-2} \alpha_i \int_a^b {G_t}^{[1]}(\eta_i, s)h(s)ds$ be a solution of (2.1), then we have that

$$\begin{aligned} u(t) \quad &= \quad \frac{1}{D}\int_a^t \theta(s)\varphi(t)h(s)ds + \frac{1}{D}\int_t^b \theta(t)\varphi(s)h(s)ds \\ &+ \quad \sum_{i=1}^{m-2} \alpha_i \int_a^b {G_t}^{[1]}(\eta_i, s)h(s)ds, \end{aligned}$$

$$\begin{aligned} p(t)u^{'}(t) &= p(t)\varphi^{'}(t)\int_a^t \frac{1}{D}\theta(s)h(s)ds + p(t)\theta^{'}(t)\int_t^b \frac{1}{D}\varphi(s)h(s)ds \\ &= \int_a^b {G_t}^{[1]}(t,s)h(s)ds, \end{aligned}$$

and

$$\begin{aligned} (p(t)u^{'}(t))^{'} \quad &= \quad (p(t)\varphi^{'}(t))^{'}\int_a^t \frac{1}{D}\theta(s)h(s)ds + p(t)\varphi^{'}(t)\frac{1}{D}\theta(t)h(t) \\ &+ \quad (p(t)\theta^{'}(t))^{'}\int_t^b \frac{1}{D}\varphi(s)h(s)ds - p(t)\theta^{'}(t)\frac{1}{D}\varphi(t)h(t) \\ &= \quad p(t)\varphi^{'}(t)\frac{1}{D}\theta(t)h(t) - p(t)\theta^{'}(t)\frac{1}{D}\varphi(t)h(t) \\ &= \quad \frac{p(t)}{D}\left[\varphi^{'}(t)\theta(t) - \theta^{'}(t)\varphi(t)\right]h(t) \\ &= \quad h(t). \end{aligned}$$

Since

$$\begin{aligned} u(a) &= \frac{1}{D}\int_a^b \theta(a)\varphi(s)h(s)ds + \sum_{i=1}^{m-2} \alpha_i \int_a^b {G_t}^{[1]}(\eta_i, s)h(s)ds \\ &= \sum_{i=1}^{m-2} \alpha_i \int_a^b {G_t}^{[1]}(\eta_i, s)h(s)ds \\ &= \sum_{i=1}^{m-2} \alpha_i p(\eta_i)u^{'}(\eta_i), \end{aligned}$$

and

$$
\begin{aligned}
u(b) &= \frac{1}{D}\int_a^b \theta(s)\varphi(b)h(s)ds + \sum_{i=1}^{m-2}\alpha_i \int_a^b {G_t}^{[1]}(\eta_i, s)h(s)ds \\
&= \sum_{i=1}^{m-2}\alpha_i \int_a^b {G_t}^{[1]}(\eta_i, s)h(s)ds \\
&= \sum_{i=1}^{m-2}\alpha_i p(\eta_i)\varphi^{'}(\eta_i)\int_a^{\eta_i}\frac{1}{D}\theta(s)h(s)ds + \sum_{i=1}^{m-2}\alpha_i p(\eta_i)\theta^{'}(\eta_i)\int_{\eta_i}^b \frac{1}{D}\varphi(s)h(s)ds \\
&= -\sum_{i=1}^{m-2}\alpha_i \int_a^{\eta_i}\frac{1}{D}\theta(s)h(s)ds + \sum_{i=1}^{m-2}\alpha_i\int_{\eta_i}^b\frac{1}{D}\varphi(s)h(s)ds \\
&= -\sum_{i=1}^{m-2}\alpha_i \int_{b+a-\eta_i}^{b}\frac{1}{D}\theta(b+a-s)h(b+a-s)d(b+a-s) \\
&\quad + \sum_{i=1}^{m-2}\alpha_i \int_a^{b+a-\eta_i}\frac{1}{D}\varphi(b+a-s)h(b+a-s)d(b+a-s) \\
&= -\sum_{i=1}^{m-2}\alpha_i \int_{\xi_i}^b \frac{1}{D}(-\varphi(s))h(s)ds + \sum_{i=1}^{m-2}\alpha_i\int_a^{\xi_i}\frac{1}{D}(-\theta(s))h(s)ds \\
&= \sum_{i=1}^{m-2}\alpha_i p(\xi_i)u^{'}(\xi_i).
\end{aligned}
$$

We are able to obtain the boundary value conditions. □

Lemma 2.2 *For $t, s \in [a, b]$, we have $G(b+a-t, b+a-s) = G(t, s)$.*

Proof. In fact, if $t \leq s$, then $1-t \geq 1-s$. In view of (2.2) and the assumption (C1), we get

$$
\begin{aligned}
G(b+a-t, b+a-s) &= \frac{1}{D}\left(\int_a^{b+a-s}\frac{1}{p(\tau)}d\tau\right)\left(\int_{b+a-t}^b \frac{1}{p(\tau)}d\tau\right) \\
&= \frac{1}{D}\left(\int_b^s \frac{1}{p(b+a-\tau)}d(b+a-\tau)\right)\left(\int_t^a \frac{1}{p(b+a-\tau)}d(b+a-\tau)\right) \\
&= \frac{1}{D}\left(\int_s^b \frac{1}{p(\tau)}d\tau\right)\left(\int_a^t \frac{1}{p(\tau)}d\tau\right) \\
&= G(t, s),\ a \leq t \leq s \leq b.
\end{aligned}
$$

Similarly, we can prove that $G(b+a-t, b+a-s) = G(t, s),\ a \leq s \leq t \leq b$. We have $G(b+a-t, b+a-s) = G(t, s)$ for all $(t, s) \in [a, b] \times [a, b]$, i.e., G(t,s) is symmetric function on $[a, b] \times [a, b]$. □

Lemma 2.3 *For $t, s \in [a, b]$, we have* $\max\limits_{(t,s)\in[a,b]\times[a,b]}|G(t, s)| \leq \int_a^b \frac{1}{p(t)}dt \max\limits_{(t,s)\in[a,b]\times[a,b]}|{G_t}^{[1]}(t, s)|$.

Proof. We apply (2.2), we get that for $t \in [a, b]$

$$
\frac{G(t,s)}{{G_t}^{[1]}(t,s)} = \begin{cases} \dfrac{\theta(t)}{p(t)\theta'(t)}, & a \leq t \leq s \leq b, \\[2ex] \dfrac{\varphi(t)}{p(t)\varphi'(t)}, & a \leq s \leq t \leq b, \end{cases} = \begin{cases} \theta(t), & a \leq t \leq s \leq b, \\ -\varphi(t), & a \leq s \leq t \leq b, \end{cases} \tag{2.5}
$$

Then we obtain that from (2.5), $\max_{(t,s)\in[a,b]\times[a,b]}|G(t, s)| \leq \int_a^b \frac{1}{p(t)}dt \max\limits_{(t,s)\in[a,b]\times[a,b]}|{G_t}^{[1]}(t, s)|$ for $t \in [a, b]$. □

4

Let $\mathcal{B} = \mathcal{C}^{'}([a,b];\mathcal{R}^n)$ then $\mathcal{B}$ is a Banach space with $||u(t)||_0 := \max\{\max_{t\in[a,b]} \|u(t)\|, \max_{t\in[a,b]} \|u^{'}(t)\|\}$, and define a cone $P \subset \mathcal{B}$ by

$$P = \{u \in \mathcal{B} : u(t) \; is \; symmetric \; on \; \text{[a,b]}\}.$$

We define the integral operator $T : P \to \mathcal{B}$ by

$$Tu(t) = \int_a^b G(t,s)f(s,u(s),u'(s))ds + \frac{1}{\alpha}\sum_{i=1}^{m-2}\alpha_i \int_a^b {G_t}^{[1]}(\eta_i,s)f(s,u(s),u'(s))ds,$$

where $G(t,s)$ is given by (2.2) and ${G_t}^{[1]}(t,s) = p(t){G_t}^{'}(t,s)$.

Lemma 2.4 *Let $(C1)$-$(C2)$ hold. Then $T : P \to P$ is completely continuous.*

Proof. For all $u \in P$, using that $f(t,u(t),u'(t))$ is symmetric on [a,b], and $p(t)$ is antisymmetric on [a,b] and by Lemma 2.2, we have

$$\begin{aligned}
Tu(b+a-t) &= \int_a^b G(b+a-t,s)f(s,u(s),u'(s))ds + \frac{1}{\alpha}\sum_{i=1}^{m-2}\alpha_i \int_a^b {G_t}^{[1]}(\eta_i,s)f(s,u(s),u'(s))ds \\
&= \int_b^a G(b+a-t,b+a-s)f(b+a-s,u(b+a-s),u'(b+a-s))d(b+a-s) \\
&+ \frac{1}{\alpha}\sum_{i=1}^{m-2}\alpha_i \int_a^b {G_t}^{[1]}(\eta_i,s)f(s,u(s),u'(s))ds \\
&= \int_a^b G(t,s)f(s,u(s),u'(s))d(s) + \frac{1}{\alpha}\sum_{i=1}^{m-2}\alpha_i \int_a^b {G_t}^{[1]}(\eta_i,s)f(s,u(s),u'(s))ds \\
&= Tu(t),
\end{aligned}$$

for every $t \in [a,b]$. This implies that $Tu(t)$ is symmetric on $[a,b]$. So, $Tu \in P$ and then $Tu \subset P$. Next, by standard methods and Arzela-Ascoli theorem, one can easily prove that operator T is completely continuous. □

3 Main result

In this section, we discuss the existence of at least one symmetric solution for the problem (1.1). The following fixed point theorem is fundamental and important to the proof of our main result.

Lemma 3.1 *(See[3]). Let B be a Banach space with $P \subseteq B$ closed and convex. Assume U is a open subset of P with $0 \in U$ and $T : \overline{U} \to P$ is a continuous and compact map. Then either*

(i) T has a fixed point in $\overline{U}$, or

(ii) there exists $u \in \partial U$ and $\lambda \in (0,1)$ such that $u = \lambda Tu$.

Lemma 3.2 *Let $f \in \mathcal{C}([a,b] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ and let $(C1)$-$(C2)$ hold with*

$$G_0 := \max_{(t,s)\in[a,b]\times[a,b]} \mid {G_t}^{[1]}(t,s) \mid \left(\int_a^b \frac{1}{p(t)}dt + \frac{1}{\alpha}\sum_{i=1}^{m-2}\alpha_i\right),$$

$$G_1 := \max_{(t,s)\in[a,b]\times[a,b]} \mid G_t^{'}(t,s) \mid,$$

$$M := \max\{G_0W(b-a), G_1W(b-a)\}.$$

If there exist non-negative constant V and W such that

$$||f(t,u,v)|| \leq V\left[\langle u(t), f(t,u(t),v(t))\rangle + p(t)||v||^2\right] + W, \tag{3.1}$$

for all $(t, u, v) \in [a, b] \times \mathbb{R}^n \times \mathbb{R}^n$. *Then all solutions* $u = u(t)$ *to the BVP* (1.1) *satisfy*

$$||u(t)||_0 \leq M,$$

for all $t \in [a, b]$.

Proof. Let $u = u(t) \in \mathrm{P}$ be any solution to the BVP (1.1). From Lemma 2.1, we get

$$u(t) = \int_a^b G(t,s)f(s,u(s),u'(s))ds + \frac{1}{\alpha}\sum_{i=1}^{m-2}\alpha_i \int_a^b {G_t}^{[1]}(\eta_i,s)f(s,u(s),u'(s))ds.$$

Then

$$\begin{aligned}
\|u(t)\| &\leq \max_{(t,s)\in[a,b]\times[a,b]} \mid G(t,s) \mid \int_a^b ||f(s,u(s),u'(s))||ds \\
&+ \max_{(t,s)\in[a,b]\times[a,b]} \mid {G_t}^{[1]}(t,s) \mid \frac{1}{\alpha}\sum_{i=1}^{m-2}\alpha_i \int_a^b ||f(s,u(s),u'(s))||ds
\end{aligned}$$

$$\begin{aligned}
&\leq \max_{(t,s)\in[a,b]\times[a,b]} \mid {G_t}^{[1]}(t,s) \mid \left(\int_a^b \frac{1}{p(t)}dt + \frac{1}{\alpha}\sum_{i=1}^{m-2}\alpha_i\right) \int_a^b ||f(s,u(s),u'(s))||ds \\
&\leq G_0 \int_a^b \Big(V\left[\langle u(s), f(s,u(s),u'(s))\rangle + p(s)||u'(s)||^2\right] + W\Big)\,ds \\
&\leq G_0 V \int_a^b \left[\langle u(s), f(s,u(s),u'(s))\rangle + p(s)||u'(s)||^2\right] ds + G_0 W(b-a) \\
&\leq G_0 V \int_a^b \langle u(t), p(t)u'(t)\rangle^{'} ds + G_0 W(b-a) \\
&\leq G_0 V\left[\langle u(b), p(b)u'(b)\rangle - \langle u(a), p(a)u'(a)\rangle\right] + G_0 W(b-a) \\
&\leq G_0 V\left[\langle u(b), p(b)u'(b) - p(b)u'(b)\rangle\right] + G_0 W(b-a) \\
&\leq G_0 W(b-a),
\end{aligned}$$

and

$$u^{'}(t) = \int_a^b {G_t}^{'}(t,s)f(s,u(s),u'(s))ds,$$

$$\begin{aligned}
\|u^{'}(t)\| &\leq \max_{(t,s)\in[a,b]\times[a,b]} \mid {G_t}^{'}(t,s) \mid \int_a^b ||f(s,u(s),u'(s))||ds, \\
&\leq G_1 \int_a^b \Big(V\left[\langle u(s), f(s,u(s),u'(s))\rangle + p(s)||u'(s)||^2\right] + W\Big)\,ds, \\
&\leq G_1 W(b-a).
\end{aligned}$$

We have

$$\begin{aligned}
||u(t)||_0 &:= \max\{\max_{t\in[a,b]}\|u(t)\|, \max_{t\in[a,b]}\|u^{'}(t)\|\}, \\
&\leq \max\{G_0 W(b-a), G_1 W(b-a)\}, \\
&\leq M.
\end{aligned}$$

Thus our claimed a priori bound has been obtained. □

Theorem 3.1 *Under the condition of Lemma* 3.2*, the BVP* (1.1) *has at least one symmetric solution.*

Proof. Let u be a possible solution of (1.1). Lemma 3.2 implies that $||u(t)||_0 \leq M$ for all $t \in [a, b]$. We now apply the priori bound result to has the existence of solution.

Let $\Omega := \{u \in P : ||u(t)||_0 < M+1\}$. By Lemma 2.4, we know that the operator $T : P \to P$ is completely continuous. Since all possible solutions of (1.1) satisfy $||u(t)||_0 \leq M$. It follows that there isn't u in $\partial\Omega$ and $\lambda \in (0, 1)$ such that $u = \lambda T u$. We conclude that (ii) of Lemma 3.1 does not hold. Therefore, the operator T has a fixed point in $\overline{U}$, which is a symmetric solution of (1.1). □

4 Example

Example 4.1 We consider the following multi-point second order boundary value problem with $n = 2$, $m = 3$,

$$\begin{cases} (p(t)u^{'}(t))^{'} = f(t, u(t), u^{'}(t)), \quad t \in (0, 1), \\ u(0) = \frac{1}{2}p\left(\frac{1}{4}\right)u^{'}\left(\frac{1}{4}\right), u(1) = \frac{1}{2}p\left(\frac{3}{4}\right)u^{'}\left(\frac{3}{4}\right), \end{cases} \tag{4.1}$$

where $p(t)$ and $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}^2$ are given by

$$\begin{aligned} p(t) &= \left(\frac{1}{2} - t\right), \\ f(t, u) &= (f_1(t, u_1, u_2), f_2(t, u_1, u_2)) \\ &= (t(1-t)u_1 + u_1^3u_2^2, t(1-t)u_2 + u_2^3u_1^2), \end{aligned}$$

and f not depending on $u^{'}$. It is easy to see that $f(t, u) = f(1 - t, u)$ and $p(t) = -p(1 - t)$. We claim that the above f satisfies the condition of Lemma 3.2, (3.1). Note that for all $(t, u) \in [0, 1] \times \mathbb{R}^2$ we have

$$\begin{aligned} ||f(t, u)|| &\leq |f_1(t, u_1, u_2)| + |f_2(t, u_1, u_2)| \\ &\leq t(1-t)|u_1| + |u_1|^3u_2^2 + t(1-t)|u_2| + |u_2|^3u_1^2. \end{aligned}$$

If we choose $V = 1$ and $W = 4$ then, for all $(t, u_1, u_2) \in [0, 1] \times \mathbb{R}^2$,

$$\begin{aligned} V\langle u(t), f(t, u(t))\rangle + W &= \left[t(1-t)u_1^2 + u_1^4u_2^2 + t(1-t)u_2^2 + u_2^4u_1^2\right] + 4 \\ &\geq \left[t(1-t)|u_1| - 1 + |u_1^3|u_2^2 - 1 + t(1-t)|u_2| - 1 + |u_2|^3u_1^2 - 1\right] + 4 \\ &\geq t(1-t)|u_1| + |u_1^3|u_2^2 + t(1-t)|u_2| + |u_2|^3u_1^2 \\ &\geq ||f(t, u)||. \end{aligned}$$

So, $f(t, u)$ satisfies Lemma 3.2. Then, Theorem 3.1 hold. The BVP (1.1) has at least one symmetric solution. □

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Cascadic Multigrid Method for The Elliptic Monge-Ampère Equation☆

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Abstract

The elliptic Monge-Ampère (M-A) equation is a fully nonlinear partial differential equation, which originated in geometric surface theory and has been widely applied in dynamic meteorology, elasticity, geometric optics, image processing and others. The numerical solution of the elliptic Monge-Ampère equation has been a subject of increasing interest recently. In this paper, the cascadic multigrid method (CMG) is used to solve numerically the M-A equation. Before the application of CMG method, an equivalent form of M-A equation is given. On each successive refinement level, weak formulation of this equivalent form can be written and finite element methods can be used successfully. We analyze the convergence and computational complexity for the cascadic multigrid method. And we find that the CMG method is optimal with respect to the energy norm. Finally, numerical experiments confirm the efficiency and robustness of CMG method.

Keywords: Cascadic multigrid, Finite element methods, Interpolation, Monge-Ampère equation
2000 MSC: 65F10, 65N30, 65N55, 35J60

1. Introduction

In this paper, we will introduce a cascadic multirid method for the fully nonlinear elliptic partial differential equation

$$\det(D^2u(z)) = f(z), \tag{1.1}$$

where z has n independent variables, and D^2u is the Hessian of the function u. When restricting it to domains $\Omega \subset \mathbb{R}^2$, we can rewrite the equation as

$$(u_{xx}u_{yy} - u_{xy}^2)(x, y) = f(x, y). \tag{1.2}$$

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The equation comes with Dirichlet boundary conditons

$$u(x, y) = g(x, y), \quad \text{on} \quad \partial\Omega \tag{1.3}$$

and the additional convexity constraint

$$u(x, y) \quad \text{is convex}, \tag{1.4}$$

which is required for the equation to be elliptic. Here $\Omega \subset \mathbb{R}^2$ is a bounded domain with boundary $\partial\Omega$ and $f : \Omega \to \mathbb{R}$ is a non-negative function. The equation (1.2) (along with boundary conditions and convexity constraint) is called the elliptic Monge-Ampère equation.

The elliptic M-A equation is a fully nonlinear partial differential equation, which originated in geometric surface theory and has been widely applied in dynamic meteorology, elasticity, geometric optics, image processing and others [17, 18, 25, 26]. And the numerical solution of the elliptic M-A equation has been a subject of increasing interest recently. The early paper introducing the numerical solution of the elliptic M-A equation is written by Oliker and Prussner [24]. They presented a discretization that converges to the generalized solution in two dimensions. Subsequently, other excellent methods also are used to solve the M-A equation. For the finite difference methods, it has been proved that consistency, stability and monotonicity are the convergence criterion for (1.2)-(1.4) in [2]. But narrow stencils won't be monotone and even consistent, and establishing a wide scheme requires more works [3, 12, 13, 23]. The implementation and the convergence theory of finite element methods for the elliptic M-A equation is less understood. Feng et al. [11] considered a fourth order problem by adding a small multiple of the biharmonic operator to (1.1). Brenner et al. [5] introduced C^0 penalty methods for the elliptic M-A equation. The crux of both methods is putting the essential boundary condition (1.3) into a weak form by addition and penalization techniques respectively. However, it will become very difficult to prove the solvability and convergence for new perturbed problems. Dean and Glowinski [7, 8, 9, 10, 14, 15] presented an augmented Lagrange multiplier method and a least squares method for the elliptic M-A equation. The convergence of these methods still remains an open problem. Liu and He [21] introduced meshfree method for the elliptic M-A equation. Although meshfree method is easy to implement, its convergence theory still remains open. Consequently, developing efficient discretizations still is challenging for the elliptic M-A equation.

In this paper, we will solve numerically the M-A equation by cascadic multigrid method. Firstly, we will provide an equivalent form of M-A equation and write its variational form. Then, the M-A equation can be solved by finite element (FE) methods based on successive refinement levels and interpolation techniques. We will analyze the convergence and computational complexity of CMG method.

The rest of paper is organized as follows. In Section 2, we design the cascadic multigrid algorithm for M-A equation. The convergence of CMG method will be proved in Section 3. In Section 4, some numerical experiments are presented to demonstrate the effectiveness of CMG method.

2. CMG method for M-A equation

We know that a function u is convex is equivalent to

$$u_{xx} \geq 0, \quad u_{yy} \geq 0, \quad u_{xx}u_{yy} - u_{xy}^2 \geq 0. \tag{2.1}$$

And we have

$$(\triangle u)^2 = (u_{xx} + u_{yy})^2 = u_{xx}^2 + u_{yy}^2 + 2u_{xx}u_{yy}, \tag{2.2}$$

or

$$u_{xx}u_{yy} = \frac{1}{2}((\triangle u)^2 - u_{xx}^2 - u_{yy}^2). \tag{2.3}$$

By substituting (2.3) into (1.2), we have

$$(\triangle u)^2 - u_{xx}^2 - u_{yy}^2 - 2u_{xy}^2 = 2f(x,y). \tag{2.4}$$

According to (2.1), we can choose a convex solution

$$\triangle u = \sqrt{2f(x,y) + u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2}. \tag{2.5}$$

So, the equation (1.2)-(1.4) can be rewritten as

$$\triangle u = \sqrt{2f(x,y) + u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2}, \quad \text{in} \quad \Omega \tag{2.6}$$

$$u = g(x,y). \quad \text{on} \quad \partial\Omega \tag{2.7}$$

Following, we design the CMG algorithm for equation (2.6)-(2.7).

2.1. Algorithm

Let $j = 0, 1, \cdots, L$ be a sequence of grid levels, where $j = 0$ denotes the coarsest grid and $j = L$ is the finest grid level. Corresponding to the sequence of grid levels, we have a nested family of partition $(\mathcal{T}_j)_{j=0}^L$ (triangles or rectangles) and the spaces of linear finite elements $(X_j)_{j=0}^L$.

Then we can design the CMG algorithm as following:
(1) First, the following equation is discretized by FE method on level $j = 0$.

$$\triangle u = \sqrt{2f(x,y)}, \quad \text{in} \quad \Omega \tag{2.8}$$

$$u = g(x,y). \quad \text{on} \quad \partial\Omega \tag{2.9}$$

Then the resulting discrete system is solved by m_0 smoothing steps with any initial value (or by direct methods).
(2) For $j = 1, \cdots, L$

First, obtaining the approximation of $u, u_{xx}, u_{yy}, u_{xy}$ on j level. This can be finished by interpolation techniques. We remark them as u^j, u^j_{xx}, u^j_{yy} and u^j_{xy} respectively. Then following equation can be discretized by FE method.

$$\triangle u = \sqrt{2f(x,y) + (u^j_{xx})^2 + (u^j_{yy})^2 + 2(u^j_{xy})^2}, \quad \text{in} \quad \Omega \tag{2.10}$$

$$u = g(x, y). \quad \text{on} \quad \partial\Omega \tag{2.11}$$

With initial value u^j, the discrete system is solved by m_j smoothing steps.

Remarks: (1) The linear finite elements spaces can be defined by

$$X_j = \{u \in C(\bar{\Omega}) : u|_e \in P_1(e) \quad \forall e \in \mathcal{T}_j, \quad u|_{\partial\Omega} = 0\},$$

where $P_1(e)$ denotes the linear polynomial on the triangle or rectangle e. Clearly, we have

$$X_0 \subset X_1 \subset \cdots \subset X_L \subset H_0^1(\Omega).$$

(2) The equation (2.6)-(2.7) is inhomogeneous. We let u be a function which coincides with g on the boundary of Ω. That is to say, there exists $Rg \in H^1(\Omega)$ with $Rg|_{\partial\Omega} = g$ and $\|Rg\|_1 \leq C\|g\|_{H^{1/2}(\Omega)}$. Then we can write the variational form of (2.10)-(2.11) as following:

Let $w_j = u_j - Rg$, find $w_j \in H_0^1(\Omega)$ such that

$$a(w_j, v_j) = F(v_j), \quad v_j \in X_j, \tag{2.12}$$

where

$$a(w_j, v_j) = \int_\Omega \nabla w_j \cdot \nabla v_j,$$

$$F(v_j) = \int_\Omega -v_j\sqrt{2f(x,y) + (u_{xx}^j)^2 + (u_{yy}^j)^2 + 2(u_{xy}^j)^2} - \int_\Omega \nabla Rg \cdot \nabla v_j.$$

2.2. Approximate to u, u_{xx}, u_{yy}, u_{xy}

Indeed, constructing effective interpolation is challenging in multigrid methods. [1, 19, 20] designed kinds of interpolation operators to transfer error residual from coarse grid to fine grid. But it is missing to construct the approximation of derivative functions, especially higher derivative functions. How to construct the approximation of u_{xx}, u_{yy} and u_{xy} on j level through information on $j-1$ level? For simplicity, in this subsection we only consider nested uniform rectangle grids. In future work, we will construct the approximation of higher derivative functions on other finite elements.

In Figure 1, we show two grid levels $j-1$ and j. Here, blue lines denote grid on $j-1$ level and red lines denote uniform refinement. And all nodes on j level are divided into cases (b)-(f). We want to approximate the values of u, u_{xx}, u_{yy} and u_{xy} at each black node for each case.

Let the meshsizes be H for $j-1$ level and h for j level. u^H denotes value of u restricted on $j-1$ level and u^h denotes the value on j level. Then applying Taylor expansion we have following expressions.

For case (b),

$$u_7^h = u_{7,}^H$$

$$(u_{xx}^h)_7 = \frac{1}{4h^2}(u_5^H + u_9^H - 2u_7^H),$$

$$(u_{yy}^h)_7 = \frac{1}{4h^2}(u_6^H + u_8^H - 2u_7^H),$$

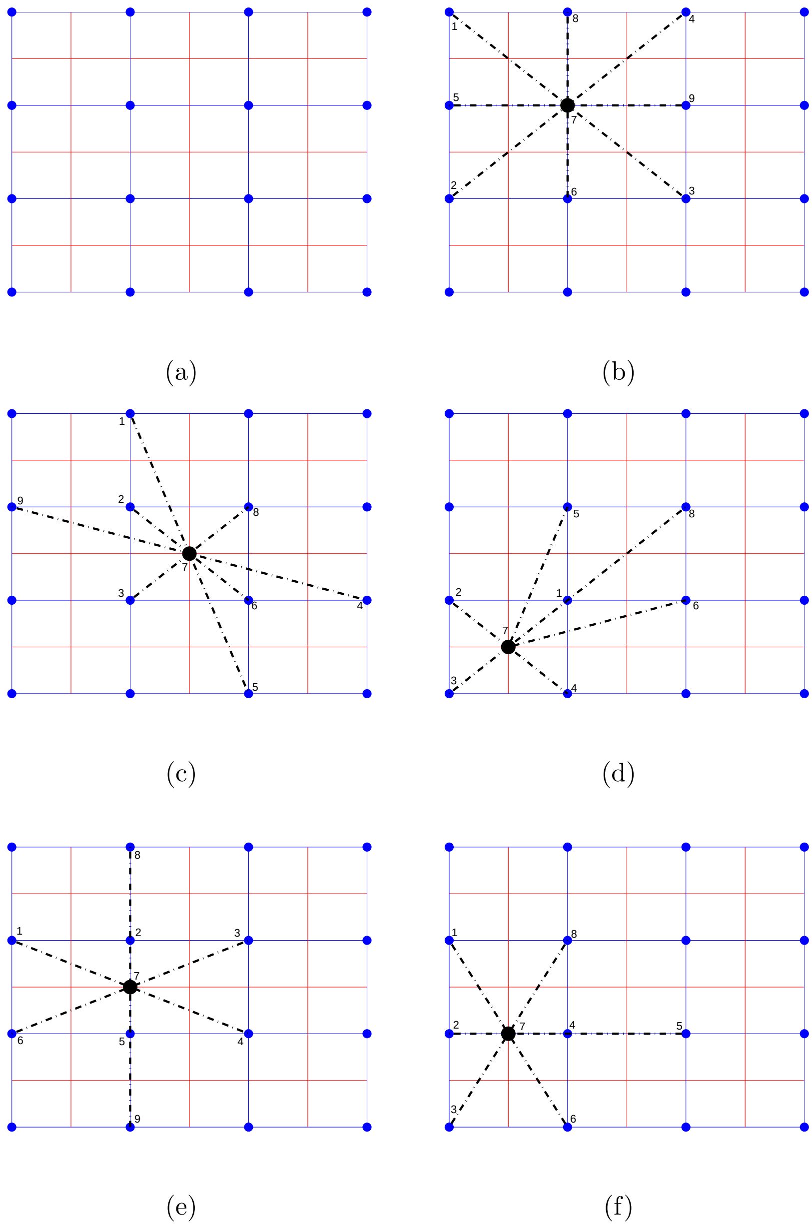

Figure 1: Two grid levels: $j-1$ and j.

$$(u_{xy}^h)_7 = \frac{1}{16h^2}(u_2^H + u_4^H - u_1^H - u_3^H).$$

For case (c),

$$u_7^h = \frac{1}{4}(u_2^H + u_3^H + u_6^H + u_8^H),$$

$$(u_{xx}^h)_7 = \frac{1}{8h^2}(u_9^H + u_4^H + u_3^H + u_8^H - 2u_2^H - 2u_6^H),$$

$$(u_{yy}^h)_7 = \frac{1}{8h^2}(u_1^H + u_5^H + u_3^H + u_8^H - 2u_2^H - 2u_6^H),$$

$$(u_{xy}^h)_7 = \frac{1}{4h^2}(u_3^H + u_8^H - u_2^H - u_6^H).$$

For case (d),

$$u_7^h = \frac{1}{4}(u_1^H + u_2^H + u_3^H + u_4^H),$$

$$(u_{xx}^h)_7 = \frac{1}{4h^2}(u_8^H + 2u_2^H + u_4^H - u_3^H - 2u_1^H - u_5^H),$$

$$(u_{yy}^h)_7 = \frac{1}{4h^2}(u_8^H + 2u_4^H + u_2^H - u_3^H - 2u_1^H - u_6^H),$$

$$(u_{xy}^h)_7 = \frac{1}{4h^2}(u_1^H + u_3^H - u_2^H - u_4^H).$$

For case (e),

$$u_7^h = \frac{1}{2}(u_2^H + u_5^H),$$

$$(u_{xx}^h)_7 = \frac{1}{8h^2}(u_1^H + u_3^H + u_4^H + u_6^H - 2u_2^H - 2u_5^H),$$

$$(u_{yy}^h)_7 = \frac{1}{8h^2}(u_8^H + u_9^H - u_2^H - u_5^H),$$

$$(u_{xy}^h)_7 = \frac{1}{8h^2}(u_3^H + u_6^H - u_1^H - u_4^H).$$

For case (f),

$$u_7^h = \frac{1}{2}(u_2^H + u_4^H),$$

$$(u_{xx}^h)_7 = \frac{1}{4h^2}(u_2^H + u_5^H - 2u_4^H),$$

$$(u_{yy}^h)_7 = \frac{1}{8h^2}(u_1^H + u_3^H + u_6^H + u_8^H - 2u_2^H - 2u_4^H),$$

$$(u_{xy}^h)_7 = \frac{1}{8h^2}(u_3^H + u_8^H - u_1^H - u_6^H).$$

3. Analysis of convergence

In this section, we prove the convergence of CMG method which is introduced in above section. C denote positive constants in this section. First, following Lemma provides the existence and uniqueness theory for variational problem (2.12).

Lemma 3.1. *Let M-A equation (1.2)-(1.4) be uniquely solvable for all* $f \in L^2(\Omega)$. *If there exists* $Rg \in H^1(\Omega)$ *with* $Rg|_{\partial\Omega} = g$ *and* $\|Rg\|_1 \le C\|g\|_{H^{1/2}(\Omega)}$, *then variational problem (2.12) has a unique solution* $u_j = w_j + Rg$.

PROOF. We have

$$a(w_j, v_j) = (\nabla w_j, \nabla v_j) \le \|w_j\|_1 \|v_j\|_1,$$

and

$$a(v_j, v_j) = |v_j|_1^2 \ge C\|v_j\|_1^2 \quad for \quad v_j \in X_j.$$

In addition,

$$\begin{aligned} |F(v_j)| &\le \|\sqrt{2f(x,y) + (u_{xx}^j)^2 + (u_{yy}^j)^2 + 2(u_{xy}^j)^2}\|_0 \|v_j\|_0 + \|\nabla Rg\|_0 \|\nabla v_j\|_0 \\ &\le \left(\|\sqrt{2f(x,y) + (u_{xx}^j)^2 + (u_{yy}^j)^2 + 2(u_{xy}^j)^2}\|_0 + \|Rg\|_1 \right) \|v_j\|_1 \\ &\le C \left(\|\sqrt{2f(x,y) + (u_{xx}^j)^2 + (u_{yy}^j)^2 + 2(u_{xy}^j)^2}\|_0 + \|g\|_{H^{1/2}(\Omega)} \right) \|v_j\|_1. \end{aligned}$$

F is bounded because u_{xx}^j, u_{yy}^j and u_{xy}^j are known. According to Lax-Milgram Lemma, variational problem has a unique solution.

The variational form of (2.6)-(2.7) can be given by (3.1).
Find $\omega = u - Rg$, $\omega \in H_0^1(\Omega)$, such that

$$a(w, v) = \int_\Omega -v\sqrt{2f(x,y) + u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2} - \int_\Omega \nabla Rg \cdot \nabla v \qquad (3.1)$$

for all $v \in H_0^1(\Omega)$.

Hence, for all $v \in X_j$ we have following error equation:

$$a(u_j - u, v) =$$

$$\left(\sqrt{2f(x,y) + u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2} - \sqrt{2f(x,y) + (u_{xx}^j)^2 + (u_{yy}^j)^2 + 2(u_{xy}^j)^2}, v \right).$$

Lemma 3.2. *Assume M-A equation (2.6)-(2.7) has unique solution* $u \in H^2(\Omega)$ *for* $f \in L^2(\Omega)$, *and* u_j *is finite element solution of equation (2.10)-(2.11). Then we have*

$$|u_j - u|_1 \le Ch_j\|u\|_2,$$

where, h_j *is the meshsize on level* j.

PROOF. Step (1) We prove $a(u_j - u, v) \leq Ch_j\|v\|_1$ for all $v \in X_j$.

$$\begin{aligned} & a(u_j - u, v) \\ \leq\ & \|\sqrt{2f(x,y) + u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2} - \sqrt{2f(x,y) + (u_{xx}^j)^2 + (u_{yy}^j)^2 + 2(u_{xy}^j)^2}\|_0\|v\|_0. \end{aligned}$$

We have

$$\begin{aligned} & \|\sqrt{2f(x,y) + u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2} - \sqrt{2f(x,y) + (u_{xx}^j)^2 + (u_{yy}^j)^2 + 2(u_{xy}^j)^2}\|_0 \\ =\ & \frac{\|(u_{xx} + u_{xx}^j)(u_{xx} - u_{xx}^j) + (u_{yy} + u_{yy}^j)(u_{yy} - u_{yy}^j) + 2(u_{xy} + u_{xy}^j)(u_{xy} - u_{xy}^j)\|_0}{\|\sqrt{2f(x,y) + u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2} + \sqrt{2f(x,y) + (u_{xx}^j)^2 + (u_{yy}^j)^2 + 2(u_{xy}^j)^2}\|_0} \\ <\ & \frac{\|u_{xx} + u_{xx}^j\|_0\|u_{xx} - u_{xx}^j\|_0 + \|u_{yy} + u_{yy}^j\|_0\|u_{yy} - u_{yy}^j\|_0 + 2\|u_{xy} + u_{xy}^j\|_0\|u_{xy} - u_{xy}^j\|_0}{\|\sqrt{u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2} + \sqrt{(u_{xx}^j)^2 + (u_{yy}^j)^2 + 2(u_{xy}^j)^2}\|_0}. \end{aligned}$$

By Cauchy inequality

$$\|u_{xx}+u_{xx}^j\|_0\|u_{xx}-u_{xx}^j\|_0+\|u_{yy}+u_{yy}^j\|_0\|u_{yy}-u_{yy}^j\|_0+2\|u_{xy}+u_{xy}^j\|_0\|u_{xy}-u_{xy}^j\|_0 \leq I_+I_-$$

$$\begin{aligned} I_+ &= (\|u_{xx} + u_{xx}^j\|_0^2 + \|u_{yy} + u_{yy}^j\|_0^2 + 2\|u_{xy} + u_{xy}^j\|_0^2)^{\frac{1}{2}}, \\ I_- &= (\|u_{xx} - u_{xx}^j\|_0^2 + \|u_{yy} - u_{yy}^j\|_0^2 + 2\|u_{xy} - u_{xy}^j\|_0^2)^{\frac{1}{2}}. \end{aligned}$$

By Minkowski inequality

$$\begin{aligned} & \|\sqrt{u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2} + \sqrt{(u_{xx}^j)^2 + (u_{yy}^j)^2 + 2(u_{xy}^j)^2}\|_0 \\ >\ & \|\sqrt{(u_{xx} + u_{xx}^j)^2 + (u_{yy} + u_{yy}^j)^2 + 2(u_{xy} + u_{xy}^j)^2}\|_0. \end{aligned}$$

Then we have

$$a(u_j - u, v) \leq I_-\|v\|_1.$$

By the construction of u_{xx}^j, we have

$$|u_{xx}(k) - u_{xx}^j(k)| \leq Ch_j,$$

k denote all nodes on j level. Similar estimate can be found for u_{yy}^j and u_{xy}^j. Hence, $a(u_j - u, v) \leq Ch_j\|v\|_1$ for all $v \in X_j$.

Step (2) H^1 semi norm of error.

Let $v_j \in X_j$, with the fact $u_j - u \in H_0^1(\Omega)$ we have

$$\begin{aligned} |u_j - u|_1^2 &= a(u_j - u, u_j - u) \\ &= a(u_j - u, u_j - v_j) + a(u_j - u, v_j - u) \\ &\leq C_1h_j\|u_j - v_j\|_1 + C_2|u_j - u|_1\|v_j - u\|_1 \\ &\leq C_1h_j|u_j - u|_1 + (C_1h_j + C_2|u_j - u|_1)\|v_j - u\|_1. \end{aligned}$$

If $C_2|u_j - u|_1 \leq C_1 h_j$, then we have the assertion directly. Otherwise,

$$|u_j - u|_1^2 \quad \leq \quad C_1 h_j |u_j - u|_1 + C_3 |u_j - u|_1 \|v_j - u\|_1.$$

After dividing by $|u_j - u|_1$, we get

$$|u_j - u|_1 \quad \leq \quad C_1 h_j + C_3 \inf_{v_j \in X_j} \|v_j - u\|_1.$$

Then, the result follows immediately from interpolation error estimate. This completes the proof.

In addition, we have following approximation property by the Aubin-Nitsche technique.

Lemma 3.3. *If u_{j-1} and u_j are the finite element solutions on levels $j-1$ and j respectively, then we have*

$$\|u_j - u_{j-1}\|_0 \leq C h_j |u_j - u_{j-1}|_1, \quad j = 1, \cdots, L. \tag{3.2}$$

Denoting the cascadic procedure on j level by operator P_{j,m_j}, CMG algorithm can be rewritten as:

(1) $u_0^* = u_0$ (by direct methods)

(2) $j = 1, \cdots, L: \qquad u_j^* = P_{j,m_j} u_{j-1}^*$.

Here P_{j,m_j} contains two processes: interpolation and smoothing (m_j steps). The smoothing can be several basic iteration methods such as Gauss-Seidel and SSOR.

As in [4], we consider following type of basic iterations started with $u_j^0 \in X_j$:

$$u_j - P_{j,m_j} u_j^0 = R_{j,m_j}(u_j - u_j^0). \tag{3.3}$$

We are accustomed to call the basic iteration an energy reducing smoother, if it satisfies following smoothing properties:

$$|R_{j,m_j} v_j|_1 \leq C \frac{h_j^{-1}}{m_j^r} \|v_j\|_0, \tag{3.4}$$

$$|R_{j,m_j} v_j|_1 \leq |v_j|_1, \tag{3.5}$$

for $\forall v_j \in X_j$ with parameter $0 < r \leq 1$. Here, m_j is the number of steps of smoothing applied on level j.

Lemma 3.4. *The symmetric Gauss-Seidel, SSOR and the damped Jacobi iteration satisfy (3.4)-(3.5) with $r = \frac{1}{2}$.*

Proof. See [16].

Then, general algebraic error of CMG method can be estimated by following Theorem.

Theorem 3.1. *If the basic iteration is an energy reducing smoother in CMG method, then the algebraic error can be estimated by*

$$|u_L - u_L^*|_1 \le C \sum_{j=1}^{L} \frac{h_j}{m_j^r} \|u\|_2.$$

PROOF.

$$\begin{aligned} |u_L - u_L^*|_1 &= |u_L - P_{L,m_L} u_{L-1}^*|_1 \\ &= |R_{L,m_L}(u_L - u_{L-1}^*)|_1 \\ &\le |R_{L,m_L}(u_L - u_{L-1})|_1 + |R_{L,m_L}(u_{L-1} - u_{L-1}^*)|_1 \\ &\le C\frac{h_L^{-1}}{m_L^r} \|u_L - u_{L-1}\|_0 + |u_{L-1} - u_{L-1}^*|_1. \end{aligned}$$

By Lemma 3.3 and Lemma 3.2,

$$\begin{aligned} |u_L - u_L^*|_1 &\le C\frac{1}{m_L^r}|u_L - u_{L-1}|_1 + |u_{L-1} - u_{L-1}^*|_1 \\ &\le C\sum_{j=1}^{L} \frac{1}{m_j^r}|u_j - u_{j-1}|_1 \\ &\le C\sum_{j=1}^{L} \frac{1}{m_j^r}(|u_j - u|_1 + |u - u_{j-1}|_1) \\ &\le C\sum_{j=1}^{L} \frac{h_j}{m_j^r}\|u\|_2. \end{aligned}$$

Let the basic iteration in CMG method be one of energy reducing smoothers with $r = \frac{1}{2}$. In order to ensure optimal accuracy, we select parameters in accordance with the following method.

(i) Meshsize h_j satisfy

$$\frac{2^{L-j} h_L}{C} \le h_j \le C 2^{L-j} h_L.$$

(ii) Number of smoothing steps m_j satisfy

$$m_j = \lceil 4^{L-j} m_L \rceil, \quad m_L = \lceil m_* L^2 \rceil.$$

Theorem 3.2. *When select parameters in accordance with (i)-(ii), we have algebraic error*

$$|u_L - u_L^*|_1 \le C\frac{h_L}{m_*^{1/2}}\|u\|_2,$$

and complexity

$$\sum_{j=1}^{L} m_j n_j \le C m_* n_L (1 + \log n_L)^3,$$

where $n_j = \dim X_j$.

Proof. The conclusion follows directly by Lemma 3.4 and Theorem 3.1.

Comparing Lemma 3.2 with Theorem 3.2, we have

$$|u_L - u|_1 \approx |u_L - u_L^*|_1 = O(h_L),$$

and

$$\text{amount of work} = O(n_L).$$

Consequently, the CMG algorithm is optimal with respect to the energy norm.

4. Numerical experiments

In this section, to demonstrate the effectiveness of the CMG method, we present some numerical experiments. We consider solving the M-A equation (1.2)-(1.4) using CMG method in the domain $[0,1]^2$ with exact solution $u = \exp(\frac{x^2+y^2}{2})$. On each grid level, the given equation is discretized by Q_1 finite element methods. We let $m_* = 10$, and choose SSOR as the basic iteration.

Table 1 shows the relative error of H^1 semi-norm, L^2 norm and Max norm on six levels respectively. We observe that the CMG algorithm is fast and robust. The numerical results coincide with Theorem 3.2.

Figure 2 shows the error $u - u_j^*$ on these six levels.

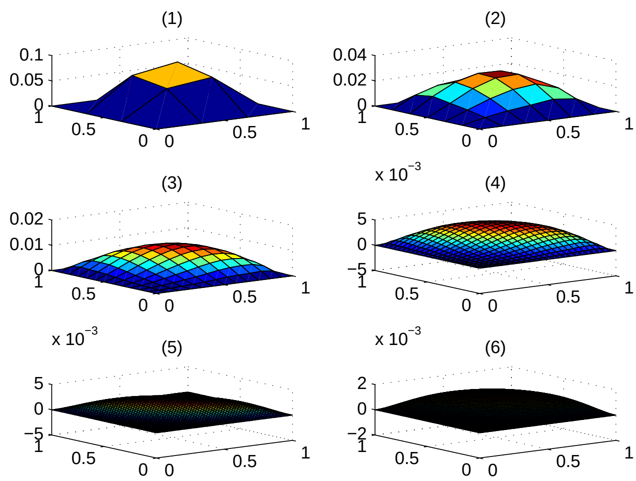

Figure 2: The error on each level.

Table 1: Relative error on $j = 0, \cdots, 5$ levels. CPU time= 240.17 seconds.

j	h_j	$\frac{\lvert u-u_j^*\rvert_1}{\lvert u\rvert_1}$	$\frac{\lVert u-u_j^*\rVert_0}{\lVert u\rVert_0}$	$\frac{\lVert u-u_j^*\rVert_\infty}{\lVert u\rVert_\infty}$
0	1/3	8.35E-02	2.06E-02	2.92E-02
1	1/6	3.95E-02	7.53E-03	9.20E-03
2	1/12	1.96E-02	3.19E-03	3.72E-03
3	1/24	1.01E-02	1.49E-03	1.65E-03
4	1/48	5.27E-03	7.53E-04	8.15E-04
5	1/96	2.97E-03	5.17E-04	5.62E-04

5. Conclusions

The elliptic Monge-Ampère equation is a fully nonlinear partial differential equation which has a wide range of applications. In this paper, we provided an cascadic multigrid method for the M-A equation. We proved the convergence of CMG method. And we found that the CMG method is optimal with respect to the energy norm (H^1 semi-norm). Finally, some numerical experiments were presented to demonstrate the the efficiency and robustness of CMG method. But we still have a lot of work to do in future work. For example, how to construct the approximation of derivative functions (especially higher derivative functions) on unstructured or adaptive grid levels is not understood.

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Iterative algorithms for zeros of accretive operators and fixed points of nonexpansive mappings in Banach spaces

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Abstract

In this paper, we introduce two new iterative algorithms (one implicit and one explicit) for finding a common point of the set of zeros of an accretive operator and the set of fixed points of a nonexpansive mapping in a real uniformly convex Banach space having a uniformly Gâteaux differentiable norm. Then under suitable control conditions, we establish strong convergence of sequence generated by proposed algorithm to a common point of above two sets, which is a solution of a ceratin variational inequality. The main theorems develop and complement some well-known results in the literature.

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Key words: Iterative algorithm; Accretive operator; Resolvent; Zeros; Nonexpansive mappings; Fixed points; Variational inequalities;

1. Introduction

Let E be a real Banach space with norm $\|\cdot\|$ and the dual space E^*. The value of $x^* \in E^*$ at $y \in E$ is denoted by $\langle y, x^* \rangle$ and the normalized duality mapping $\mathcal{J}$ from E into 2^{E^*} is defined by

$$\mathcal{J}(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\|, \ \|x\| = \|x^*\|\}, \quad \forall x \in E.$$

Recall that a (possibly multivalued) operator $A \subset E \times E$ with the domain $D(A)$ and the range $R(A)$ in E is *accretive* if , for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1,\ 2)$, there exists a $j \in \mathcal{J}(x_1 - x_2)$ such that $\langle y_1 - y_2, j \rangle \geq 0$. (Here $\mathcal{J}$ is the normalized duality mapping.) In a Hilbert space, an accretive operator is also called monotone operator.

Interest in accretive operators stems mainly from their firm connection with evolution equations. It is well-known that many physically significant problems can be modeled by initial-value problems of the form

$$\frac{dx(t)}{dt} + Ax(t) \ni 0, \quad x(0) = x_0, \tag{1.1}$$

where A is an an accretive operator in a ceratin Banach space. Typical examples where such evolution equations occurs can be found in the heat, wave, or Schrodinger equations. If in (1.1), $x(t)$ is independent of t, then (1.1) reduces $Az \ni 0$ whose solutions correspond to the equilibrium points of system (1.1). Consequently, the iterative algorithms of Halpern type, Mann type, and Rockafellar type have extensively been studied over the last forty years for constructions of zeros of accretive operators (see, e,g., [1–17] and the references therein). As an original one, the following iterative algorithm in Hilbert spaces or Banach spaces was considered by many authors: for resolvent J_{r_n} of m-accretive operator A,

$$x_{n+1} = J_{r_n} x_n, \quad \forall n \geq 0,$$

where the initial guess $x_0 \in E$ is chosen arbitrarily (see, e.g., [4,5,12] and the references therein). In particular, in order to find a zero of a monotone operator A, Rockafellar [13] introduced a powerful and successful algorithm which is recognized as Rockafellar proximal point algorithm in Hilbert space H : for any initial point $x_0 \in H$, a sequence $\{x_n\}$ is generated by

$$x_{n+1} = J_{r_n}(x_n + e_n), \quad \forall n \geq 0,$$

where $J_r = (I + rA)^{-1}$, for $r > 0$, is the resolvent of A and $\{e_n\}$ is an error sequence in H.

Xu [18] in 2006 and Song and Yang [19] in 2009 obtained the strong convergence of the regularization method for Rockafellar's proximal point algorithm in a Hilbert space H: for any initial point $x_0 \in H$

$$x_{n+1} = J_{r_n}(\alpha_n u + (1 - \alpha_n)x_n + e_n), \quad \forall n \geq 0,$$

where $\{\alpha_n\} \subset (0, 1)$, $\{e_n\} \subset H$ and $\{r_n\} \subset (0, \infty)$.

On the other hand, in 2011, He *et al.* [20] studied the following iterative algorithm for finding a common point of the set of zeros of accretive operator A such that $A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$ and the set of fixed points of a nonexpansive mapping S in a real reflexive Banach space E having a weakly sequentially continuous duality mapping:

$$\begin{cases} x_0 = x \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S J_{r_n} x_n, \quad \forall n \geq 0, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\} \subset [0, 1]$, $\lim_{n\to\infty} r_n = r$ and $f : C \to C$ is a contractive mapping. Under the suitable conditions $\{\alpha_n\}$ and $\{\beta_n\}$, they also showed that the sequence $\{x_n\}$ generated by (1.2) converges strongly to a common point in $F(S) \cap A^{-1}0$, which is a solution of a certain variational inequality.

Inspired and motivated by the above-mentioned results, in this paper, we introduce new implicit and explicit algorithms for finding a common point of the set of zeros of accretive operator A and the set of fixed points of a nonexpansive mapping S in a real uniformly convex Banach space E having a uniformly Gâteaux differentiable norm. Under suitable control conditions, we prove that the sequence generated by proposed iterative algorithm converge strongly to a common point in $A^{-1}0 \cap F(S)$, which is a solution of a certain variational inequality. The main results develop and supplement the corresponding results of He *et al.* [20] as well as Xu [18] and Song and Yang [19] and the reference therein.

2. Preliminaries and Lemmas

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be its dual. Let C be a nonempty subset of E. The value of $f \in E^*$ at $x \in E$ will be denoted by $\langle x, f\rangle$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ ($x_n \rightharpoonup x$) will denote strong (weak) convergence of the sequence $\{x_n\}$ to x. For the mapping $S : C \to C$, $F(S)$ will denote the set of fixed point of S; that is, $F(S) = \{x \in C : Sx = x\}$.

A Banach space E is said to be *uniformly convex* if for all $\varepsilon \in [0, 2]$, there exists $\delta_\varepsilon > 0$ such that

$$\|x\| = \|y\| = 1 \text{ implies } \frac{\|x + y\|}{2} < 1 - \delta_\varepsilon \text{ whenever } \|x - y\| \geq \varepsilon.$$

Let $l > 1$ and $M > 0$ be two fixed real numbers. Then a Banach space is uniformly convex if and only if there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$\|\lambda x + (1 - \lambda)y\|^l \leq \lambda\|x\|^l + (1 - \lambda)\|y\|^l - \omega(\lambda)g(\|x - y\|), \tag{2.1}$$

for all $x,\ y \in B_M(0) = \{x \in E : \|x\| \leq M\}$, where $\omega(\lambda) = \lambda^l(1 - \lambda) + \lambda(1 - \lambda)^l$. For more detail, see Xu [21].

The norm of E is said to be *Gâteaux differentiable* if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.2}$$

exists for each $x,\ y$ in its unit sphere $U = \{x \in E : \|x\| = 1\}$. Such an E is said to be *smooth* Banach space. The norm is said to be *uniformly Gâteaux differentiable* if for $y \in U$, the limit is attained uniformly for $x \in U$. The space E is said to have a *uniformly Fréchet differentiable norm* (and E is said to be *uniformly smooth*) if the limit in (2.2) is attained uniformly for $(x, y) \in U \times U$. It is known that E is smooth if and only if the normalized duality mapping $\mathcal{J}$ is single-valued. Also, it is well-known that if E has a uniformly Gâteaux differentiable norm, $\mathcal{J}$ is norm to weak* uniformly continuous on each bounded subsets of E. The following property of the normalized duality mapping $\mathcal{J}$ is well-known: $\mathcal{J}(-x) = -\mathcal{J}(x)$ for all $x \in E$ ([22]).

An accretive operator A is said to satisfy *the range condition* if $\overline{D(A)} \subset R(I + rA)$ for all $r > 0$, where I is an identity operator of E and $\overline{D(A)}$ denotes the closure of the domain $D(A)$ of A. An accretive operator A is called *m-accretive* if $R(I + rA) = E$ for each $r > 0$. If A is an accretive operator which satisfies the range condition, then we can define, for each $r > 0$ a mapping $J_r : R(I + rA) \to D(A)$ defined by $J_r = (I + rA)^{-1}$, which is called the *resolvent* of A. We know that J_r is nonexpansive (i.e., $\|J_r x - J_r y\| \leq \|x - y\|,\ \ \forall x, y \in R(I + rA)$) and $A^{-1}0 = F(J_r) = \{x \in D(J_r) : J_r x = x\}$ for all $r > 0$. For these facts, see [22].

We need the following lemmas for the proof of our main results. We refer to [22] for Lemma 2.1, Lemma 2.2, and Lemma 2.3.

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Lemma 2.1. *If E be a real smooth Banach space, then one has*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, \mathcal{J}(x+y)\rangle, \quad \forall x,\ y \in E,$$

where $\mathcal{J}$ is the normalized duality mapping of E.

Lemma 2.2 (The Resolvent Identity)**.** *For $\lambda > 0$, $\mu > 0$ and $x \in H$,*

$$J_\lambda x = J_\mu\left(\frac{\mu}{\lambda} + \left(1 - \frac{\mu}{\lambda}\right) J_\lambda x\right).$$

Lemma 2.3. *Let E be a real Banach space having a uniformly Gâteaux differentiable norm, let C be a nonempty closed convex subset of E, and let $\{y_n\}$ be a bounded sequence in E. Let* LIM *be a Banach limit and $q \in C$. Then*

$$\mathrm{LIM}\|y_n - q\|^2 = \min_{x \in C} \mathrm{LIM}\|x_n - x\|^2$$

if and only if

$$\mathrm{LIM}\langle x - q, \mathcal{J}(y_n - q)\rangle \leq 0, \quad \forall x \in C,$$

where $\mathcal{J}$ is the normalized duality mapping of E.

The following lemma is given in [23].

Lemma 2.4 ([23])**.** *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1 - \lambda_n)s_n + \lambda_n\delta_n + \gamma_n, \quad \forall n \geq 0,$$

where $\{\lambda_n\}$, $\{\delta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\{\lambda_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \lambda_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} \lambda_n|\delta_n| < \infty$;

(iii) $\gamma_n \geq 0$ $(n \geq 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Finally, we will use the next lemma which is of fundamental importance for our proof.

Lemma 2.5 ([24])**.** *Let $\{s_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{s_{n_i}\}$ of $\{s_n\}$ such that $s_{n_i} < s_{n_i+1}$ for all $i \geq 0$. For every $n \geq n_0$, define the sequence of integers $\{\tau(n)\}$ by*

$$\tau(n) := \max\{k \leq n : s_k < s_{k+1}\}.$$

Then $\{\tau(n)\}_{n\geq n_0}$ is a nondecreasing sequence verifying

$$\lim_{n\to\infty} \tau(n) = \infty,$$

and, for all $n \geq n_0$, the following two estimates hold:

$$s_{\tau(n)} \leq s_{\tau(n)+1}, \quad s_n \leq s_{\tau(n)+1}.$$

3. Iterative algorithms

Throughout the rest of this paper, we always assume the following:

- E is a real Banach space;
- $\mathcal{J}$ is the normalized duality mapping of E;
- C is a nonempty closed convex subset of E;
- $A \subset E \times E$ is an accretive operator in E such that $A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$;
- J_r is the resolvent of A for each $r > 0$;
- $S : C \to C$ is a nonexpansive mapping with $F(S) \cap A^{-1}0 \neq \emptyset$;
- $f : C \to C$ is a contractive mapping with a constant $k \in (0, 1)$.

In this section, we introduce the following algorithm that generates a net $\{x_t\}_{t\in(0,1)}$ in an implicit way:

$$x_t = J_r(tfx_t + (1-t)Sx_t). \tag{3.1}$$

We prove strong convergence of $\{x_t\}$ as $t \to 0$ to a point q in $A^{-1}0 \cap F(S)$ which is a solution of the following variational inequality:

$$\langle (I-f)q, \mathcal{J}(q-p)\rangle \geq 0, \quad \forall p \in A^{-1}0 \cap F(S). \tag{3.2}$$

We also propose the following algorithm which generates a sequence in an explicit way:

$$x_{n+1} = J_{r_n}(\alpha_n f x_n + (1-\alpha_n)Sx_n), \quad \forall n \geq 0, \tag{3.3}$$

where $\{\alpha_n\} \subset (0, 1)$, $\{r_n\} \subset (0, \infty)$ and $x_0 \in C$ is an arbitrary initial guess, and establish the strong convergence of this sequence to a point q in $A^{-1}0 \cap F(S)$, which is also a solution of the variational inequality (3.2).

3.1. Strong convergence of the implicit algorithm

Now, for $t \in (0, 1)$, consider a mapping $Q_t : C \to C$ defined by

$$Q_t x = J_r(tfx + (1-t)Sx), \quad \forall x \in C.$$

It is easy to see that Q_t is a contractive mapping with constant $1 - (1-k)t$. Indeed, we have

$$\begin{aligned}\|Q_t x - Q_t y\| &\leq t\|fx - fy\| + \|(1-t)Sx - (1-t)Sy\|\\ &\leq tk\|x-y\| + (1-t)\|x-y\|\\ &= (1-(1-k)t)\|x-y\|.\end{aligned}$$

Hence Q_t has a unique fixed point, denoted x_t, which uniquely solves the fixed point equation (3.1).

We summary the basic properties of $\{x_t\}$ and $\{y_t\}$, where $y_t = tfx_t + (1-t)Sx_t$ for $t \in (0,1)$.

Proposition 3.1. *Let E be a uniformly convex Banach space. Let the net $\{x_t\}$ be defined by (3.1), and let $\{y_t\}$ be a net defined by $y_t = tfx_t + (1-t)Sx_t$ for $t \in (0,1)$. Then*

(1) *$\{x_t\}$ and $\{y_t\}$ are bounded for $t \in (0,1)$;*

(2) *x_t defines a continuous path from $(0,1)$ in C and so does y_t;*

(3) $\lim_{t\to 0}\|y_t - Sx_t\| = 0$*;*

(4) $\lim_{t\to 0}\|y_t - J_r y_t\| = 0$*;*

(5) $\lim_{t\to 0}\|x_t - y_t\| = 0$*;*

(6) $\lim_{t\to 0}\|y_t - Sy_t\| = 0$*;*

Proof. (1) Let $p \in F(S) \cap A^{-1}0$. Observing $p = Sp = J_r p$, we have

$$\begin{aligned}\|x_t - p\| &= \|J_r(tfx_t + (1-t)Sx_t) - J_r p\| = \|J_r y_t - J_r p\| \\ &\le \|y_t - p\| \\ &= \|t(fx_t - fp) + t(fp - p) + (1-t)(Sx_t - Sp)\| \\ &\le tk\|x_t - p\| + t\|fp - p\| + (1-t)\|x_t - p\|.\end{aligned}$$

So, it follows that

$$\|x_t - p\| \le \frac{\|fp - p\|}{1-k} \text{ and } \|y_t - p\| \le \frac{\|fp - p\|}{1-k}.$$

Hence $\{x_t\}$ and $\{y_t\}$ are bounded and so are $\{fx_t\}$, $\{Sx_t\}$, $\{Sy_t\}$, and $\{J_r y_t\}$.

(2) Let $t,\ t_0 \in (0,1)$ and calculate

$$\begin{aligned}\|x_t - x_{t_0}\| &= \|J_r(tfx_t + (1-t)Sx_t) - J_r(t_0 f x_{t_0} + (1-t_0)Sx_{t_0})\| \\ &\le \|(t-t_0)fx_t + t_0(fx_t - fx_{t_0}) \\ &\qquad - (t-t_0)Sx_t + (1-t_0)Sx_t - (1-t_0)J_r x_{t_0}\| \\ &\le |t-t_0|\|fx_t\| + t_0 k\|x_t - x_{t_0}\| \\ &\qquad + |t-t_0|\|Sx_t\| + (1-t_0)\|x_t - x_{t_0}\|.\end{aligned}$$

It follows that

$$\|x_t - x_{t_0}\| \le \frac{\|fx_t\| + \|Sx_t\|}{t_0(1-k)}|t-t_0|.$$

This show that x_t is locally Lipschitzian and hence continuous. Also we have

$$\|y_t - y_{t_0}\| \le \frac{\|fx_t\| + \|Sx_t\|}{t_0(1-k)}|t-t_0|,$$

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and hence y_t is a continuous path.

(3) By the boundedness of $\{fx_t\}$ and $\{Sx_t\}$ in (1), we have

$$\begin{aligned}\|y_t - Sx_t\| &= \|tfx_t + (1-t)Sx_t - Sx_t\| \\ &\le t\|fx_t - Sx_t\| \to 0 \quad \text{as} \quad t \to 0.\end{aligned}$$

(4) Let $p \in A^{-1}0 \cap F(S)$. Then, it follows from Lemma 2.2 (Resolvent Identity) that

$$J_r y_t = J_{\frac{r}{2}}\left(\frac{1}{2}y_t + \frac{1}{2}J_r y_t\right).$$

Then we have

$$\|J_r y_t - p\| = \left\|J_{\frac{r}{2}}\left(\frac{1}{2}y_t + \frac{1}{2}J_r y_t\right) - p\right\| \le \left\|\frac{1}{2}\left(y_t - p\right) + \frac{1}{2}\left(J_r y_t - p\right)\right\|.$$

By the inequality (2.1) ($l = 2, \lambda = \frac{1}{2}$), we obtain that

$$\begin{aligned}\|J_r y_t - p\|^2 &\le \left\|J_{\frac{r}{2}}\left(\frac{1}{2}y_t + \frac{1}{2}J_r y_t\right) - p\right\|^2 \\ &\le \left\|\frac{1}{2}\left(y_t - p\right) + \frac{1}{2}\left(J_r y_t - p\right)\right\|^2 \\ &\le \frac{1}{2}\|y_t - p\|^2 + \frac{1}{2}\|J_r y_t - p\|^2 - \frac{1}{4}g(\|y_t - J_r y_t\|) \\ &\le \frac{1}{2}\|y_t - p\|^2 + \frac{1}{2}\|y_t - p\|^2 - \frac{1}{4}g(\|y_t - J_r y_t\|) \\ &= \|y_t - p\|^2 - \frac{1}{4}g(\|y_t - J_r y_t\|)\end{aligned} \tag{3.4}$$

Thus, from (3.1), the convexity of the real function $\psi(t) = t^2$ ($t \in (-\infty, \infty)$) and the inequality (3.4), we have

$$\begin{aligned}\|x_t - p\|^2 &= \|J_r y_t - p\|^2 \\ &\le \|y_t - p\|^2 - \frac{1}{4}g(\|y_t - J_r y_t\|) \\ &= \|t(fx_t - p) + (1-t)(Sx_t - p)\|^2 - \frac{1}{4}g(\|y_t - J_r y_t\|) \\ &\le t\|fx_t - p\|^2 + (1-t)\|x_t - p\|^2 - \frac{1}{4}g(\|y_t - J_r y_t\|)\end{aligned}$$

and hence

$$\frac{1}{4}g(\|y_t - J_r y_t\|)) \le t(\|fx_t - p\|^2 - \|x_t - p\|^2).$$

By boundedness of $\{fx_t\}$ and $\{x_t\}$, letting $t \to 0$ yields

$$\lim_{t \to 0} g(\|y_t - J_r y_t\|) = 0.$$

Thus, from the property of the function g in (2.1), it follows that

$$\lim_{t \to 0} \|y_t - J_r y_t\| = 0.$$

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(5) By (4), we have

$$\|x_t - y_t\| \le \|x_t - J_r y_t\| + \|J_r y_t - y_t\| = \|J_r y_t - y_t\| \to 0 \quad (t \to 0).$$

(6) By (3) and (5), we have

$$\begin{aligned}\|y_t - Sy_t\| &\le \|y_t - Sx_t\| + \|Sx_t - Sy_t\| \\ &\le \|y_t - Sx_t\| + \|x_t - y_t\| \to 0 \quad (t \to 0). \qquad \square\end{aligned}$$

We establish the strong convergence of the net $\{x_t\}$ as $t \to 0$, which guarantees the existence of solutions of the variational inequality (3.2).

Theorem 3.2. Let E be a uniformly convex Banach space having a uniformly Gâteaux differentiable norm. Let $\{x_t\}$ be a net defined by (3.1), and let $\{y_t\}$ be a net defined by $y_t = tfx_t + (1-t)Sx_t$ for $t \in (0,1)$. Then the nets $\{x_t\}$ and $\{y_t\}$ converge strongly to a point $q \in A^{-1}0 \cap F(S)$ as $t \to 0$, which is the unique solution of the variational inequality (3.2).

Proof. By (1) in Proposition 3.1, we see that $\{x_t\}$ and $\{y_t\}$ are bounded. Assume $t_n \to 0$. Set $x_n := x_{t_n}$ and $y_n := y_{t_n}$. We use the so-called optimization method (see [25]). Define $\phi : C \to \mathbb{R}$ by

$$\phi(x) = \mathrm{LIM}\|y_n - x\|^2, \quad x \in C,$$

where LIM is a Banach limit on l^∞. Since ϕ is continuous and convex, $\phi(z) \to \infty$ as $\|z\| \to \infty$ and E is reflexive, ϕ attain its infimum over C. Let

$$K = \{x \in C : \phi(x) = \min_{x\in C} \mathrm{LIM}\|y_n - x\|^2\}.$$

It is easily seen that K is a nonempty closed convex bounded subset of E. Moreover, K is invariant under J_r. Indeed, since $\|y_t - J_r y_t\| \to 0$ by (4) in Proposition 3.1, it follows that for each $z \in K$

$$\phi(J_r z) = \mathrm{LIM}\|y_n - J_r z\|^2 = \mathrm{LIM}\|J_r y_n - J_r z\|^2 \le \mathrm{LIM}\|y_n - z\|^2 = \phi(z),$$

so that $J_r K \subset K$. By the fixed point property for nonexpansive mappings of a uniformly convex Banach space E (cf. Theorem 5.1 in [26]), J_r has a fixed point, say q, in K. Also by (6) in Proposition 3.1, K is invariant under S, that is, $SK \subset K$. Also S has a fixed point $\widehat{q}$ in K. By uniform convexity of E, we have $q = \widehat{q}$ (cf. Theorem 2.9.11 in [22]) and hence $q \in A^{-1}0 \cap F(S)$. By Lemma 2.3, we obtain

$$\mathrm{LIM}\langle x - q, \mathcal{J}(y_n - q)\rangle \le 0, \quad \forall x \in C. \tag{3.5}$$

Since

$$y_t - q = t(fx_t - q) + (1-t)(Sx_t - q),$$

we have

$$\begin{aligned}\|y_t - q\|^2 &= t\langle fx_t - q, \mathcal{J}(y_t - q)\rangle + (1-t)\langle Sx_t - q, \mathcal{J}(y_t - q)\rangle \\ &\le t\langle fx_t - q, \mathcal{J}(y_t - q)\rangle + (1-t)\|x_t - q\|\|y_t - q\| \\ &\le t\langle fx_t - q, \mathcal{J}(y_t - q)\rangle + (1-t)\|y_t - q\|^2.\end{aligned}$$

Hence

$$\begin{aligned}\|y_t - q\|^2 &\le \langle fx_t - q, \mathcal{J}(y_t - q)\rangle \\ &= \langle fx_t - x, \mathcal{J}(y_t - q)\rangle + \langle x - q, \mathcal{J}(y_t - q)\rangle.\end{aligned} \tag{3.6}$$

Thus, by (3.5), for $x \in C$

$$\begin{aligned}\mathrm{LIM}\|y_t - q\|^2 &\le \mathrm{LIM}\langle fx_n - x, \mathcal{J}(y_n - q)\rangle + \mathrm{LIM}\langle x - q, \mathcal{J}(y_n - q)\rangle \\ &= \mathrm{LIM}\langle fx_n - x, \mathcal{J}(y_n - q)\rangle \\ &\le \mathrm{LIM}\|fx_n - x\|\|y_n - q\|.\end{aligned}$$

In particular,

$$\begin{aligned}\mathrm{LIM}\|y_n - q\|^2 &\le \mathrm{LIM}\|fx_n - fq\|\|y_n - q\| \\ &\le k\mathrm{LIM}\|x_n - q\|\|y_n - q\| \le k\mathrm{LIM}\|y_n - q\|^2.\end{aligned}$$

Hence

$$\mathrm{LIM}\|y_n - q\|^2 = 0,$$

and there exists a subsequence which is still denoted by $\{y_n\}$ such that $y_n \to q$.

Now assume that there exists another subsequence $\{y_m\}$ of $\{y_t\}$ such that $y_m \to \overline{q} \in A^{-1}0 \cap F(S)$. Then, by (5) in Proposition 3.1, $x_m \to \overline{q}$. So, it follows from (3.6) that

$$\|\overline{q} - q\|^2 \le \langle f\overline{q} - q, \mathcal{J}(\overline{q} - q)\rangle. \tag{3.7}$$

Interchanging $\overline{q}$ and q, we obtain

$$\|q - \overline{q}\|^2 \le \langle fq - \overline{q}, \mathcal{J}(q - \overline{q})\rangle. \tag{3.8}$$

Adding up (3.7) and (3.8) yields

$$2\|\overline{q} - q\|^2 \le \langle f\overline{q} - fq, \mathcal{J}(\overline{q} - q)\rangle + \langle \overline{q} - q, \mathcal{J}(\overline{q} - q)\rangle \le (1 + k)\|\overline{q} - q\|^2.$$

Since $k \in (0, 1)$, this implies that $\overline{q} = q$. Hence $y_t \to q$ as $t \to 0$ and by (5) in Proposition 3.1, also $x_t \to q$ as $t \to 0$.

Finally, we show that q is the unique solution of the variational inequality (3.2). To this end, noting

$$\begin{aligned}y_t - fx_t &= -\frac{1-t}{t}(y_t - Sx_t) \\ &= -\frac{1-t}{t}(x_t - Sx_t) + \left(1 - \frac{1}{t}\right)(y_t - x_t),\end{aligned}$$

and $\langle (I-S)x_t - (I-S)p, \mathcal{J}(x_t - p)\rangle \ge 0$ by nonexpansivity of S, we have for $p \in A^{-1}0 \cap F(S)$,

$$\begin{aligned}\langle y_t - fx_t, \mathcal{J}(x_t - p)\rangle &= -\frac{1-t}{t}\langle (I - S)x_t - (I - S)p, \mathcal{J}(x_t - p)\rangle \\ &\quad + \left(1 - \frac{1}{t}\right)\langle y_t - x_t, \mathcal{J}(x_t - p)\rangle \\ &\le \left(1 - \frac{1}{t}\right)\|y_t - x_t\|\|x_t - p\| \\ &\le \|y_t - x_t\|\|x_t - q\|.\end{aligned}$$

Since $x_t,\ y_t \to q$ and $fx_t \to fq$ as $t \to 0$, by (5) in Proposition 3.1, letting $t \to 0$ yields

$$\langle (I-f)q, \mathcal{J}(q-p)\rangle \leq 0.$$

This implies that q is a solution of the variational inequality (3.2). If $\widetilde{q} \in A^{-1}0 \cap F(S)$ is other solution of the variational inequality (3.2), then

$$\langle (I-f)\widetilde{q}, \mathcal{J}(\widetilde{q}-q)\rangle \leq 0. \tag{3.9}$$

Interchanging $\overline{q}$ and q, we obtain

$$\langle (I-f)q, \mathcal{J}(q-\widetilde{q})\rangle \leq 0. \tag{3.10}$$

Adding up (3.9) and (3.10) yields

$$(1-k)\|\widetilde{q}-q\|^2 \leq 0.$$

That is, $q = \widetilde{q}$. Hence q is the unique solution of the variational inequality (3.2). This completes the proof. ☐

Corollary 3.3. Let E be a uniformly convex and uniformly smooth Banach space. Let $\{x_t\}$ be a net defined by (3.1), and let $\{y_t\}$ be a net defined by $y_t = tfx_t + (1-t)Sx_t$ for $t \in (0,1)$. Then the nets $\{x_t\}$ and $\{y_t\}$ converge strongly to a point $q \in A^{-1}0 \cap F(S)$ as $t \to 0$, which is the unique solution of the variational inequality (3.2).

3.2. Strong convergence of the explicit algorithm

Now, using Theorem 3.2, we show the strong convergence of the sequence generated by the explicit algorithm (3.3) to a point $q \in A^{-1}0 \cap F(S)$, which is the unique solution of the variational inequality (3.2).

Theorem 3.4. *Let E be a uniformly convex Banach space having a uniformly Gâteaux differentiable norm. Let $\{\alpha_n\} \in (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfy the conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C3) $|\alpha_{n+1} - \alpha_n| \leq o(\alpha_{n+1}) + \sigma_n,\quad \sum_{n=0}^{\infty} \sigma_n < \infty$ *(the perturbed control condition);*

(C4) $r_n \geq \varepsilon > 0$ *for* $n \geq 0$ *and* $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$.

Let $x_0 = x \in C$ be chosen arbitrarily, and let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = J_{r_n}(\alpha_n f x_n + (1-\alpha_n)Sx_n), \quad \forall n \geq 0. \tag{3.11}$$

Let $\{y_n\}$ be a sequence defined by $y_n = \alpha_n fx_n + (1-\alpha_n)Sx_n$. Then $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in A^{-1}0 \cap F(S)$, where q is the unique solution of the variational inequality (3.2).

Proof. First, we note that by Theorem 3.2, there exists the unique solution q of the variational inequality

$$\langle (I-f)q, \mathcal{J}(q-p)\rangle \le 0, \quad \forall p \in A^{-1}0 \cap F(S),$$

where $q = \lim_{t\to 0} x_t = \lim_{t\to 0} y_t$ being defined by $x_t = J_r(tfx_t + (1-t)Sx_t)$ and $y_t = tfx_t + (1-t)Sx_t$ for $0 < t < 1$, respectively.

We divide the proof into several steps.

Step 1. We show that $\|x_n - p\| \le \max\{\|x_0 - p\|, \frac{1}{1-k}\|fp - p\|\}$ for all $n \ge 0$ and all $p \in A^{-1}0 \cap F(S)$, and so $\{x_n\}$, $\{y_n\}$, $\{J_{r_n}x_n\}$, $\{Sx_n\}$, $\{J_{r_n}y_n\}$, $\{Sy_n\}$ and $\{fx_n\}$ are bounded. Indeed, let $p \in A^{-1}0 \cap F(S)$. From $A^{-1}0 = F(J_r)$ for each $r > 0$, we know $p = Sp = J_{r_n}p$. Then we have

$$\begin{aligned}
\|x_{n+1} - p\| \le \|y_n - p\| &= \|\alpha_n(fx_n - p) + (1-\alpha_n)(Sx_n - Sp)\| \\
&\le \alpha_n\|fx_n - p\| + (1-\alpha_n)\|x_n - p\| \\
&\le \alpha_n(\|fx_n - fp\| + \|fp - p\|) + (1-\alpha_n)\|x_n - p\| \\
&\le \alpha_n k\|x_n - p\| + \alpha_n\|fp - p\| + (1-\alpha_n)\|x_n - p\| \\
&= (1-(1-k)\alpha_n)\|x_n - p\| + (1-k)\alpha_n\frac{\|fp - p\|}{1-k} \\
&\le \max\left\{\|x_n - p\|, \frac{1}{1-k}\|f(p) - p\|\right\}.
\end{aligned}$$

Using an induction, we obtain

$$\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{1}{1-k}\|fp - p\|\right\}.$$

Hence $\{x_n\}$ is bounded. Also for $p \in A^{-1}0 \cap F(S)$, we get

$$\begin{aligned}
\|y_n - p\| &\le \alpha_n\|fx_n - fp\| + (1-\alpha_n)\|Sx_n - Sp\| + \alpha_n\|fp - p\| \\
&\le \alpha_n k\|x_n - p\| + (1-\alpha_n)\|x_n - p\| + \alpha_n\|fp - p\| \\
&= (1-(1-k)\alpha_n)\|x_n - p\| + (1-k)\alpha_n\frac{\|fp - p\|}{1-k} \\
&\le \max\left\{\|x_n - p\|, \frac{\|fp - p\|}{1-k}\right\},
\end{aligned}$$

and so $\{y_n\}$ is bounded, and so are $\{y_n\}$, $\{J_{r_n}y_n\}$, $\{Sx_n\}$, $\{Sy_n\}$ and $\{fx_n\}$. Moreover, it follows from condition (C1) that

$$\|y_n - Sx_n\| = \alpha_n\|fx_n - Sx_n\| \le \alpha_n(\|fx_n\| + \|Sx_n\|) \to 0 \quad (n \to \infty). \tag{3.12}$$

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. First, from Lemma 2.2 (Resolvent

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identity), we observe that

$$\begin{aligned}
&\|J_{r_n}y_n - J_{r_{n-1}}y_{n-1}\| \\
&= \left\|J_{r_{n-1}}\left(\frac{r_{n-1}}{r_n}y_n + \left(1-\frac{r_{n-1}}{r_n}\right)J_{r_n}y_n\right) - J_{r_{n-1}}y_{n-1}\right\| \\
&\le \left\|\frac{r_{n-1}}{r_n}y_n + \left(1-\frac{r_{n-1}}{r_n}\right)J_{r_n}y_n\right) - y_{n-1}\right\| \\
&\le \|y_n - y_{n-1}\| + \left|1-\frac{r_{n-1}}{r_n}\right|(\|y_n - y_{n-1}\| + \|J_{r_n}y_n - y_{n-1}\|) \\
&\le \|y_n - y_{n-1}\| + \left|\frac{r_n - r_{n-1}}{\varepsilon}\right|M_1,
\end{aligned} \tag{3.13}$$

where $M_1 = \sup_{n\ge 0}\{\|J_{r_n}y_n - y_{n-1}\| + \|y_n - y_{n-1}\|\}$. Since

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)Sx_n, \\ y_{n-1} = \alpha_{n-1}f(x_{n-1}) + (1-\alpha_{n-1})Sx_{n-1}, \quad \forall n \ge 1, \end{cases}$$

by (3.13), we have for $n \ge 1$,

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|J_{r_n}y_n - J_{r_{n-1}}y_{n-1}\| \le \ \|y_n - y_{n-1}\| + \left|\frac{r_n - r_{n-1}}{\varepsilon}\right|M_1 \\
&= \|(1-\alpha_n)(Sx_n - Sx_{n-1}) + \alpha_n(fx_n - fx_{n-1}) \\
&\quad + (\alpha_n - \alpha_{n-1})(fx_{n-1} - Sx_{n-1})\| + \left|\frac{r_n - r_{n-1}}{\varepsilon}\right|M_1 \\
&\le (1-\alpha_n)\|x_n - x_{n-1}\| + k\alpha_n\|x_n - x_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|M_2 + \left|1 - \frac{r_{n-1}}{r_n}\right|M_1 \\
&\le (1-(1-k)\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M_2 + \left|\frac{r_n - r_{n-1}}{\varepsilon}\right|M_1,
\end{aligned} \tag{3.14}$$

where $M_2 = \sup\{\|f(x_n) - Sx_n\| : n \ge 0\}$. Thus, by (C3) we have

$$\|x_{n+1} - x_n\| \le (1-(1-k)\alpha_n)\|x_n - x_{n-1}\| + M_2(o(\alpha_n) + \sigma_{n-1}) + M_1\left|\frac{r_n - r_{n-1}}{\varepsilon}\right|.$$

In (3.14), by taking $s_{n+1} = \|x_{n+1} - x_n\|$, $\lambda_n = (1-k)\alpha_n$, $\lambda_n\delta_n = M_2 o(\alpha_n)$ and

$$\gamma_n = M_1\left|\frac{r_n - r_{n-1}}{\varepsilon}\right| + M_2\sigma_{n-1},$$

we have

$$s_{n+1} \le (1-\lambda_n)s_n + \lambda_n\delta_n + \gamma_n.$$

Hence, by the conditions (C1), (C2), (C3), (C4) and Lemma 2.4, we obtain

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.$$

Now, in order to prove that $\lim_{n\to\infty}\|x_n - q\| = 0$, we consider two possible cases as in [8] and [16].

Case 1. Assume that $\{\|x_n - q\|\}$ is a monotone sequence. In other words, for n_0 large enough, $\{\|x_n - q\|\}$ is either nondecreasing or nonincreasing. Hence $\{\|x_n - q\|\}$ converges (since $\{\|x_n - q\|\}$ is bounded).

Step 3. We show that $\lim_{n\to\infty} \|y_n - J_{r_n} y_n\| = 0$. First, from Lemma 2.2 (Resolvent Identity), we know that

$$J_{r_n} y_t = J_{\frac{r_n}{2}}\left(\frac{1}{2}y_n + \frac{1}{2}J_{r_n}y_n\right).$$

Then we have

$$\|J_{r_n}y_n - q\| = \left\|J_{\frac{r}{2}}\left(\frac{1}{2}y_n + \frac{1}{2}J_{r_n}y_n\right) - q\right\| \le \left\|\frac{1}{2}\Big(y_n - q\Big) + \frac{1}{2}\Big(J_{r_n}y_n - q\Big)\right\|.$$

By the inequality (2.1) ($l = 2, \lambda = \frac{1}{2}$), we obtain that

$$\begin{aligned}
\|J_{r_n}y_n - q\|^2 &\le \left\|J_{\frac{r_n}{2}}\left(\frac{1}{2}y_n + \frac{1}{2}J_{r_n}y_n\right) - q\right\|^2 \\
&\le \left\|\frac{1}{2}\Big(y_n - q\Big) + \frac{1}{2}\Big(J_{r_n}y_n - q\Big)\right\|^2 \\
&\le \frac{1}{2}\|y_n - q\|^2 + \frac{1}{2}\|J_{r_n}y_n - q\|^2 - \frac{1}{4}g(\|y_n - J_{r_n}y_n\|) \\
&\le \frac{1}{2}\|y_n - q\|^2 + \frac{1}{2}\|y_n - q\|^2 - \frac{1}{4}g(\|y_n - J_{r_n}y_n\|) \\
&= \|y_n - q\|^2 - \frac{1}{4}g(\|y_n - J_{r_n}y_n\|)
\end{aligned} \tag{3.15}$$

Thus, from (3.11), the convexity of the real function $\psi(t) = t^2$ $(t \in (-\infty, \infty))$ and the inequality (3.15), we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|J_{r_n}y_n - q\|^2 \\
&\le \|y_n - q\|^2 - \frac{1}{4}g(\|y_n - J_{r_n}y_n\|) \\
&= \|\alpha_n(fx_n - q) + (1-\alpha_n)(Sx_n - q)\|^2 - \frac{1}{4}g(\|y_n - J_{r_n}y_n\|) \\
&\le \alpha_n\|fx_n - q\|^2 + (1-\alpha_n)\|x_n - q\|^2 - \frac{1}{4}g(\|y_n - J_{r_n}y_n\|)
\end{aligned}$$

and hence

$$\frac{1}{4}g(\|y_n - J_{r_n}y_n\|)) - \alpha_n\|fx_n - q\|^2 \le \|x_n - q\|^2 - \|x_{n+1} - q\|^2.$$

Since $\{\|x_n - q\|\}$ converges, by condition (C1), we obtain

$$\lim_{n\to\infty} g(\|y_n - J_{r_n}y_n\|) = 0.$$

Thus, from the property of the function g in (2.1), it follows that

$$\lim_{n\to\infty} \|y_n - J_{r_n}y_n\| = 0.$$

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Step 4. We show that $\lim_{n\to\infty}\|x_n - y_n\| = 0$. Indeed, from Step 2 and Step 3, it follows that

$$\begin{aligned}\|x_n - y_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \\ &\le \|x_n - x_{n+1}\| + \|J_{r_n}y_n - y_n\| \to 0, \quad (n\to\infty).\end{aligned}$$

Step 5. We show that $\lim_{n\to\infty}\|y_n - Sy_n\| = 0$. In fact, by (3.12) and Step 4, we have

$$\begin{aligned}\|y_n - Sy_n\| &\le \|y_n - Sx_n\| + \|Sx_n - Sy_n\| \\ &\le \|y_n - Sx_n\| + \|x_n - y_n\| \to 0 \quad (n\to\infty).\end{aligned}$$

Step 6. We show that $\lim_{n\to\infty}\|y_n - J_ry_n\| = 0$ for $r > 0$. Indeed, from Lemma 2.2 (Resolvent identity), we obtain

$$\begin{aligned}\|J_{r_n}y_n - J_ry_n\| &= \left\|J_r\left(\frac{r}{r_n}y_n + \left(1-\frac{r}{r_n}\right)J_{r_n}y_n\right) - J_ry_n\right\| \\ &\le \left\|\left(\frac{r}{r_n}y_n + \left(1-\frac{r}{r_n}\right)J_{r_n}y_n\right) - y_n\right\| \\ &\le \left|1-\frac{r}{r_n}\right|\|y_n - J_{r_n}y_n\| \to 0 \quad (n\to\infty).\end{aligned} \tag{3.16}$$

Hence, by Step 3 and (3.16) we have

$$\|y_n - J_ry_n\| \le \|y_n - J_{r_n}y_n\| + \|J_{r_n}y_n - J_ry_n\| \to 0 \ \ (n\to\infty).$$

Step 7. We show that $\limsup_{n\to\infty}\langle (I-f)q, \mathcal{J}(q-y_n)\rangle \le 0$. To prove this, let a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ be such that

$$\limsup_{n\to\infty}\langle (I-f)q, \mathcal{J}(q-y_n)\rangle = \lim_{j\to\infty}\langle (I-f)q, \mathcal{J}(q-y_{n_j})\rangle$$

and $y_{n_j} \rightharpoonup z$ for some $z \in E$. From Step 5 and Step 6, it follows that $\lim_{j\to\infty}\|y_{n_j} - Sy_{n_j}\| = 0$ and $\lim_{j\to\infty}\|y_{n_j} - J_ry_{n_j}\| = 0$ for $r > 0$.

Now let $q = \lim_{t\to 0} x_t = \lim_{t\to 0} y_t$ where $y_t = tfx_t + (1-t)Sx_t$ and $x_t = J_ry_t$ for $r > 0$. Then we can write

$$y_t - y_{n_j} = t(fx_t - y_{n_j}) + (1-t)(Sx_t - y_{n_j})$$

and

$$\|x_t - y_{n_j}\| = \|J_ry_t - y_{n_j}\| \le \|y_t - y_{n_j}\| + \|J_ry_{n_j} - y_{n_j}\|.$$

Putting

$$a_j(t) = (1-t)^2\|Sy_{n_j} - y_{n_j}\|(2\|x_t - y_{n_j}\| + \|Sy_{n_j} - y_{n_j}\|) \to 0 \ \ (j\to\infty)$$

and

$$b_j(t) = \|J_ry_{n_j} - y_{n_j}\|(2\|y_t - y_{n_j}\| + \|J_ry_{n_j} - y_{n_j}\|) \to 0 \ \ (j\to\infty)$$

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by Step 5 and Step 6, and using Lemma 2.1, we obtain

$$\begin{aligned}
\|x_t - y_{n_j}\|^2 &\le \|y_t - y_{n_j}\|^2 + b_j(t) \\
&\le (1-t)^2\|Sx_t - y_{n_j}\|^2 + 2t\langle fx_t - y_{n_j}, \mathcal{J}(y_t - y_{n_j})\rangle + b_j(t) \\
&\le (1-t)^2(\|Sx_t - Sy_{n_j}\| + \|Sy_{n_j} - y_{n_j}\|)^2 \\
&\quad + 2t\langle fx_t - x_t, \mathcal{J}(y_t - y_{n_j})\rangle + 2t\|x_t - y_{n_j}\|\|y_t - y_{n_j}\| \\
&\le (1-t)^2\|x_t - y_{n_j}\|^2 + a_j(t) + b_j(t) \\
&\quad + 2t\langle fx_t - x_t, \mathcal{J}(y_t - y_{n_j})\rangle + 2t\|x_t - y_{n_j}\|^2 + 2t\|x_t - y_{n_j}\|\|y_t - x_t\|.
\end{aligned}$$

The last inequality implies

$$\langle (I-f)x_t, \mathcal{J}(y_t - y_{n_j})\rangle \le \frac{t}{2}\|x_t - y_{n_j}\|^2 + \frac{1}{2t}(a_j(t) + b_j(t)) + \|x_t - y_t\|\|x_t - y_{n_j}\|.$$

It follows that

$$\limsup_{j\to\infty}\langle (I-f)x_t, \mathcal{J}(y_t - y_{n_j})\rangle \le \frac{t}{2}M^2 + \|x_t - y_t\|M, \tag{3.17}$$

where $M = \sup\{\|x_t - y_n\| : n \ge 0 \text{ and } t \in (0,1)\}$. Recalling (5) in Proposition 3.1, taking the $\limsup$ as $t \to 0$ in (3.17), and noticing the fact that the two limits are interchangeable due to the fact that J is uniformly continuous on bounded subsets of E from the strong topology of E to the weak* topology of E^*, we have

$$\limsup_{j\to\infty}\langle (I-f)q, \mathcal{J}(q - y_{n_j})\rangle \le 0.$$

Step 8. We show that $\lim_{n\to\infty}\|x_n - q\| = 0$. By using (3.11), we have

$$\|x_{n+1} - q\| \le \|y_n - q\| = \|\alpha_n(fx_n - q) + (1-\alpha_n)(Sx_n - q)\|.$$

Applying Lemma 2.1, we obtain

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\le \|y_n - q\|^2 \\
&\le (1-\alpha_n)^2\|Sx_n - q\|^2 + 2\alpha_n\langle fx_n - q, \mathcal{J}(y_n - q)\rangle \\
&\le (1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\langle fx_n - fq, \mathcal{J}(y_n - q)\rangle \\
&\quad + 2\alpha_n\langle fq - q, \mathcal{J}(y_n - q)\rangle \\
&\le (1-\alpha_n)^2\|x_n - q\|^2 + 2k\alpha_n\|x_n - q\|\|y_n - q\| \\
&\quad + 2\alpha_n\langle fq - q, \mathcal{J}(y_n - q)\rangle \\
&\le (1-\alpha_n)^2\|x_n - q\|^2 + 2k\alpha_n\|x_n - q\|^2 \\
&\quad + 2k\alpha_n\|x_n - q\|\|y_n - x_n\| + 2\alpha_n\langle fq - q, \mathcal{J}(y_n - q)\rangle.
\end{aligned}$$

It then follows that

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\le (1 - 2(1-k)\alpha_n + \alpha_n^2)\|x_n - q\|^2 \\
&\quad + 2k\alpha_n\|x_n - q\|\|y_n - x_n\| + 2\alpha_n\langle fq - q, \mathcal{J}(y_n - q)\rangle \\
&\le (1 - 2(1-k)\alpha_n)\|x_n - q\|^2 + \alpha_n^2 L^2 \\
&\quad + 2kL\alpha_n\|y_n - x_n\| + 2\alpha_n\langle (I-f)q, \mathcal{J}(q - y_n)\rangle,
\end{aligned} \tag{3.18}$$

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where $L = \sup\{\|x_n - q\| : n \geq 0\}$. Put

$$\lambda_n = 2(1-k)\alpha_n \quad \text{and}$$
$$\delta_n = \frac{\alpha_n L^2}{2(1-k)} + \frac{kL}{(1-k)}\|y_n - x_n\| + \frac{1}{1-k}\langle (I-f)q, \mathcal{J}(q-y_n)\rangle.$$

From (C1), (C2), Step 4 and Step 7, it follows that have $\lambda_n \to 0$, $\sum_{n=0}^{\infty} \lambda_n = \infty$ and $\limsup_{n\to\infty} \delta_n \leq 0$. Since (3.18) reduces to

$$\|x_{n+1} - q\|^2 \leq (1-\lambda_n)\|x_n - q\|^2 + \lambda_n \delta_n,$$

from Lemma 2.4 with $\gamma_n = 0$, we conclude that $\lim_{n\to\infty} \|x_n - q\| = 0$. By Step 4, we also have $\lim_{n\to\infty} y_n = q$.

Case 2. Assume that $\{\|x_n - q\|\}$ is not a monotone sequence. Then, we can define a sequence of integers $\{\tau(n)\}$ for all $n \geq n_0$ (for some n_0 large enough) by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n,\ \|x_k - q\| < \|x_{k+1} - q\|\}.$$

Clearly, $\{\tau(n)\}$ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$\|x_{\tau(n)} - q\| \leq \|x_{\tau(n)+1} - q\|$$

for all $n \geq n_0$. In this case, by using the same argument as in Step 2 – Step 8 with $\{x_{\tau(n)}\}$, $\{y_{\tau(n)}\}$, $\{J_{r_{\tau(n)}} y_{\tau(n)}\}$, $\{J_r y_{\tau(n)}\}$, $\{Sx_{\tau(n)}\}$, $\{Sy_{\tau(n)}\}$, and $\{fx_{\tau(n)}\}$, we obtain the following:

Step 2′ $\lim_{n\to\infty} \|x_{\tau(n)+1} - x_{\tau(n)}\| = 0$;

Step 3′ $\lim_{n\to\infty} \|y_{\tau(n)} - J_{r_{\tau(n)}} y_{\tau(n)}\| = 0$.

Step 4′ $\lim_{n\to\infty} \|x_{\tau(n)} - y_{\tau(n)}\| = 0$.

Step 5′ $\lim_{n\to\infty} \|y_{\tau(n)} - Sy_{\tau(n)}\| = 0$.

Step 6′ $\lim_{n\to\infty} \|y_{\tau(n)} - J_r y_{\tau(n)}\| = 0$ for $r > 0$.

Step 7′ $\limsup_{n\to\infty} \langle (I-f)q, \mathcal{J}(q - y_{\tau(n)})\rangle \leq 0$.

Step 8′ $\lim_{n\to\infty} \|x_{\tau(n)} - q\| = 0$ and $\lim_{n\to\infty} \|x_{\tau(n)+1} - q\| = 0$.

Thus, from Lemma 2.5, we have

$$\|x_n - q\| \leq \|x_{\tau(n)+1} - q\|.$$

Therefore, $\lim_{n\to\infty} \|x_n - q\| = 0$. This completes the proof. □

Corollary 3.5. *Let E be a uniformly convex and uniformly smooth Banach space. Let C, A, J_{r_n}, S, and f be as in Theorem 3.4. Let $\{\alpha_n\} \in (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfy the conditions* (C1), (C2), (C3) *and* (C4) *in Theorem 3.4. Let $x_0 = x \in C$ be chosen arbitrarily, and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = J_{r_n}(\alpha_n f x_n + (1-\alpha_n)Sx_n), \quad \forall n \geq 0.$$

Let $\{y_n\}$ *be a sequence defined by* $y_n = \alpha_n f x_n + (1-\alpha_n)Sx_n$. *Then* $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $q \in A^{-1}0 \cap F(S)$, *where* q *is the unique solution of the variational inequality* (3.2).

Corollary 3.6. *Let* E, C, A, J_{r_n}, S, *and* f *be as in Theorem 3.4. Let* $\{\alpha_n\} \in (0,1)$ *and* $\{r_n\} \subset (0,\infty)$ *satisfy the conditions* (C1), (C2), (C3) and (C4) *in Theorem 3.4. Let* $x_0 = x \in C$ *be chosen arbitrarily, and let* $\{x_n\}$ *be a sequence generated by*

$$x_{n+1} = J_{r_n}(\alpha_n f x_n + (1-\alpha_n)Sx_n + e_n), \quad \forall n \geq 0,$$

where $\{e_n\} \subset E$ *satisfies* $\sum_{n=0}^{\infty} \|e_n\| < \infty$ *or* $\lim_{n\to\infty} \frac{\|e_n\|}{\alpha_n} = 0$. *Let* $\{y_n\}$ *be a sequence defined by* $y_n = \alpha_n f x_n + (1-\alpha_n)Sx_n + e_n$. *Then* $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $q \in A^{-1}0 \cap F(S)$, *where* q *is the unique solution of the variational inequality* (3.2).

Proof. Let $z_{n+1} = J_{r_n}(\alpha_n f z_n + (1-\alpha_n)Sz_n)$ for $n \geq 0$. Then by Theorem 3.4, $\{z_n\}$ converges strongly to a point $q \in A^{-1}0 \cap F(S)$ where q is the unique solution of the variational inequality (3.2), and

$$\begin{aligned}\|x_{n+1} - z_{n+1}\| &\leq \|\alpha_n f x_n + (1-\alpha_n)Sx_n - (\alpha_n z_n + (1-\alpha_n)Sz_n + e_n)\| \\ &\leq \alpha_n \|f x_n - f z_n\| + (1-\alpha_n)\|Sx_n - Sz_n\| + \|e_n\| \\ &\leq (1-(1-k)\alpha_n)\|x_n - z_n\| + \|e_n\|.\end{aligned}$$

By Lemma 2.4, we obtain

$$\lim_{n\to\infty} \|x_n - z_n\| = 0,$$

and hence the desired result follows. $\square$

Remark 3.7. (1) We point out that our iterative algorithms (3.1) and (3.3) for finding common point in the set of zeros of an accretive operator and the set of fixed points of a nonexpansive mapping are new ones different from those in the literature (see [20] and others in References). Thus Theorem 3.2 and Theorem 3.4 develop, and complement the recent corresponding results studied by many authors in this direction.

(2) If we take $fx = u$, $\forall x \in C$, as a constant function and $Sx = x$, $\forall x \in C$, as the identity mapping in Corollary 3.6, then the result extends corresponding results of Xu [18] and Song and Yang [19] in Hilbert spaces to a Banach space setting.

(3) The control condition (C3) in Theorem 3.4 can be replaced by the condition $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$; or the condition $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$, which are not comparable ([27]).

(4) The results in this paper apply to all L^p spaces, $1 < p < \infty$.

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Fixed points by some iterative algorithms in Banach and Hilbert spaces with some applications

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Abstract

In this paper, we apply a fixed point approach to derive an iterative method that converges to a general iteration scheme with errors in Hilbert and Banach spaces. Further, we obtain necessary and sufficient conditions for this sequence to converge to a common fixed point of two self mappings in normed spaces. Finally, we apply this iteration process to obtain a solution of a nonlinear equation.

1 Introduction

The iteration technique is a topic of great interest for a long time in the fixed point theory. During the last half of a century significant efforts have been applied to study fixed points by some iteration schemes. Indeed, such iterations based on the convergence of the sequence of iteration scheme and the kind of the mappings in each certain iteration. Therefore, this study is of a great importance for applications. This is the main motivation of the present paper and here in this paper, we prove some fixed point theorems using the so called, doubly G-iteration process with errors, then we apply this iteration to prove the existence theorem for the solution of a certain functional equation.

Let N be a normed linear space, $K \subset N$. A mapping $T : K \to K$ is said to be strongly pseudocontractive if there exists $t > 1$ such that the inequality

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\|$$

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holds for every $x, y \in K$ and $r > 0$. A mapping U with domain $D(U)$ and range $R(U)$ in N is called accretive if the following inequality

$$\|x - y\| \leq \|x - y + s(Ux - Uy)\|$$

holds for every $x, y \in D(U)$ and for all $s > 0$. Browder [5] proved that T is pseudocontractive if and only if $(I - T)$ is accretive, where I denotes the identity operator.
Let X be a real Banach space and X^* its dual. For $1 < p < \infty$, the duality mapping $J_p : X \to 2^{X^*}$, is defined by

$$J_p(x) = \{f^* \in X^* : \langle x\ ,\ f^* \rangle = \|x\|^p, \|f^*\|^p = \|x\|^{p-1}\}, \quad x \in X,$$

where $\langle .,. \rangle$ denotes the generalized duality pairing between X and X^*. Recall that a mapping $A : X \to X$ is said to be accretive if for all $x, y\ \in D(A)$ there exists $j_p(x - y)\ \in\ J_p(x - y)$ such that

$$\langle Ax - Ay\ ,\ j_p(x - y) \rangle \geq 0,$$

and is said to be strongly accretive if $A - kI$ is accretive where $k \in (0, 1)$ is a constant and I denotes the identity operator on X. Let $S(T) = \{x^* \in D(A) : Ax^* = f\} \neq \emptyset$ denote the solution set of the equation $Ax = f$. If $\langle Ax - Ay\ ,\ j_p(x - y) \rangle \geq 0$ for all $x \in D(A)$ and $y = x^* \in S(T)$, then A is said to be quasi-accretive. The notion of strongly quasi-accretive is similarly defined. A mapping $T : X \to X$ is said to be pseudo-contractive if for all $x, y\ \in D(T)$, there exists $j_p(x - y) \in J_p(x - y)$ such that

$$\langle (I - T)x - (I - T)y\ ,\ j_p(x - y) \rangle \geq 0.$$

Observe that T is pseudo-contractive if and only if $A = (I - T)$ is accretive. A map T is called hemicontractive if and only if $A = (I - T)$ is quasi-accretive.
Let X be a real Banach space of dimension $\dim X > 2$. The modulus of smoothness of X is defined by

$$\rho_X(\tau) = \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau \right\}; \quad \tau > 0.$$

If $\rho_X(\tau) > 0$ for all $\tau > 0$, then X is said to be smooth. If there exists a constant $c > 0$ and a real number $1 < p < \infty$ such that $\rho_X(\tau) \leq C\tau^p$, then X is said to p-uniformly smooth Banach space, then the following geometric inequality holds (see e.g., [4, 6]):

$$\|x + y\|^p \leq \|x\|^p + p\ \langle y\ ,\ j_p(x + y) \rangle + C_p \|y\|^p, \quad x, y \in X, \tag{1}$$

for some real positive constant $C_p \geq 1$. If T is a self-mapping of a closed convex subset E of X and I the identity of X, then T is a nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \text{ for all } x, y \in E.$$

Krasnoselskii [12] proved that the sequence of iteration $\{T^n(x_0)\}$, starting from a given point $x_0 \in E$, does not converge necessarily to a fixed point of T, whereas the sequence $\{T^n{}_\lambda(x_0)\}$, where

$$T_\lambda = (1 - \lambda)I + \lambda T, \;\; 0 < \lambda < 1,$$

may converge to a fixed point of T, as shown by Krasnoselskii [12] which assumed $\lambda = \frac{1}{2}$, E compact and X uniformly convex. The above scheme has been extended by means of so-called,

Mann iterative process (see [15]), associated with T and described in the following way:
Let $x_0 \in E$ and $\{x_n\}$ be a sequence defined by

$$x_{n+1} = (1 - c_n)x_n + c_n T x_n, \tag{2}$$

for $n = 0, 1, 2, \ldots,$ where
(a) $0 \leq c_n < 1, \quad n \geq 0,$
(b) $\lim\limits_{n\to\infty} c_n = 0,$
(c) $\sum\limits_{n=1}^{\infty} c_n = \infty.$
The scheme (2) has been studied by many authors (see for example [4, 11, 13, 16, 17, 20, 23, 25, 26, 27, 29]) and others. See also the work in double sequence setting [1, 2].
The scheme (2) has been extended by means of the so-called G-iteration process (see [21, 22]) associated with a single mapping T and described in the following manner:
Let $x_0 \in E$ and $\{x_n\}$ be a sequence defined by

$$x_{n+1} = (\mu_n - \lambda_n)x_n + \lambda_n T x_n + (1 - \mu_n) T x_{n-1} \ \text{ for } \ n \geq 0, \tag{3}$$

where $\{\mu_n\}$ and $\{\lambda_n\}$ satisfy
(i) $\lambda_0 = \mu_0 = 1,$
(ii) $0 < \lambda_n < 1, \ 0 \leq \mu_n \leq 1$ such that $\mu_n \geq \lambda_n$, $n > 0,$
(iii) $\lim\limits_{n\to\infty} \lambda_n = h > 0,$
(iv) $\lim\limits_{n\to\infty} \mu_n = 1.$
We note that when $\mu_n = 1$ for all $n \in \mathbb{N}$, the G-iteration process reduces to Mann iteration (1). It should be noted that the iteration scheme (3) is called G-iteration because its more general than Mann iteration process.
Let us compute the roots of equations numerically. That is, we would like to find numeric solutions to equations of one variable that can be written in the form $T(x) = 0$. Note that we do not put any restriction on the mapping T, only that it is a reasonably well-behaved mapping that we know how to evaluate. This kind of assumptions about the mapping turns out to be very important in numerical computation. In general, if we can place certain kinds of restrictions on the mapping, we will be able to use better and better methods to calculate its properties. A certain process using an algorithm for solving equations called fixed point iterations. In order to use fixed point iterations, we need the following information:
1. We need to know that there is a solution to the equation
2. We need to know approximately where the solution is (i.e. an approximation to the solution).
However, if the numerical method involves iteration then the root can be approximated to whatever accuracy we desire. One good way to measure the speed of the convergence is to use the ratio of the errors between successive iterations. In most cases the root must be obtained by numerical methods using a recipe or algorithm.
The idea of considering fixed point iteration procedures with errors comes from practical numerical computations. This topic of research play an important role in the stability problem of fixed point iterations. In 1995, Liu [14] initiated a study of fixed point iterations with errors. Several authors have proved some fixed point theorems for some certain iterations using several classes of mappings (see [3, 4, 9, 7, 8, 10, 18, 19, 25, 28] and others).
Now, we give the following iteration process with errors.

For $x_0 \in N$ and $n \in \mathbb{N} \cup \{0\}$, set

$$\begin{aligned} x_{n+1} &= (\mu_n - \lambda_n)x_n + \eta_n\lambda_n T x_n + \eta_n(1-\mu_n)Tx_{n-1} \\ &+ (1-\eta_n)\lambda_n S x_n + (1-\eta_n)(1-\mu_n)Sx_{n-1} + (1-\eta_n)(1-\mu_n)u_n \end{aligned} \quad (4)$$

where $\{\mu_n\}$ and $\{\lambda_n\}$ satisfy (i), (ii),(iii) and (iv) and $0 \leq \eta_n \leq 1$.

Remark 1.1 *It should be remarked that the above iteration scheme (4) is more general than some other scheme from literature. If $\mu_n = 1$ and $\eta_n = 1$ or $\eta_n = 0$ for all $n \in \mathbb{N}$, we obtain the Mann iteration process as defined by (2). Also, if $\eta_n = 1$ or $\eta_n = 0$, then we obtain the G-iteration process as defined by (3).*

2 Fixed Points in Hilbert Spaces

In this section, we give an implicit fixed point iterations associated with general hemicontractive mappings in Hilbert spaces. Some examples are also given to clear the role of the conditions on parameters of the defined iterations.

Definition 2.1 *Let $F(T) := \{x \in H : Tx = x\}$, $F(S) := \{x \in H : Sx = x\}$ and let K be a nonempty subset of H. Two mappings $S, T : K \to K$ are called general hemicontractive if $F(T) \cap F(S) \neq \varnothing$; and*

$$\|Tx - Sx^*\|^2 \leq \|x - Sx^*\|^2 + \|x - Tx\|^2 \textit{ for all } x \in H, x^* \in F(T) \bigcap F(S).$$

In the above definition if $S = I$, where I denotes the identity mapping, we obtain the definition of hemicontractive mapping (see [24]).

Theorem 2.1 *Let K be a compact convex subset of a real Hilbert space H and $S, T : K \to K$ be continuous general hemicontractive mappings. Let $\{\alpha_n\}$ be a real sequence in [0,1] satisfying $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$. For arbitrary $x_0 \in K$ and $\{v_n\}$ in K, define the sequence $\{x_n\}$ by*

$$\begin{aligned} &x_0 \in K \\ &Sx_n = \alpha_n x_{n-1} + (1-\alpha_n)(\lambda T\nu_n + (1-\lambda)S\nu_n) \end{aligned}$$

satisfying

$$\sum_{n \geq 1} \|S\nu_n - Sx_n\| < \infty.$$

Then $\{x_n\}$ converges strongly to a coincidence point of S, T.

Proof: The proof is very similar to the corresponding result in [24], by using Definition 2.1, so it will be omitted.
The next examples reveal that conditions on α_n must be imposed for the alodiality of Theorem 2.1.

Example 2.1 *Let (α_n) be a sequence in $(0,1)$ defined by $\alpha_n = 1 - \frac{5}{n+5}$. Define a sequence (x_n) in K by*

$$\begin{aligned} &x_0 = \frac{1}{5} \\ &x_n = \alpha_n x_{n-1} + (1-\alpha_n)(\lambda T\nu_n + (1-\lambda)S\nu_n). \end{aligned}$$

Let $\nu_n = \frac{1}{5}$, for all $n \geq 1$. Consider that

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + (1-\alpha_n)(\lambda T\nu_n + (1-\lambda)S\nu_n) \\ &= \alpha_n x_{n-1} + 1 - \alpha_n \nu_n \\ &= \alpha_n x_{n-1} + \frac{1-\alpha_n}{5} = \left(1 - \frac{5}{n+5}\right)x_{n-1} + \frac{1}{n+5}. \end{aligned}$$

By induction, it follows that $x_n = \frac{1}{5}$ for all $n \geq 1$.

Example 2.2 *Let $X = \mathbb{R}^2$, $S, T : K \times K \to K \times K$, where $K = [-1, 1]$. Define*

$$T(x, y) = (-x, -y);\ x, y \in K.$$

Let S be the identity mapping I. Then, $(0, 0)$ is the only fixed point of and T. Let (α_n) be a sequence in $(0, 1)$. Fix $\delta > 1$ and define a sequence (α_n) in K by

$$\alpha_n = \begin{cases} (\frac{1}{n\delta}, 0) & \text{if } n \text{ is odd} \\ \\ (\frac{1}{n}, 0) & \text{if } n \text{ is even.} \end{cases}$$

Take the initial point $x_1 = (\frac{\delta-1}{\delta+1}, 0)$. Then it is shown easily by induction that $x_{2n+1} = (\frac{\delta-1}{\delta+1}, 0)$ for all $n \geq 1$. Thus, the sequence (x_n) does not converge to $(0, 0)$.

3 Convergence Theorem

In this section, it is proved that for two mappings S and T which satisfy condition (5) below, if the sequence of iteration associated with S, T as defined in (4), then it converges to a common fixed point of S and T.
The contractive condition to be used is the following:
For all $x, y \in N$,

$$\begin{aligned} &\lambda\|Tx - Ty\| + (1-\lambda)\|Sx - Sy\| \\ &\leq \lambda\Big[\alpha\|x-y\| + \beta\|x - Tx\| + \gamma\|y - Tx\| + \delta\max\{\|y - Ty\|, \|x - Ty\|\}\Big] \\ &+(1-\lambda)\Big[\alpha\|x-y\| + \beta\|x - Sx\| + \gamma\|y - Sx\| + \delta\max\{\|y - Sy\|, \|x - Sy\|\}\Big], \qquad (5) \end{aligned}$$

where, $0 < \lambda < 1$ and $\alpha, \beta, \gamma \geq 0$ with $0 < \alpha + \beta + \gamma + \delta < 1$.
First of all we prove the following theorem:

Theorem 3.1 *Let K be a nonempty closed convex subset of a normed space N. Let $S, T : K \to K$ be mappings satisfying condition (5) and the following condition:*

$$S^2 = T^2 = I, \text{ where } I \text{ denotes the identity mapping.} \qquad (6)$$

Let $\{x_n\}$ be the sequence of iteration as defined by (4). If the sequence $\{x_n\}$ converges to a point $z \in K$, then z is the unique common fixed point of S and T.

Proof: For each $n \geq 0$, we have

$$\begin{aligned}
&\eta_n\|x_{n+1} - Tz\| + (1-\eta_n)\|x_{n+1} - Sz\| \\
&\leq \eta_n[(\mu_n - \lambda_n)\|x_n - Tz\| + \lambda_n\|Tx_n - Tz\| + (1-\mu_n)\|Tx_{n-1} - Tz\|] + \eta_n(1-\mu_n)\|u_n\| \\
&+(1-\eta_n)[(\mu_n - \lambda_n)\|x_n - Sz\| + \lambda_n\|Sx_n - Sz\| + (1-\mu_n)\|Sx_{n-1} - Sz\|]. \qquad (7)
\end{aligned}$$

Since S and T satisfy (6), then by using (7) we have

$$\begin{aligned}
&\eta_n\|Tx_n - Tz\| + (1-\eta_n)\|Sx_n - Sz\| \\
&\leq \alpha\|x_n - z\| + \beta\Big(\eta_n\|x_n - Tx_n\| + (1-\eta_n)\|x_n - Sx_n\|\Big) \\
&+\gamma\Big(\eta_n\|z - Tx_n\| + (1-\eta_n)\|z - Sx_n\|\Big) \\
&+\delta\Big(\eta_n \max\{\|z - Tz\|, \|x_n - Tz\|\} + (1-\eta_n)\max\{\|z - Sz\|, \|x_n - Sz\|\}\Big). \qquad (8)
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\eta_n\|x_{n+1} - Tz\| + (1-\eta_n)\|x_{n+1} - Sz\| \\
&\leq (1-\mu_n)\|u_n\| + (\mu_n - \lambda_n)\Big(\eta_n\|x_n - Tz\| + (1-\eta_n)\|x_n - Sz\|\Big) \\
&+(1-\mu_n)\Big(\eta_n\|Tx_{n-1} - Tz\| + (1-\eta_n)\|Sx_{n-1} - Sz\|\Big) \\
&+\alpha\lambda_n\|x_n - z\| + \beta\lambda_n\Big(\eta_n\|x_n - Tx_n\| + (1-\eta_n)\|x_n - Sx_n\|\Big) \\
&+\gamma\lambda_n\Big(\|z - x_n\| + \eta_n\|x_n - Tx_n\| + (1-\eta_n)\|x_n - Sx_n\|\Big) \\
&+\delta\lambda_n\Big(\eta_n \max\{\|z - Tz\|, \|x_n - Tz\|\} + (1-\eta_n)\max\{\|z - Sz\|, \|x_n - Sz\|\}\Big). \qquad (9)
\end{aligned}$$

From (6), one gets

$$\begin{aligned}
&\eta_n\|x_n - Tx_n\| + (1-\eta_n)\|x_n - Sx_n\| \\
&\leq \frac{1}{\lambda_n}[\|\mu_n x_n - x_{n+1}\|] + \frac{1-\mu_n}{\lambda_n}\Big(\eta_n\|Tx_{n-1}\| + (1-\eta_n)\|Sx_{n-1}\|\Big) + \frac{1-\mu_n}{\lambda_n}\|u_n\|. \quad (10)
\end{aligned}$$

Then, $\|x_n - Tx_n\| \to 0$ and $\|x_n - Sx_n\| \to 0$ as $n \to \infty$. Substituting (10) in (9) and letting $n \to \infty$, we obtain

$$\lambda\|z - Tz\| + (1-\lambda)\|z - Sz\| \leq (1 - h(1-\delta))\Big(\lambda\|z - Tz\| + (1-\lambda)\|z - Sz\|\Big).$$

Since $0 \leq [1 - h(1-\delta)] < 1$ and $0 < \lambda < 1$, we obtain that

$$\lambda\|z - Tz\| + (1-\lambda)\|z - Sz\| = 0.$$

Then, $Tz = z$ and $Sz = z$. Hence $Tz = Sz = z$, therefore z is a common fixed point of S and T. Now using (9), we have

$$Tz = z \text{ and hence, } Sz = z. \qquad (11)$$

It follows that

$$Sz = Tz = z$$

and z is a common fixed point of S and T.
Now to prove the uniqueness of z, let $w(w \neq z)$ be another common fixed point of S and T. Then, we have

$$\begin{aligned}
\|z-w\| &= \lambda\|T(Tz)-T(Tw)\| + (1-\lambda)\|S(Sz)-S(Tw)\| \\
&\le \alpha\Big(\lambda\|STz-STw\| + (1-\lambda)\|TSz-TSw\|\Big) \\
&+ \beta\Big(\lambda\|STz-T^2z\| + (1-\lambda)\|TST-S^2z\|\Big) \\
&+ \gamma\Big(\lambda\|STw-T^2z\| + (1-\lambda)\|TSw-S^2z\|\Big) \\
&+ \delta\Big(\lambda\max\{\|STw-T^2w\|, \|STz-T^2w\|\} \\
&+ (1-\lambda)\max\{\|TSw-S^2w\|, \|TSz-S^2w\|\}\Big) \\
&\le \eta\|z-w\|,
\end{aligned}$$

a contradiction, since $0 < \eta = \alpha+\gamma+\delta < 1$, then $z = w$. This completes the proof of the theorem.
Now, we give an example to discuss the validity of the hypothesis and degree of generality of the above theorem.

Example 3.1 *Let $N = \mathbb{R}^n$, the set of all n-tuples i.e., $x = (x_1, x_2, \dots, x_n)$ of real numbers and the norm $\|x\|$ is defined by*

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}, \; x \in \mathbb{R}^n.$$

Further, let $K = \{x : \|x\| \le 1, \, x \in \mathbb{R}^n\}$ and define the mappings $S, T : K \to K$ such that for arbitrary $x = (x_1, x_2, x_3, , , \dots, x_n) \in K$,

$$Sx = (x_2, x_1, 0, 0 \dots, 0),$$

and

$$Tx = (-x_1, -x_2, 0, 0, \dots, 0).$$

Suppose $\{x_n\}$ be a sequence of elements of K satisfying condition (4) with $u_n = 0$, where

$$\eta_n = 1, \;\; \lambda_n = 1 - \frac{n}{2n+1} \;\; \text{and} \;\; \mu_n = \frac{1}{2} - \frac{n+3}{2n+3} \; \text{for} \;\; n \ge 0.$$

Consider, $x_1 = (0.5, 0, 0, \dots, 0) \in K$, then it is easy to see that

$$x_2 = (0.166667, -0.333333, 0, 0, \dots, 0) \;\; \text{and} \;\; x_3 = (0.21903, -0.2810981, 0, 0, \dots, 0) \; \text{etc.}$$

Now it is easy to see that all conditions of Theorem 2.1 are satisfied, for instance taking $x = x_1$ and $y = x_2$ then, we have $0.4713996 \le \alpha+\gamma+\delta < 1$, which is true, since $0 \le \alpha+\gamma+\delta < 1$. Also, $0 = (\underbrace{0, 0, \dots, 0}_{n})$ is the unique common fixed point of S and T.

4 An application

In this section we will apply the iteration process as defined below to find the solution of the equation $Tx = f$. For this purpose we let X be a Banach space and let $T : D(T) \subset X \to X$ and $S : D(S) \subset X \to X$ be locally Lipschitz and strongly quasi-accretive mappings. It is proved that an iteration process (4) converges strongly to the unique solution of the equation $Tx = Sx = f,\ f \in R(T) \bigcap R(S)$.

Theorem 4.1 *Let X be a real $p-$uniformly smooth Banach space. Also, suppose that the mapping $S : D(T) \subset X \to X$ and let $T : D(T) \subset X \to X$ be nonexpansive, locally Lipschitz and strongly quasi-accretive operators with open domains $D(T)$ and $D(S)$ in X such that the equation $Tx = Sx = f$ has a solution $x^* \in D(T) \bigcap D(S)$ for $f \in R(T) \bigcap R(S)$ arbitrary but fixed. Define $T_\lambda : D(T) \to X$ and $S_\lambda : D(S) \to X$ by*

$$T_\lambda x = x - \lambda(Tx - f) \ \text{ for all } \ x \in D(T)$$

and

$$S_\lambda x = x - \lambda(Sx - f) \ \text{ for all } \ x \in D(S).$$

Then there exists a neighborhood B of x^* and a real number $\lambda \in (0, 1)$ such that starting with an arbitrary $x_0 \in B$ the iteration sequence $\{x_n\}$ generated by (4) remains in B and converges strongly to x^* with convergence being at least as fast as geometric progression.

Proof: Since S, T are locally Lipschitz, there is an $r > 0$ such that T is Lipschitz on

$$B = \bar{B}_r(x_0) = \{x \in X : \|x - x^*\| \leq r\} \subset D(T)$$

and S is Lipschitz on

$$B = \bar{B}_r(x_0) = \{x \in X : \|x - x^*\| \leq r\} \subset D(S).$$

Let $k \in (0, 1)$ and $L > 1, p > 1$ denote the strong accretivity and Lipschitz constant of A respectively. Observe that $f = Tx^*$. Pick an arbitrary $x_0 \in B$, choose

$$h\lambda = \left(\frac{k}{L^p C_p}\right)^{\frac{1}{p-1}}$$

and generate the sequence $\{x_n\}_{n\geq 0}$ as in (4). We now prove that $x_n \in B$, for all $n \geq 0$.
Suppose that $x_0 \in B$. Then

$$\begin{aligned}
&\|x_{n+1} - x^*\|^p = \|(\mu_n - \lambda_n)x_n + \lambda_n\eta_n Tx_n + (1 - \mu_n)\eta_n(Tx_{n-1} + u_n) \\
&+(1 - \eta_n)\lambda_n Sx_n + (1 - \mu_n)(1 - \eta_n)(Sx_{n-1}) - x^*\|^p \\
&= \|(\mu_n - \lambda_n)x_n + \lambda_n\eta_n[x_n - \lambda(Tx_n - f)] \\
&+(1 - \mu_n)\eta_n[x_{n-1} - \lambda(Tx_{n-1} - f) + u_n] \\
&+\lambda_n(1 - \eta_n)[x_n - \lambda(Sx_n - f) + u_n] \\
&+(1 - \mu_n)(1 - \eta_n)[x_{n-1} - \lambda(Sx_{n-1} - f) + u_n] - x^*\|^p \\
&= \|x_n - \lambda\lambda_n\Big(\eta_n(Tx_n - Tx^*) + (1 - \eta_n)(Sx_n - Sx^*)\Big) \\
&-(1 - \mu_n)\lambda\Big(\eta_n(Tx_n - Tx^*) + (1 - \eta_n)(Sx_n - Sx^*)\Big) + (1 - \mu_n)u_n - x^*\|^p \\
&= \|x_n - x^* - (\lambda\lambda_n + (1 - \mu_n)\lambda)(\eta_n(Tx_n - Tx^*) + (1 - \eta_n)(Sx_n - Sx^*)) + (1 - \mu_n)u_n\|^p.
\end{aligned}$$

Now using (1), we obtain

$$\begin{aligned}
&\|x_{n+1} - x^*\|^p \\
&\leq \|x_n - x^* + (1-\mu_n)u_n\|^p \\
&+p[(\mu_n - 1)\lambda - \lambda\lambda_n]\langle \eta_n(Tx_n - Tx^*) + (1-\eta_n)(Sx_n - Sx^*), J_p(x_n - x^*)\rangle \\
&+C_p[\lambda\lambda_n + (1-\mu_n)\lambda]^p \|\eta_n(Tx_n - Tx^*) + (1-\eta_n)(Sx_n - Sx^*)\|^p.
\end{aligned}$$

Since S and T are nonexpansive, then

$$\begin{aligned}
&\|x_{n+1} - x^*\|^p \\
&\leq (1 - ph\lambda k + L^p C_p (h\lambda)^p)\|x_n - x^*\|^p + (1-\mu_n)\,\eta(u_n, p) \\
&\leq (1 - (pk - L^p C_p (h\lambda)^{p-1})h\lambda)\|x_n - x^*\|^p + (1-\mu_n)\,\eta(u_n, p) \\
&= (1 - (p-1)k(\frac{k}{L^p C_p})^{\frac{1}{p-1}})\|x_n - x^*\|^p + (1-\mu_n)\,\eta(\|u_n\|, p) \leq r^p + (1-\mu_n)\,\eta(u_n, p),
\end{aligned}$$

where $\eta(u_n, p)$ is a function depends on u_n and p. Now, since $x_0 \in B$ by choice of the initial guess, it follows by the inductive hypothesis that the sequence $\{x_n\}$ remains in B. Set

$$\delta^* = \left(1 - (p-1)k(\frac{k}{L^p C_p})^{\frac{1}{p-1}}\right)^{\frac{1}{p}}$$

and observe that $\delta^* \in (0,1)$ since

$$k < \frac{LC_p^{\frac{1}{p}}}{(p-1)^{\frac{p-1}{p}}}\ , \quad \text{where} \ \ 1 < p < \infty$$

Hence, we obtain

$$\|x_n - x^*\|^p \leq (\delta^*)^{np}\|x_0 - x^*\|^p + (1-\mu_n)\,\eta(u_n, p),$$

since $\delta^{*np} \to 0$ as $n \to \infty$ the assertions of the theorem follows and the proof is complete.

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A variant of second-order Arnoldi method for solving the quadratic eigenvalue problem☆

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Abstract

In this paper, we give a variant of second-order Krylov subspace $R_n(A, B; u)$ based on a pair of square matrices A and B and a vector u, which is a modification of second-order Krylov subspace presented by Bai and Su [SIAM J. Matrix Anal. Appl., 26(2005) 640-659]. Then we can compute an orthonormal basis of $R_n(A, B; u)$ by using second-order Arnoldi procedure. By applying the standard Rayleigh-Ritz orthogonal projection technique, a variant of second-order Arnoldi method (VSOAR) for solving large-scale quadratic eigenvalue problems (QEPs) has been presented. Finally, numerical experiments are given to show the efficiency of the new method.

Keywords: variant of second-order Arnoldi method (VSOAR), Krylov subspace, quadratic eigenvalue problem (QEP), Arnoldi procedure, Rayleigh-Ritz orthogonal projection
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1. Introduction

The large-scale quadratic eigenvalue problem (QEP)

$$Q(\lambda)x = (\lambda^2 M + \lambda D + K)x = 0, \tag{1}$$

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(where $M, D, K \in \mathbb{R}^{n\times n}$,$\lambda \in \mathbb{C}$ and $x \in \mathbb{R}^n$) arises in many scientific and engineering applications, see [14] for a survey. Krylov subspace methods for the solution of quadratic eigenvalue problems have been studied by many authors, such as Parlett and Chen [14], Saad [18], Mehrmann and Watkins [11], Bai and Su [3] and the references therein. A Krylov subspace-based method is often the method of choice due to its simplicity, its availability of reliable and efficient processes for generating its orthonormal basis, and the superiority of convergence [3, 7, 8, 13, 18]. Many state-of-the-art Krylov subspace methods for solving large-scale eigenvalue problems are presented in [4]. Moreover, the solution methods for quadratic eigenvalue problem are reviewed by Tisseur and Meerbergen in [23].

As well known, the generalized eigenvalue problem of the form $Ax = \lambda Bx$ can be reduced to the linear eigenvalue problem in a form such as $B^{-1}Ax = x$, explicitly or implicitly, and then a Krylov subspace-based method can be applied [3]. The quadratic eigenvalue problem (QEP) of the following form

$$(\lambda^2 M + \lambda D + K)x = 0$$

is usually processed in two stages, as recommended in most literature such as [5, 6, 8, 1], public domain packages, and proprietary software today. At the first stage, the QEP is transformed into an equivalent generalized eigenvalue problem:

$$Cy = \lambda Gy \tag{2}$$

where $y^T = [\lambda x^T, x^T]$, and C and G are in forms as follows

$$C = \begin{pmatrix} -D & -K \\ I & 0 \end{pmatrix}, G = \begin{pmatrix} M & 0 \\ 0 & I \end{pmatrix},$$

where the matrix M is assumed to be nonsingular throughout the paper. At the second stage, it transforms the generalized eigenvalue problem (2) to a standard eigenvalue problem $Ax = \lambda x$ and then a Krylov subspace-based methods can be used for this problem[2, 12, 16, 17, 19, 20, 24]. This kind of approach can take advantages of Krylov subspace-based methods, such as the fast convergence rate and the simultaneous convergence of a group of eigenvalues, but it also suffers some disadvantages, such as having to solve the generalized eigenvalue problem, which has twice the dimension of the original QEP and, more importantly, will lose the original structures of the

QEP in the process of linearization [3]. The reader is referred to [23] for a recent survey on theory, applications, and algorithms of the QEP.

For years, researchers have been studying numerical methods which can be applied to the large-scale QEP directly, such as the Jacobi-Davidson method[21, 22], Arnoldi and Lanczos-type process[9]. Instead of using the linearization technique, the QEP can be projected onto a properly chosen low-dimensional subspace to reduce to a QEP directly with matrix dimension of lower order. Unfortunately, the method is strongly dependent on the initial approximation. Then the reduced QEP problem can then be solved by a standard dense matrix technique. The method computes one eigenvalue each time with local convergence versus Krylov subspace methods in which a group of eigenvalues are approximated with global convergence. In [15], a direct Krylov-type subspace method with a generalized Arnoldi procedure has been briefly described, but the procedure presented does not compute an orthonormal basis of the desired Krylov-type subspace. In [9], Arnoldi and Lanczos-type processes are developed to construct projections of the QEP. The convergence of these methods is usually slower than a Krylov subspace method applied to the mathematically equivalent linear eigenvalue problem. Recently, a subspace approximation based method was recently presented in [10], by using the perturbation theory of the QEP. The success of the method is strongly dependent on the initial approximation, although Rayleigh quotient iteration can be used for acceleration. In [3], Bai and Su developed a projection method, named by SOAR, which not only can be applied to the QEP directly to preserve the essential structures of the QEP, but also achieves the superior global convergence behaviors of Krylov subspace methods by linearization.

Motivated by the idea of [3], we present a new and efficient variant of SOAR in this paper, denoted by VSOAR mehtod. Firstly, we introduce a variant of second-order Krylov subspace $R_n(A, B; u)$ based on a pair of square matrices A and B and a vector u . The basis vectors of the subspace are defined via a homogenous recurrence of degree 2 with coefficient matrices A and B . Then a variant of second-order Arnoldi (VSOAR) procedure is presented for generating an basis of $R_n(A, B; u)$. As an application of the VSOAR procedure, a Rayleigh-Ritz orthogonal projection technique based on $R_n(A, B; u)$ is discussed for finding the largest magnitude eigenvalue and the corresponding eigenvector of the large-scale QEP (1).

The rest of this paper is organized as follows. In Section 2, we introduce a variant of second-order Krylov subspace $R_n(A, B; u)$ and a simple VSOAR

procedure for generating an orthonormal basis of subspace. In Section 3 we discuss the deflation and the convergence of the VSOAR. In Section 4, a Rayleigh-Ritz procedure for solving the QEP (1) is presented. Numerical examples are presented in Section 5 to show the efficiency of this new method.

2. A variant of second-order Krylov subspace.

In this section, we first define a variant of second-order Krylov subspace induced by a pair of matrices A and B and a vector u. Then we discuss the motivation for such a generalization.

Definition 1. *Let A and B be square matrices of order N, and let $u \neq 0$ be a vector of order N. Then the sequence*

$$r_0, r_1, r_2, ..., r_{n-1}, \cdots \tag{3}$$

where

$$\begin{cases} r_0 = u \\ r_1 = Br_0 \\ r_2 = Ar_1 \\ r_j = Ar_{j-1} + Br_{j-2}, j \geq 3 \end{cases} \tag{4}$$

is called a variant of second-order Krylov sequence based on A,B and u. The space

$$R_n(A, B; u) = span\{r_0, r_1, r_2, ...r_{n-1}\}$$

is called a variant of the n-th second-order Krylov subspace.

First, just like second-order Krylov sequence, the variant of second-order Krylov sequence also has the important characterization in terms of matrix polynomials, i.e.,

$$\begin{cases} r_0 = u, \\ r_1 = Br_0 = Bu, \\ r_2 = Ar_1 = ABu, \\ r_3 = Ar_2 + Br_1 = (A^2B + B^2)u, \\ r_4 = Ar_3 + Br_2 = (A^3B + AB^2 + BAB)u, \\ r_5 = Ar_4 + Br_3 = (A^4B + A^2B^2 + ABAB + BA^2B + B^3)u. \end{cases} \tag{5}$$

Second, we note that the subspace $R_n(A, B; u)$ generalizes the standard Krylov subspace $K_n(B; u)$ in the way that when A is a zero matrix, that is,

$$R_n(0, B; u) = K_n(B; u).$$

We now discuss the motivation for the definition of the variant of second-order Krylov subspace $R_n(A, B; u)$ in the context of solving QEP (1). Recall that the QEP (1) can transformed into an equivalent generalized eigenvalue problem (2). If one applies a Krylov subspace technique to (2), then an associated Krylov subspace would naturally be

$$K_n(H; v) = span\{v, Hv, H^2v, ..., H^{n-1}v\}, \tag{6}$$

where v is a starting vector of length $2N$, and

$$H = G^{-1}C = \begin{pmatrix} -M^{-1}D & -M^{-1}K \\ I & 0 \end{pmatrix}.$$

Let $A = -M^{-1}D$, $B = -M^{-1}K$ and $v = [u^T, 0]^T$; then the second-order Krylov vectors $\{r_j\}$ of length N and the standard Krylov vectors $\{H^jv\}$ of length $2N$ defined in (6) have the following relation:

$$H^jv = \begin{pmatrix} r_j \\ r_{j-1} \end{pmatrix}, forj \geq 1.$$

Note that, in the second-order Krylov subspace, we first used the matrix A , i.e.,$r_0 = u, r_1 = Au$. However, in this paper the matrix B was used firstly to construct the variant of second-order Krylov subspace. If we define $v = [0, u^T]^T$, then we can derive the variant of second-order Krylov vectors $\{r_j\}$ of length N defined in (3) and the standard Krylov vectors H^jv of length $2N$ defined in (6) are related as the following form:

$$\begin{cases} H^0v = \begin{pmatrix} 0 \\ r_0 \end{pmatrix}, \\ Hv = \begin{pmatrix} r_1 \\ 0 \end{pmatrix}, \\ H^jv = \begin{pmatrix} r_j \\ r_{j-1} \end{pmatrix}, forj \geq 3. \end{cases} \tag{7}$$

Equation (7) indicates that the subspace $R_n(A, B; u)$ of R^n should be able to provide sufficient information to let us directly work with the QEP, instead of using the subspace $K_n(H; v)$ of R^{2N} for the linearized eigenvalue problem (2).

We turn to the question of how to construct an orthonormal basis q_n of $R_n(A, B; u)$. Namely

$$span\{q_1, q_2, q_3, ..., q_n\} = R_n(A, B; u) \; for \; n \geq 1.$$

The following is a procedure to implicitly apply to the sequence of variant of second-order Krylov vector r_n to generate an orthonormal basis $\{q_1, q_2, q_3, ..., q_n\}$. We call it a VSOAR (variant of second-order Arnoldi) procedure.

Algorithm 1: (VSOAR procedure)

$q_2 = Bu/\|Bu\|_2$
$p_2 = 0$
for $j = 2, 3, ..., n$
    $r = Aq_j + Bp_j$
    $s = q_j$
    for $i = 2, 3, ...j$
        $t_{ij} = q_i^T r$
        $r := r - q_i t_{ij}$
        $s := s - q_i t_{ij}$
    end for
    $t_{j+1,j} = \|r\|_2$
    if $t_{j+1,j} = 0$ **stop**
    $q_{j+1} = r/t_{j+1,j}$
    $p_{j+1} = s/t_{j+1,j}$
end for
for $i = 2, 3, ..., n$
    $t_{i1} = q_i^T u$
    $q_1 = u - q_i t_{i1}$
end for
$t_{11} = \|q_1\|_2$
$q_1 = q_1/t_{11}$

Let us recall the following SOAR procedure for generating an orthonormal basis $\{\bar{q}_1, \bar{q}_2, \bar{q}_3, ..., \bar{q}_n\}$ of the second-order Krylov subspace $G_n(A, B; u)$,where H and v are defined in (7).

Algorithm 2:[3] (SOAR procedure)

1. $\bar{q}_1 = u/\|u\|_2$
2. $\bar{p}_1 = 0$
3. for $j = 1, 2, 3, ..., n$
4. $\quad \bar{r} = A\bar{q}_j + B\bar{p}_j$
5. $\quad \bar{s} = \bar{q}_j$
6. $\quad$ **for** $i = 1, 2, 3, ...j$
7. $\qquad \bar{t}_{ij} = \bar{q}_i^T \bar{r}$
8. $\qquad \bar{r} := \bar{r} - \bar{q}_i \bar{t}_{ij}$
9. $\qquad \bar{s} := \bar{s} - \bar{q}_i \bar{t}_{ij}$
10. $\quad$ **end for**
11. $\quad \bar{t}_{j+1,j} = \|\bar{r}\|_2$
12. $\quad$ **if** $\bar{t}_{j+1,j} = 0$ **stop**
13. $\quad \bar{q}_{j+1} = \bar{r}/\bar{t}_{j+1,j}$
14. $\quad \bar{p}_{j+1} = \bar{s}/\bar{t}_{j+1,j}$
15. end for

If Q_n,$\bar{Q}_n$ respectively denotes the $N \times n$ matrix with column vectors $\{q_1, q_2, q_3, ..., q_n\}$ and $\{\bar{q}_1, \bar{q}_2, \bar{q}_3, ..., \bar{q}_n\}$; P_n,$\bar{P}_n$ respectively denotes the $N \times n$ matrix with column vectors $\{p_1, p_2, p_3, ..., p_n\}$ and $\{\bar{p}_1, \bar{p}_2, \bar{p}_3, ..., \bar{p}_n\}$; T_n,$\bar{T}_n$ respectively denotes the $n \times n$ upper Hessenberg matrix with nonzero entries t_{ij},$\bar{t}_{ij}$ as defined in the Algorithm 1 and Algorithm 2. In [3], the following relations hold:

$$A\bar{Q}_n + B\bar{P}_n = \bar{Q}_n\bar{T}_n + \bar{q}_{n+1}e_n^T\bar{t}_{n+1,n}, \tag{8}$$

$$\bar{Q}_n = \bar{P}_n\bar{T}_n + \bar{q}_{n+1}e_n^T\bar{t}_{n+1,n}, \tag{9}$$

with the orthonormality of the vector sequence $\{\bar{q}_1, \bar{q}_2, \bar{q}_3, ..., \bar{q}_n, \bar{q}_{n+1}\}$. Let $\tilde{T}_n$ be an $(n+1) \times n$ upper Hessenberg matrix of the form $\tilde{T}_n = \begin{pmatrix} \bar{T}_n \\ e^T\bar{t}_{n+1,n} \end{pmatrix}$. Then equations (8) and (9) can be rewritten in the compact form

$$\begin{pmatrix} A & B \\ I & 0 \end{pmatrix} \begin{pmatrix} \bar{Q}_n \\ \bar{P}_n \end{pmatrix} = \begin{pmatrix} \bar{Q}_{n+1} \\ \bar{P}_{n+1} \end{pmatrix} \tilde{T}_n. \tag{10}$$

We note that if we let Bu be an initial vector , then in Algorithm 1 lines 1-14 is an SOAR procudere which is based on matrix A and B with the initial Bu . Let $\hat{Q}_{n-1} = [q_2, q_3, q_4, ..., q_n]$, $\hat{P}_{n-1} = [p_2, p_3, p_4, ..., p_n]$,

$$\hat{T}_{n-1} = \begin{pmatrix} t_{22} & t_{23} & t_{24} & \cdots & t_{2n} \\ t_{32} & t_{33} & t_{34} & \cdots & t_{3n} \\ 0 & t_{43} & t_{44} & \cdots & t_{4n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & t_{nn} \end{pmatrix}, \tilde{T}_{n-1} = \begin{pmatrix} \hat{T}_{n-1} \\ e_n^T t_{n+1,n} \end{pmatrix}.$$

Then the following relations hold:

$$A\hat{Q}_{n-1} + B\hat{P}_{n-1} = \hat{Q}_{n-1}\hat{T}_{n-1} + q_{n+1}e_n^T t_{n+1,n}, \tag{11}$$

$$\hat{Q}_{n-1} = \hat{P}_{n-1}\hat{T}_{n-1} + q_{n+1}e_n^T t_{n+1,n}. \tag{12}$$

Lines 15-22 are one step of Arnoldi procedure, and so we can get

$$t_{11}q_1 = u - \sum_{i=2}^{n} t_{i1}q_i. \tag{13}$$

For the rest of this section, we prove that the vector sequence $\{q_1, q_2, q_3, ..., q_n\}$ indeed is an orthonormal basis of the generalized second-order Krylov subspace $R_n(A, B; u)$. First, we introduce the following theorem:

Theorem 2. *[3] If $t_{j+1,j} \neq 0$ for $j \geq 1$ in Algorithm 2, then the vector sequence $\{\bar{q}_1, \bar{q}_2, \bar{q}_3, ..., \bar{q}_n\}$ forms an orthonormal basis of the second Krylov subspace $G_n(A, B; u)$,i.e.,*

$$span\{\bar{Q}_n\} = G_n(A, B; u) \ for \ j \geq 1$$

and $\bar{q}_i^T \bar{q}_k = 0$ if $i \neq k$ and $\bar{q}_i^T \bar{q}_i = 1$ for $i, k = 1, 2, ..., n$.

Similar to the Theorem 2 in [3], we can get the following theorem.

Theorem 3. *If $t_{j+1,j} \neq 0$ for $j \geq 1$ in Algorithm 1, then the vector sequence $\{q_1, q_2, q_3, ..., q_n\}$ forms an orthonormal basis of the variant second Krylov subspace $R_n(A, B; u)$,i.e.,*

$$span\{Q_n\} = R_n(A, B; u) \ for \ j \geq 1$$

and $q_i^T q_k = 0$ if $i \neq k$ and $q_i^T q_i = 1$ for $i, k = 1, 2, ..., n$.

Proof. From Theorem 2, we know that

$$span\{q_2, q_3, ..., q_n\} = span\{r_1, r_2, ..., r_{n-1}\}$$

and $q_i^T q_k = 0$ if $i \neq k$ and $q_i^T q_i = 1$ for $i, k = 2, 3, ..., n$. From Lines 16-22 of Algorithm 1 we can get

$$span\{q_1, q_2, q_3, ..., q_n\} = span\{r_0, r_1, r_2, ..., r_{n-1}\}$$

and $q_i^T q_k = 0$ if $i \neq k$ and $q_i^T q_i = 1$ for $i, k = 1, 2, ..., n$.

Remark: In [3], the authors discussed deflation SOAR process in detail. In fact, we can apply it directly to the VSOAR process without any modifications.

Now let us discuss the situation where breakdown occurs. According to [3], we can get the theorem as follows.

Theorem 4. *The VSOAR procedure (Algorithm 1) with matrices A and B and starting vector u breaks down at a certain step j if and only if the Arnoldi procedure with matrix H and starting vector v breaks down at the same step j.*

Proof. As the essence of VSOAR procedure with matrix A ,B and initial vector u is the SOAR procedure with matrix A ,B and initial vector Bu, adding a step for orthogonalization with u. Correspondingly, we decompose the sencond Arnoldi procedure with matrix H and initial vector v into two stages. The first stage is an Arnoldi procedure with matrix H and initial vector Hv; the second stage is orthogonal with v. The second stage of two procedures is equivalent. So we only need to prove the first stage of two procedures breaks down at the same step.

The essence of SOAR procedure with matrix A, B and initial vector Bu is solving the orthogonal basis of the following vectors:

$$r_1 = Bu, r_2 = ABu = Ar_0, r_j = Ar_{j-1} + Br_{j-2}, j \geq 3.$$

Correspondingly, the essence of Arnoldi procedure with matrix H and initial vector Hv is solving the orthogonal basis of the following vectors:

$$Hv = \begin{pmatrix} r_1 \\ 0 \end{pmatrix}, H^j v = \begin{pmatrix} r_j \\ r_{j-1} \end{pmatrix}, j \geq 2.$$

From Theorem 3, we can know that SOAR procedure with matrices A and B and starting vector Bu breaks down at a certain step j if and only if the Arnoldi procedure with matrix H and starting vector Hv breaks down at the same step j.

3. A projection method applied directly to the QEP.

In this section, we apply the variant of the second-order Krylov subspace and its orthonormal basis generated by the VSOAR procedure to develop a projection technique for solving the QEP (1). The technique is completely the same as that in [3], except for the new second-order Krylov subspace and VSOAR procedure. For completeness, we present the algorithm as follows:

Algorithm 5: (VSOAR method for solving the QEP directly)

1. Run the VSOAR procedure (Algorithm 4) with $A = -M^{-1}D$,$B = -M^{-1}K$and a starting vector u to generate an $N \times m$ orthogonal matrix Q_m whose columns span an orthonormal basis of $R_n(A, B; u)$.

2. Compute M_m,D_m,K_m as defined in (17)

3. Solve the reduced QEP (16) for (λ, g) and obtain the Ritz pairs (λ, z) , where $z = Q_m g/\|Q_m g\|_2$.

4. Test the accuracy of Ritz pairs (λ, z) as approximate eigenvalues and eigenvectors of the QEP (1) by the norms of residual vectors:

$$\|(\lambda^2 M + \lambda D + K)z\|_2 \tag{14}$$

Remark: At Step 4, we use the residual norms (14) as the accuracy assessment to indicate the errors of the approximate eigenpairs (λ, z). Also you can use the relative residual norms. For detail discussion, the reader is referred to [3].

4. Numerical examples

In this section, we present some examples to illustrate the performance of the VSOAR method for solving quadratic eigenvalue problem(QEP): $Q(\lambda)x = (\lambda^2 M + \lambda D + K)x = 0$. The numerical experiments are performed in Matlab on an Inter dual core processor (1.40GHz, 2GB RAM). All experiments of this section are with starting vector $u = [1, 1, \cdots, 1]^T$, subspace dimension

Table 1: IT and CPU for VSOAR and SOAR

	VSOAR		SOAR			VSOAR		SOAR	
n	IT	CPU	IT	CPU	n	IT	CPU	IT	CPU
400	28	0.9405	34	1.0369	450	28	1.2077	62	2.4297
500	26	1.3271	66	3.1764	550	26	1.7206	68	4.1856
600	26	2.1541	70	5.2887	700	24	2.8617	25	2.7034
800	22	3.3652	23	3.1755	1000	19	4.4574	21	4.4463

Table 2: IT and CPU for VSOAR and SOAR

	VSOAR		SOAR	
n	IT	CPU	IT	CPU
1000	95	22.7176	270	58.8461
1500	96	49.2231	1000	497.6421

$m = 10$ and stopped once the number of iterations is over 1000 or current residual norm satisfies the following condition

$$\|r_k\| = \|(\lambda^2 M + \lambda D + K)z\|_2 \le 10^{-5}.$$

Example 1.We consider the quadratic eigenvalue problem(QEP): $Q(\lambda)x = (\lambda^2 M + \lambda D + K)x = 0$ with

$$M = 0.1 \times I, K = I,$$

and

$$D = \begin{pmatrix} 0.2 & -0.1 & & & \\ -0.1 & 0.2 & -0.1 & & \\ & \ddots & \ddots & \ddots & \\ & & -0.1 & 0.2 & -0.1 \\ & & & -0.1 & 0.1 \end{pmatrix}$$

In this example, we test the dimension (denoted as **n**) of matrix with $n = 400 \sim n = 1000$. The iteration steps (denoted as IT) and the computing time (denoted as **CPU**) of both VSOAR method and SOAR method are

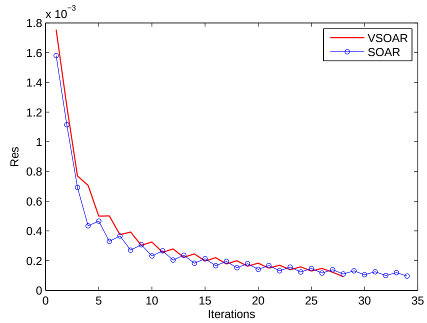

Figure 1: Example 1 for $n = 400$

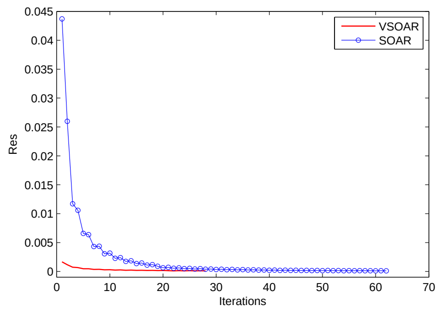

Figure 2: Example 1 for $n = 450$

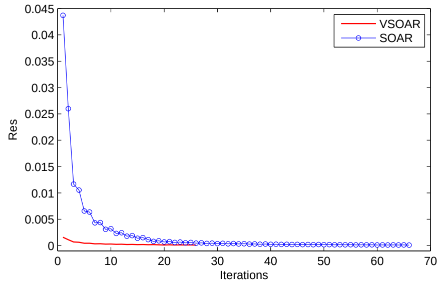

Figure 3: Example 1 for $n = 500$

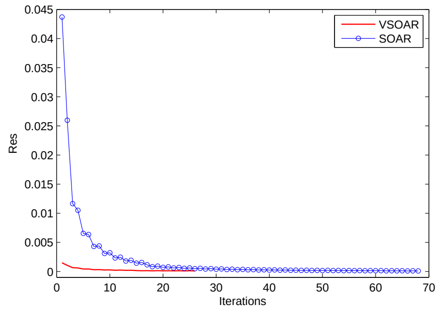

Figure 4: Example 1 for $n = 550$

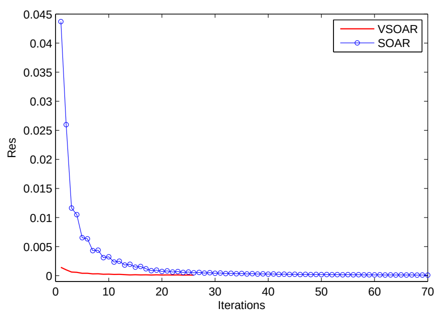

Figure 5: Example 1 for $n = 600$

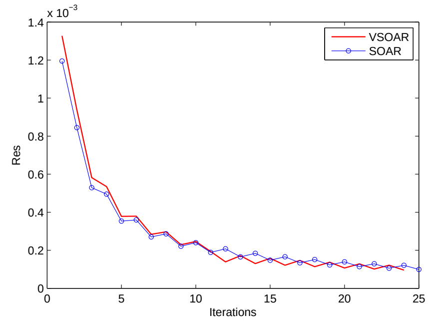

Figure 6: Example 1 for $n = 700$

listed in Table 1. The residual variation trend is showed from Figure 1 $\sim$ Figure 8.

Example 2. We consider the quadratic eigenvalue problem(QEP): $Q(\lambda)x = (\lambda^2 M + \lambda D + K)x = 0$ with

$$M = 0.1 \times I, K = I$$

$$D = \begin{pmatrix} 0 & -0.1 & & & 0.1 \\ -0.1 & 0 & -0.1 & & \\ & \ddots & \ddots & \ddots & \\ & & -0.1 & 0 & -0.1 \\ 0.1 & & & -0.1 & 0 \end{pmatrix}$$

In this example, we test the dimension of matrix $n = 1000$ and $n = 1500$. The iteration steps (denoted as IT) and the computing time (denoted as **CPU**) of both VSOAR method and SOAR method are listed in Table 2. From the Table 2 and Figure 9 and Figure 10, we know that the VSOAR is better than SOAR. Moreover, when the dimension of matrix is 1500×1500 , the SOAR will not converge.

As the above numerical experiments show, we observe that when SOAR method fails in some cases versus the VSOAR can do it well.

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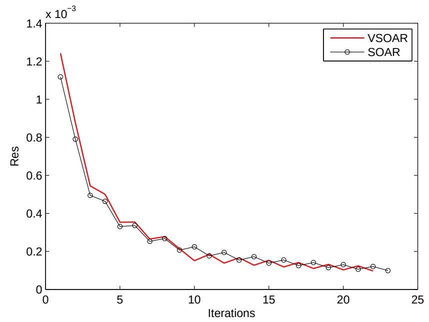

Figure 7: Example 1 for $n = 800$

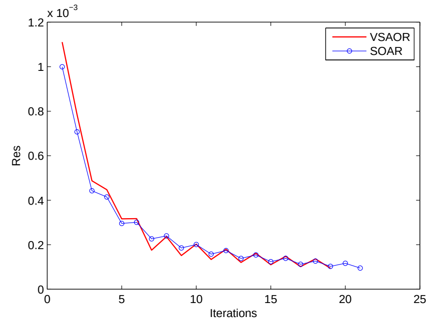

Figure 8: Example 1 for $n = 1000$

Table 3: IT and CPU for VSOAR and SOAR

	VSOAR		SOAR			VSOAR		SOAR	
n	IT	CPU	IT	CPU	n	IT	CPU	IT	CPU
300	53	0.8781	76	1.0774	500	93	4.6989	132	5.5053
800	154	20.8691	223	26.7822	1000	203	49.908	284	53.9909

Table 4: IT and CPU for VSOAR and SOAR

	VSOAR		SOAR			VSOAR		SOAR	
n	IT	CPU	IT	CPU	n	IT	CPU	IT	CPU
300	56	0.9128	80	1.1459	500	96	4.2466	138	5.4234
800	165	22.5349	230	27.4151	1000	213	46.0053	297	54.0763

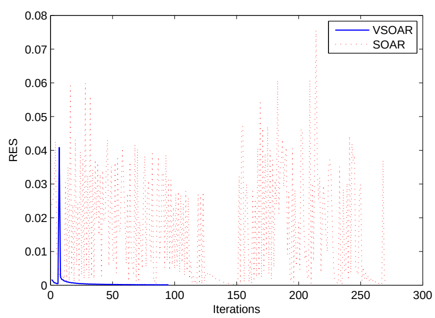

Figure 9: Example 2 for $n = 1000$

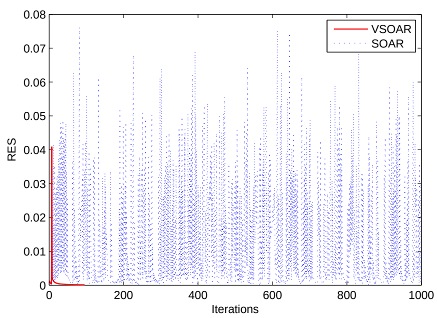

Figure 10: Example 2 for $n = 1500$

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Solvability for fractional differential inclusions with fractional nonseparated boundary conditions*

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Abstract

In this paper, a class of fractional differential inclusions with fractional non-separated (integral) boundary conditions is investigated under both convexity and non-convexity conditions on the multivalued term. Some new existence results are obtained by using standard fixed point theorems. Examples are given to illustrate the results.

Key words: *Fractional differential inclusions, boundary value problems, existence results, multivalued maps*

1 Introduction

Fractional differential equations have recently gained much importance and attention due to the fact that they have been proved to be valuable tools in the modeling of many physical phenomena [1, 2, 3]. For some recent developments on the existence results of fractional differential equations, we can refer to, for instance, [4, 5, 6, 7, 8, 9, 10, 11, 12, 13] and the references therein.

Differential inclusions arise in the mathematical modeling of certain problems in economics, optimal control, etc. and are widely studied by many authors; see [14, 15] and the references therein. For some recent works on differential inclusions of fractional order, we refer the reader to the references [4, 5, 16, 17, 18, 19, 20, 21, 22].

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Motivated by the above papers, in this article, we study a new class of fractional boundary value problems, i.e., the following fractional differential inclusions with fractional non-separated boundary conditions

$$\begin{cases} {}^cD^\alpha x(t) \in F(t, x(t)),\ t \in [0, T],\ 1 < \alpha \le 2,\ T > 0, \\ a_1 x(0) + b_1 x(T) = c_1, a_2({}^cD^\gamma x(0)) + b_2({}^cD^\gamma x(T)) = c_2, 0 < \gamma < 1, \end{cases} \tag{1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative (see [23]) of order q, $F : [0, T] \times \mathbb{R} \to 2^{\mathbb{R}}$ is a multifunction and $a_i, b_i, c_i,\ i = 1, 2$ are real constants such that $a_1 + b_1 \neq 0$ and $b_2 \neq 0$.

Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multi-point and non-local boundary value problems as special cases. Integral boundary conditions appear in the study of population dynamics and cellular systems etc. We can see the papers [24, 25], etc., for fractional differential equations with integral boundary conditions.

Along with the problem (1), we also consider the following fractional differential inclusions with fractional non-separated integral boundary conditions

$$\begin{cases} {}^cD^\alpha x(t) \in F(t, x(t)),\ t \in [0, T],\ 1 < \alpha \le 2,\ T > 0, \\ a_1 x(0) + b_1 x(T) = c_1 \int_0^T g(s, x(s))ds, \\ a_2({}^cD^\gamma x(0)) + b_2({}^cD^\gamma x(T)) = c_2 \int_0^T h(s, x(s))ds,\ 0 < \gamma < 1, \end{cases} \tag{2}$$

where $g, h : [0, T] \times \mathbb{R} \to \mathbb{R}$ are given functions.

We shall give some existence results for the problems (1) and (2) when the multivalued term F is convex as well as nonconvex valued. The main tools used in this paper are nonlinear alternative of Leray and Schauder type for multivalued (single-valued) maps, a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with decomposable values and a fixed point theorem for multivalued contraction maps due to Covitz and Nadler. Our approachs used are standard, however their exposition in the framework of the problems (1) and (2) is new.

We remark that when $a_1 = 1$, $b_1 = 1$, $c_1 = 0$, $a_2 = 1$, $b_2 = 1$ and $c_2 = 0$, the problem (1) reduces to an anti-periodic fractional boundary value problem (see [9] with $F = f$ a given continuous function). Our results extend some results from the literature cited above and constitute a contribution to this emerging field of research. In the next section for the convenience of the reader we recall some of the main preliminary facts which we will use in this paper.

2 Preliminaries

We denote by $\mathcal{C} = C([0, T], \mathbb{R})$ the Banach space of all continuous functions from $[0, T]$ into $\mathbb{R}$ with the norm $\|x\| = \sup_{t \in [0,T]} |x(t)|$.

Let $(X, \|\cdot\|)$ be a normed space. We use the notations: $P(X) = \{Y \subseteq X : Y \neq \emptyset\}$, $P_{cl}(X) = \{Y \in P(X) : Y \text{ closed}\}$, $P_b(X) = \{Y \in P(X) : Y \text{ bounded}\}$, $P_{cp}(X) = \{Y \in P(X) : Y \text{ compact}\}$, $P_{cp,c}(X) = \{Y \in P(X) :$

Y compact, convex$\}$ and so on. Let A, $B \in P_{cl}(X)$, the Pompeiu-Hausdorff distance of A, B is defined as:

$$h(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}.$$

A multivalued map $F : X \to P(X)$ is convex (closed) valued if $F(x)$ is convex (closed) for all $x \in X$. F is said to be completely continuous if $F(B)$ is relatively compact for every $B \in P_b(X)$. F is called upper semicontinuous on X, if for every $x_0 \in X$, the set $F(x_0)$ is a nonempty closed subset of X, and for every open set O of X containing $F(x_0)$, there exists an open neighborhood U_0 of x_0 such that $F(U_0) \subseteq O$. Equivalently, F is upper semicontinuous if the set $\{x \in X : F(x) \subseteq O\}$ is open for any open set O of X. F is called lower semicontinuous if the set $\{x \in X : F(x) \cap O \neq \emptyset\}$ is open for each open set O in X. If a multivalued map F is completely continuous with nonempty compact values, then F is upper semicontinuous if and only if F has a closed graph, i.e., if $x_n \to x_*$ and $y_n \to y_*$, then $y_n \in F(x_n)$ implies $y_* \in F(x_*)$ [26].

A multivalued map $F : [0,T] \to P_{cl}(X)$ is said to be measurable, if for every $x \in X$, the function $t \to d(x, F(t)) = \inf\{d(x,y) : y \in F(t)\}$ is measurable.

Definition 2.1. *A multivalued map $F : X \to P_{cl}(X)$ is called*
(1) γ-Lipschitz if there exists $\gamma > 0$ such that

$$h(F(x), F(y)) \leq \gamma d(x,y), \text{ for each } x, y \in X.$$

(2) a contraction if it is γ-Lipschitz with $\gamma < 1$.

Definition 2.2. *A multivalued map $F : [0,T] \times \mathbb{R} \to P(\mathbb{R})$ is said to be Carathéodory if:*
(1) $t \to F(t,x)$ is measurable for each $x \in \mathbb{R}$;
(2) $x \to F(t,x)$ is upper semicontinuous for a.e. $t \in [0,T]$.
Further, a Carathéodory function F is said to be L^1- Carathéodory if:
(3) for each $l > 0$, there exists $\varphi_l \in L^1([0,T], \mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq \varphi_l(t)$$

for all $|x| \leq l$ and a.e. $t \in [0,T]$.

For each $x \in \mathcal{C}$, define the set of selections of F by

$$S_{F,x} = \{v \in L^1([0,T], \mathbb{R}) : v(t) \in F(t, x(t)) \text{ for a.e. } t \in [0,T]\}.$$

Lemma 2.1 (see [27])**.** *Let X be a Banach space. Let $F : [0,T] \times X \to P_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and Γ be a linear continuous map from $L^1([0,T], X)$ to $C([0,T], X)$, then the operator*

$$\Gamma \circ S_F : C([0,T], X) \to P_{cp,c}(C([0,T], X)), \; y \mapsto (\Gamma \circ S_F)(y) = \Gamma(S_{F,y})$$

is a closed graph operator in $C([0,T], X) \times C([0,T], X)$.

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Lemma 2.2 ([13]). *Let $\alpha > 0$, then the differential equation*

$$^{c}D^{\alpha}h(t) = 0$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1}$ and

$$I^{\alpha}\,{}^{c}D^{\alpha}h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1},$$

here $c_i \in \mathbb{R}$, $i = 0, 1, 2, \cdots, n-1$, $n = [\alpha] + 1$.

Lemma 2.3. *For any $y \in C([0,T], \mathbb{R})$, the unique solution of the fractional non-separated boundary value problem*

$$\begin{cases} {}^{c}D^{\alpha}x(t) = y(t), \ t \in [0,T], \ 1 < \alpha \le 2, \\ a_1 x(0) + b_1 x(T) = c_1, a_2({}^{c}D^{\gamma}x(0)) + b_2({}^{c}D^{\gamma}x(T)) = c_2, 0 < \gamma < 1, \end{cases} \tag{3}$$

is given by

$$\begin{aligned} x(t) = & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - \frac{t\Gamma(2-\gamma)}{T^{1-\gamma}} \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds \\ & + \frac{t\Gamma(2-\gamma)c_2}{T^{1-\gamma}b_2} - \frac{b_1}{a_1+b_1}\Big(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds \\ & - T^{\gamma}\Gamma(2-\gamma) \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds \Big) \\ & - \frac{1}{a_1+b_1}\Big(\frac{b_1 c_2 T^{\gamma}\Gamma(2-\gamma)}{b_2} - c_1 \Big). \end{aligned} \tag{4}$$

Proof. For $1 < \alpha \le 2$, by Lemma 2.2, we know that the general solution of the equation ${}^{c}D^{\alpha}x(t) = y(t)$ can be written as

$$x(t) = I^{\alpha}y(t) - k_1 - k_2 t = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - k_1 - k_2 t, \tag{5}$$

where k_1, $k_2 \in \mathbb{R}$ are arbitrary constants. Since ${}^{c}D^{\gamma}k = 0$ (k is a constant), ${}^{c}D^{\gamma}t = \frac{t^{1-\gamma}}{\Gamma(2-\gamma)}$, ${}^{c}D^{\gamma}I^{\alpha}y(t) = I^{\alpha-\gamma}y(t)$ (see [23]), from (5) we have

$${}^{c}D^{\gamma}x(t) = I^{\alpha-\gamma}y(t) - \frac{k_2 t^{1-\gamma}}{\Gamma(2-\gamma)} = \int_0^t \frac{(t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds - \frac{k_2 t^{1-\gamma}}{\Gamma(2-\gamma)}.$$

Using the boundary conditions, we obtain

$$a_1(-k_1) + b_1\Big(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - k_1 - k_2 T \Big) = c_1,$$

$$a_2 \times 0 + b_2\Big(\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds - \frac{k_2 T^{1-\gamma}}{\Gamma(2-\gamma)} \Big) = c_2.$$

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Therefore we have

$$\begin{aligned} k_1 =& \frac{1}{a_1+b_1}\Big(\frac{b_1c_2T^\gamma\Gamma(2-\gamma)}{b_2}-c_1\Big)+\frac{b_1}{a_1+b_1} \\ &\times\Big(\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds-T^\gamma\Gamma(2-\gamma)\int_0^T\frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}y(s)ds\Big), \end{aligned}$$

$$k_2=\frac{\Gamma(2-\gamma)}{T^{1-\gamma}}\Big(\int_0^T\frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}y(s)ds-\frac{c_2}{b_2}\Big).$$

Substituting the values of k_1, k_2 in (5), we obtain (4). This completes the proof. □

From the proof of the above lemma, we notice that the solution (4) of the problem (3) does not depend on the parameter a_2, that is to say, the parameter a_2 is of arbitrary nature for this problem.

Definition 2.3. *A function $x \in \mathcal{C}$ is a solution of the problem (1) if it satisfies the boundary conditions in (1) and there exists a function $f \in L^1([0,T],\mathbb{R})$ such that $f(t) \in F(t,x(t))$ a.e. on $t \in [0,T]$ and*

$$\begin{aligned} x(t)=&\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds-\frac{t\Gamma(2-\gamma)}{T^{1-\gamma}}\int_0^T\frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}f(s)ds \\ &+\frac{t\Gamma(2-\gamma)c_2}{T^{1-\gamma}b_2}-\frac{b_1}{a_1+b_1}\Big(\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds \\ &-T^\gamma\Gamma(2-\gamma)\int_0^T\frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}f(s)ds\Big) \\ &-\frac{1}{a_1+b_1}\Big(\frac{b_1c_2T^\gamma\Gamma(2-\gamma)}{b_2}-c_1\Big). \end{aligned}$$

We end this section with two fixed point theorems.

Theorem 2.1 (Nonlinear alternative of Leray-Schauder type [28])**.** *Let X be a Banach space, C a closed convex subset of X, U an open subset of C with $0 \in U$. Suppose that $F : \overline{U} \to P_{cp,c}(C)$ is an upper semicontinuous compact map. Then either (1) F has a fixed point in $\overline{U}$, or (2) there is a $x \in \partial U$ and $\lambda \in (0,1)$ such that $x \in \lambda F(x)$.*

Theorem 2.2 (Covitz and Nadler [29])**.** *Let (X,d) be a complete metric space. If $F : X \to P_{cl}(X)$ is a contraction, then F has a fixed point.*

3 Existence results

In this section, three existence results of the problem (1) are presented. The first one concerns with the convex valued case, and the others relate to the nonconvex valued case.

Theorem 3.1. *Suppose that the following (H1), (H2) and (H3) are satisfied.*
(H1) $F : [0,T] \times \mathbb{R} \to P_{cp,c}(\mathbb{R})$ is a Carathéodory multivalued map;
(H2) there exist $m \in L^\infty([0,T], \mathbb{R}^+)$ and $\varphi : [0,\infty) \to (0,\infty)$ continuous, nondecreasing such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq m(t)\varphi(|x|) \text{ for } x \in \mathbb{R}, t \in [0,T];$$

(H3) there exists a constant $M > 0$ such that

$$\frac{M}{O + \varphi(M)Q} > 1, \tag{6}$$

here

$$O = \frac{T^\gamma \Gamma(2-\gamma)|c_2|}{|b_2|} + \Big|\frac{b_1 c_2 T^\gamma \Gamma(2-\gamma)}{(a_1+b_1)b_2} - \frac{c_1}{a_1+b_1}\Big|,$$
$$Q = \|m\|_{L^\infty} T^\alpha \Big(1 + \frac{|b_1|}{|a_1+b_1|}\Big)\Big(\frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)}\Big).$$

Then the boundary value problem (1) has at least one solution on $[0,T]$.

Proof. Consider the multivalued operator $N : \mathcal{C} \to P(\mathcal{C})$ defined as

$$N(x) = \{h \in \mathcal{C} : h = Sv, v \in S_{F,x}\}, \tag{7}$$

with

$$\begin{aligned}(Sv)(t) = &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds - \frac{t\Gamma(2-\gamma)}{T^{1-\gamma}} \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} v(s)ds \\ &+ \frac{t\Gamma(2-\gamma)c_2}{T^{1-\gamma}b_2} - \frac{b_1}{a_1+b_1}\Big(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \\ &- T^\gamma \Gamma(2-\gamma) \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} v(s)ds\Big) \\ &- \frac{1}{a_1+b_1}\Big(\frac{b_1 c_2 T^\gamma \Gamma(2-\gamma)}{b_2} - c_1\Big).\end{aligned}$$

We put $Sv = S_1 v + S_2 v$ and

$$(S_1 v)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds, \quad (S_2 v)(t) = -k_2^v t - k_1^v,$$

here k_1^v and k_2^v are constants given by

$$\begin{aligned}k_1^v = &\frac{b_1 c_2 T^\gamma \Gamma(2-\gamma)}{(a_1+b_1)b_2} - \frac{c_1}{a_1+b_1} + \frac{b_1}{a_1+b_1}\Big(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \\ &- T^\gamma \Gamma(2-\gamma) \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} v(s)ds\Big),\end{aligned}$$

$$k_2^v = \frac{\Gamma(2-\gamma)}{T^{1-\gamma}}\Big(\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}v(s)ds - \frac{c_2}{b_2}\Big).$$

Clearly, from Lemma 2.3, the fixed points of N are solutions of the problem (1). From (H1) and (H2), we have for each $x \in \mathcal{C}$, the set $S_{F,x}$ is nonempty [27]. Next we will show that N satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof is given in five steps.

Step 1: $N(x)$ is convex valued. Since F is convex valued, we know that $S_{F,x}$ is convex and therefore it is obvious that $N(x)$ is convex for each $x \in \mathcal{C}$.

Step 2: N maps bounded sets into bounded sets in $\mathcal{C}$. Let B_r be a bounded subset of $\mathcal{C}$ such that for any $x \in B_r$, $\|x\| \le r$. We prove that there exists a constant $l > 0$ such that for each $x \in B_r$, one has $\|h\| \le l$ for each $h \in N(x)$. Let $x \in B_r$ and $h \in N(x)$, then there exists $v \in S_{F,x}$ such that

$$h(t) = (Sv)(t) \quad \text{for} \quad t \in [0, T].$$

By simple calculations, we have

$$|(S_1v)(t)| \le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|v(s)|ds \le \varphi(r)\|m\|_{L^\infty}\frac{T^\alpha}{\Gamma(\alpha+1)},$$

$$|(S_2v)(t)| \le T|k_2^v| + |k_1^v|,$$

$$T|k_2^v| \le \varphi(r)\|m\|_{L^\infty}\frac{\Gamma(2-\gamma)T^\alpha}{\Gamma(\alpha-\gamma+1)} + \frac{T^\gamma\Gamma(2-\gamma)|c_2|}{|b_2|},$$

$$\begin{aligned}|k_1^v| \le &\frac{|b_1|}{|a_1+b_1|}\Big(\varphi(r)\|m\|_{L^\infty}\frac{T^\alpha}{\Gamma(\alpha+1)} + \varphi(r)\|m\|_{L^\infty}\frac{\Gamma(2-\gamma)T^\alpha}{\Gamma(\alpha-\gamma+1)}\Big)\\ &+ \Big|\frac{b_1c_2T^\gamma\Gamma(2-\gamma)}{(a_1+b_1)b_2} - \frac{c_1}{a_1+b_1}\Big|.\end{aligned}$$

Hence we obtain

$$\begin{aligned}\|h\| \le &\frac{T^\gamma\Gamma(2-\gamma)|c_2|}{|b_2|} + \Big|\frac{b_1c_2T^\gamma\Gamma(2-\gamma)}{(a_1+b_1)b_2} - \frac{c_1}{a_1+b_1}\Big|\\ &+ \varphi(r)\|m\|_{L^\infty}T^\alpha\Big(1 + \frac{|b_1|}{|a_1+b_1|}\Big)\Big(\frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)}\Big)\\ =&O + \varphi(r)Q = l.\end{aligned}$$

Step 3: N maps bounded sets into equicontinuous sets in $\mathcal{C}$. Let B_r be as in Step 2 and $0 \le t_1 < t_2 \le T$. For each $x \in B_r$ and $h \in N(x)$, there exists $v \in S_{F,x}$ such that $h(t) = (Sv)(t)$ for $t \in [0, T]$. Since we have

$$\begin{aligned}&|(S_1v)(t_2) - (S_1v)(t_1)|\\ \le&\Big|\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds\Big| + \Big|\int_0^{t_1}\frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds\Big|\\ \le&\frac{\varphi(r)\|m\|_{L^\infty}(t_2-t_1)^\alpha}{\Gamma(\alpha+1)} + \frac{\varphi(r)\|m\|_{L^\infty}\big(t_2^\alpha-(t_2-t_1)^\alpha-t_1^\alpha\big)}{\Gamma(\alpha+1)}\\ \le&\frac{\varphi(r)\|m\|_{L^\infty}(t_2^\alpha-t_1^\alpha)}{\Gamma(\alpha+1)}\end{aligned}$$

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and

$$
\begin{aligned}
|(S_2v)(t_2) - (S_2v)(t_1)| \leq & |k_2^v|(t_2 - t_1) \\
\leq & \frac{\Gamma(2-\gamma)}{T^{1-\gamma}}\Big(\frac{\varphi(r)\|m\|_{L^\infty}T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{|c_2|}{|b_2|}\Big)(t_2 - t_1),
\end{aligned}
$$

we deduce that

$$|h(t_2) - h(t_1)| \to 0 \text{ as } t_2 \to t_1$$

independently of $x \in B_r$ and $h \in N(x)$.

Step 4: N has a closed graph. Let $x_n \to x_*$, $h_n \in N(x_n)$ and $h_n \to h_*$, we need to show that $h_* \in N(x_*)$. Since $h_n \in N(x_n)$, there exists $v_n \in S_{F,x_n}$ such that $h_n(t) = (Sv_n)(t)$ for $t \in [0, T]$. We must prove that there exists $v_* \in S_{F,x_*}$ such that $h_*(t) = (Sv_*)(t)$ for $t \in [0, T]$.

Consider the continuous linear operator $\Gamma : L^1([0, T], \mathbb{R}) \to \mathcal{C}$ given by

$$
\begin{aligned}
v \to \Gamma(v)(t) = & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds - \frac{t\Gamma(2-\gamma)}{T^{1-\gamma}}\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}v(s)ds \\
& - \frac{b_1}{a_1+b_1}\Big(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds - T^\gamma\Gamma(2-\gamma)\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}v(s)ds\Big).
\end{aligned}
$$

And let

$$w(t) = \frac{t\Gamma(2-\gamma)c_2}{T^{1-\gamma}b_2} - \frac{1}{a_1+b_1}\Big(\frac{b_1c_2T^\gamma\Gamma(2-\gamma)}{b_2} - c_1\Big).$$

Clearly, we have $Sv = \Gamma v + w$ and

$$\Gamma(v_n)(t) = h_n(t) - w(t) \to h_*(t) - w(t) \text{ in } \mathcal{C}.$$

Also, by the definition of Γ, we have

$$h_n - w \in \Gamma(S_{F,x_n}).$$

It follows from Lemma 2.1 that $\Gamma \circ S_F$ is a closed graph operator. Since $x_n \to x_*$, we obtain

$$h_*(t) - w(t) = \Gamma(v_*)(t)$$

for some $v_* \in S_{F,x_*}$. This implies that $h_* \in N(x_*)$.

Step 5: A priori bounds for solutions. Let $x \in \lambda N(x)$ for some $\lambda \in (0, 1)$. Then there exists $v \in S_{F,x}$ such that $x(t) = \lambda(Sv)(t)$ for $t \in [0, T]$. By the similar computations as in the step 2, we have

$$|x(t)| \leq O + \varphi(\|x\|)Q \quad \text{for} \quad t \in [0, T].$$

In view of (H3), there exists M such that $\|x\| \neq M$. Let us set

$$U = \{x \in \mathcal{C} : \|x\| < M\}.$$

As a consequence of Steps 1-4, together with the Arzela-Ascoli theorem, we can conclude that $N : \overline{U} \to P_{cp,c}(\mathcal{C})$ is a upper semicontinuous and completely continuous map. From the choice of U, there is no $x \in \partial U$ such that $x \in \lambda N(x)$ for some $\lambda \in (0, 1)$. Hence by Theorem 2.1, we deduce that N has a fixed point $x \in \overline{U}$ which is a solution of the problem (1). This is the end of the proof. $\square$

Let A be a subset of $[0,T]\times\mathbb{R}$. A is said to be $\Sigma\otimes\mathcal{B}_{\mathbb{R}}$ measurable if A belongs to the σ-algebra generated by all sets of the form $J\times D$, where J is Lebesgue measurable in $[0,T]$ and D is a Borel set of $\mathbb{R}$. A subset A of $L^1([0,T],\mathbb{R})$ is said to be decomposable if for all u, $v\in A$ and $J\subseteq[0,T]$ Lebesgue measurable, then $u\chi_J+v\chi_{[0,T]-J}\in A$, where χ stands for the characteristic function.

Theorem 3.2. *Let (H2) and (H3) hold and assume:*
(H4) $F:[0,T]\times\mathbb{R}\to P_{cp}(\mathbb{R})$ is such that: (1) $(t,x)\to F(t,x)$ is $\Sigma\otimes\mathcal{B}_{\mathbb{R}}$ measurable; (2) the map $x\to F(t,x)$ is lower semicontinuous for a.e. $t\in[0,T]$. Then the problem (1) has at least one solution on $[0,T]$.

Proof. From (H2), (H4) and Lemma 4.4 of [22], the map

$$\mathcal{F}:\mathcal{C}\to P(L^1([0,T],\mathbb{R})),\quad x\to\mathcal{F}(x)=S_{F,x} \tag{8}$$

is lower semicontinuous and has nonempty closed and decomposable values. Then from a selection theorem due to Bressan and Colombo [30], there exists a continuous function $f:\mathcal{C}\to L^1([0,T],\mathbb{R})$ such that for all $x\in\mathcal{C}$, $f(x)(t)\in F(t,x(t))$ a.e. $t\in[0,T]$. Now consider the problem

$${}^cD^\alpha x(t)=f(x)(t),\ t\in[0,T] \tag{9}$$

with the boundary conditions in (1). Note that if $x\in\mathcal{C}$ is a solution of the problem (9), then x is a solution to the problem (1).

Problem (9) is then reformulated as a fixed point problem for the operator $N_1:\mathcal{C}\to\mathcal{C}$ defined by

$$N_1(x)(t)=(Sf(x))(t).$$

It can easily be shown that N_1 is continuous and completely continuous and satisfies all conditions of the Leray-Schauder nonlinear alternative for single-valued maps [28]. The proof is similar to that of Theorem 3.1, so we omit it here. This completes the proof. □

Theorem 3.3. *We assume that:*
(H5) Let $F:[0,T]\times\mathbb{R}\to P_{cp}(\mathbb{R})$ is such that: (1) the map $t\to F(t,x)$ is measurable for each $x\in\mathbb{R}$, (2) there exists $m\in L^\infty([0,T],\mathbb{R}^+)$ such that for a.e. $t\in[0,T]$ and all $x,y\in\mathbb{R}$,

$$h(F(t,x),F(t,y))\le m(t)|x-y|,\quad d(0,F(t,0))\le m(t).$$

If

$$\|m\|_{L^\infty}T^\alpha\Big(1+\frac{|b_1|}{|a_1+b_1|}\Big)\Big(\frac{1}{\Gamma(\alpha+1)}+\frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)}\Big)<1, \tag{10}$$

then the problem (1) has at least one solution on $[0,T]$.

Proof. From (H5), for each $x\in\mathcal{C}$, the multivalued map $t\to F(t,x(t))$ is measurable and closed valued. Hence it has measurable selection (Theorem 2.2.1 [26]) and the set $S_{F,x}$ is nonempty. Let N be defined in (7). We will show that, under this situation, N satisfies the requirements of Theorem 2.2.

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Step 1: For each $x \in \mathcal{C}$, $N(x) \in P_{cl}(\mathcal{C})$. Let $h_n \in N(x)$, $n \geq 1$, such that $h_n \to h$ in $\mathcal{C}$. Then $h \in \mathcal{C}$ and there exists $v_n \in S_{F,x}$, $n \geq 1$, such that

$$h_n(t) = (Sv_n)(t) \quad t \in [0,T].$$

By (H5), the sequence v_n is integrable bounded. Since F has compact values, we may pass to a subsequence if necessary to get that v_n converges to v in $L^1([0,T],\mathbb{R})$. Thus $v \in S_{F,x}$ and for each $t \in [0,T]$,

$$h_n(t) \to h(t) = (Sv)(t).$$

This means that $h \in N(x)$ and $N(x)$ is closed.

Step 2: There exists $\gamma < 1$ such that

$$h(N(x), N(y)) \leq \gamma \|x - y\| \text{ for all } x, y \in \mathcal{C}.$$

Let x, $y \in \mathcal{C}$ and $h_1 \in N(y)$, then there exists $v_1 \in S_{F,y}$ such that

$$h_1(t) = (Sv_1)(t),\ t \in [0,T].$$

From (H5)(2), we deduce

$$h(F(t,x(t)), F(t,y(t))) \leq m(t)|x(t) - y(t)|.$$

Hence, for a.e. $t \in [0,T]$, there exists $u \in F(t,x(t))$ such that

$$|v_1(t) - u| \leq m(t)|x(t) - y(t)|. \tag{11}$$

Consider the multivalued map $V : [0,T] \to P(\mathbb{R})$ given by

$$V(t) = \{u \in \mathbb{R} : |v_1(t) - u| \leq m(t)|x(t) - y(t)|\}.$$

Since $v_1(t)$, $\alpha(t) = m(t)|x(t) - y(t)|$ are measurable, Theorem III.41 in [31] implies that V is measurable. It follows from (H5) that the map $t \to F(t,x(t))$ is measurable. Hence by (11) and Proposition 2.1.43 in [26], the multivalued map $t \to V(t) \cap F(t,x(t))$ with nonempty closed values is measurable. Therefore, we can find $v_2(t) \in F(t,x(t))$ and

$$|v_1(t) - v_2(t)| \leq m(t)|x(t) - y(t)| \text{ for a.e. } t \in [0,T].$$

Let $h_2(t) = (Sv_2)(t)$, i.e., $h_2 \in N(x)$. Since

$$\begin{aligned}
&|h_1(t) - h_2(t)| \\
\leq& \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \big|v_1(s) - v_2(s)\big| ds \\
&+ T^\gamma \Gamma(2-\gamma) \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} \big|v_1(s) - v_2(s)\big| ds \\
&+ \frac{|b_1|}{|a_1 + b_1|} \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \big|v_1(s) - v_2(s)\big| ds \\
&+ \frac{|b_1| T^\gamma \Gamma(2-\gamma)}{|a_1 + b_1|} \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} \big|v_1(s) - v_2(s)\big| ds \\
\leq& \|m\|_{L^\infty} T^\alpha \Big(1 + \frac{|b_1|}{|a_1 + b_1|}\Big) \Big(\frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)}\Big) \|x - y\|.
\end{aligned}$$

Denote $\gamma = \|m\|_{L^\infty} T^\alpha \Big(1+\frac{|b_1|}{|a_1+b_1|}\Big)\Big(\frac{1}{\Gamma(\alpha+1)}+\frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)}\Big)$. By using an analogous relation obtained by interchanging the roles of x and y, we get

$$h(N(x), N(y)) \leq \gamma \|x-y\|.$$

Now in view of (10), Theorem 2.2 implies that N has a fixed point which is a solution of the problem (1). This completes the proof. □

4 Integral boundary problems

In this section the existence results of the problem (1) obtained above will be extended to the case of integral boundary conditions.

Lemma 4.1. *For any y, ξ, $\chi \in C([0,T],\mathbb{R})$, the unique solution of the fractional non-separated integral boundary value problem*

$$\begin{cases} {}^cD^\alpha x(t) = y(t), \ t\in[0,T], \ 1<\alpha\leq 2, \\ a_1x(0)+b_1x(T) = c_1\int_0^T \xi(s)ds, \\ a_2({}^cD^\gamma x(0)) + b_2({}^cD^\gamma x(T)) = c_2\int_0^T \chi(s)ds, \ 0<\gamma<1, \end{cases}$$

is given by

$$\begin{aligned} x(t) = &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - \frac{t\Gamma(2-\gamma)}{T^{1-\gamma}}\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds \\ &+ \frac{t\Gamma(2-\gamma)c_2}{T^{1-\gamma}b_2}\int_0^T \chi(s)ds - \frac{b_1}{a_1+b_1}\Big(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds \\ &- T^\gamma\Gamma(2-\gamma)\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}y(s)ds\Big) \\ &- \frac{b_1T^\gamma\Gamma(2-\gamma)c_2}{b_2(a_1+b_1)}\int_0^T \chi(s)ds + \frac{c_1}{a_1+b_1}\int_0^T \xi(s)ds. \end{aligned}$$

To obtain the existence results of the problem (2), in view of Lemma 4.1, we define an operator $\Omega : \mathcal{C} \to P(\mathcal{C})$ as

$$\Omega(x) = \{h \in \mathcal{C} : h = Zv, v \in S_{F,x}\} \tag{12}$$

with

$$\begin{aligned} (Zv)(t) = &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds - \frac{t\Gamma(2-\gamma)}{T^{1-\gamma}}\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} v(s)ds \\ &+ \frac{t\Gamma(2-\gamma)c_2}{T^{1-\gamma}b_2}\int_0^T h(s,x(s))ds - \frac{b_1}{a_1+b_1}\Big(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds \\ &- T^\gamma\Gamma(2-\gamma)\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}v(s)ds\Big) \\ &- \frac{b_1T^\gamma\Gamma(2-\gamma)c_2}{b_2(a_1+b_1)}\int_0^T h(s,x(s))ds + \frac{c_1}{a_1+b_1}\int_0^T g(s,x(s))ds. \end{aligned}$$

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Observe that if $x \in \mathcal{C}$ is a fixed point of the operator Ω, i.e., $x \in \Omega(x)$, then x is a solution of the problem (2).

From the definitions of the operators N and Ω, we know that the difference between them is very apparent, i.e., c_1, c_2 in (7) were replaced by $c_1 \int_0^T g(s, x(s))ds$ and $c_2 \int_0^T h(s, x(s))ds$ in (12). We omit the proofs of the following theorems, since they are similar to the ones obtained in Section 3.

Theorem 4.1. *Let (H1), (H2) hold and g, $h : [0,T] \times \mathbb{R} \to \mathbb{R}$ be continuous functions. Assume that:*
(A1) there exist functions m_2, $m_3 \in L^1([0,T], \mathbb{R}^+)$ and φ_2, $\varphi_3 : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that for $t \in [0,T]$, $x \in \mathbb{R}$,

$$|g(t,x)| \leq m_2(t)\varphi_2(|x|), \quad |h(t,x)| \leq m_3(t)\varphi_3(|x|);$$

(A2) there exists a constant $M > 0$ such that

$$\frac{M}{\varphi(M)Q + \varphi_3(M)\|m_3\|_{L^1}O + \frac{|c_1|}{|a_1+b_1|}\varphi_2(M)\|m_2\|_{L^1}} > 1,$$

here

$$O = \frac{T^\gamma \Gamma(2-\gamma)|c_2|}{|b_2|}\Big(1 + \frac{|b_1|}{|a_1+b_1|}\Big),$$

$$Q = \|m\|_{L^\infty} T^\alpha \Big(1 + \frac{|b_1|}{|a_1+b_1|}\Big)\Big(\frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)}\Big).$$

Then the problem (2) has at least one solution on $[0,T]$.

Theorem 4.2. *Let g, $h : [0,T] \times \mathbb{R} \to \mathbb{R}$ be continuous functions. Assume (H2), (H4), (A1) and (A2) hold, then the problem (2) has at least one solution on $[0,T]$.*

Theorem 4.3. *Let $F : [0,T] \times \mathbb{R} \to P_{cp}(\mathbb{R})$ be as in Theorem 3.3. In addition, we suppose that g, $h : [0,T] \times \mathbb{R} \to \mathbb{R}$ are continuous and satisfy*

$$|g(t,x) - g(t,y)| \leq m_2(t)|x-y|,$$

$$|h(t,x) - h(t,y)| \leq m_3(t)|x-y|,$$

for each $t \in [0,T]$ and all x, $y \in \mathbb{R}$ with m_2, $m_3 \in L^1([0,T], \mathbb{R}^+)$. If

$$\|m\|_{L^\infty} T^\alpha \Big(1 + \frac{|b_1|}{|a_1+b_1|}\Big)\Big(\frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)}\Big) + \frac{T^\gamma \Gamma(2-\gamma)|c_2|\|m_3\|_{L^1}}{|b_2|}\Big(1 + \frac{|b_1|}{|a_1+b_1|}\Big) + \frac{|c_1|\|m_2\|_{L^1}}{|a_1+b_1|} < 1,$$

then the problem (2) has at least one solution on $[0,T]$.

5 Examples

In this section, we give two examples to illustrate the results.

Example 1: Consider the fractional boundary value problem

$$\begin{cases} {}^cD^{\frac{3}{2}}x(t) \in F(t, x(t)), \ t \in [0,1], \\ x(0) - \frac{1}{2}x(1) = 2.5, \\ 2({}^cD^{\frac{1}{2}}x(0)) + \frac{1}{3}({}^cD^{\frac{1}{2}}x(1)) = -\frac{1}{3}, \end{cases} \tag{13}$$

where $\alpha = \frac{3}{2}$, $\gamma = \frac{1}{2}$, $a_1 = 1$, $b_1 = -\frac{1}{2}$, $c_1 = 2.5$, $a_2 = 2$, $b_2 = \frac{1}{3}$, $c_2 = -\frac{1}{3}$, $T = 1$ and $F : [0,1] \times \mathbb{R} \to P(\mathbb{R})$ is a multivalued map given by

$$F(t, x) = \{y \in \mathbb{R} : e^{-|x|} + \sin t + t^2 \le y \le 5 + \frac{|x|}{1 + x^2} + 6t^3\}.$$

In the context of this problem, we have

$$\|F(t, x)\| = \sup\{|v| : v \in F(t, x)\} \le 6 + 6t^2 \le 12, \ \text{for } t \in [0,1], x \in \mathbb{R}.$$

It is clear that F is convex compact valued and is of Carathéodory type. Let $m(t) \equiv 1$ and $\varphi(|x|) \equiv 12$, we get

$$\|F(t, x)\| = \sup\{|v| : v \in F(t, x)\} \le m(t)\varphi(|x|), \ \text{for } t \in [0,1], x \in \mathbb{R}.$$

As for the condition (6), since $O + \varphi(|x|)Q = O + 12Q$ (see O, Q in (H3)) is a constant, we can choose M large enough so that

$$\frac{M}{O + \varphi(M)Q} > 1.$$

Thus, by the conclusion of Theorem 3.1, the boundary value problem (13) has at least one solution on $[0,1]$.

Example 2: Consider the fractional integral boundary value problem

$$\begin{cases} {}^cD^{\frac{5}{4}}x(t) \in F(t, x(t)), \ t \in [0,1], \\ 3x(0) + \frac{1}{3}x(1) = \int_0^1 g(s, x(s))ds, \\ 2({}^cD^{\frac{1}{4}}x(0)) + 3({}^cD^{\frac{1}{4}}x(1)) = \frac{1}{4}\int_0^1 h(s, x(s))ds, \end{cases} \tag{14}$$

where $\alpha = \frac{5}{4}$, $\gamma = \frac{1}{4}$, $T = 1$, $a_1 = 3$, $b_1 = \frac{1}{3}$, $c_1 = 1$, $a_2 = 2$, $b_2 = 3$, $c_2 = \frac{1}{4}$,

$$F(t, x) = \Big[-\frac{1}{(4+t)^2}|x| - \frac{1}{8}, -\frac{1}{10}\Big] \bigcup \Big[0, \frac{e^{-t}}{16}\sin^2 x\Big],$$

$$g(t, x) = \frac{1}{(3+t)^2}\cos x, \ h(t, x) = \frac{|x|}{1 + |x|}.$$

From the data given above, we have

$$\sup\{|v| : v \in F(t, x)\} \le \frac{1}{8} + \frac{1}{(4+t)^2}|x|, \ t \in [0,1], \ x \in \mathbb{R},$$

$$h(F(t, x), F(t, y)) \le \max\{\frac{e^{-t}}{8}, \frac{1}{(4+t)^2}\}|x - y|, \ t \in [0,1], \ x, y \in \mathbb{R},$$

$$|g(t, x) - g(t, y)| \le \frac{1}{(3+t)^2}|x - y|, |h(t, x) - h(t, y)| \le |x - y|, t \in [0,1], x, y \in \mathbb{R}.$$

Then let $m_2(t) = \frac{1}{(3+t)^2}$, $m_3(t) = 1$ and $m(t) = \frac{1}{8} + \frac{e^{-t}}{8} + \frac{1}{(4+t)^2}$, we have

$$h(F(t,x), F(t,y)) \le m(t)|x-y|, \quad d(0, F(t,0)) \le m(t),$$

and

$$\begin{aligned}
&\|m\|_{L^\infty} T^\alpha \Big(1 + \frac{|b_1|}{|a_1+b_1|}\Big)\Big(\frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)}\Big) \\
&+ \frac{T^\gamma \Gamma(2-\gamma)|c_2|\|m_3\|_{L^1}}{|b_2|}\Big(1 + \frac{|b_1|}{|a_1+b_1|}\Big) + \frac{|c_1|\|m_2\|_{L^1}}{|a_1+b_1|} \\
\le& \frac{5}{16} \times \frac{11}{10} \times 1.8017 + \frac{0.9191}{12} \times \frac{11}{10} + \frac{3}{10} \times \frac{1}{9} \approx 0.7369 < 1.
\end{aligned}$$

Hence it follows from Theorem 4.3 that the fractional boundary value problem (14) has at least one solution on $[0, 1]$.

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Generalized intuitionistic fuzzy soft rough set and its application in decision making

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Abstract

Intuitionistic fuzzy set theory, soft set theory and rough set theory are mathematical tools for dealing with uncertainties and are closely related. This paper is devoted to the discussion of the combinations of intuitionistic fuzzy set, rough set and soft set. The concept of generalized intuitionistic fuzzy soft rough sets is proposed, and its properties are investigated. Furthermore, classical representations of generalized intuitionistic fuzzy soft rough approximation operators are presented. Finally, we develop an approach to generalized intuitionistic fuzzy soft rough sets based decision making and a practical example is provided to illustrate the developed approach.

Key words: Soft set; Rough set; Generalized intuitionistic fuzzy soft rough set; Decision making

1 Introduction

To solve complicated problems in economics, engineering, environmental science and social science, methods in classical mathematics are not always successful because of various types of uncertainties presented in these problems. There are several well-known theories to describe uncertainty. For instance, fuzzy set theory [1], intuitionistic fuzzy set theory [2,3], rough set theory [4,5] and other mathematical tools. But each of these theories has its inherent difficulties as pointed out in [6]. Perhaps above mentioned these theories are due to lack of parametrization tools. Theory of soft sets presented by Molodtsov [6] has enough parameters, so that it is free from inherent difficulties of above mentioned

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these theories. Soft set theory deals with uncertainty and vagueness on the one hand while on the other it has enough parametrization tools. These qualities of soft set theory make it popular among researchers and experts working in diverse areas. Soft set theory has potential applications in many different fields including the smoothness of functions, game theory, operational research, Perron integration, probability theory, and measurement theory [6, 7]. Presently, works on soft set theory are progressing rapidly. Maji et al. [8] defined several operations on soft sets and made a theoretical study on the theory of soft sets. Furthermore, based on [8], Ali et al. [9] introduced some new operations on soft sets and improved the notion of complement of soft set. They proved that certain De Morgans laws hold in soft set theory. Park et al [10] discussed some properties of equivalence soft set relations. The study of hybrid models combining soft sets with other mathematical structures is also emerging as an active research topic of soft set theory. Maji et al. [11] initiated the study on hybrid structures involving fuzzy sets and soft sets. They introduced the notion of fuzzy soft sets, which can be seen as a fuzzy generalization of soft sets. Furthermore, based on [11], Maji et al [12] modified definition of fuzzy soft sets, and presented the notion of generalized fuzzy soft sets theory. Yang et al. [13] presented the concept of the interval-valued fuzzy soft sets by combining interval-valued fuzzy set [14,15] and soft set models. By combining the concept of trapezoidal fuzzy set and soft set models, Xiao et al. [16] presented the concept of the trapezoidal fuzzy soft set which can deal with certain linguistic assessments. Yang et al. [17] presented the concept of the multi-fuzzy soft set by combining the multi-fuzzy set and soft set models, and provided its application in decision making under an imprecise environment.

The concept of rough sets, proposed by Pawlak [4, 5] as a framework for the construction of approximations of concepts, is a formal tool for modeling and processing insufficient and incomplete information. In order to handle vagueness and imprecision in the data equivalence relations play an important role in this theory. This theory has been applied successfully to solve many problems, but in daily life, it is very difficult to find an equivalence relation among the elements of a set under consideration. Therefore many authors have generalized the notion of Pawlak rough set by using non-equivalence binary relations. This has led to various other generalized rough set models [18–27].

Soft set theory, fuzzy set theory and rough set theory are all mathematical tools to deal with uncertainty. It has been found that soft set, fuzzy set and rough set are closely related concepts [28]. Feng et al. [29] provided a framework to combine fuzzy sets, rough sets and soft sets all together, which gives rise to several interesting new concepts such as rough soft sets, soft rough sets and soft rough fuzzy sets. The combination of soft set and rough set models was also discussed by some researchers [30–32].

In this paper, we devote to the discussion of the combinations of intuitionistic fuzzy set, rough set and soft set. The traditional intuitionistic fuzzy rough set [33–35] and soft set theory are two different tools to deal with uncertainty. Apparently there is no direct connection between these two theories. The major criticism on rough set theory is that

it lacks parametrization tools. In order to make parametrization tools available in rough sets, we construct an intuitionistic fuzzy soft relation between the universe set U and the parameter set E in intuitionistic fuzzy soft set so that it is natural to combine intuitionistic fuzzy rough set and soft set theory. So the concept of generalized intuitionistic fuzzy soft rough sets is propose, and its some properties are discussed. Then the relationships between generalized intuitionistic fuzzy soft rough sets and the existing generalized soft rough sets are also established. We finally present an illustrative example which show that the decision making method of generalized intuitionistic fuzzy soft rough sets can be successfully applied to many problems that contain uncertainties.

The rest of this paper is organized as follows. Section 2 briefly reviews some preliminaries. In section 3, we construct the crisp soft rough approximation operators, and discuss their some interesting properties. In Section 4, an intuitionistic fuzzy soft relation is first defined by us. By combining the intuitionistic fuzzy soft relation with intuitionistic fuzzy rough sets, then the concept of generalized intuitionistic fuzzy soft rough approximation operators is presented and the properties of the generalized upper and lower intuitionistic fuzzy soft rough approximation operators are examined. Furthermore, classical representations of generalized intuitionistic fuzzy soft rough approximation operators are presented. Section 5 is devoted to studying the application of generalized intuitionistic fuzzy soft rough sets. Some conclusions and outlooks for further research are given in Section 6.

2 Preliminaries

In this section, we shall briefly recall some basic notions being used in the study.

Definition 2.1 ([36]) *Let* $L^* = \{(\mu, \nu) \in [0,1] \times [0,1] | \mu + \nu \leq 1\}$ *and denote* $(\mu_1, \nu_1) \leq_{L^*} (\mu_2, \nu_2) \Leftrightarrow \mu_1 \leq \mu_2$ *and* $\nu_1 \geq \nu_2, \forall (\mu_1, \nu_1), (\mu_2, \nu_2) \in L^*$. *Then the pair* $(L^*, \leq_{L^*})$ *is called a complete lattice. The operators* $\wedge$ *and* $\vee$ *on* $(L^*, \leq_{L^*})$ *are defined as follows:for* $(\mu_1, \nu_1), (\mu_2, \nu_2) \in L^*$,

$(\mu_1, \nu_1) \wedge (\mu_2, \nu_2) = (min\{\mu_1, \mu_2\}, max\{\nu_1, \nu_2\})$,

$(\mu_1, \nu_1) \vee (\mu_2, \nu_2) = (max\{\mu_1, \mu_2\}, min\{\nu_1, \nu_2\})$,

Obviously, a complete lattice on L^* has the smallest element $0_{L^*} = (0, 1)$ and the greatest element $1_{L^*} = (1, 0)$. The definitions of fuzzy logical operators can be straightforwardly extended to the intuitionistic fuzzy case. The strict partial order $<_{L^*}$ is defined by

$$(\mu_1, \nu_1) <_{L^*} (\mu_2, \nu_2) \Leftrightarrow (\mu_1, \nu_1) \leq_{L^*} (\mu_2, \nu_2) \text{ and } (\mu_1, \nu_1) \geq_{L^*} (\mu_2, \nu_2).$$

In the following, we review the concept of the intuitionistic fuzzy set introduced by Atanassov [2, 3].

Definition 2.2 ([2, 3]) *Let a set* U *be fixed. An intuitionistic fuzzy (IF, for short) set* A *in* U *is an object having the form*

$A = \{< x, \mu_A(x), \gamma_A(x) > | x \in U\}$,
where $\mu_A : U \to [0, 1]$, *and* $\gamma_A : U \to [0, 1]$, *satisfy* $0 \le \mu_A(x) + \gamma_A(x) \le 1$ *for all* $x \in U$, *and* $\mu_A(x)$ *and* $\gamma_A(x)$ *are, respectively, called the degree of membership and the degree of non-membership of the element* $x \in U$ *to* A.

The family of all intuitionistic fuzzy subsets in U is denoted by $IF(U)$. The complement of an IF set A is denoted by $\sim A = \{< x, \gamma_A(x), \mu_A(x) > | x \in U\}$.

Obviously, every fuzzy set $A = \{< x, A(x) > | x \in U\} = \{< x, \mu_A(x) > | x \in U\}$ can be identified with the IF set of the form $\{< x, \mu_A(x), 1 - \mu_A(x) > | x \in U\}$ and is thus an IF set.

The basic operations on $IF(U)$ are defined as follows [2,3,37–39]: for all $A, B \in IF(U)$,
(1) $A \subseteq B$ iff $\mu_A(x) \le \mu_B(x)$ and $\gamma_A(x) \ge \gamma_B(x)$ for all $x \in U$,
(2) $A = B$ iff $A \subseteq B$ and $B \subseteq A$,
(3) $A \cap B = \{< x, min\{\mu_A(x), \mu_B(x)\}, max\{\gamma_A(x), \gamma_B(x) > | x \in U\}$,
(4) $A \cup B = \{< x, max\{\mu_A(x), \mu_B(x)\}, min\{\gamma_A(x), \gamma_B(x) > | x \in U\}$.

For $(\alpha, \beta) \in L^*$, $\widehat{(\alpha, \beta)}$ denotes a constant IF set: $\widehat{(\alpha, \beta)}(x) = \{< x, \alpha, \beta > | x \in U\}$,where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \le 1$; The IF universe set is $U = 1_U = \widehat{(1, 0)} = \{< x, 1, 0 > | x \in U\}$ and the IF empty set is $\emptyset = 0_U = \widehat{(0, 1)} = \{< x, 0, 1 > | x \in U\}$.

Definition 2.3 ([33, 35]) *Let* $A = \{< x, \mu_A(x), \gamma_A(x) > | x \in U\} \in IF(U)$, *and* $(\alpha, \beta) \in L^*$. *The* (α, β)*-level cut set of* A, *denoted by* A_α^β , *is defined as follows:*

$$A_\alpha^\beta = \{x \in U | \mu_A(x) \ge \alpha, \gamma_A(x) \le \beta\}.$$

$A_\alpha = \{x \in U | \mu_A(x) \ge \alpha\}$, *and* $A_{\alpha+} = \{x \in U | \mu_A(x) > \alpha\}$, *are, respectively, called the* α*-level cut set and the strong* α*-level cut set of membership generated by* A. *And* $A^\beta = \{x \in U | \gamma_A(x) \le \beta\}$ *and* $A^{\beta+} = \{x \in U | \gamma_A(x) < \beta\}$ *are, respectively, referred to as the* β*-level cut set and the strong* β*-level cut set of non-membership generated by* A.

At the same time, other types of cut sets of the IF set A *are denoted as follows:*
$A_{\alpha+}^\beta = \{x \in U | \mu_A(x) > \alpha, \gamma_A(x) \le \beta\}$, *which is called the* $(\alpha+, \beta)$*-level cut set of* A;
$A_\alpha^{\beta+} = \{x \in U | \mu_A(x) \ge \alpha, \gamma_A(x) < \beta\}$, *which is called the* $(\alpha, \beta+)$*-level cut set of* A;
$A_{\alpha+}^{\beta+} = \{x \in U | \mu_A(x) > \alpha, \gamma_A(x) < \beta\}$, *which is called the* $(\alpha+, \beta+)$*-level cut set of* A.

Theorem 2.4 ([33, 35]) *The cut sets of IF sets satisfy the following properties:* $\forall A \in IF(U), \alpha, \beta \in [0, 1]$ *with* $\alpha + \beta \le 1$,
(1) $A_\alpha^\beta = A_\alpha \cap A^\beta$,
(2) $(\sim A)_\alpha = \sim A^{\alpha+}, (\sim A)^\beta = \sim A_{\beta+}$.

Definition 2.5 ([22, 24]) *Let* U *be a nonempty and finite universe of discourse and* $R \subseteq U \times U$ *an arbitrary crisp relation on* U. *We define a set-valued function* $R_s : U \to P(U)$ *by* $R_s(x) = \{y \in U | (x, y) \in R\}, x \in U$.

The pair (U, R) is called a crisp approximation space. For any $A \subseteq U$, the upper and lower approximations of A with respect to (U, R) , denoted by $\overline{R}(A)$ and $\underline{R}(A)$, are defined, respectively, as follows:

$\overline{R}(A) = \{x \in U | R_s(x) \cap A \neq \emptyset\}$, $\underline{R}(A) = \{x \in U | R_s(x) \subseteq A\}$.

The pair $(\overline{R}(A), \underline{R}(A))$ is referred to as a crisp rough set, and $\overline{R}, \underline{R} : P(U) \to P(U)$ are, respectively, referred to as upper and lower crisp approximation operators induced from (U, R) .

3 Construction of crisp soft rough sets

In this section, we will introduce the concept of crisp soft rough sets by combining the crisp soft relation from U to E with crisp rough sets.

Definition 3.1 ([6]) *Let U be an initial universe set and E be a universe set of parameters. A pair (F, E) is called a soft set over U if $F : E \to P(U)$, where $P(U)$ is the set of all subsets of U.*

Definition 3.2 ([11]) *Let U be an initial universe set and E be a universe set of parameters. A pair (F, E) is called a fuzzy soft set over U if $F : E \to F(U)$, where $F(U)$ is the set of all fuzzy subsets of U.*

By using the concepts of the above soft set and fuzzy soft set, Cagman et al. [40, 41] introduce the definitions of crisp soft relation and fuzzy soft relation, respectively.

Definition 3.3 ([40]) *Let (F, E) be a soft set over U. Then a subset of $U \times E$ called a crisp soft relation from U to E is uniquely defined by*

$R = \{< (u, x), \mu_R(u, x) > | (u, x) \in U \times E\}$,

where $\mu_R : U \times E \to \{0, 1\}$, $\mu_R(u, x) = \begin{cases} 1, & (u, x) \in R \\ 0, & (u, x) \notin R. \end{cases}$

Definition 3.4 ([41]) *Let (F, E) be a fuzzy soft set over U. Then a fuzzy subset of $U \times E$ called a fuzzy soft relation from U to E is uniquely defined by*

$R = \{< (u, x), \mu_R(u, x) > | (u, x) \in U \times E\}$,

where $\mu_R : U \times E \to [0, 1]$, $\mu_R(u, x) = \mu_{F(x)}(u)$.

Based the crisp soft relation proposed by Cagman, we can construct the following crisp soft rough sets.

Definition 3.5 *Let U be an initial universe set and E be a universe set of parameters. For an arbitrary crisp soft relation R over $U \times E$, we can define a set-valued function $R_s : U \to P(E)$ by $R_s(u) = \{x \in E | (u, x) \in R\}, u \in U$.*

R is referred to as serial if for all $u \in U, R_s(u) \neq \emptyset$. The pair (U, E, R) is called a crisp soft approximation space. For any $A \subseteq E$, the upper and lower soft approximations of A with respect to (U, E, R), denoted by $\overline{R}(A)$ and $\underline{R}(A)$, are defined, respectively, as follows:

$$\overline{R}(A) = \{u \in U | R_s(u) \cap A \neq \emptyset\}, \tag{1}$$

$$\underline{R}(A) = \{u \in U | R_s(u) \subseteq A\}. \tag{2}$$

The pair $(\overline{R}(A), \underline{R}(A))$ is referred to as a crisp soft rough set, and $\overline{R}, \underline{R} : P(E) \to P(U)$ are, referred to as upper and lower crisp soft rough approximation operators, respectively.

Example 3.6 *Let U be a universal set, which is denoted by $U = \{u_1, u_2, u_3, u_4, u_5\}$. Let E be a set of parameters, where $E = \{e_1, e_2, e_3, e_4\}$. Suppose that a soft set over U is defined as follows:*

$F(e_1) = \{u_1, u_3, u_4\}$, $F(e_2) = \{u_2, u_4\}$, $F(e_3) = \emptyset$, $F(e_4) = U$.

Then the crisp soft relation on $U \times E$ is written by

$R = \{(u_1, e_1), (u_3, e_1), (u_4, e_1), (u_2, e_2), (u_4, e_2), (u_1, e_4), (u_2, e_4), (u_3, e_4), (u_4, e_4), (u_5, e_4)\}$.

If the set of parameter $A = \{e_2, e_3, e_4\}$, by Equations (1) and (2), we have $\underline{R}(A) = \{u_2, u_5\}$, and $\overline{R}(A) = U$.

Theorem 3.7 *Let (U, E, R) be a crisp soft approximation space. Then upper and lower crisp soft approximation operators $\overline{R}(A)$ and $\underline{R}(A)$ in Definition 3.5 satisfy the following properties: for all $A, B \in P(E)$*

(CSL1) $\underline{R}(A) = \sim \overline{R}(\sim A)$, *(CSU1)* $\overline{R}(A) = \sim \underline{R}(\sim A)$;

(CSL2) $\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$, *(CSU2)* $\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$;

(CSL3) $A \subseteq B \Rightarrow \underline{R}(A) \subseteq \underline{R}(B)$, *(CSU3)* $A \subseteq B \Rightarrow \overline{R}(A) \subseteq \overline{R}(B)$;

(CSL4) $\underline{R}(A \cup B) \supseteq \underline{R}(A) \cup \underline{R}(B)$, *(CSU4)* $\overline{R}(A \cap B) = \overline{R}(A) \cap \overline{R}(B)$.

Proof. The proof can be directly followed from Definition 3.5. □

From Definition 3.5, the following theorem can be easily derived.

Theorem 3.8 *Let (U, E, R) be a crisp soft approximation space, and $\overline{R}(A)$ and $\underline{R}(A)$ be the upper and lower soft approximations operators in Definition 3.5. Then*

R is serial $\Leftrightarrow \underline{R}(A) \subseteq \overline{R}(A), \forall A \in E \Leftrightarrow \underline{R}(\emptyset) = \emptyset \Leftrightarrow \overline{R}(E) = U$.

4 Construction of generalized intuitionistic fuzzy soft rough sets

In this section, inspired by the constructive method regard to generalized intuitionistic fuzzy rough sets in [35], we will present the concept of generalized intuitionistic fuzzy soft rough sets by combining the intuitionistic fuzzy soft relation from U to E with generalized intuitionistic fuzzy rough sets.

Definition 4.1 ([42]) *Let U be an initial universe set and E be a universe set of parameters. A pair (F, E) is called an intuitionistic fuzzy soft set over U if $F : E \to IF(U)$, where $IF(U)$ is the set of all intuitionistic fuzzy subsets of U. That is, for $\forall x \in E, F(x) = \{< u, \mu_{F(x)}(u), \gamma_{F(x)}(u) > |u \in U\} \in IF(U)$, where $\mu_{F(x)}(u) \in [0,1]$ and $\gamma_{F(x)}(u) \in [0,1]$ denote membership and non-membership degrees of an element u regard to intuitionistic fuzzy set $F(x)$ respectively, which satisfy the condition $\mu_{F(x)}(u) + \gamma_{F(x)}(u) \leq 1$.*

In the following, an intuitionistic fuzzy soft relation will be presented by us which is important for us to construct generalized intuitionistic fuzzy soft rough sets.

Definition 4.2 *Let (F, E) be an intuitionistic fuzzy soft set over U. Then an intuitionistic fuzzy subset of $U \times E$ called an intuitionistic fuzzy soft relation from U to E is uniquely defined by*

$R = \{< (u, x), \mu_R(u, x), \gamma_R(u, x) > |(u, x) \in U \times E\}$,

where $\mu_R : U \times E \to [0,1]$ and $\gamma_R : U \times E \to [0,1]$ satisfy the condition $0 \leq \mu_R(u, x) + \gamma_R(u, x) \leq 1$ for all $(u, x) \in U \times E$.

If $U = \{u_1, u_2, \cdots, u_m\}, E = \{x_1, x_2, \cdots, x_n\}$ then the intuitionistic fuzzy soft relation R from U to E can be presented by a table as in the following form

R	x_1	x_2	$\cdots$	x_n
u_1	$(\mu_R(u_1, x_1), \gamma_R(u_1, x_1))$	$(\mu_R(u_1, x_2), \gamma_R(u_1, x_2))$	$\cdots$	$(\mu_R(u_1, x_n), \gamma_R(u_1, x_n))$
u_2	$(\mu_R(u_2, x_1), \gamma_R(u_2, x_1))$	$(\mu_R(u_2, x_2), \gamma_R(u_2, x_2))$	$\cdots$	$(\mu_R(u_2, x_n), \gamma_R(u_2, x_n))$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
u_m	$(\mu_R(u_m, x_1), \gamma_R(u_m, x_1))$	$(\mu_R(u_m, x_2), \gamma_R(u_m, x_2))$	$\cdots$	$(\mu_R(u_m, x_n), \gamma_R(u_m, x_n))$

From the above form and according to the definition of IF soft relation, we could find that every IF soft set (F, E) is uniquely characterized by the IF soft relation, namely they are mutual determined. It means that an IF soft set (F, E) is formally equal to its IF soft relation. Therefore, we shall identify any IF soft set with its IF soft relation and view these two concepts as interchangeable. Now, any discussion regard to IF soft set could be converted into analysis about IF soft relation, which will bring great convenience for our future researches.

In this case, according to the definition IF soft relation, we can construct generalized intuitionistic fuzzy soft rough sets.

Definition 4.3 *Let U be an initial universe set and E be a universe set of parameters. For an arbitrary intuitionistic fuzzy soft relation R over $U \times E$, the pair (U, E, R) is called an intuitionistic fuzzy soft approximation space. For any $A \in IF(E)$, we define the upper and lower soft approximations of A with respect to (U, E, R), denoted by $\overline{R}(A)$ and $\underline{R}(A)$,*

respectively, as follows:

$$\overline{R}(A) = \{< u, \mu_{\overline{R}(A)}(u), \gamma_{\overline{R}(A)}(u) > | u \in U\}, \tag{3}$$

$$\underline{R}(A) = \{< u, \mu_{\underline{R}(A)}(u), \gamma_{\underline{R}(A)}(u) > | u \in U\}. \tag{4}$$

where

$\mu_{\overline{R}(A)}(u) = \bigvee_{x\in E}[\mu_R(u,x) \wedge \mu_A(x)], \quad \gamma_{\overline{R}(A)}(u) = \bigwedge_{x\in E}[\gamma_R(u,x)) \vee \gamma_A(x)];$

$\mu_{\underline{R}(A)}(u) = \bigwedge_{x\in E}[\gamma_R(u,x) \vee \mu_A(x)], \quad \gamma_{\underline{R}(A)}(u) = \bigvee_{x\in E}[\mu_R(u,x) \wedge \gamma_A(x)].$

The pair $(\overline{R}(A), \underline{R}(A))$ is referred to as a generalized IF soft rough set of A with respect to (U, E, R).

By $\mu_R(u,x) + \gamma_R(u,x) \leq 1$ and $\mu_A(x) + \gamma_A(x) \leq 1$, it can be easily verified that $\overline{R}(A)$ and $\underline{R}(A) \in IF(U)$. In fact,

$$\begin{aligned}
\mu_{\overline{R}(A)}(u) + \gamma_{\overline{R}(A)}(u) &= \bigvee_{x\in E}[\mu_R(u,x) \wedge \mu_A(x)] + \bigwedge_{x\in E}[(\gamma_R(u,x)) \vee \gamma_A(x)] \\
&\leq \bigvee_{x\in E}[(1-\gamma_R(u,x)) \wedge (1-\gamma_A(x))] + \bigwedge_{x\in E}[(\gamma_R(u,x)) \vee \gamma_A(x)] \\
&= 1 - \bigwedge_{x\in E}[(\gamma_R(u,x)) \vee \gamma_A(x)] + \bigwedge_{x\in E}[(\gamma_R(u,x)) \vee \gamma_A(x)] \\
&= 1.
\end{aligned}$$

Hence, $\overline{R}(A) \in IF(U)$. Similarly, we can obtain $\underline{R}(A) \in IF(U)$. So we call $\overline{R}, \underline{R} : IF(E) \to IF(U)$ generalized upper and lower IF soft rough approximation operators, respectively.

Remark 4.4 *If $\gamma_R(u,x) = 1-\mu_R(u,x)$ in Definition 4.2, then R is a fuzzy soft relation on $U \times E$ (see Definition 3.4), that is, $R = \{< (u,x), \mu_R(u,x), 1-\mu_R(u,x) > | (u,x) \in U \times E\}$. In this case, we call (U, E, R) a fuzzy soft approximation space. Let (U, E, R) be the fuzzy soft approximation space and $A \in IF(E)$, then generalized IF soft rough approximation operators $\overline{R}(A)$ and $\underline{R}(A)$ in Definition 4.3 degenerate to the following forms:*

$$\overline{R}(A) = \{< u, \mu_{\overline{R}(A)}(u), \gamma_{\overline{R}(A)}(u) > | u \in U\},$$

$$\underline{R}(A) = \{< u, \mu_{\underline{R}(A)}(u), \gamma_{\underline{R}(A)}(u) > | u \in U\}.$$

where

$\mu_{\overline{R}(A)}(u) = \bigvee_{x\in E}[\mu_R(u,x) \wedge \mu_A(x)], \quad \gamma_{\overline{R}(A)}(u) = \bigwedge_{x\in E}[(1-\mu_R(u,x)) \vee \gamma_A(x)];$

$\mu_{\underline{R}(A)}(u) = \bigwedge_{x\in E}[(1-\mu_R(u,x)) \vee \mu_A(x)], \quad \gamma_{\underline{R}(A)}(u) = \bigvee_{x\in E}[\mu_R(u,x) \wedge \gamma_A(x)].$

In this case, the pair $(\overline{R}(A), \underline{R}(A))$ is referred to as an IF soft rough set of A with respect to (U, E, R). That is, generalized IF soft rough set in Definition 4.3 has included IF soft rough set.

Remark 4.5 *Let (U, E, R) be a fuzzy soft approximation space. If $A \in F(E)$, then generalized IF soft rough approximation operators $\overline{R}(A)$ and $\underline{R}(A)$ degenerate to the following forms:*

$$\overline{R}(A) = \{< u, \mu_{\overline{R}(A)}(u) > | u \in U\}, \quad \underline{R}(A) = \{< u, \mu_{\underline{R}(A)}(u) > | u \in U\}.$$

where $\mu_{\overline{R}(A)}(u) = \bigvee_{x \in E} [\mu_R(u, x) \wedge \mu_A(x)]$, $\mu_{\underline{R}(A)}(u) = \bigwedge_{x \in E} [(1 - \mu_R(u, x)) \vee \mu_A(x)$.

In this case, generalized IF soft rough approximation operators $\overline{R}(A)$ and $\underline{R}(A)$ are identical with the soft fuzzy rough approximation operators defined by Sun [32]. That is, generalized IF soft rough approximation operators in Definition 4.3 are an extension of the soft fuzzy rough approximation operators defined by Sun [32].

In order to better understand the concept of generalized IF soft rough approximation operators, let us consider the following example.

Example 4.6 *Suppose that $U = \{u_1, u_2, u_3, u_4, u_5\}$ is the set of five houses under consideration of a decision maker to purchase. Let E be a parameter set, where $E = \{e_1, e_2, e_3, e_4\}$=\{expensive; beautiful; size; location\}. Mr. X wants to buy the house which qualifies with the parameters of E to the utmost extent from available houses in U. Assume that Mr. X describes the "attractiveness of the houses" by constructing an IF soft relation R from U to E. And it is presented by a table as in the following form.*

R	e_1	e_2	e_3	e_4
u_1	$(0.7, 0.2)$	$(0.6, 0.3)$	$(0.1, 0.9)$	$(0.1, 0.6)$
u_2	$(0.2, 0.6)$	$(0.6, 0.4)$	$(0.5, 0.4)$	$(0.7, 0.3)$
u_3	$(0.4, 0.6)$	$(0.8, 0.1)$	$(0.2, 0.7)$	$(0.6, 0.3)$
u_4	$(0.2, 0.5)$	$(0.3, 0.6)$	$(0.8, 0.1)$	$(0.1, 0.7)$
u_5	$(0.6, 0.4)$	$(0.5, 0.2)$	$(0.3, 0.5)$	$(0.2, 0.5)$

As a generalization of Zadeh's fuzzy set, intuitionistic fuzzy set is characterized by a membership function and a nonmembership function, and thus can depict the fuzzy character of data more detailedly and comprehensively than Zadehs fuzzy set which is only characterized by a membership function. Therefore, the characteristics of the five houses with respect to the four parameters can be represented by the intuitionistic fuzzy sets. For example, the characteristics of the house u_1 under the parameter e_1 is (0.7,0.2). The values of 0.7 and 0.2 are the degrees of membership and non-membership of the house u_1

with respect to the parameter e_1 *, respectively. In other words, house* u_1 *is expensive on the membership degree of 0.7 and it is not expensive on the non-membership degree of 0.2.*

Now suppose that Mr X gives the optimum normal decision object A which an IF subset over the parameter set E as follows:

$A = \{< e_1, 0.7, 0.1 >, < e_2, 0.3, 0.6 >, < e_3, 0.8, 0.2 >, < e_4, 0.3, 0.5 >\}$

By Equations (3) and (4), we have

$\mu_{\overline{R}(A)}(u_1) = 0.7,\ \gamma_{\overline{R}(A)}(u_1) = 0.2,\ \mu_{\overline{R}(A)}(u_2) = 0.5,\ \gamma_{\overline{R}(A)}(u_2) = 0.4,$
$\mu_{\overline{R}(A)}(u_3) = 0.4,\ \gamma_{\overline{R}(A)}(u_3) = 0.5,\ \mu_{\overline{R}(A)}(u_4) = 0.8,\ \gamma_{\overline{R}(A)}(u_4) = 0.2,$
$\mu_{\overline{R}(A)}(u_5) = 0.6,\ \gamma_{\overline{R}(A)}(u_5) = 0.4;\ \mu_{\underline{R}(A)}(u_1) = 0.3,\ \gamma_{\underline{R}(A)}(u_1) = 0.6,$
$\mu_{\underline{R}(A)}(u_2) = 0.3,\ \gamma_{\underline{R}(A)}(u_2) = 0.6,\ \mu_{\underline{R}(A)}(u_3) = 0.3,\ \gamma_{\underline{R}(A)}(u_3) = 0.6,$
$\mu_{\underline{R}(A)}(u_4) = 0.3,\ \gamma_{\underline{R}(A)}(u_4) = 0.3,\ \mu_{\underline{R}(A)}(u_5) = 0.3,\ \gamma_{\underline{R}(A)}(u_5) = 0.5.$

Thus

$$\begin{aligned}\overline{R}(A) = \{ &< u_1, 0.7, 0.2 >, < u_2, 0.5, 0.4 >, < u_3, 0.4, 0.5 >,\\ &< u_4, 0.8, 0.2 >, < u_5, 0.6, 0.4 >\}\end{aligned}$$

and

$$\begin{aligned}\underline{R}(A) = \{ &< u_1, 0.3, 0.6 >, < u_2, 0.3, 0.6 >, < u_3, 0.3, 0.6 >,\\ &< u_4, 0.3, 0.3 >, < u_5, 0.3, 0.5 >\}.\end{aligned}$$

Theorem 4.7 *Let* (U, E, R) *be an intuitionistic fuzzy soft approximation space. Then the generalized upper and lower IF soft rough approximation operators* $\overline{R}(A)$ *and* $\underline{R}(A)$ *in Definition 4.3 satisfy the following properties:* $\forall A, B \in IF(E), \forall(\alpha, \beta) \in L^*$ *,*

(GIFSL1) $\underline{R}(A) = \sim \overline{R}(\sim A)$, *(GIFSU1)* $\overline{R}(A) = \sim \underline{R}(\sim A)$;
(GIFSL2) $\underline{R}(A \cup \widehat{(\alpha, \beta)}) = \underline{R}(A) \cup \widehat{(\alpha, \beta)}$, *(GIFSU2)* $\overline{R}(A \cap \widehat{(\alpha, \beta)}) = \overline{R}(A) \cap \widehat{(\alpha, \beta)}$;
(GIFSL3) $\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$, *(GIFSU3)* $\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$;
(GIFSL4) $A \subseteq B \Rightarrow \underline{R}(A) \subseteq \underline{R}(B)$, *(GIFSU4)* $A \subseteq B \Rightarrow \overline{R}(A) \subseteq \overline{R}(B)$;
(GIFSL5) $\underline{R}(A \cup B) \supseteq \underline{R}(A) \cup \underline{R}(B)$, *(GIFSU5)* $\overline{R}(A \cap B) = \overline{R}(A) \cap \overline{R}(B)$;

Proof. It can be easily followed from Definition 4.3. □

In Theorem 4.7, properties (GIFSL1) and (GIFSU1) show that the generalized upper lower IF soft rough approximation operators $\overline{R}$ and $\underline{R}$ are dual to each other.

Assume that R is an intuitionistic fuzzy soft relation from U to E, denote

$R_\alpha = \{(u, x) \in U \times E | \mu_R(u, x) \geq \alpha\}$, $R_\alpha(u) = \{x \in E | \mu_R(u, x) \geq \alpha\}$, $\alpha \in [0, 1]$,
$R_{\alpha+} = \{(u, x) \in U \times E | \mu_R(u, x) > \alpha\}$, $R_{\alpha+}(u) = \{x \in E | \mu_R(u, x) > \alpha\}, \alpha \in [0, 1)$,
$R^\alpha = \{(u, x) \in U \times E | \gamma_R(u, x) \leq \alpha\}$, $R^\alpha(u) = \{x \in E | \gamma_R(u, x) \leq \alpha\}, \alpha \in [0, 1]$,
$R^{\alpha+} = \{(u, x) \in U \times E | \gamma_R(u, x) < \alpha\}$, $R^{\alpha+}(u) = \{x \in E | \gamma_R(u, x) < \alpha\}, \alpha \in (0, 1]$.

Then R_α, $R_{\alpha+}$, R^α, and $R^{\alpha+}$ are crisp soft relations on $U \times E$.

The following Theorems 4.8 and 4.9 show that the generalized IF soft rough approximation operators can be represented by crisp soft rough approximation operators.

Theorem 4.8 *Let (U, E, R) be an intuitionistic fuzzy soft approximation space, and $A \in IF(E)$. Then the generalized upper IF soft rough approximation operator can be represented as follows: $\forall u \in U$*

(1)

$$\mu_{\overline{R}(A)}(u) = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R_\alpha}(A_\alpha)(u)] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R_\alpha}(A_{\alpha+})(u)]$$
$$= \bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R_{\alpha+}}(A_\alpha)(u)] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R_{\alpha+}}(A_{\alpha+})(u)],$$

(2)

$$\gamma_{\overline{R}(A)}(u) = \bigwedge_{\alpha\in[0,1]} [\alpha \vee \overline{R^\alpha}(A^\alpha)(u)] = \bigwedge_{\alpha\in[0,1]} [\alpha \vee \overline{R^\alpha}(A^{\alpha+})(u)]$$
$$= \bigwedge_{\alpha\in[0,1]} [\alpha \vee \overline{R^{\alpha+}}(A^\alpha)(u)] = \bigwedge_{\alpha\in[0,1]} [\alpha \vee \overline{R^{\alpha+}}(A^{\alpha+})(u)]$$

and moreover, for any $\alpha \in [0,1]$,

(3) $[\overline{R}(A)]_{\alpha+} \subseteq \overline{R_{\alpha+}}(A_{\alpha+}) \subseteq \overline{R_{\alpha+}}(A_\alpha) \subseteq \overline{R_\alpha}(A_\alpha) \subseteq [\overline{R}(A)]_\alpha$,

(4) $[\overline{R}(A)]^{\alpha+} \subseteq \overline{R^{\alpha+}}(A^{\alpha+}) \subseteq \overline{R^{\alpha+}}(A^\alpha) \subseteq \overline{R^\alpha}(A^\alpha) \subseteq [\overline{R}(A)]^\alpha$.

Proof. (1) For any $u \in U$, we have that

$$\begin{aligned}\bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R_\alpha}(A_\alpha)(u)] &= sup\{\alpha \in [0,1] | u \in \overline{R_\alpha}(A_\alpha)\} \\ &= sup\{\alpha \in [0,1] | R_\alpha(u) \cap A_\alpha \neq \emptyset\} \\ &= sup\{\alpha \in [0,1] | \exists x \in E[x \in R_\alpha(u), x \in A_\alpha]\} \\ &= sup\{\alpha \in [0,1] | \exists x \in E[\mu_R(u,x) \geq \alpha, \mu_A(x) \geq \alpha]\} \\ &= \bigvee_{x\in E} [\mu_R(u,x) \wedge \mu_A(x)] = \mu_{\overline{R}(A)}(u)\end{aligned}$$

Similarly, we can prove

$$\mu_{\overline{R}(A)}(u) = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R_\alpha}(A_{\alpha+})(u)] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R_{\alpha+}}(A_\alpha)(u)] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R_{\alpha+}}(A_{\alpha+})(u)].$$

(2) By the definition of upper crisp soft rough approximation operator in Definition 3.5, we have that

$$\begin{aligned}\bigwedge_{\alpha\in[0,1]}[\alpha\vee\overline{R^{\alpha}}(A^{\alpha})(u))] &= inf\{\alpha\in[0,1]|u\in\overline{R^{\alpha}}(A^{\alpha})\}\\ &= inf\{\alpha\in[0,1]|R^{\alpha}(u)\cap A^{\alpha}\neq\emptyset\}\\ &= inf\{\alpha\in[0,1]|\exists x\in E[x\in R^{\alpha}(u),x\in A^{\alpha}]\}\\ &= inf\{\alpha\in[0,1]|\exists x\in E[\gamma_R(u,x)\leq\alpha,\gamma_A(x)\leq\alpha]\}\\ &= \bigwedge_{x\in E}[\gamma_R(u,x)\vee\gamma_A(x)]=\gamma_{\overline{R}(A)}(u).\end{aligned}$$

Likewise, we can prove that

$$\begin{aligned}\gamma_{\overline{R}(A)}(u) &= \bigwedge_{\alpha\in[0,1]}[\alpha\vee\overline{R^{\alpha}}(A^{\alpha+})(u)]=\bigwedge_{\alpha\in[0,1]}[\alpha\vee\overline{R^{\alpha+}}(A^{\alpha})(u)]\\ &= \bigwedge_{\alpha\in[0,1]}[\alpha\vee\overline{R^{\alpha+}}(A^{\alpha+})(u)].\end{aligned}$$

(3) It is easily verified that $\overline{R_{\alpha+}}(A_{\alpha+})\subseteq\overline{R_{\alpha+}}(A_{\alpha})\subseteq\overline{R_{\alpha}}(A_{\alpha})$. We only need to prove that $[\overline{R}(A)]_{\alpha+}\subseteq\overline{R_{\alpha+}}(A_{\alpha+})$ and $\overline{R_{\alpha}}(A_{\alpha})\subseteq[\overline{R}(A)]_{\alpha}$.

In fact, $\forall u\in[\overline{R}(A)]_{\alpha+}$, we have $\mu_{\overline{R}(A)}(u)>\alpha$. According to Definition 4.3, $\bigvee_{x\in E}[\mu_R(u,x)\wedge\mu_A(x)]>\alpha$ holds. Then $\exists x_0\in E$, such that $\mu_R(u,x_0)\wedge\mu_A(x_0)>\alpha$, that is, $\mu_R(u,x_0)>\alpha$ and $\mu_A(x_0)>\alpha$. Thus $x_0\in R_{\alpha+}(u)$ and $x_0\in A_{\alpha+}$. Consequently, $R_{\alpha+}(u)\cap A_{\alpha+}\neq\emptyset$. By Definition 3.5, we have $u\in\overline{R_{\alpha+}}(A_{\alpha+})$. Hence $[\overline{R}(A)]_{\alpha+}\subseteq\overline{R_{\alpha+}}(A_{\alpha+})$.

On the other hand, for any $u\in\overline{R_{\alpha}}(A_{\alpha})$, we have $\overline{R_{\alpha}}(A_{\alpha})(u)=1$. Since $\mu_{\overline{R}(A)}(u)=\bigvee_{\beta\in[0,1]}[\beta\wedge\overline{R_{\beta}}(A_{\beta})(u)]\geq\alpha\wedge\overline{R_{\alpha}}(A_{\alpha})(u)=\alpha$, we obtain $u\in[\overline{R}(A)]_{\alpha}$. Hence, $\overline{R_{\alpha}}(A_{\alpha})\subseteq[\overline{R}(A)]_{\alpha}$.

(4) Similar to the proof of (3), it can be easily verified. □

Theorem 4.9 *Let (U,E,R) be an intuitionistic fuzzy soft approximation space, and $A\in IF(E)$. Then the generalized lower IF soft rough approximation operator can be represented as follows: $\forall u\in U$*

(1)

$$\begin{aligned}\mu_{\underline{R}(A)}(u) &= \bigwedge_{\alpha\in[0,1]}[\alpha\vee(1-\underline{R^{\alpha}}(A_{\alpha+})(u))]=\bigwedge_{\alpha\in[0,1]}[\alpha\vee(1-\underline{R^{\alpha}}(A_{\alpha})(u))]\\ &= \bigwedge_{\alpha\in[0,1]}[\alpha\vee(1-\underline{R^{\alpha+}}(A_{\alpha+})(u))]=\bigwedge_{\alpha\in[0,1]}[\alpha\vee(1-\underline{R^{\alpha+}}(A_{\alpha})(u))],\end{aligned}$$

(2)

$$\gamma_{\underline{R}(A)}(u) = \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - \underline{R_\alpha}(A^{\alpha+})(u))] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - \underline{R_\alpha}(A^{\alpha})(u))]$$
$$= \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - \underline{R_{\alpha+}}(A^{\alpha+})(u))] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - \underline{R_{\alpha+}}(A^{\alpha})(u))]$$

and moreover, for any $\alpha \in [0,1]$,
(3) $[\underline{R}(A)]_{\alpha+} \subseteq \underline{R^\alpha}(A_{\alpha+}) \subseteq \underline{R^{\alpha+}}(A_{\alpha+}) \subseteq \underline{R^{\alpha+}}(A_\alpha) \subseteq [\underline{R}(A)]_\alpha$,
(4) $[\underline{R}(A)]^{\alpha+} \subseteq \underline{R_\alpha}(A^{\alpha+}) \subseteq \underline{R_{\alpha+}}(A^{\alpha+}) \subseteq \underline{R_{\alpha+}}(A^\alpha) \subseteq [\underline{R}(A)]^\alpha$.

Proof. (1) and (2) According to Theorem 4.8 and 2.4, for all $u \in U$ we have

$$\mu_{\overline{R}(\sim A)}(u) = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R_\alpha}(\sim A)_\alpha(u)] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R_\alpha}(\sim A^{\alpha+})(u)]$$
$$= \bigvee_{\alpha\in[0,1]} [\alpha \wedge (\sim \underline{R_\alpha}(A^{\alpha+}))(u)] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - \underline{R_\alpha}(A^{\alpha+})(u))]$$

and

$$\gamma_{\overline{R}(\sim A)}(u) = \bigwedge_{\alpha\in[0,1]} [\alpha \vee \overline{R^\alpha}(\sim A)^\alpha(u)] = \bigwedge_{\alpha\in[0,1]} [\alpha \vee \overline{R^\alpha}(\sim A_{\alpha+})(u)]$$
$$= \bigwedge_{\alpha\in[0,1]} [\alpha \vee (\sim \underline{R^\alpha}(A_{\alpha+}))(u)] = \bigwedge_{\alpha\in[0,1]} [\alpha \vee (1 - \underline{R^\alpha}(A_{\alpha+})(u))]$$

Hence, by the duality of generalized upper and lower IF soft rough approximation operators (see Theorem 4.7), we can conclude
$\mu_{\underline{R}(A)}(u) = \gamma_{\overline{R}(\sim A)}(u) = \bigwedge_{\alpha\in[0,1]} [\alpha \vee (1 - \underline{R^\alpha}(A_{\alpha+})(u))]$,
$\gamma_{\underline{R}(A)}(u) = \mu_{\overline{R}(\sim A)}(u) = \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - \underline{R_\alpha}(A^{\alpha+})(u))]$.
Similar to the above proof, we can obtain

$$\mu_{\underline{R}(A)}(u) = \bigwedge_{\alpha\in[0,1]} [\alpha \vee (1 - \underline{R^\alpha}(A_\alpha)(u))] = \bigwedge_{\alpha\in[0,1]} [\alpha \vee (1 - \underline{R^{\alpha+}}(A_{\alpha+})(u))]$$
$$= \bigwedge_{\alpha\in[0,1]} [\alpha \vee (1 - \underline{R^{\alpha+}}(A_\alpha)(u))],$$

$$\gamma_{\underline{R}(A)}(u) = \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - \underline{R_\alpha}(A^\alpha)(u))] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - \underline{R_{\alpha+}}(A^{\alpha+})(u))]$$
$$= \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - \underline{R_{\alpha+}}(A^\alpha)(u))].$$

(3) and (4) It is similar to the proof of Theorem 4.8(3). □

5 Application of IF soft rough sets in decision making

In this section, we shall develop an approach to generalized IF soft rough sets based decision making. In the following, we will define the ring sum operation of IF sets. By the operation, an approach to generalized intuitionistic fuzzy soft rough sets based decision making will be presented.

Definition 5.1 *Let $F, G \in IF(U)$. The ring sum operation about IF sets F and G can be defined as follows:*
$F \oplus G = \{< u, \mu_F(u) + \mu_G(u) - \mu_F(u) \cdot \mu_G(u), \gamma_F(u) \cdot \gamma_G(u) > | u \in U\}$.

Let (U, E, R) be an intuitionistic fuzzy soft approximation space, where U is the universe of the discourse, E is the parameter set, and R is an intuitionistic fuzzy soft relation on $U \times E$. Then we can give an algorithm based on generalized IF soft rough sets as follows:

1. Input the intuitionistic fuzzy soft relation R from U to E, or the intuitionistic fuzzy soft set (F, E) over U.

2. Give the optimum normal decision object A which is an IF set over E, according to different needs to decision maker.

3. Compute the generalized IF soft rough approximation operators $\overline{R}(A)$ and $\underline{R}(A)$ by Equations (3) and (4).

4. Compute the choice set

$$H = \overline{R}(A) \oplus \underline{R}(A) = \{< u, \mu_{\overline{R}(A)}(u) + \mu_{\underline{R}(A)}(u) - \mu_{\overline{R}(A)}(u) \cdot \mu_{\underline{R}(A)}(u), \gamma_{\overline{R}(A)}(u) \cdot \gamma_{\underline{R}(A)}(u) > | u \in U\}.$$

5. Choose the top-level threshold value $\lambda = (\mu, \gamma) \in L^*$, where $\mu = \max_{1 \le i \le n} \mu_H(u_i), \gamma = \min_{1 \le i \le n} \gamma_H(u_i)$.

6. The decision is u, if IF set $H(u) \ge_{L^*} \lambda$, that is, $\mu_H(u) \ge \mu$ and $\lambda_H(u) \le \gamma$.

In the last step of the above algorithm, one may go back to the second step and change decision object so that the final decision is only one, when there exist too many "optimal choices" to be chosen.

To illustrate the idea of algorithm given above, let us consider the example as follows.

Example 5.2 *Reconsider Example 4.6. In Example 4.6, we have computed generalized IF soft rough approximation operators $\overline{R}(A)$ and $\underline{R}(A)$ of the optimum normal decision object A. Now by using the fourth step of algorithm for generalized IF soft rough sets in decision making presented in this section, we can obtain*

$$H = \overline{R}(A) \oplus \underline{R}(A) = \{ < u_1, 0.79, 0.12 >, < u_2, 0.65, 0.24 >, < u_3, 0.58, 0.30 >, < u_4, 0.86, 0.06 >, < u_5, 0.72, 0.20 > \}.$$

Obviously, the optimal decision is u_4. *Hence, Mr X will buy the house* u_4.

6 Conclusion

Intuitionistic fuzzy set theory, soft set theory and rough set theory are all mathematical tools for dealing with uncertainties. This paper is devoted to the discussion of the combinations of intuitionistic fuzzy set, rough set and soft set. Based on the models presented in [35], by combining soft set theory with generalized IF rough set theory, a new soft rough set model called generalized IF soft rough set is proposed and its properties are derived. Furthermore, the relationships between generalized intuitionistic fuzzy soft rough sets with the existing generalized soft rough sets are established. Finally, a practical application based on generalized intuitionistic fuzzy soft rough sets is applied to show the validity.

Actually, there are at least two aspects in the study of rough set theory: constructive and axiomatic approaches. In further research, the axiomatization of generalized intuitionistic fuzzy soft rough approximation operators is an important and interesting issue to be addressed.

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Global Exponential Dissipativity of Static Neural Networks with Time Delay and Impulses

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Abstract

In this paper, we investigate the problem of global exponential dissipativity of static neural networks with time delay and impulses. The impulses are classified into two classes: the ones are input disturbances and the ones are stabilizing. For each type of impulses, by adopting proper Lyapunov function, sufficient conditions for global exponential dissipativity are established in terms of linear matrix inequalities (LMIs). The new sufficient conditions can explicitly reveal the influence of time delay, impulses, etc., on the dissipativity. We show that these conditions can be straightforwardly reduced to exponential stability conditions and that these stability conditions are remarkably less conservative than the existing ones. Numerical results are given to show the less conservatism of the obtained criteria compared to the existing ones.

Keywords: dissipativity, stability, static neural network, impulse, time delay, LMI.

1. Introduction.

In the past few years, neural networks (NNs) have been extensively studied duo to their applications in many areas, such as signal processing, associative memory, pattern recognition, combination optimization and so on. As reported in [1] and [2], NNs can be classified as local field neural networks and static neural networks. For both types of NNs, time delay and impulses occur unavoidably during implementation of the corresponding artificial circuits. Time delay occurs due to the finite switching speeds of the amplifiers and impulses arise from the abrupt changes in the voltages (which can affect the dynamical behaviors of the system) produced by faulty circuit elements. Research of the local field NNs with both time delay and impulses has received lots of attention in recent years, and several important and interesting sufficient conditions ensuring the existence and global exponential stability of a unique equilibrium solution are given; see, e.g., [3–7] and references therein.

Nevertheless, there is only a few theoretical results for the impulsive static NNs with time delay. As reviewed in [8–11], many neural networks exhibiting short-term memory are modeled by non-invertible networks (such as the oculomotor integrator or the head-direction system [12]) and this implies that the local field NNs and the static ones are not always equivalent. Therefore, it is necessary to pay special attention to static NNs with both time delay and impulses. Zhao

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and Wang [13] presented some stability results of the static NNs with time delay and impulses. However, the criteria presented in [13] impose strict conditions on the model parameters and the impulsive effects (for example, the impulses are assumed to be stabilizing), and thus these criteria are less applicable. Recently, the authors in [14] further investigated the problem of global exponential stability of the static NNs with time delay and impulses. The results presented in [14] were shown less conservative and more applicative than the ones given by Zhao and Wang [13].

However, all of the aforementioned results concentrate on stability of the studied NNs, that is obtaining sufficient conditions under which the trajectories starting from any initial value tend to equilibrium points. Moreover, due to the mathematical difficulties in dealing with impulses, most of these results are based on a rigorous assumption— the impulsive operator, namely $\mathcal{I}$, satisfies $\mathcal{I}(x^*) = 0$, where x^* is the equilibrium point of the interested neural networks. But, from a practical point of view, it is not always the case that the orbits of a NNs converge to equilibrium points and they can behave arbitrarily in a bounded set. Moreover, the assumption $\mathcal{I}(x^*) = 0$ is too strong, since it means that we need to know the value of the equilibrium points in advance. It is possible that there are no equilibrium points and/or $\mathcal{I}(x^*) \neq 0$ in some situations (in fact, $\mathcal{I}(x^*) \neq 0$ is more general than $\mathcal{I}(x^*) = 0$). Therefore, the concept *dissipativity* has been introduced and investigated; see, e.g., [15–22] and references therein. Dissipativity states that the trajectories of a dynamic system starting from any initial value go to a bounded set (and never go away this set) if the evolution time is sufficient long and in this set the trajectories can behave arbitrarily. Clearly, dissipativity generalizes the notion of stability. Nowadays, dissipativity has found applications in many areas, such as stability theory, chaos and synchronization theory, system norm estimation, and robust control(see [16, 19–22]).

In this paper, we consider the global exponential dissipativity of the static NNs with both time delay and impulses. According to our best knowledge, this problem has not been studied by other authors. Hence, it is our intention in this paper to tackle such an important yet challenging problem. We consider three types of impulses: the ones are input disturbances, the ones are stabilizing and the ones are "neural" type, which are neither helpful for stabilizing nor destabilizing the neural networks. Since the treatment of neutral type impulses is similar to the input disturbances ones, we divide the impulses into two classes— input disturbances impulses and stabilizing ones, and then we adopt the following guiding ideology to derive conditions of exponential dissipativity:

1. for disturbances impulses, we explore on what conditions the exponential damping rate of the used Lyapunov function overcomes the influence of impulsive disturbance;
2. for stabilizing impulses, we suppose the Lyapunov function can be growing with some exponential rate instead of assuming it must be damping (many authors require this in the stability analysis; see, e.g., [4–6, 23]), and then we explore under what conditions the effect of the stabilizing impulses can offset the growth of the Lyapunov function.

By using the above guiding ideology and some novel analysis techniques, sufficient conditions concerning the upper bound of the time delay, the magnitude of impulses, the distance between two consecutive impulsive instants (or say frequency of impulses) are derived to maintain the exponential dissipativity, when the *impulse-free* static NNs are dissipative but the impulses are input disturbances. When the impulse-free static NNs are not dissipative, sufficient conditions that utilize impulsive effects to stabilize the static NNs to be dissipative are also given.

We remark that the guiding ideology mentioned above is not our original invention, and it should be credited to Chen and Zheng [3]. In [3], this ideology is utilized to study the exponential

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stability of the local field NNs with impulses and time delay by using a very simple Lyapunov function—$V(t) = e^{2\gamma t}x^T(t)Px(t)$ ($\gamma > 0$ denotes the exponential convergence rate, P is a positive definite matrix), and it is shown that the obtained stability conditions are much less conservative than the existing ones. The simple form of the used Lyapunov function is very important to perform the stability analysis and for complex Lyapunov function, those analysis can not be straightforwardly generalized. On the other hand, it is a common sense that complex Lyapunov function may result in less conservative results. Therefore, it is a meaningful work to realize the excellent ideology by using more general Lyapunov functions. In our previous paper [14], we have used this ideology to study the exponential stability of the static NNs with impulses and time delay, but the utilized Lyapunov function is still the aforementioned simplest one.

In this paper, with a more general Lyapunov function, new analysis techniques are proposed to realize the ideology and benefiting from this Lyapunov function much less conservative conditions are derived. These conditions can be easily reduced to the exponential stability conditions and we show that the deduced stability conditions are also much less conservative than the existing ones. The reminder of this paper is organized as follows. In Section 2, we introduce the impulsive static NNs with time delay discussed in this paper. Some definitions and lemmas are also given in this section. In Section 3, sufficient conditions for exponential dissipativity are established in terms of LMIs. In Section 4, several numerical examples are given to show the usefulness of our results. The comparison of our results with the existing ones is the main topic. Finally, Section 5 concludes the work of this paper.

2. Problem description and preliminaries.

The impulsive static NNs with time delay can be described by the following impulsive delay differential equations:

$$\begin{cases} x'(t) = -Ax(t) + f(Cx(t-\tau(t))) + \psi(t), & t \neq t_k,\ k \in \mathbb{N}, \\ \Delta x(t) = W_k x(t^-), & t = t_k,\ k \in \mathbb{N}, \\ x(t) = \phi(t), & t \in [-\tau,\ 0], \end{cases} \tag{2.1}$$

where $x(t) = (x_1(t),\ \ldots,\ x_n(t))^T$ denotes the state variables associated with the neurons; A is a positive diagonal matrix representing the self-feedback term; $C \in \mathbb{R}^{n\times n}$ is the internal delayed connection weigh matrix; $f(Cx(t)) = (f_1(C_1x(t)),\ f_2(C_2x(t)),\ \ldots,\ f_n(C_nx(t)))^T$ is the neuron activation function, where C_i is the $i-$th row of the matrix C; $\psi(t) = (\psi_1(t),\ \ldots,\ \psi_n(t))^T$ is the external input function and each component $\psi_i(t)$ is bounded. The time delay $\tau(t)$ is bounded as $0 \le \tau(t) \le \tau$ and $\phi : [-\tau,\ 0] \to \mathbb{R}^n$ is a piecewise right continuous function. For the impulsive parameters, $\Delta x(t) = x(t^+) - x(t^-)$ denotes the state jumping at impulsive time instant $t = t_k$, where $x(t^+)$ and $x(t^-)$ are the right-hand and left-hand limits of the functions $x(t)$, respectively; $W_k \in \mathbb{R}^{n\times n}$ represents the abrupt change of the state at t_k; the impulsive time instants $\{t_k\}_{k=1}^{+\infty}$ satisfy $0 < t_1 < t_2 < \cdots$ and $\lim_{k\to+\infty} t_k = +\infty$.

Throughout this paper, we assume that the following hypotheses are satisfied:

$$f_i(0) = 0, \quad l_i^- \le \frac{f_i(s_1) - f_i(s_2)}{s_1 - s_2} \le l_i^+, \quad \forall s_1,\ s_2 \in \mathbb{R} \text{ and } s_1 \neq s_2, \quad i = 1,\ 2,\ \ldots,\ n. \tag{2.2}$$

Set $L_0 = \mathrm{diag}(l_1^-,\ l_2^-,\ \ldots,\ l_n^-)$ and $L_1 = \mathrm{diag}(l_1^+,\ l_2^+,\ \ldots,\ l_n^+)$. Moreover, we will use the notation $P > 0$ (or $P < 0$) to denote that P is a symmetric and positive definite (or negative definite)

matrix. If P_1, P_2 are symmetric matrices, then $P_1 > P_2$ (or $P_1 \geq P_2$) means that $P_1 - P_2$ is a positive definite (positive semi-definite) matrix. For any matrix $P \in \mathbb{R}^{n\times n}$, we use $\lambda_m(P)$ and $\lambda_M(P)$ to denote its minimal and maximal eigenvalues respectively. For any vector $z \in \mathbb{R}^n$ and matrix $P \in \mathbb{R}^{n\times n}$, $\|z\|$ denotes the Euclidean norm of z and $\|P\|$ denotes the induced norm of the matrix P, that is $\|P\| = \sqrt{\rho(P^T P)}$, where $\rho(P^T P)$ denotes the spectral radius of matrix $P^T P$. Moreover, for any initial function $\phi : (-\infty, 0] \to \mathbb{R}^n$ which is piecewise right continuous, we let $\|\phi\|_0 = \max_{s\in(-\infty,\ 0]} \|\phi(s)\|$.

Lemma 2.1 (Berman and Plemmons [24]) *For any symmetric matrix $P \in \mathbb{R}^{n\times n}$, it holds*

$$\lambda_m(P)x^T x \leq x^T Px \leq \lambda_M(P)x^T x, \quad \forall x \in \mathbb{R}^n.$$

Definiton 2.1 *A neural network (2.1) is said to be global exponential dissipative system if there exists a compact set $\mathbb{S}$ in $\mathbb{R}^n$ such that $\forall \phi(t) \in \mathbb{R}^n \backslash \mathbb{S}$, there exist constants $M(\phi) > 0$ and $\gamma > 0$ such that*

$$\inf_{x(t)\in\mathbb{R}^n\backslash\mathbb{S}} \{\|x(t) - \tilde{x}\| : \tilde{x} \in \mathbb{S}\} \leq M(\phi)e^{-\gamma t}.$$

The argument γ is called dissipativity rate and the set $\mathbb{S}$ is called global exponential attractive set, where $x(t) \in \mathbb{R}^n\backslash\mathbb{S}$ means $x(t) \in \mathbb{R}^n$ but $x(t) \notin \mathbb{S}$.

Definiton 2.2 *For any function $u(t)$, we define its right-hand derivative as*

$$\mathcal{D}^+ u(t) = \lim_{s\to 0^+} \frac{u(t+s) - u(t)}{s}.$$

3. Analysis of exponential dissipativity.

In this section, we analyze the exponential dissipativity of (2.1). We utilize the following Lyapunov function

$$V(t) = e^{2\gamma t} x(t)^T Px(t) + e^{2\gamma t} x^T(t-\tau)Qx(t-\tau). \tag{3.1}$$

We first provide a lemma which estimates the jump of $V(t)$ at each impulsive time instant. Note that, for $Q = 0$ this Lyapunov function reduces to $V(t) = e^{2\gamma t} x^T(t)Px(t)$, which is used in [3] and [14].

Lemma 3.1 *If there exist matrices P, $Q > 0$ and positive scalar $\mu > 0$ such that the following matrix inequalities hold:*

$$(I + W_k)^T P(I + W_k) - \mu P < 0, \quad (I + W_k)^T Q(I + W_k) - \mu Q < 0,\ \forall k \geq 1,$$

then we have for every integer $k \geq 1$ that

$$V(t_k) \leq \begin{cases} \mu V(t_k^-), & \textit{if } t_k \textit{ is an impulsive instant,} \\ \mu V(t_k^-) + (1-\mu)e^{2\gamma t_k} x^T(t_k - \tau)Qx(t_k - \tau), & \textit{if } t_k \textit{ is not an impulsive instant} \\ & \textit{and } \mu < 1. \end{cases} \tag{3.2}$$

Proof. We consider the following two cases:
(a) if $t_k - \tau = t_{k-m}$, i.e., $t_k - \tau$ is also an impulsive time instant, we have

$$\begin{aligned}V(t_k) - \mu V(t_k^-) =& e^{2\gamma t_k} x^T(t_k^-) \left[(I + W_k)^T P(I + W_k) - \mu P\right] x(t_k^-) + \\ & e^{2\gamma t_k} x^T(t_{k-m}^-) \left[(I + W_{k-m})^T Q(I + W_{k-m}) - \mu Q\right] x(t_{k-m}^-) \\ \leq& 0,\end{aligned}$$

which gives for any $\mu \geq 0$ that $V(t_k) \leq \mu V(t_k^-)$;
(b) if $t_k - \tau$ is not an impulsive time instant, we have

$$V(t_k) - \mu V(t_k^-) = e^{2\gamma k} x^T(t_k^-) \left[(I + W_k)^T P(I + W_k) - \mu P\right] x(t_k^-) + (1 - \mu) e^{2\gamma t_k} x^T(t_k - \tau) Q x(t_k - \tau),$$

which gives $V(t_k) \leq \mu V(t_k^-)$ for $\mu \geq 1$ and $V(t_k) \leq \mu V(t_k^-) + (1 - \mu) e^{2\gamma t_k} x^T(t_k - \tau) Q x(t_k - \tau)$ for $\mu < 1$, since $(I + W_k)^T P(I + W_k) - \mu P < 0$. ■

Because the dissipativity analysis for the case of "neutral-type" impulses (i.e., $\mu = 1$) is similar to that of input disturbances, we therefore perform the analysis by distinguishing two types of impulses: $\mu \geq 1$ and $\mu < 1$.

3.1. Disturbances impulses: $\mu \geq 1$.

When the static NNs (2.1) without impulses (i.e., $W_k = 0$, $\forall k \geq 1$) are dissipative but the impulses are input disturbances, we try to derive conditions with respect to the magnitude of the impulses and the distance between two consecutive impulsive instants, under which the NNs remain dissipative. Moreover, we want to know how fast the solutions of (2.1) converge to the attractive set.

Theorem 3.1 *Suppose there exist symmetric positive matrices P, Q, D_1, D_2, positive diagonal matrices U_{ij} $(1 \leq i \leq 3,\ 1 \leq j \leq 4)$ and scalar numbers $\mu \geq 1$, $\gamma > 0$, $\alpha > 0$ such that*

$$\begin{aligned}&\Psi_1 = (I + W_k)^T P(I + W_k) - \mu P < 0,\ \forall k \geq 1, \\ &\Psi_2 = (I + W_k)^T Q(I + W_k) - \mu P < 0,\ \forall k \geq 1, \\ &\Omega = \begin{bmatrix} \Omega_{11} & 0 & 0 & 0 & \Omega_{15} & 0 & \Omega_{17} & 0 \\ \star & \Omega_{22} & 0 & 0 & 0 & \Omega_{26} & 0 & \Omega_{28} \\ \star & \star & \Omega_{33} & 0 & 0 & 0 & \Omega_{37} & 0 \\ \star & \star & \star & \Omega_{44} & 0 & 0 & 0 & \Omega_{48} \\ \star & \star & \star & \star & \Omega_{55} & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \Omega_{66} & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \Omega_{77} & 0 \\ \star & \star & \star & \star & \star & \star & \star & \Omega_{88} \end{bmatrix} < 0,\end{aligned} \tag{3.3a}$$

where $\star$ denotes the symmetric terms in a symmetric matrix and

$$
\begin{aligned}
\Omega_{11} &= 2D_1 - PA - AP + \left(2\gamma + \alpha\mu + \frac{\ln\mu}{\beta}\right)P + C^T\left[2U_{11}L_1 + (L_1 - 2L_0)^T U_{21}L_1 - 2L_0^T U_{31}L_1\right]C,\\
\Omega_{15} &= -C^T U_{11} + C^T L_0^T U_{21} + C^T\left(L_0^T + L_1^T\right)U_{31},\\
\Omega_{17} &= P,\\
\Omega_{22} &= 2D_2 - QA - AQ + \left(2\gamma + \alpha\mu + \frac{\ln\mu}{\beta}\right)Q + C^T\left[2U_{12}L_1 + (L_1 - 2L_0)^T U_{22}L_1 - 2L_0^T U_{32}L_1\right]C,\\
\Omega_{26} &= -C^T U_{12} + C^T L_0^T U_{22} + C^T\left(L_0^T + L_1^T\right)U_{32},\\
\Omega_{28} &= Q,\\
\Omega_{33} &= -\alpha e^{-2\gamma\tau}P + C^T\left[2U_{13}L_1 + (L_1 - 2L_0)^T U_{23}L_1 - 2L_0^T U_{33}L_1\right]C,\\
\Omega_{37} &= -C^T U_{13} + C^T L_0^T U_{23} + C^T\left(L_0^T + L_1^T\right)U_{33},\\
\Omega_{44} &= -\alpha e^{-2\gamma\tau}Q + C^T\left[2U_{14}L_1 + (L_1 - 2L_0)^T U_{24}L_1 - 2L_0^T U_{34}L_1\right]C,\\
\Omega_{48} &= -C^T U_{14} + C^T L_0^T U_{24} + C^T\left(L_0^T + L_1^T\right)U_{34},\\
\Omega_{55} &= -U_{21} - 2U_{31},\\
\Omega_{66} &= -U_{22} - 2U_{32},\\
\Omega_{77} &= -U_{23} - 2U_{33},\\
\Omega_{88} &= -U_{24} - 2U_{34},\\
\beta &= \inf_{k\geq 1}\{t_k - t_{k-1}\},\quad t_0 = 0.
\end{aligned}
\tag{3.3b}
$$

Then the static NNs (2.1) is global exponential dissipative with dissipativity rate γ, and

$$
\mathbb{S} = \left\{x : \|x\| \leq \max\left\{\frac{\sup_{t\in\mathbb{R}}\|P\psi(t)\|}{\lambda_m(D_1)},\ \frac{\sup_{t\in\mathbb{R}}\|Q\psi(t)\|}{\lambda_m(D_2)}\right\}\right\} \tag{3.4}
$$

is a positive invariant and global attractive set. In particular, we have

$$
\inf_{x(t)\in\mathbb{R}^n\setminus\mathbb{S}}\{\|x(t) - \tilde{x}\| : \tilde{x}\in\mathbb{S}\} < M(\phi)e^{-\gamma t}, \tag{3.5}
$$

where $M(\phi) = \sqrt{\frac{\mu\lambda_1}{\lambda_0}}\|\phi\|_0$, $\lambda_1 = \max\{\lambda_M(P),\ \lambda_M(Q)\}$, $\lambda_0 = \min\{\lambda_m(P),\ \lambda_m(Q)\}$.

Proof. The proof is divided into two parts.
(A) Since $\Omega < 0$, it holds for sufficient small positive number η that

$$
\tilde{\Omega} = \begin{bmatrix}
\tilde{\Omega}_{11} & 0 & 0 & 0 & \Omega_{15} & 0 & \Omega_{17} & 0\\
\star & \tilde{\Omega}_{22} & 0 & 0 & 0 & \Omega_{26} & 0 & \Omega_{28}\\
\star & \star & \Omega_{33} & 0 & 0 & 0 & \Omega_{37} & 0\\
\star & \star & \star & \Omega_{44} & 0 & 0 & 0 & \Omega_{48}\\
\star & \star & \star & \star & \Omega_{55} & 0 & 0 & 0\\
\star & \star & \star & \star & \star & \Omega_{66} & 0 & 0\\
\star & \star & \star & \star & \star & \star & \Omega_{77} & 0\\
\star & \star & \star & \star & \star & \star & \star & \Omega_{88}
\end{bmatrix} < 0,
$$

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where

$$\tilde{\Omega}_{11} = 2D_1 + 2\gamma P - PA - AP + (\alpha\mu_1 + \mu_2)P + C^T\left[2U_{11}L_1 + (L_1 - 2L_0)^T U_{21}L_1 - 2L_0^T U_{31}L_1\right]C,$$
$$\tilde{\Omega}_{22} = 2D_2 + 2\gamma Q - QA - AQ + (\alpha\mu_1 + \mu_2)Q + C^T\left[2U_{12}L_1 + (L_1 - 2L_0)^T U_{22}L_1 - 2L_0^T U_{32}L_1\right]C,$$
$$\mu_1 = \mu + \eta, \ \ \mu_2 = \frac{\ln(\mu + 2\eta)}{\beta}.$$

By routine calculations we get

$$\begin{aligned}
&\mathcal{D}^+V(t) + (\alpha\mu_1 + \mu_2)V(t) - \alpha V(t - \tau(t))\\
&= e^{2\gamma t}\left(2\gamma x^T(t)Px(t) + 2x^T(t)P\left[-Ax(t) + f(Cx(t - \tau(t)))\right]\right) + (\alpha\mu_1 + \mu_2)e^{2\gamma t}x^T(t)Px(t)-\\
&\quad e^{2\gamma(t-\tau(t))}\alpha x^T(t - \tau(t))Px(t - \tau(t)) + e^{2\gamma t}\left(2\gamma x^T(t - \tau)Qx(t - \tau) + 2x^T(t - \tau)Q\left[-Ax(t - \tau)\right.\right.\\
&\quad \left.\left.+f(Cx(t - \tau - \tau(t)))\right]\right) + (\alpha\mu_1 + \mu_2)e^{2\gamma t}x^T(t - \tau)Qx(t - \tau) - e^{2\gamma(t-\tau(t))}\alpha x^T(t - \tau - \tau(t))Q\\
&\quad x(t - \tau - \tau(t)) + 2e^{2\gamma t}\left(x^T(t)P\psi(t) + x^T(t - \tau)Q\psi(t - \tau)\right)\\
&\le e^{2\gamma t}\left(x^T(t)\left[2\gamma P - 2PA + (\alpha\mu_1 + \mu_2)P\right]x(t) + 2x^TPf(Cx(t - \tau(t))) - \alpha e^{-2\gamma\tau}x^T(t - \tau(t))P\right.\\
&\quad \left.x(t - \tau(t))\right) + e^{2\gamma t}\left(x^T(t - \tau)\left[2\gamma Q - 2QA + (\alpha\mu_1 + \mu_2)Q\right]x(t - \tau) + 2x^T(t - \tau)Q\right.\\
&\quad \left.f(Cx(t - \tau - \tau(t))) - \alpha e^{-2\gamma\tau}x^T(t - \tau - \tau(t))Qx(t - \tau - \tau(t))\right) +\\
&\quad 2e^{2\gamma t}\left(x^T(t)P\psi(t) + x^T(t - \tau)Q\psi(t - \tau)\right).
\end{aligned} \tag{3.6}$$

Moreover, by Lipschitz condition (2.2) and the fact that U_{ij} $(1 \le i \le 3,\ 1 \le j \le 4)$ are positive diagonal matrices, it is easy to get the following inequalities

$$\begin{aligned}
&0 \le 2e^{2\gamma t}\left[Cz_j\right]^T U_{1j}[L_1\left(Cz_j\right) - f(Cz_j)] = e^{2\gamma t}z_j^T[2C^TU_{1j}L_1C]z_j + 2e^{2\gamma t}z_j^T[-C^TU_{1j}]f(Cz_j),\\
&0 \le e^{2\gamma t}\left[(Cz_j)^T(L_1^T - L_0^T) + f^T(Cz_j) - (Cz_j)^TL_0^T\right]U_{2j}[L_1Cz_j - f(Cz_j)]\\
&\quad = e^{2\gamma t}\left(z_j^T\left[C^T(L_1 - 2L_0)^TU_{2j}L_1C\right]z_j + 2z_j^T[C^TL_0^TU_{2j}]f(Cz_j) + f^T(Cz_j)[-U_{2j}]f(Cz_j)\right),\\
&0 \le 2e^{2\gamma t}\left[f^T(Cz_j) - (Cz_j)^TL_0^T\right]U_{3j}[L_1(Cz_j) - f(Cz_j)]\\
&\quad = z_j^T(t)[-2C^TL_0^TU_{3j}L_1C]z_j + 2z_j^T[C^TL_0^TU_{3j} + C^TL_1^TU_{3j}^T]f(Cz_j) + f^T(Cz_j)[-2U_{3j}]f(Cz_j),
\end{aligned} \tag{3.7}$$

where $j = 1,\ 2,\ 3,\ 4$ and

$$z_1 = x(t),\ z_2 = x(t - \tau), \quad z_3 = x(t - \tau(t)),\ z_4 = x(t - \tau - \tau(t)). \tag{3.8}$$

Combining (3.6) and (3.7) leads to

$$\begin{aligned}
&\mathcal{D}^+V(t) + (\alpha\mu_1 + \mu_2)V(t) - \alpha V(t - \tau(t))\\
&\le 2e^{2\gamma t}\left[\left(x^T(t)P\psi(t) - x^T(t)D_1x(t)\right) + \left(x^T(t - \tau)Q\psi(t - \tau) - x^T(t - \tau)D_2x(t - \tau)\right)\right] + e^{2\gamma t}X^T\tilde{\Omega}X\\
&\le 2e^{2\gamma t}\left(\|x(t)\|\left[\|P\psi(t)\| - \lambda_m(D_1)\|x(t)\|\right] + \|x(t - \tau)\|\left[\|Q\psi(t - \tau)\| - \lambda_m(D_2)\|x(t - \tau)\|\right]\right.\\
&\quad \left.+\frac{1}{2}X^T\tilde{\Omega}X\right),
\end{aligned} \tag{3.9}$$

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where $X = \left(z_1^T,\ z_2^T,\ z_3^T,\ z_4^T,\ f^T(Cz_1),\ f^T(Cz_2),\ f^T(Cz_3),\ f^T(Cz_4)\right)^T$ and z_j $(1 \le j \le 4)$ are defined by (3.8). Now, by using $\tilde{\Omega} < 0$ we have

$$\mathcal{D}^+V(t) + (\alpha\mu_1 + \mu_2)V(t) - \alpha V(t - \tau(t)) \le 0, \tag{3.10}$$

whenever $x(t) \in \mathbb{R}^n \backslash \mathbb{S}$.

(B) We next prove (3.5). To this end, we first prove the following inequality

$$V(t) < \lambda_0\zeta^2,\ t \ge -\tau, \tag{3.11}$$

where $\zeta = \sqrt{\frac{(\mu+\eta)\lambda_1}{\lambda_0}}\|\phi\|_0$ and $x(t) \in \mathbb{R}^n \backslash \mathbb{S}$.

(b1) For $t \in [-\tau,\ 0]$, by using Lemma 2.1 and $\mu_1 = \mu + \eta > 1$ we get

$$\begin{aligned} V(t) &= e^{2\gamma t}\left(x^T(t)Px(t) + x^T(t-\tau)Qx(t-\tau)\right) \\ &\le x^T(t)Px(t) + x^T(t-\tau)Qx(t-\tau) \\ &\le \lambda_1\|\phi\|_0^2 = \frac{1}{\mu_1}\lambda_0\zeta^2 < \lambda_0\zeta^2. \end{aligned} \tag{3.12}$$

(b2) We next prove $V(t) < \lambda_0\zeta^2$ for $t \in [0,\ t_1)$. If not, there exists $t \in (0,\ t_1)$ such that $V(t) \ge \lambda_0\zeta^2$. Set $\bar{t} = \inf\{t : V(t) \ge \lambda_0\zeta^2 \text{ and } t \in [0,\ t_1)\}$. Clearly, $\bar{t} \in (0,\ t_1)$ and $V(\bar{t}) = \lambda_0\zeta^2$. Since $V(0) \le \frac{1}{\mu_1}\lambda_0\zeta^2$, there exists $\underline{t} = \sup\left\{t : V(t) \le \frac{1}{\mu_1}\lambda_0\zeta^2 \text{and } t \in [0,\ \bar{t})\right\}$. Hence, for $t \in [\underline{t},\ \bar{t}]$ we have

$$V(t) \ge \frac{1}{\mu_1}\lambda_0\zeta^2 \ \text{ and } \ V(t+\theta) \le \lambda_0\zeta^2,\ \ \theta \in [-\tau,\ 0], \tag{3.13}$$

which leads to

$$\mathcal{D}^+V(t) \le \mathcal{D}^+V(t) + \alpha(\mu_1V(t) - V(t-\tau(t))),\ t \in [\underline{t},\ \ \bar{t}]. \tag{3.14}$$

By using (3.10) we have

$$\mathcal{D}^+V(t) + \alpha\mu_1V(t) - \alpha V(t-\tau(t)) \le -\mu_2V(t) \le 0, \tag{3.15}$$

and this together with (3.14) gives $\mathcal{D}^+V(t) \le 0$ for $x(t) \in \mathbb{R}^n \backslash \mathbb{S}$ and $t \in [\underline{t},\ \ \bar{t}]$. It then follows that $V(\bar{t}) \le V(\underline{t}) = \frac{1}{\mu_1}\lambda_0\zeta^2$. This is a contradiction. Therefore, $V(t) < \lambda_0\zeta^2$ for $t \in [0,\ t_1)$.

(b3) Suppose for any integer $k \ge 1$ that

$$V(t) < \lambda_0\zeta^2,\ \ t \in [-\tau,\ t_k). \tag{3.16}$$

We try to prove

$$V(t) < \lambda_0\zeta^2,\ \ t \in [t_k,\ t_{k+1}). \tag{3.17}$$

To this end, we first claim $V(t_k^-) \le \frac{1}{\mu_1}\lambda_0\zeta^2$. By the contrary, we have $V(t_k^-) > \frac{1}{\mu_1}\lambda_0\zeta^2$. For this situation, we need to consider two cases.

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Case 1: $V(t) > \frac{1}{\mu_1}\lambda_0\zeta^2$ for $t \in [t_{k-1},\ t_k)$. In this case, by assumption (3.16) we have

$$V(t) > \frac{1}{\mu_1}\lambda_0\zeta^2 \ \text{ and } \ V(t+\theta) < \lambda_0\zeta^2,\ \ \theta \in [-\tau,\ 0], \tag{3.18}$$

which gives

$$\mathcal{D}^+V(t) \le \mathcal{D}^+V(t) + \alpha[\mu_1 V(t) - V(t-\tau(t))],\ t \in [t_{k-1},\ t_k). \tag{3.19}$$

Hence, by (3.10) we have $\mathcal{D}^+V(t) \le -\mu_2 V(t)$ for $x(t) \in \mathbb{R}^n \backslash \mathbb{S}$ and $t \in [t_{k-1},\ t_k)$. This gives

$$V(t) \le V(t_{k-1})e^{-\mu_2(t-t_{k-1})},\ \ t \in [t_{k-1},\ t_k), \tag{3.20}$$

which yields

$$V(t_k^-) \le V(t_{k-1})e^{-\mu_2(t_k-t_{k-1})} \le \lambda_0\zeta^2 e^{-\mu_2\beta} = \frac{1}{\mu+2\eta}\lambda_0\zeta^2 = \frac{1}{\mu_1+\eta}\lambda_0\zeta^2 < \frac{1}{\mu_1}\lambda_0\zeta^2, \tag{3.21}$$

since $\eta > 0$. Clearly, this is a contradiction.
Case 2: There exists some $t \in [t_{k-1},\ t_k)$ such that $V(t) \le \frac{1}{\mu_1}\lambda_0\zeta^2$. Since $V(t_k^-) > \frac{1}{\mu_1}\lambda_0\xi^2$, we may set $\bar{t} = \sup\{t : V(t) \le \frac{1}{\mu_1}\lambda_0\zeta^2$ and $t \in [t_{k-1},\ t_k)\}$. Clearly, $\bar{t} \in (t_{k-1},\ t_k)$ and $V(\bar{t}) = \frac{1}{\mu_1}\lambda_0\zeta^2$. Therefore, we have

$$V(t) > \frac{1}{\mu_1}\lambda_0\zeta^2 \ \text{ and } \ V(t+\theta) \le \lambda_0\zeta^2,\ t \in [\bar{t},\ t_k),\ \ \theta \in [-\tau,\ 0], \tag{3.22}$$

which gives

$$\mathcal{D}^+V(t) \le \mathcal{D}^+V(t) + \alpha(\mu_1 V(t) - V(t-\tau(t))),\ \ t \in [\bar{t},\ t_k). \tag{3.23}$$

Hence, by (3.10) we get $\mathcal{D}^+V(t) \le 0$ for $x(t) \in \mathbb{R}^n \backslash \mathbb{S}$ and $t \in [\bar{t},\ t_k)$. This implies $V(t_k^-) \le V(\bar{t}) = \frac{1}{\mu_1}\lambda_0\zeta^2$. This again leads to contradiction.

So we have $V(t_k^-) \le \frac{1}{\mu_1}\lambda_0\zeta^2$. From Lemma 3.1 we have

$$V(t_k) \le \mu V(t_k^-) = \frac{\mu}{\mu_1}\lambda_0\zeta^2 < \lambda_0\zeta^2. \tag{3.24}$$

This together with (3.16) gives $V(t) < \lambda_0\zeta^2,\ \ t \in [-\tau,\ t_k]$. Now, suppose $V(t) < \lambda_0\zeta^2$ is not true for $t \in (t_k,\ t_{k+1})$. If so, we set $t^* = \inf\{t : V(t) \ge \lambda_0\zeta^2$ and $t \in [t_k,\ t_{k+1})\}$. Then $t^* \in (t_k,\ t_{k+1})$ and $V(t^*) = \lambda_0\zeta^2$. Set $\bar{t} = t_k$ if $V(t) > \frac{1}{\mu_1}\lambda_0\zeta^2$ for all $t \in [t_k,\ t^*]$; otherwise, set $\bar{t} = \sup\{t : V(t) \le \frac{1}{\mu_1}\lambda_0\zeta^2$ and $t \in [t_k,\ t^*)\}$. Therefore, for $t \in [\bar{t},\ t^*]$ we have

$$V(t) \ge \frac{1}{\mu_1}\lambda_0\zeta^2 \ \text{ and } \ V(t+\theta) \le \lambda_0\zeta^2,\ \ \theta \in [-\tau,\ 0]. \tag{3.25}$$

It then follows

$$\mathcal{D}^+V(t) \le \mathcal{D}^+V(t) + \alpha(\mu_1 V(t) - V(t-\tau(t))),\ t \in [\bar{t},\ t^*]. \tag{3.26}$$

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By (3.10) we have $\mathcal{D}^+V(t) \le 0$ for $x(t) \in \mathbb{R}^n \backslash \mathbb{S}$ and $t \in [\bar{t},\ t^*]$, which implies $V(t^*) \le V(\bar{t}) < \lambda_0\zeta^2$ (for the situation $\bar{t} = t_k$, (3.24) gives the last inequality). This obviously contradicts the fact $V(t^*) = \lambda_0\zeta^2$.

Thus, by the method of mathematical induction, (3.17) holds for any integer k. So (3.11) is true. It therefore follows by applying Lemma 2.1 that

$$\lambda_0 e^{2\gamma t}\|x(t)\|^2 \le V(t) < \lambda_0\zeta^2, \quad \text{when } x(t) \in \mathbb{R}^n \backslash \mathbb{S}, \tag{3.27}$$

which implies

$$\inf_{x(t)\in\mathbb{R}^n\backslash\mathbb{S}} \{\|x(t) - \tilde{x}\| : \tilde{x} \in \mathbb{S}\} \le \|x(t)\| < M(\phi)e^{-\gamma t}, \tag{3.28}$$

since the set $\mathbb{S}$ includes origin and the argument $\eta > 0$ in ζ can be sufficient small. ■

The following result gives a corollary for the case of constant delay, i.e., $\tau(t) \equiv \tau$. In this case, $x(t - \tau(t)) = x(t - \tau)$ and $x(t - \tau - \tau(t)) = x(t - 2\tau)$, and therefore the matrix Ω in (3.3a) can be compacted into an 6-block matrix.

Corollary 3.1 *Suppose $\tau(t) \equiv \tau$ and there exist symmetric positive matrices P, Q, D_1, D_2, positive diagonal matrices $U_{ij}(1 \le i,\ j \le 3)$, scalar numbers $\mu \ge 1$, $\gamma > 0$, $\alpha > 0$ such that*

$$\begin{aligned}
&\Psi_1 = (I + W_k)^T P(I + W_k) - \mu P < 0,\ \forall k \ge 1,\\
&\Psi_2 = (I + W_k)^T Q(I + W_k) - \mu Q < 0,\ \forall k \ge 1,\\
&\Omega = \begin{bmatrix} \Omega_{11} & 0 & 0 & \Omega_{14} & \Omega_{15} & 0 \\ \star & \Omega_{22} & 0 & 0 & \Omega_{25} & \Omega_{26} \\ \star & \star & \Omega_{33} & 0 & 0 & \Omega_{36} \\ \star & \star & \star & \Omega_{44} & 0 & 0 \\ \star & \star & \star & \star & \Omega_{55} & 0 \\ \star & \star & \star & \star & \star & \Omega_{66} \end{bmatrix} < 0,
\end{aligned} \tag{3.29}$$

where

$$\begin{aligned}
\Omega_{11} &= 2D_1 + \left(2\gamma + \alpha\mu + \frac{\ln\mu}{\beta}\right)P - PA - A^T P^T + C^T[2U_{11}L_1 + (L_1 - 2L_0)^T U_{21}L_1 - 2L_0^T U_{31}L_1]C,\\
\Omega_{14} &= -C^T U_{11} + C^T L_0^T U_{21} + C^T L_0^T U_{31} + C^T L_1^T U_{31}^T,\\
\Omega_{15} &= P,\\
\Omega_{22} &= 2D_2 + \left(2\gamma + \alpha\mu + \frac{\ln\mu}{\beta}\right)Q - \alpha e^{-2\gamma\tau}P - QA - A^T Q^T + C^T[2U_{12}L_1 + (L_1 - 2L_0)^T U_{22}L_1\\
&\quad - 2L_0^T U_{32}L_1]C,\\
\Omega_{25} &= -C^T U_{12} + C^T L_0^T U_{22} + C^T L_0^T U_{32} + C^T L_1^T U_{32}^T,\\
\Omega_{26} &= Q,\\
\Omega_{33} &= -\alpha e^{-2\gamma\tau}Q + C^T[2U_{13}L_1 + (L_1 - 2L_0)^T U_{23}L_1 - 2L_0^T U_{33}L_1]C,\\
\Omega_{36} &= -C^T U_{13} + C^T L_0^T U_{23} + C^T L_0^T U_{33} + C^T L_1^T U_{33}^T,\\
\Omega_{44} &= -U_{21} - 2U_{31},\\
\Omega_{55} &= -U_{22} - 2U_{32},\\
\Omega_{66} &= -U_{23} - 2U_{33},\\
\beta &= \inf_{k\geq 1}\{t_k - t_{k-1}\}, \quad t_0 = 0.
\end{aligned} \tag{3.30}$$

Then the static neural network (2.1) is global exponential dissipative with dissipativity rate γ, and in particular (3.4) holds.

Remark 3.1 *Theorem 3.1 and Corollary 3.1 require that there exists a positive lower bound β on the time instants between two adjacent impulses. This condition ensures that the impulses, which destabilize the neural networks, do not occur too frequently. When α, β, τ, γ are chosen, inequalities (3.3a) and (3.29) are linear matrix inequalities (LMIs), which can be solved numerically and very efficiently using the interior point algorithms [25].*

3.2. Stabilizing Impulses: $\mu < 1$.

In this subsection, we study the exponential dissipativity of the static neural network (2.1) for the case $\mu < 1$, i.e., the impulses are stabilizing.

Theorem 3.2 *Suppose there exist symmetric positive matrices P, Q, D_1, D_2, positive diagonal matrices $U_{ij}(1 \leq i \leq 3,\ 1 \leq j \leq 4)$ and scalar numbers $\mu < 1$, $\gamma > 0$, $\alpha > 0$ such that*

$$\begin{aligned}
&\Psi_1 = (I + W_k)^T P(I + W_k) - \mu P < 0,\ \forall k \geq 1,\\
&\Psi_2 = (I + W_k)^T Q(I + W_k) - \mu Q < 0,\ \forall k \geq 1,\\
&\Omega = \begin{bmatrix}
\Omega_{11} & 0 & 0 & 0 & \Omega_{15} & 0 & \Omega_{17} & 0\\
\star & \Omega_{22} & 0 & 0 & 0 & \Omega_{26} & 0 & \Omega_{28}\\
\star & \star & \Omega_{33} & 0 & 0 & 0 & \Omega_{37} & 0\\
\star & \star & \star & \Omega_{44} & 0 & 0 & 0 & \Omega_{48}\\
\star & \star & \star & \star & \Omega_{55} & 0 & 0 & 0\\
\star & \star & \star & \star & \star & \Omega_{66} & 0 & 0\\
\star & \star & \star & \star & \star & \star & \Omega_{77} & 0\\
\star & \star & \star & \star & \star & \star & \star & \Omega_{88}
\end{bmatrix} < 0,
\end{aligned} \tag{3.31}$$

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where

$$
\begin{aligned}
\Omega_{11} &= 2D_1 - PA - AP + (2\gamma + \sigma) P + C^T \left[2U_{11}L_1 + (L_1 - 2L_0)^T U_{21}L_1 - 2L_0^T U_{31}L_1\right] C,\\
\Omega_{15} &= -C^T U_{11} + C^T L_0^T U_{21} + C^T \left(L_0^T + L_1^T\right) U_{31},\\
\Omega_{17} &= P,\\
\Omega_{22} &= 2D_2 - QA - AQ + (2\gamma + \sigma) Q + C^T \left[2U_{12}L_1 + (L_1 - 2L_0)^T U_{22}L_1 - 2L_0^T U_{32}L_1\right] C,\\
\Omega_{26} &= -C^T U_{12} + C^T L_0^T U_{22} + C^T \left(L_0^T + L_1^T\right) U_{32},\\
\Omega_{28} &= Q,\\
\Omega_{33} &= -\alpha e^{-2\gamma\tau} P + C^T \left[2U_{13}L_1 + (L_1 - 2L_0)^T U_{23}L_1 - 2L_0^T U_{33}L_1\right] C,\\
\Omega_{37} &= -C^T U_{13} + C^T L_0^T U_{23} + C^T \left(L_0^T + L_1^T\right) U_{33},\\
\Omega_{44} &= -\alpha e^{-2\gamma\tau} Q + C^T \left[2U_{14}L_1 + (L_1 - 2L_0)^T U_{24}L_1 - 2L_0^T U_{34}L_1\right] C,\\
\Omega_{48} &= -C^T U_{14} + C^T L_0^T U_{24} + C^T \left(L_0^T + L_1^T\right) U_{34},\\
\Omega_{55} &= -U_{21} - 2U_{31},\\
\Omega_{66} &= -U_{22} - 2U_{32},\\
\Omega_{77} &= -U_{23} - 2U_{33},\\
\Omega_{88} &= -U_{24} - 2U_{34},\\
\beta &= \sup_{k\geq 1}\{t_k - t_{k-1}\}, \;\; t_0 = 0,\\
\sigma &= \begin{cases} \alpha, & \text{if } 1 \leq \alpha\beta,\\ \frac{\alpha}{\mu} + \frac{\ln \mu}{\beta}, & \text{if } \alpha\beta \leq \mu,\\ \frac{1+\ln(\alpha\beta)}{\beta}, & \text{if } \mu < \alpha\beta < 1. \end{cases}
\end{aligned} \tag{3.32}
$$

Then the static neural network (2.1) *is global exponential dissipative with dissipativity rate* γ, *and*

$$
\mathbb{S} = \left\{ x : \|x\| \leq \min \left\{ \frac{\sup_{t\in\mathbb{R}} \|P\psi(t)\|}{\lambda_m(D_1)},\; \frac{\sup_{t\in\mathbb{R}} \|Q\psi(t)\|}{\lambda_m(D_2)} \right\} \right\} \tag{3.33}
$$

is a positive invariant and global attractive set. In particular, we have

$$
\inf_{x(t)\in\mathbb{R}^n\setminus\mathbb{S}} \{\|x(t) - \tilde{x}\| : \tilde{x} \in \mathbb{S}\} < M(\phi) e^{-\gamma t}, \tag{3.34}
$$

where $M(\phi) = \sqrt{\frac{\mu\lambda_1}{\lambda_0}}\|\phi\|_0$, $\lambda_1 = \max\{\lambda_M(P),\ \lambda_M(Q)\}$, $\lambda_0 = \min\{\lambda_m(P),\ \lambda_m(Q)\}$.

Proof. The proof is also divided into two parts.
(A) Set $\mathcal{R}(c) = \frac{\alpha}{c} + \frac{\ln c}{\beta}$. It is easy to get $\inf_{\mu\leq c<1} \mathcal{R}(c) = \sigma$ and

$$
\inf_{\mu\leq c<1} \mathcal{R}(c) = \begin{cases} \mathcal{R}(1), & \text{if } \alpha\beta \geq 1,\\ \mathcal{R}(\alpha\beta), & \text{if } \mu < \alpha\beta < 1,\\ \mathcal{R}(\mu), & \text{if } \alpha\beta \leq \mu. \end{cases} \tag{3.35}
$$

Therefore, we can always choose a proper $\hat{\mu} \in [\mu,\ 1)$ such that

$$\hat{\Omega} = \begin{bmatrix} \hat{\Omega}_{11} & 0 & 0 & 0 & \Omega_{15} & 0 & \Omega_{17} & 0 \\ \star & \hat{\Omega}_{22} & 0 & 0 & 0 & \Omega_{26} & 0 & \Omega_{28} \\ \star & \star & \Omega_{33} & 0 & 0 & 0 & \Omega_{37} & 0 \\ \star & \star & \star & \Omega_{44} & 0 & 0 & 0 & \Omega_{48} \\ \star & \star & \star & \star & \Omega_{55} & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \Omega_{66} & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \Omega_{77} & 0 \\ \star & \star & \star & \star & \star & \star & \star & \Omega_{88} \end{bmatrix} < 0, \tag{3.36}$$

since $\Omega < 0$, where

$$\hat{\Omega}_{11} = 2D_1 - PA - AP + \left(2\gamma + \frac{\alpha}{\hat{\mu}} + \frac{\ln\hat{\mu}}{\beta}\right)P + C^T\left[2U_{11}L_1 + (L_1 - 2L_0)^T U_{21}L_1 - 2L_0^T U_{31}L_1\right]C,$$
$$\hat{\Omega}_{22} = 2D_2 - QA - AQ + \left(2\gamma + \frac{\alpha}{\hat{\mu}} + \frac{\ln\hat{\mu}}{\beta}\right)Q + C^T\left[2U_{12}L_1 + (L_1 - 2L_0)^T U_{22}L_1 - 2L_0^T U_{32}L_1\right]C. \tag{3.37}$$

Similarly, it holds for sufficient small positive number $\eta > 0$ that

$$\tilde{\Omega} = \begin{bmatrix} \tilde{\Omega}_{11} & 0 & 0 & 0 & \Omega_{15} & 0 & \Omega_{17} & 0 \\ \star & \tilde{\Omega}_{22} & 0 & 0 & 0 & \Omega_{26} & 0 & \Omega_{28} \\ \star & \star & \Omega_{33} & 0 & 0 & 0 & \Omega_{37} & 0 \\ \star & \star & \star & \Omega_{44} & 0 & 0 & 0 & \Omega_{48} \\ \star & \star & \star & \star & \Omega_{55} & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \Omega_{66} & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \Omega_{77} & 0 \\ \star & \star & \star & \star & \star & \star & \star & \Omega_{88} \end{bmatrix} < 0, \tag{3.38}$$

since $\hat{\Omega} < 0$, where

$$\tilde{\Omega}_{11} = 2D_1 - PA - AP + \left(2\gamma + \frac{\alpha}{\hat{\mu}} - \mu_1\right)P + C^T\left[2U_{11}L_1 + (L_1 - 2L_0)^T U_{21}L_1 - 2L_0^T U_{31}L_1\right]C,$$
$$\tilde{\Omega}_{22} = 2D_2 - QA - AQ + \left(2\gamma + \frac{\alpha}{\hat{\mu}} - \mu_1\right)Q + C^T\left[2U_{12}L_1 + (L_1 - 2L_0)^T U_{22}L_1 - 2L_0^T U_{32}L_1\right]C,$$
$$\mu_1 = -\frac{\ln(\hat{\mu} + \eta)}{\beta} > 0. \tag{3.39}$$

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By routine calculations we have

$$
\begin{aligned}
&\mathcal{D}^{+}V(t)+\left(\frac{\alpha}{\hat{\mu}}-\mu_1\right)V(t)-\alpha V(t-\tau(t))\\
&=e^{2\gamma t}\left(2\gamma x^T(t)Px(t)+2x^T(t)P\left[-Ax(t)+f(Cx(t-\tau(t)))\right]\right)+\left(\frac{\alpha}{\hat{\mu}}-\mu_1\right)e^{2\gamma t}x^T(t)Px(t)-\\
&\quad e^{2\gamma(t-\tau(t))}\alpha x^T(t-\tau(t))Px(t-\tau(t))+e^{2\gamma t}\left(2\gamma x^T(t-\tau)Qx(t-\tau)+2x^T(t-\tau)Q\left[-Ax(t-\tau)\right.\right.\\
&\quad \left.\left.+f(Cx(t-\tau-\tau(t)))\right]\right)+\left(\frac{\alpha}{\hat{\mu}}-\mu_1\right)e^{2\gamma t}x^T(t-\tau)Qx(t-\tau)-e^{2\gamma(t-\tau(t))}\alpha x^T(t-\tau-\tau(t))\\
&\quad Qx(t-\tau-\tau(t))\\
&\le e^{2\gamma t}\left(x^T(t)\left[2\gamma P-2PA+\left(\frac{\alpha}{\hat{\mu}}-\mu_1\right)P\right]x(t)+2x^TPf(Cx(t-\tau(t)))-\alpha e^{-2\gamma\tau}x^T(t-\tau(t))\right.\\
&\quad \left.Px(t-\tau(t))\right)+e^{2\gamma t}\left(x^T(t-\tau)\left[2\gamma Q-2QA+\left(\frac{\alpha}{\hat{\mu}}-\mu_1\right)Q\right]x(t-\tau)+2x^T(t-\tau)Q\right.\\
&\quad \left.f(Cx(t-\tau-\tau(t)))-\alpha e^{-2\gamma\tau}x^T(t-\tau-\tau(t))Qx(t-\tau-\tau(t))\right).
\end{aligned}
\tag{3.40}
$$

It then follows by combining (3.7) and (3.40) that

$$
\begin{aligned}
&\mathcal{D}^{+}V(t)+\left(\frac{\alpha}{\hat{\mu}}-\mu_1\right)V(t)-\alpha V(t-\tau(t))\\
&\le 2e^{2\gamma t}\left[\left(x^T(t)P\psi(t)-x^T(t)D_1x(t)\right)+\left(x^T(t-\tau)Q\psi(t-\tau)-x^T(t-\tau)D_2x(t-\tau)\right)\right]+e^{2\gamma t}X^T\tilde{\Omega}X\\
&\le 2e^{2\gamma t}\left(\|x(t)\|\left(\|P\psi(t)\|-\lambda_m(D)\|x(t)\|\right)+\frac{1}{2}X^T\tilde{\Omega}X\right),
\end{aligned}
\tag{3.41}
$$

where X is the same vector as used in Theorem 3.1. Therefore, we get by using $\tilde{\Omega}<0$ that

$$\mathcal{D}^{+}V(t)+\left(\frac{\alpha}{\hat{\mu}}-\mu_1\right)V(t)-\alpha V(t-\tau(t))\le 0, \tag{3.42}$$

when $x(t)\in\mathbb{R}^n\backslash\mathbb{S}$. This implies

$$\mathcal{D}^{+}V(t)+\alpha\left(\frac{1}{\hat{\mu}}V(t)-V(t-\tau(t))\right)\le\mu_1V(t). \tag{3.43}$$

(B) We next prove (3.34). To this end, it is sufficient to prove

$$V(t)<\lambda_0M^2(\phi),\quad \text{for } x(t)\in\mathbb{R}^n\backslash\mathbb{S} \text{ and } t\ge-\tau. \tag{3.44}$$

(b1) For $t\in[-\tau,\ 0]$, by using Lemma 2.1 and $\mu\le\hat{\mu}<1$ it is easy to get

$$
\begin{aligned}
V(t)&=e^{2\gamma t}\left(x^T(t)Px(t)+x^T(t-\tau)Px(t-\tau)\right)\\
&\le\lambda_1\|\phi\|_0^2=\mu\lambda_0M^2(\phi)\le\hat{\mu}\lambda_0M^2(\phi)\\
&<\lambda_0M^2(\phi).
\end{aligned}
\tag{3.45}
$$

(b2) We next prove $V(t) < \lambda_0 M^2(\phi)$ for $t \in [0,\ t_1)$. If not, there exists $t \in (0,\ t_1)$ such that $V(t) \geq \lambda_0 M^2(\phi)$. Set $\bar{t} = \inf\{t : V(t) \geq \lambda_0 M^2(\phi)$ and $t \in [0,\ t_1)\}$. Clearly, $\bar{t} \in (0,\ t_1)$ and $V(\bar{t}) = \lambda_0 M^2(\phi)$. Since $V(0) \leq \hat{\mu}\lambda_0 M^2(\phi)$, there exists $\underline{t} = \sup\left\{t : V(t) \leq \hat{\mu}\lambda_0 M^2(\phi) \text{ and } t \in [0,\ \bar{t})\right\}$. Hence, for $t \in [\underline{t},\ \bar{t}]$ we have

$$V(t) \geq \hat{\mu}\lambda_0 M^2(\phi) \text{ and } V(t+\theta) \leq \lambda_0 M^2(\phi), \quad \theta \in [-\tau,\ 0], \tag{3.46}$$

which leads to

$$\mathcal{D}^+ V(t) \leq \mathcal{D}^+ V(t) + \alpha\left(\frac{1}{\hat{\mu}} V(t) - V(t-\tau(t))\right), \quad t \in [\underline{t},\ \ \bar{t}]. \tag{3.47}$$

By using (3.43) we have $\mathcal{D}^+ V(t) \leq \mu_1 V(t)$ for $x(t) \in \mathbb{R}^n \backslash \mathbb{S}$ and $t \in [\underline{t},\ \ \bar{t}]$ and this gives

$$V(t) \leq V(\underline{t}) e^{\mu_1 (t-\underline{t})}, \quad t \in [\underline{t},\ \ \bar{t}], \tag{3.48}$$

From (3.48), we know

$$V(\bar{t}) \leq V(\underline{t}) e^{\mu_1(\bar{t}-\underline{t})} \leq \hat{\mu}\lambda_0 M^2(\phi) e^{\mu_1 \beta} = \frac{\hat{\mu}}{\hat{\mu}+\eta}\lambda_0 M^2(\phi) < \lambda_0 M^2(\phi), \tag{3.49}$$

since $\eta > 0$. This leads to a contradiction and therefore (3.44) holds for $t \in [0,\ t_1)$.
(b3) Suppose for any integer $k \geq 1$ that

$$V(t) < \lambda_0 M^2(\phi), \quad t \in [-\tau,\ t_k). \tag{3.50}$$

We try to prove

$$V(t) < \lambda_0 M^2(\phi), \quad t \in [t_k,\ t_{k+1}). \tag{3.51}$$

If not, there exists $t \in [t_k,\ t_{k+1})$ such that $V(t) \geq \lambda_0 M^2(\phi)$. Set $t^* = \inf\{t : V(t) \geq \lambda_0 M^2(\phi)$ and $t \in [t_k,\ t_{k+1})\}$. From the assumption $V(t) < \lambda_0 M^2(\phi)$ for $t \in [-\tau,\ t_k)$ and Lemma 3.1, we have

$$\begin{aligned}
V(t_k) &\leq \mu V(t_k^-) + (1-\mu) e^{2\gamma t} x^T(t_k^- - \tau) Q x(t_k^- - \tau) \\
&\leq \mu V(t_k^-) + (1-\mu) V(t_k - \tau) \\
&< \mu\lambda_0 M^2(\phi) + (1-\mu)\lambda_0 M^2(\phi) \\
&= \lambda_0 M^2(\phi),
\end{aligned} \tag{3.52}$$

if $t_k - \tau$ is not an impulsive instant. For the case that $t_k - \tau$ is an impulsive instant, Lemma 3.1 gives $V(t_k) \leq \mu V(t_k^-) < \lambda_0 M^2(\phi)$ since $\mu < 1$. Therefore, for both cases we have $V(t_k) < \lambda_0 M^2(\phi)$ and this means $t^* \in (t_k,\ t_{k+1})$ and $V(t^*) = \lambda_0 M^2(\phi)$. Set $\bar{t} = \sup\{t : V(t) \leq \hat{\mu}\lambda_0 M^2(\phi)$ and $t \in [t_k,\ t^*)\}$, then $V(\bar{t}) = \hat{\mu}\lambda_0 M^2(\phi)$. Moreover, for $t \in [\bar{t},\ t^*]$, we have

$$\mathcal{D}^+ V(t) \leq \mathcal{D}^+ V(t) + \alpha\left(\frac{1}{\hat{\mu}} V(t) - V(t-\tau(t))\right). \tag{3.53}$$

Then, by using the same treatment as in **(b2)** we will arrive at $V(t^*) < \lambda_0 M^2(\phi)$, which obviously contradicts the fact $V(t^*) = \lambda_0 M^2(\phi)$.

Thus, by the method of mathematical induction, (3.51) holds for any integer k. So (3.44) is true and (3.34) follows by applying Lemma 2.1. ■

Remark 3.2 *Since $\mu < 1$ means $\mu_1 = -\frac{\ln\hat{\mu}}{\beta} < 0$ (this is because we have assumed $\hat{\mu} \in [\mu, 1)$ and $\hat{\mu} + \eta < 1$), condition (3.43) implies that the static neural network (2.1) without impulses may be not dissipative in the set $\mathbb{S}$. Thus, Theorem 3.2 can be applied to design an impulsive control law to let the static neural network to be a dissipative one.*

Similar to Corollary 3.1, we can get the following result for the case $\mu < 1$ and $\tau(t) \equiv \tau$, i.e., the delay argument is a constant number.

Corollary 3.2 *Suppose $\tau(t) \equiv \tau$ and there exist symmetric positive matrices P, Q, D_1, D_2, positive diagonal matrices $U_{ij}(1 \le i, j \le 3)$, scalar numbers $\mu < 1$, $\gamma > 0$, $\alpha > 0$ such that*

$$\begin{aligned}
&\Psi_1 = (I + W_k)^T P(I + W_k) - \mu P < 0, \ \forall k \ge 1,\\
&\Psi_2 = (I + W_k)^T Q(I + W_k) - \mu Q < 0, \ \forall k \ge 1,\\
&\Omega = \begin{bmatrix} \Omega_{11} & 0 & 0 & \Omega_{14} & \Omega_{15} & 0 \\ \star & \Omega_{22} & 0 & 0 & \Omega_{25} & \Omega_{26} \\ \star & \star & \Omega_{33} & 0 & 0 & \Omega_{36} \\ \star & \star & \star & \Omega_{44} & 0 & 0 \\ \star & \star & \star & \star & \Omega_{55} & 0 \\ \star & \star & \star & \star & \star & \Omega_{66} \end{bmatrix} < 0,
\end{aligned} \tag{3.54}$$

where

$$\begin{aligned}
\Omega_{11} &= 2D_1 + (2\gamma + \sigma) P - PA - A^T P^T + C^T[2U_{11}L_1 + (L_1 - 2L_0)^T U_{21}L_1 - 2L_0^T U_{31}L_1]C,\\
\Omega_{14} &= -C^T U_{11} + C^T L_0^T U_{21} + C^T L_0^T U_{31} + C^T L_1^T U_{31}^T,\\
\Omega_{15} &= P,\\
\Omega_{22} &= 2D_2 + (2\gamma + \sigma) Q - \alpha e^{-2\gamma\tau} P - QA - A^T Q^T + C^T[2U_{12}L_1 + (L_1 - 2L_0)^T U_{22}L_1\\
&\quad - 2L_0^T U_{32}L_1]C,\\
\Omega_{25} &= -C^T U_{12} + C^T L_0^T U_{22} + C^T L_0^T U_{32} + C^T L_1^T U_{32}^T,\\
\Omega_{26} &= Q,\\
\Omega_{33} &= -\alpha e^{-2\gamma\tau} Q + C^T[2U_{13}L_1 + (L_1 - 2L_0)^T U_{23}L_1 - 2L_0^T U_{33}L_1]C,\\
\Omega_{36} &= -C^T U_{13} + C^T L_0^T U_{23} + C^T L_0^T U_{33} + C^T L_1^T U_{33}^T,\\
\Omega_{44} &= -U_{21} - 2U_{31},\\
\Omega_{55} &= -U_{22} - 2U_{32},\\
\Omega_{66} &= -U_{23} - 2U_{33},\\
\beta &= \sup_{k\ge1}\{t_k - t_{k-1}\}, \quad t_0 = 0,\\
\sigma &= \begin{cases} \alpha, & \text{if } 1 \le \alpha\beta, \\ \frac{\alpha}{\mu} + \frac{\ln\mu}{\beta}, & \text{if } \alpha\beta \le \mu, \\ \frac{1+\ln(\alpha\beta)}{\beta}, & \text{if } \mu < \alpha\beta < 1. \end{cases}
\end{aligned}$$

Then neural network (2.1) is global exponential dissipative with dissipativity rate γ, and in particular (3.34) holds.

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Remark 3.3 *For the case $\mu \geq 1$, the analysis in Theorem 3.1 is similar to Theorem 3.1 of [14] and Theorem 1 of [3]. This mainly benefits from the fact that $V(t_k) \leq \mu V(t_k^-)$ holds for whether $t_k - \tau$ is an impulsive instant or not. However, for stabilizing impulses, i.e., $\mu < 1$, if $Q \neq 0$ the inequality $V(t_k) \leq \mu V(t_k^-)$ which is very important for the proof of Theorem 3.3 in [14] and Theorem 5 in [3] does not hold, provided $t_k - \tau$ is not an impulsive instant. In our proof, we give a close look at this problem and find that $V(t_k) \leq \mu V(t_k^-)$ is just an interim step, while $V(t_k) < \lambda_0 M^2(\phi)$ is the final goal. In [3, 14], because $Q = 0$, i.e., $V(t) = e^{2\gamma t} x^T(t) P x(t)$, this inequality can be deduced straightforwardly as $V(t_k) \leq \mu V(t_k^-) < \lambda_0 M^2(\phi)$, since $\mu < 1$. For $Q \neq 0$, we prove this inequality by using the key relation $V(t_k) \leq \mu V(t_k^-) + (1-\mu) e^{2\gamma t} x^T(t_k^- - \tau) Q x(t_k^- - \tau)$ (see(3.52)).*

Remark 3.4 *A natural question is why we do not use other Lyapunov functions to perform the dissipativity analysis, such as $V(t) = e^{2\gamma t} x^T(t) P x(t) + \int_{-\tau}^{0} \int_{t+\theta}^{t} e^{2\gamma s} \dot{x}^T(s) Q \dot{x}(s) ds d\theta$, where $\int_{-\tau}^{0} \int_{t+\theta}^{t} e^{2\gamma s} \dot{x}^T(s) Q \dot{x}(s) ds d\theta$ (with $Q > 0$) is a commonly used component in the stability analysis of NNs with time delay (see, e.g., [10–13, 26, 27]). In such case, one may check that it is difficult to repeat the proof of Theorems 3.1 and 3.2. In particular, for $\mu \geq 1$ the inequality $V(t_k) \leq \mu V(t_k^-)$ still holds (the proof is similar to Lemma 3.1) but it is difficult to get a quadratic form (like (3.6)) of the left hand of (3.10). For the case $\mu < 1$, an inequality like $V(t_k) \leq \mu V(t_k^-) + (1-\mu) \int_{-\tau}^{0} \int_{t_k+\theta}^{t_k} e^{2\gamma s} \dot{x}^T(s) Q \dot{x}(s) ds d\theta$ still holds, and $V(t_k) < \lambda_0 M^2(\phi)$ can be proved. However, it is difficult to get a quadratic form (like (3.40)) of the left hand of (3.43). We remark that the quadratic forms shown in (3.6) and (3.40) play an important role to establish LMI criteria.*

Remark 3.5 *We note that the inequalities (3.7) are very important to derive efficient LMIs for static NNs with time delay and impulses. Without (3.7), the obtained LMIs do not contain the system matrix C in (2.1). Therefore, (3.7) is the tie between the matrix C and the obtained LMIs. This is the main difference between the analysis of the static NNs and the local field NNs.*

3.3. Application to stability analysis.

We next show that the LMI criteria given in Theorems 3.1, 3.2 and Corollaries 3.1, 3.2 can be easily reduced to exponential stability conditions. For concise, we assume in (2.1) that $\psi(t) \equiv 0$ and therefore the attractive set $\mathbb{S}$ defined by Theorems 3.1 and 3.2 shrinks to the origin $x^* = 0$. In this case, we can let $D_1 = D_2 = 0$ in (3.3a), (3.29), (3.31) and (3.54), and it is clear that the reduced LMI criteria can be regarded as exponential stability conditions. In the next section, we provide numerical results to show that the deduced stability conditions are mush less conservative than the existing ones.

Moreover, we can also consider the following static NNs

$$x'(t) = -Ax(t) + g(Bx(t)) + f(Cx(t - \tau(t))), \tag{3.55}$$

which is obviously more general than the one discussed in this paper. Similar to (3.7), by

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additionally introducing the following inequalities

$$
\begin{aligned}
&0 \le 2e^{2\gamma t}\left[Bz_j\right]^T V_{1j}[H_1\left(Bz_j\right) - g(Bz_j)] = e^{2\gamma t} z_j^T[2B^T V_{1j} H_1 B]z_j + 2e^{2\gamma t} z_j^T[-B^T V_{1j}]g(Bz_j),\\
&0 \le e^{2\gamma t}\left[(Bz_j)^T(H_1^T - H_0^T) + g^T(Bz_j) - (Bz_j)^T H_0^T\right] V_{2j}[H_1 Bz_j - g(Bz_j)]\\
&\quad = e^{2\gamma t}\left(z_j^T\left[B^T(H_1 - 2H_0)^T V_{2j} H_1 B\right]z_j + 2z_j^T[B^T H_0^T V_{2j}]g(Bz_j) + g^T(Bz_j)[-V_{2j}]g(Bz_j)\right),\\
&0 \le 2e^{2\gamma t}\left[g^T(Bz_j) - (Bz_j)^T H_0^T\right] V_{3j}[H_1(Bz_j) - g(Bz_j)]\\
&\quad = z_j^T(t)[-2B^T L_0^T V_{3j} H_1 B]z_j + 2z_j^T[C^T H_0^T V_{3j} + B^T H_1^T V_{3j}^T]g(Bz_j) + g^T(Bz_j)[-2V_{3j}]g(Bz_j),
\end{aligned}
\tag{3.56}
$$

it can be shown that the results obtained in Theorems 3.1, 3.2 and Corollaries 3.1, 3.2 can be straightforwardly generalized to (3.55) and therefore this is a trivial difference. Here, V_{ij} ($i = 1, 2, 3,\ j = 1, 2, 3, 4$) are positive diagonal matrices, $H_0 = \mathrm{diag}(h_1^-, h_2^-, \ldots, h_n^-)$ and $H_1 = \mathrm{diag}(h_1^+, h_2^+, \ldots, h_n^+)$ which satisfy

$$
g_i(0) = 0, \quad h_i^- \le \frac{g_i(s_1) - g_i(s_2)}{s_1 - s_2} \le h_i^+, \quad \forall s_1,\ s_2 \in \mathbb{R} \text{ and } s_1 \ne s_2, \quad i = 1, 2, \ldots, n. \tag{3.57}
$$

4. Numerical results.

In this section, we do several numerical simulations to validate the effectiveness of our results presented in Section 3.

Example 1. Consider the following static neural network

$$
\begin{cases}
\mathcal{D}^+ x(t) = -\begin{bmatrix} 2.2 & 0 \\ 0 & 2.8 \end{bmatrix} x(t) + f\left(\begin{bmatrix} 0.75 & 0.05 \\ 0.1 & 1.3 \end{bmatrix} x(t-\tau)\right) + \psi(t), & t \ne t_k,\\
\Delta x(t_k) = W_k x(t_k^-), & t = t_k,\\
x(t) = \phi, & t \le 0,
\end{cases}
\tag{4.1}
$$

where $\psi(t) = (0.5\sin(t), 0.5\cos(t))^T$ and $f(x) = \frac{|x+1|-|x-1|}{2}$. Then, we have $L_1 = \mathrm{diag}(1, 1)$ and $L_0 = \mathrm{diag}(0, 0)$. We set $W_k = \mathrm{diag}(\omega, \omega)$ and the argument $\omega > 0$ can be viewed as the magnitude of the impulses. With $\omega > 0$, the linear matrix inequalities $(I + W_k)^T P(I + W_k) - \mu P < 0$ and $(I + W_k)^T Q(I + W_k) - \mu Q < 0$ imply $\mu > (1 + \omega)^2 > 1$ and therefore the impulses are disturbances.

Let the exponential dissipative rate be $\gamma = 10^{-4}$ and the distance between two consecutive impulsive instants be 0.6, i.e., $t_k = t_{k-1} + 0.6$. In Table 4.1, we list the maximum impulse magnitude (denoted by $\omega_{\max}$) corresponding to different τ. The $\omega_{\max}$ is obtained by solving the LMIs in Theorem 3.1 and Corollary 3.1. For each $\omega_{\max}$, the corresponding parameter α is also given in the table.

We see from Table 4.1 that the results predicted by Corollary 3.1 is significantly less conservative than the ones predicted by Theorem 3.1, when the delay argument is a constant number. For example, if $\tau = 1$, the maximum impulse magnitude $\omega_{\max}$ obtained from Corollary 3.1 is 0.6021, while it is only 0.5463 from Theorem 3.1. Besides this, it is interesting to find in Table 4.1 that the maximum impulse magnitude $\omega_{\max}$ seems to be robust with respect to τ.

For $\tau = 1$, the dynamic behavior of impulse-free system (4.1) with different initial vector ϕ is shown in Figure 4.1 on the left, where we see that the impulse-free system is dissipative. With

Table 4.1: $\omega_{\max}$ corresponding to different τ

Method	$\tau = 0.5$	$\tau = 1$	$\tau = 2$	$\tau = 4$	$\tau = 5$
Theorem 3.1	0.5463	0.5463	0.5462	0.5460	0.5459
α	*0.8038*	*0.8038*	*0.8036*	*0.8037*	*0.8039*
Corollary 3.1	0.6022	0.6021	0.6021	0.6020	0.6018
α	*0.6101*	*0.6102*	*0.6101*	*0.6102*	*0.6101*

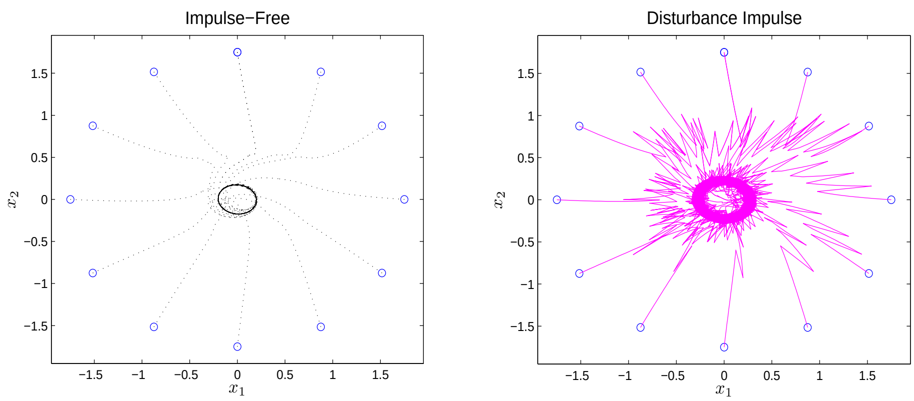

Figure 4.1: Behavior of $x(t)$ in system (4.1) without (left) and with (right) impulses. In each panel, the marker '$\circ$' denotes the initial state of the solution trajectory.

impulse matrices $W_k = \mathrm{diag}(0.6021, 0.6021)$, impulsive system (4.1) converges to an irregular *circle*, which can be see clearly in Figure 4.1 on the right.

Example 2. We next consider the following system

$$\begin{cases} x'(t) = -\begin{bmatrix} 1e-3 & 0 \\ 0 & 2e-3 \end{bmatrix} x(t) + \begin{bmatrix} 1.08 & -2 \\ 6 & 0.92 \end{bmatrix} f(x(t-\tau(t))) + \psi(t), & t \neq t_k, \\ \Delta x(t_k) = W_k x(t_k^-), & t = t_k, \\ x(t) = \phi, & t \leq 0, \end{cases} \tag{4.2}$$

where f is the same function as we has used in Example 1, $\tau(t) = \frac{2}{1+\sin^2(10t)}$, $W_k = \mathrm{diag}(-0.08, -0.08)$ and $\psi(t) = \left(\frac{-1}{4}\sin\left(\frac{\sqrt{t}}{(1+0.28\sqrt{t})} + t\right), \frac{\cos(t)}{4}\right)^T$. Then, we know $\max_{t\geq 0}\tau(t) = 2$ and the impulses are stabilizing. The simulations of impulse-free system (4.2) with 8 different initial values are shown in the left column of Figure 4.2 corresponding to three different time t, where we see clearly that the solution $x(t)$ diverges to infinity as the evolution time t increases. Let the exponential dissipative rate be $\gamma = 2e-4$. By solving the LMIs in Theorem 3.2, we get the maximal distance between two consecutive impulsive instants, $\beta_{\max} = 0.0442$. With $t_k = t_{k-1} + 0.0442$ and the same initial values, we simulate the impulsive system (4.2) and the solution trajectories are plotted in the right column of Figure 4.2. We see clearly in these three panels that each trajectory goes to

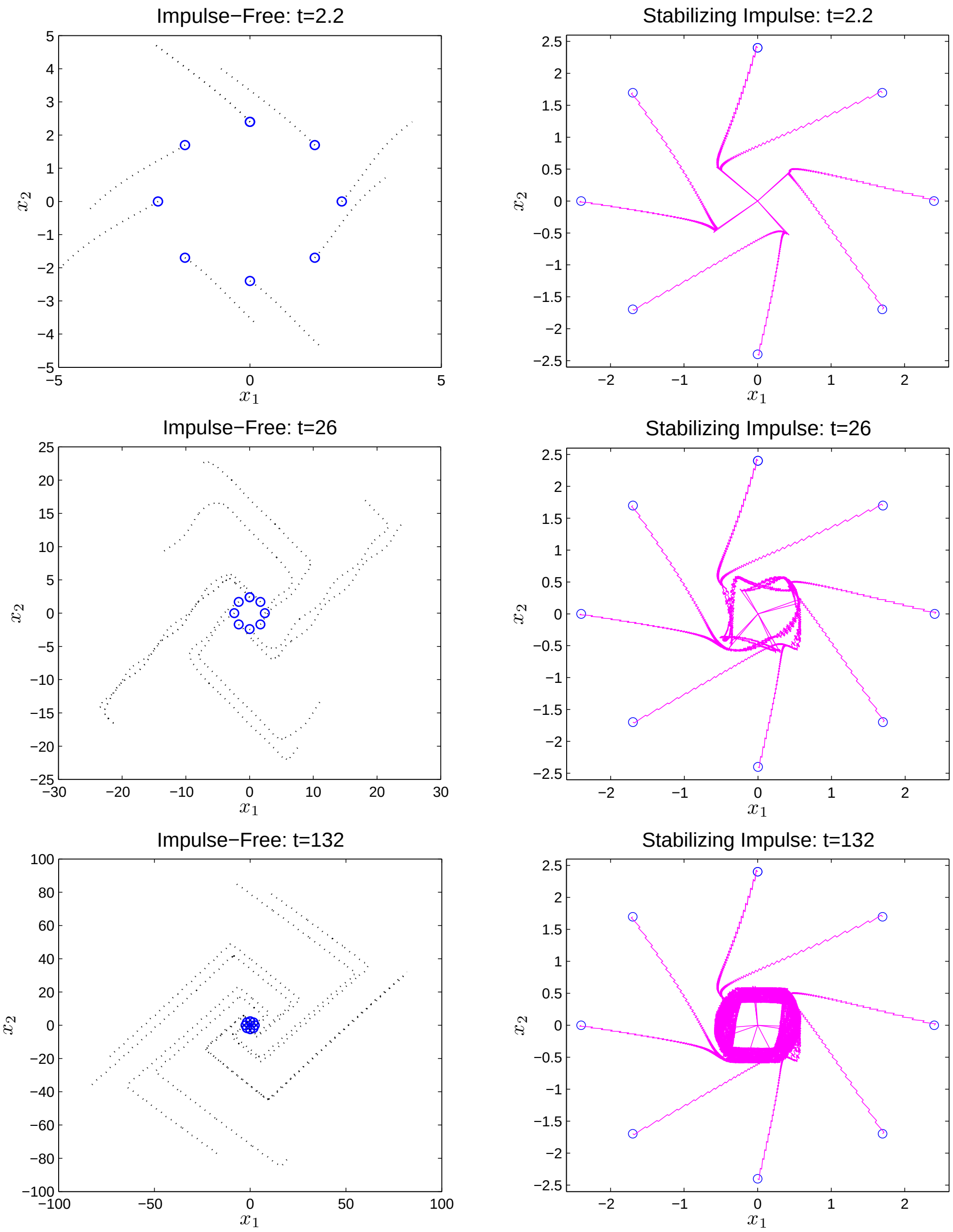

Figure 4.2: Behavior of $x(t)$ in system (4.2) without (left column) and with (right column) impulses. In each panel, the marker '∘' denotes the initial state of the solution trajectory.

a bounded domain as the evolution time t increases and therefore system (4.2) with stabilizing

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impulses is really dissipative.
Example 3. Our last example is the following static neutral network

$$\begin{cases} \mathcal{D}^+ x(t) = -\begin{bmatrix} 0.2 & 0 \\ 0 & 1.3 \end{bmatrix} x(t) + f\left(\begin{bmatrix} -1.5 & -0.12 \\ -0.26 & -2.5 \end{bmatrix} x(t-\tau)\right), & t \neq t_k, \\ \Delta x(t_k) = \begin{bmatrix} -0.3 & 0 \\ 0 & -0.3 \end{bmatrix} x(t_k^-), & t = t_k, \\ x(t) = \phi, & t \leq 0, \end{cases} \tag{4.3}$$

where $f(x) = \tanh(x)$ and $t_k = t_{k-1} + \beta$. Then, we know $L_1 = \mathrm{diag}(1,1)$ and $L_0 = \mathrm{diag}(0,0)$. Clearly, the impulses are stabilizing. Since the external input function is chosen $\psi(t) \equiv 0$, we consider the global exponential stability of (4.3). The criteria given by Zhao and Wang [13] require

$$\min_{1\leq i\leq n}\left[2A_{i,i} - L_i \sum_{j=1}^{n} |C_{i,j}|\right] > \max_{1\leq i\leq n}\left[L_i \sum_{j=1}^{n} |C_{j,i}|\right] > 0.$$

However, for system (4.3), we have

$$\min_{1\leq i\leq 2}\left[2A_{i,i} - L_i \sum_{j=1}^{2} |C_{i,j}|\right] = -0.16 \text{ and } \max_{1\leq i\leq 2}\left[L_i \sum_{j=1}^{2} |C_{j,i}|\right] = 2.62.$$

Therefore, the criteria given by Zhao and Wang [13] are invalid.

Let $t_k - t_{k-1} = \beta$. The quantity β can be viewed as the distance between two consecutive impulsive instants. In Table 4.2, for different exponential convergence rate γ, we list the maximum distance (denoted by $\beta_{\max}$) by using the criterion obtained in this paper (by letting $D_1 = D_2 = 0$ in Theorem 3.2 and Corollary 3.2) and the one given in [14]. From the results listed in Table 4.2

Table 4.2: $\beta_{\max}$ corresponding to different γ

Method	$\gamma = 0.001$	$\gamma = 0.005$	$\gamma = 0.01$	$\gamma = 0.05$
Theorem 3.2	0.0616	0.0601	0.0532	0.0497
α	*0.6322*	*0.6542*	*0.6534*	*0.6904*
Corollary 3.2	0.0746	0.0702	0.0639	0.0627
α	*0.6262*	*0.6292*	*0.6356*	*0.6378*
Thm. 3.3 in [14]	0.0521	0.0489	0.0498	0.0443
α	*0.5257*	*0.5456*	*0.5578*	*0.5894*

we see that the stability criterion deduced from Corollary 3.2 is significantly less conservative than the one given in [14]. For example, for $\gamma = 0.001$, the quantity $\beta_{\max}$ predicted by Corollary 3.2 and Theorem 3.3 in [14] is 0.0746 and 0.0521 respectively, and 43.19% improvement[1] is obtained by using Corollary 3.2. For $\tau = 1.6$ and $\phi(t) = (-0.07, 0.07)^T$, we plot in Figure 4.3 on the left the solutions $x_1(t)$ and $x_2(t)$ without impulses, where we see that the solutions behave like chaos. After imposing stabilizing impulses to (4.3) with $t_k = t_{k-1} + 0.0746$, we show in Figure 4.3 on the right the behavior of the solutions and it is clear that the system is really stabilized.

[1] For two different quantities a and b, the improvement of b against a is defined by percentage $\left(100 \times \frac{b-a}{a}\right)\%$.

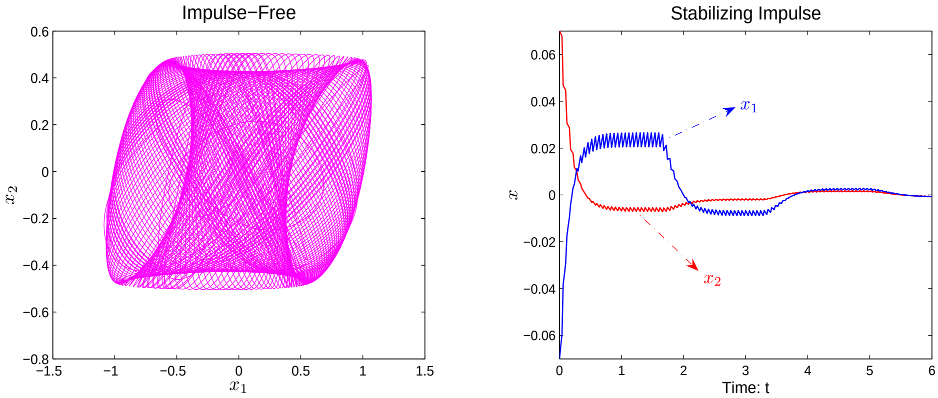

Figure 4.3: Left: chaos-like dynamic behavior of the solutions of (4.3) without impulses; Right: the solution profiles after impulsive stabilization.

5. Conclusion.

The local field neural networks and the static neural networks typically represent two fundamental models in the research of neural networks. The former has been investigated widely and deeply by many authors, while the latter has not received so much attention and systematic analysis is still rare, particularly when both the time delay and impulses are taken into account. In this paper, we propose new analysis method to study the problem of global exponential dissipativity of static neural networks with impulses and time delays. This method can be regarded as a generalization of the one used in [3] and [14], while the original method can only handle very simple Lyapunov function. Several sufficient conditions concerning global exponential dissipativity were established in terms of LMIs and therefore they can be checked efficiently via the LMI toolbox in MATLAB. Moreover, we show that the dissipativity conditions can be straightforwardly reduced to stability conditions for the static neural networks with impulses and time delay. Benefitting from the new Lyapunov function, the deduced stability conditions are much less conservative than the existing ones.

Conflict of interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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Barnes-type Peters of the first kind and poly-Cauchy of the first kind mixed-type polynomials

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Abstract

In this paper, by considering Barnes-type Peters polynomials of the first kind as well as poly-Cauchy polynomials of the first kind, we define and investigate the mixed-type polynomials of these polynomials. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

1 Introduction

In this paper, we consider the polynomials

$$s_n^{(k)}(x) = s_n^{(k)}(x|\lambda;\mu) = s_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)$$

1

called the Barnes-type Peters of the first kind and poly-Cauchy of the first kind mixed-type polynomials, whose generating function is given by

$$\prod_{j=1}^{r}\bigl(1+(1+t)^{\lambda_j}\bigr)^{-\mu_j}\mathrm{Lif}_k\bigl(\ln(1+t)\bigr)(1+t)^x=\sum_{n=0}^{\infty}s_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\frac{t^n}{n!}\,, \quad (1)$$

where $\lambda_1,\dots,\lambda_r,\mu_1,\dots,\mu_r\in\mathbb{C}$ with $\lambda_1,\dots,\lambda_r\neq 0$. Here, $\mathrm{Lif}_k(x)$ $(k\in\mathbb{Z})$ is the polyfactorial function ([6]) defined by

$$\mathrm{Lif}_k(x)=\sum_{m=0}^{\infty}\frac{x^m}{m!(m+1)^k}\,.$$

When $x=0$, $s_n^{(k)}=s_n^{(k)}(0)=s_n^{(k)}(0|\lambda;\mu)=s_n^{(k)}(0;\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)$ is called the the Barnes-type Peters of the first kind and poly-Cauchy of the first kind mixed-type number.

Recall that the Barnes-type Peters polynomials of the first kind, denoted by $s_n(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)$, are given by the generating function as

$$\prod_{j=1}^{r}\bigl(1+(1+t)^{\lambda_j}\bigr)^{-\mu_j}(1+t)^x=\sum_{n=0}^{\infty}s_n(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\frac{t^n}{n!}\,.$$

If $r=1$, then $s_n(x|\lambda;\mu)$ are the Peters polynomials of the first kind. Peters polynomials were mentioned in [9, p.128] and have been investigated in e.g. [5].

The poly-Cauchy polynomials of the first kind, denoted by $c_n^{(k)}(x)$ ([3, 7]), are given by the generating function as

$$\mathrm{Lif}_k\bigl(\ln(1+t)\bigr)(1+t)^x=\sum_{n=0}^{\infty}c_n^{(k)}(-x)\frac{t^n}{n!}\,.$$

The generalized Barnes-type Euler polynomials $E_n(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)$ are defined by the generating function

$$\prod_{j=1}^{r}\left(\frac{2}{1+e^{\lambda_j t}}\right)^{\mu_j}e^{xt}=\sum_{n=0}^{\infty}E_n(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\frac{t^n}{n!}\,.$$

If $\mu_1=\cdots=\mu_r=1$, then $E_n(x|\lambda_1,\dots,\lambda_r)=E_n(x|\lambda_1,\dots,\lambda_r;1,\dots,1)$ are called the Barnes-type Euler polynomials. If further $\lambda_1=\cdots=\lambda_r=1$, then $E_n^{(r)}(x)=E_n(x|1,\dots,1;1,\dots,1)$ are called the Euler polynomials of order r.

In this paper, by considering Barnes-type Peters polynomials of the first kind as well as poly-Cauchy polynomials of the first kind, we define and investigate the mixed-type polynomials of these polynomials. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

2

2 Umbral calculus

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable t:

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \middle| a_k \in \mathbb{C} \right\}. \tag{2}$$

Let $\mathbb{P} = \mathbb{C}[x]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L|p(x)\rangle$ is the action of the linear functional L on the polynomial $p(x)$, and we recall that the vector space operations on $\mathbb{P}^*$ are defined by $\langle L+M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle$, $\langle cL|p(x)\rangle = c\langle L|p(x)\rangle$, where c is a complex constant in $\mathbb{C}$. For $f(t) \in \mathcal{F}$, let us define the linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle = a_n, \quad (n \geq 0). \tag{3}$$

In particular,

$$\left\langle t^k|x^n\right\rangle = n!\delta_{n,k} \quad (n,k \geq 0), \tag{4}$$

where $\delta_{n,k}$ is the Kronecker's symbol.

For $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$, we have $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. That is, $L = f_L(t)$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the *umbral algebra* and the *umbral calculus* is the study of umbral algebra. The order $O\big(f(t)\big)$ of a power series $f(t)(\neq 0)$ is the smallest integer k for which the coefficient of t^k does not vanish. If $O\big(f(t)\big) = 1$, then $f(t)$ is called a *delta series*; if $O\big(f(t)\big) = 0$, then $f(t)$ is called an *invertible series*. For $f(t), g(t) \in \mathcal{F}$ with $O\big(f(t)\big) = 1$ and $O\big(g(t)\big) = 0$, there exists a unique sequence $s_n(x)$ ($\deg s_n(x) = n$) such that $\left\langle g(t)f(t)^k|s_n(x)\right\rangle = n!\delta_{n,k}$, for $n, k \geq 0$. Such a sequence $s_n(x)$ is called the *Sheffer sequence* for $\big(g(t), f(t)\big)$ which is denoted by $s_n(x) \sim \big(g(t), f(t)\big)$.

For $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$\langle f(t)g(t)|p(x)\rangle = \langle f(t)|g(t)p(x)\rangle = \langle g(t)|f(t)p(x)\rangle \tag{5}$$

and

$$f(t) = \sum_{k=0}^{\infty} \left\langle f(t)|x^k\right\rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \left\langle t^k|p(x)\right\rangle \frac{x^k}{k!} \tag{6}$$

([9, Theorem 2.2.5]). Thus, by (6), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad \text{and} \quad e^{yt}p(x) = p(x+y). \tag{7}$$

Sheffer sequences are characterized in the generating function ([9, Theorem 2.3.4]).

Lemma 1 *The sequence $s_n(x)$ is Sheffer for $\big(g(t), f(t)\big)$ if and only if*

$$\frac{1}{g\big(\bar{f}(t)\big)} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{s_k(y)}{k!} t^k \quad (y \in \mathbb{C}),$$

3

where $\bar{f}(t)$ is the compositional inverse of $f(t)$.

For $s_n(x) \sim \big(g(t), f(t)\big)$, we have the following equations ([9, Theorem 2.3.7, Theorem 2.3.5, Theorem 2.3.9]):

$$f(t)s_n(x) = ns_{n-1}(x) \quad (n \geq 0), \tag{8}$$

$$s_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \left\langle g\big(\bar{f}(t)\big)^{-1} \bar{f}(t)^j | x^n \right\rangle x^j, \tag{9}$$

$$s_n(x+y) = \sum_{j=0}^{n} \binom{n}{j} s_j(x) p_{n-j}(y) \,, \tag{10}$$

where $p_n(x) = g(t)s_n(x)$.

Assume that $p_n(x) \sim \big(1, f(t)\big)$ and $q_n(x) \sim \big(1, g(t)\big)$. Then the transfer formula ([9, Corollary 3.8.2]) is given by

$$q_n(x) = x \left(\frac{f(t)}{g(t)}\right)^n x^{-1} p_n(x) \quad (n \geq 1).$$

For $s_n(x) \sim \big(g(t), f(t)\big)$ and $r_n(x) \sim \big(h(t), l(t)\big)$, assume that

$$s_n(x) = \sum_{m=0}^{n} C_{n,m} r_m(x) \quad (n \geq 0) \,.$$

Then we have ([9, p.132])

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{h\big(\bar{f}(t)\big)}{g\big(\bar{f}(t)\big)} l\big(\bar{f}(t)\big)^m \middle| x^n \right\rangle . \tag{11}$$

3 Main results

From the definition (1), $s_n^{(k)}(x|\lambda_1, \dots, \lambda_r; \mu_1, \dots, \mu_r)$ is the Sheffer sequence for the pair

$$g(t) = \prod_{j=1}^{r} (1 + e^{\lambda_j t})^{\mu_j} \frac{1}{\mathrm{Lif}_k(t)} \quad \text{and} \quad f(t) = e^t - 1.$$

So,

$$s_n^{(k)}(x|\lambda_1, \dots, \lambda_r; \mu_1, \dots, \mu_r) \sim \left(\prod_{j=1}^{r} (1 + e^{\lambda_j t})^{\mu_j} \frac{1}{\mathrm{Lif}_k(t)}, e^t - 1 \right) . \tag{12}$$

3.1 Explicit expressions

Let $(n)_j = n(n-1)\cdots(n-j+1)$ $(j \ge 1)$ with $(n)_0 = 1$. The (signed) Stirling numbers of the first kind $S_1(n,m)$ are defined by

$$(x)_n = \sum_{m=0}^{n} S_1(n,m)x^m \,.$$

Theorem 1

$$\begin{aligned}
&s_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\
&= 2^{-\sum_{j=1}^r \mu_j} \sum_{m=0}^{n} S_1(n,m) \sum_{l=0}^{m} \frac{\binom{m}{l}}{(l+1)^k} E_{m-l}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) && (13)\\
&= \sum_{j=0}^{n}\sum_{l=0}^{n-j} \binom{n}{l} S_1(n-l,j) s_l^{(k)} x^j && (14)\\
&= \sum_{j=0}^{n}\sum_{l=0}^{n-j}\sum_{i=0}^{l} \binom{n}{l}\binom{l}{i} S_1(n-l,j) c_i^{(k)} s_{l-i}(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) x^j && (15)\\
&= \sum_{l=0}^{n} \binom{n}{l} s_{n-l}(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) c_l^{(k)}(-x)\,, && (16)\\
&= \sum_{l=0}^{n} \binom{n}{l} s_l(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) c_{n-l}^{(k)}\,. && (17)
\end{aligned}$$

Proof. Since

$$\prod_{j=1}^{r}(1+e^{\lambda_j t})^{\mu_j} \frac{1}{\mathrm{Lif}_k(t)} s_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) \sim (1, e^t-1) \tag{18}$$

and

$$(x)_n \sim (1, e^t-1)\,, \tag{19}$$

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we have

$$
\begin{aligned}
&s_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)=s_n^{(k)}(x)\\
&=\prod_{j=1}^{r}(1+e^{\lambda_j t})^{-\mu_j}\mathrm{Lif}_k(t)(x)_n\\
&=\sum_{m=0}^{n}S_1(n,m)\prod_{j=1}^{r}(1+e^{\lambda_j t})^{-\mu_j}\mathrm{Lif}_k(t)x^m\\
&=\sum_{m=0}^{n}S_1(n,m)\prod_{j=1}^{r}(1+e^{\lambda_j t})^{-\mu_j}\sum_{l=0}^{m}\frac{t^l}{l!(l+1)^k}x^m\\
&=\sum_{m=0}^{n}S_1(n,m)\prod_{j=1}^{r}(1+e^{\lambda_j t})^{-\mu_j}\sum_{l=0}^{m}\frac{(m)_l}{l!(l+1)^k}x^{m-l}\\
&=\sum_{m=0}^{n}S_1(n,m)\sum_{l=0}^{m}\frac{\binom{m}{l}}{(l+1)^k}\prod_{j=1}^{r}(1+e^{\lambda_j t})^{-\mu_j}x^{m-l}\\
&=2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}S_1(n,m)\sum_{l=0}^{m}\frac{\binom{m}{l}}{(l+1)^k}\prod_{j=1}^{r}\left(\frac{2}{1+e^{\lambda_j t}}\right)^{\mu_j}x^{m-l}\\
&=2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}S_1(n,m)\sum_{l=0}^{m}\frac{\binom{m}{l}}{(l+1)^k}E_{m-l}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\,.
\end{aligned}
$$

So, we get (13).
By (9) with (12), we get

$$
\begin{aligned}
&\left\langle g\big(\bar f(t)\big)^{-1}\bar f(t)^j|x^n\right\rangle\\
&=\left\langle\prod_{j=1}^{r}\big(1+(1+t)^{\lambda_j}\big)^{-\mu_j}\mathrm{Lif}_k\big(\ln(1+t)\big)\big(\ln(1+t)\big)^j\Big|x^n\right\rangle\\
&=\left\langle\prod_{j=1}^{r}\big(1+(1+t)^{\lambda_j}\big)^{-\mu_j}\mathrm{Lif}_k\big(\ln(1+t)\big)\Big|j!\sum_{l=j}^{\infty}S_1(l,j)\frac{t^l}{l!}x^n\right\rangle\\
&=j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\left\langle\prod_{j=1}^{r}\big(1+(1+t)^{\lambda_j}\big)^{-\mu_j}\mathrm{Lif}_k\big(\ln(1+t)\big)\Big|x^{n-l}\right\rangle\\
&=j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\left\langle\sum_{i=0}^{\infty}s_i^{(k)}\frac{t^i}{i!}\Big|x^{n-l}\right\rangle\\
&=j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)s_{n-l}^{(k)}\,.
\end{aligned}
$$

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On the other hand,

$$
\begin{aligned}
&\left\langle g\big(\bar{f}(t)\big)^{-1}\bar{f}(t)^j \middle| x^n\right\rangle \\
&= j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\left\langle \prod_{j=1}^{r}\big(1+(1+t)^{\lambda_j}\big)^{-\mu_j}\middle|\mathrm{Lif}_k\big(\ln(1+t)\big)x^{n-l}\right\rangle \\
&= j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\sum_{i=0}^{n-l}\binom{n-l}{i}c_i^{(k)}\left\langle \prod_{j=1}^{r}\big(1+(1+t)^{\lambda_j}\big)^{-\mu_j}\middle|x^{n-l-i}\right\rangle \\
&= j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\sum_{i=0}^{n-l}\binom{n-l}{i}c_i^{(k)}\left\langle \sum_{m=0}^{\infty}s_m(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\frac{t^m}{m!}\middle|x^{n-l-i}\right\rangle \\
&= j!\sum_{l=j}^{n}\sum_{i=0}^{n-l}\binom{n}{l}\binom{n-l}{i}S_1(l,j)c_i^{(k)}s_{n-l-i}(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\,.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
s_n^{(k)}(x) &= \sum_{j=0}^{n}\sum_{l=0}^{n-j}\binom{n}{l}S_1(n-l,j)s_l^{(k)}x^j \\
&= \sum_{j=0}^{n}\sum_{l=0}^{n-j}\sum_{i=0}^{l}\binom{n}{l}\binom{l}{i}S_1(n-l,j)c_i^{(k)}s_{l-i}(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)x^j\,,
\end{aligned}
$$

which are the identities (14) and (15).

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Next,

$$\begin{aligned}
s_n^{(k)}(y|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) &= \left\langle \sum_{i=0}^{\infty} s_i^{(k)}(y|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\frac{t^i}{i!}\middle| x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r}\big(1+(1+t)^{\lambda_j}\big)^{-\mu_j}\mathrm{Lif}_k\big(\ln(1+t)\big)(1+t)^y \middle| x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r}\big(1+(1+t)^{\lambda_j}\big)^{-\mu_j} \middle| \mathrm{Lif}_k\big(\ln(1+t)\big)(1+t)^y x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r}\big(1+(1+t)^{\lambda_j}\big)^{-\mu_j} \middle| \sum_{l=0}^{\infty} c_l^{(k)}(-y)\frac{t^l}{l!}x^n \right\rangle \\
&= \sum_{l=0}^{n}\binom{n}{l}c_l^{(k)}(-y)\left\langle \prod_{j=1}^{r}\big(1+(1+t)^{\lambda_j}\big)^{-\mu_j} \middle| x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n}\binom{n}{l}c_l^{(k)}(-y)\left\langle \sum_{i=0}^{\infty} s_i(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\frac{t^i}{i!} \middle| x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n}\binom{n}{l}c_l^{(k)}(-y)s_{n-l}(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\,.
\end{aligned}$$

Thus, we obtain (16).
Finally, we obtain that

$$\begin{aligned}
s_n^{(k)}(y|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) &= \left\langle \sum_{i=0}^{\infty} s_i^{(k)}(y|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\frac{t^i}{i!}\middle| x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r}\big(1+(1+t)^{\lambda_j}\big)^{-\mu_j}\mathrm{Lif}_k\big(\ln(1+t)\big)(1+t)^y \middle| x^n \right\rangle \\
&= \left\langle \mathrm{Lif}_k\big(\ln(1+t)\big) \middle| \prod_{j=1}^{r}\big(1+(1+t)^{\lambda_j}\big)^{-\mu_j}(1+t)^y x^n \right\rangle \\
&= \left\langle \mathrm{Lif}_k\big(\ln(1+t)\big) \middle| \sum_{l=0}^{\infty} s_l(y|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\frac{t^l}{l!}x^n \right\rangle \\
&= \sum_{l=0}^{n} s_l(y|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\binom{n}{l}\left\langle \mathrm{Lif}_k\big(\ln(1+t)\big) \middle| x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n} s_l(y|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\binom{n}{l}\left\langle \sum_{i=0}^{\infty} c_i^{(k)}\frac{t^i}{i!} \middle| x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n}\binom{n}{l}s_l(y|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)c_{n-l}^{(k)}\,.
\end{aligned}$$

Thus, we get the identity (17). ∎

3.2 Sheffer identity

Theorem 2

$$s_n^{(k)}(x+y|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)=\sum_{j=0}^{n}\binom{n}{j}s_j^{(k)}(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)(y)_{n-j}\,. \tag{20}$$

Proof. By (12) with

$$\begin{aligned}p_n(x)&=\prod_{j=1}^{r}(1+e^{\lambda_j t})^{\mu_j}\frac{1}{\mathrm{Lif}_k(t)}s_n(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)\\&=(x)_n\sim(1,e^t-1)\,,\end{aligned}$$

using (10), we have (20). ∎

3.3 Difference relations

Theorem 3

$$\begin{aligned}s_n^{(k)}(x+1|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)-s_n^{(k)}(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)&\\=ns_{n-1}^{(k)}(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)\,.&\end{aligned} \tag{21}$$

Proof. By (8) with (12), we get

$$(e^t-1)s_n^{(k)}(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)=ns_{n-1}^{(k)}(x|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)\,.$$

By (7), we have (21). ∎

3.4 Recurrence

Theorem 4

$$\begin{aligned}
& s_{n+1}^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) \\
&= x s_n^{(k)}(x-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) \\
&\quad - 2^{-1-\sum_{j=1}^r \mu_j} \sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{i=1}^{r} S_1(n,m)\frac{\binom{m}{l}}{(l+1)^k}\mu_i\lambda_i E_{m-l}(x+\lambda_i-1|\lambda;\mu+e_i) \\
&\quad + 2^{-\sum_{j=1}^r \mu_j} \sum_{m=0}^{n}\sum_{l=0}^{m} \frac{\binom{m}{l}}{(m+2-l)^k} S_1(n,m)E_l(x-1|\lambda;\mu) \qquad (22) \\
&= x s_n^{(k)}(x-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) \\
&\quad - \frac{1}{2}\sum_{j=0}^{n}\sum_{l=0}^{n-j}\sum_{i=1}^{r}\binom{n}{l}\mu_i\lambda_i^{j+1}S_1(n-l,j)s_l^{(k)}E_j\left(\frac{x+\lambda_i-1}{\lambda_i}\right) \\
&\quad + 2^{-\sum_{j=1}^r \mu_j} \sum_{m=0}^{n}\sum_{l=0}^{m} \frac{\binom{m}{l}}{(m+2-l)^k} S_1(n,m)E_l(x-1|\lambda;\mu)\,. \qquad (23)
\end{aligned}$$

Proof. By applying

$$s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)}\right)\frac{1}{f'(t)}s_n(x) \qquad (24)$$

([9, Corollary 3.7.2]) with (12), we get

$$\begin{aligned}
& s_{n+1}^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) \\
&\qquad = x s_n^{(k)}(x-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) - e^{-t}\frac{g'(t)}{g(t)} s_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\,.
\end{aligned}$$

Since

$$\begin{aligned}
\frac{g'(t)}{g(t)} &= \big(\ln g(t)\big)' \\
&= \left(\sum_{i=1}^{r}\mu_i\ln(1+e^{\lambda_i t}) - \ln \mathrm{Lif}_k(t)\right)' \\
&= \sum_{i=1}^{r}\frac{\mu_i\lambda_i e^{\lambda_i t}}{1+e^{\lambda_i t}} - \frac{\mathrm{Lif}_k'(t)}{\mathrm{Lif}_k(t)}\,,
\end{aligned}$$

by (13), we have

$$
\begin{aligned}
&\frac{g'(t)}{g(t)} s_n^{(k)}(x) \\
&= \left(\sum_{i=1}^{r} \frac{\mu_i \lambda_i e^{\lambda_i t}}{1+e^{\lambda_i t}} - \frac{\mathrm{Lif}'_k(t)}{\mathrm{Lif}_k(t)} \right) s_n^{(k)}(x) \\
&= 2^{-1-\sum_{j=1}^{r}\mu_j} \sum_{m=0}^{n} \sum_{l=0}^{m} S_1(n,m) \frac{\binom{m}{l}}{(l+1)^k} \sum_{i=1}^{r} \mu_i \lambda_i e^{\lambda_i t} \frac{2}{1+e^{\lambda_i t}} \prod_{j=1}^{r} \left(\frac{2}{1+e^{\lambda_j t}} \right)^{\mu_j} x^{m-l} \\
&\quad - \sum_{m=0}^{n} S_1(n,m) \prod_{j=1}^{r} (1+e^{\lambda_j t})^{-\mu_j} \mathrm{Lif}'_k(t) x^m \,. \qquad (25)
\end{aligned}
$$

The first term in (25) is

$$
2^{-1-\sum_{j=1}^{r}\mu_j} \sum_{m=0}^{n} \sum_{l=0}^{m} \sum_{i=1}^{r} S_1(n,m) \frac{\binom{m}{l}}{(l+1)^k} \mu_i \lambda_i E_{m-l}(x+\lambda_i|\lambda;\mu+e_i) \,,
$$

where $\lambda=(\lambda_1,\dots,\lambda_r)$, $\mu=(\mu_1,\dots,\mu_r)$ and $e_i=(\underbrace{0,\dots,0}_{i-1},1,\underbrace{0,\dots,0}_{r-i})$ $(i=1,2,\dots,r)$.

Since

$$
\mathrm{Lif}_{k-1}(t) - \mathrm{Lif}_k(t) = \left(\frac{1}{2^{k-1}} - \frac{1}{2^k} \right) t + \cdots \,,
$$

the second term in (25) is

$$
\begin{aligned}
&2^{-\sum_{j=1}^{r}\mu_j} \sum_{m=0}^{n} S_1(n,m) \mathrm{Lif}'_k(t) \prod_{j=1}^{r} \left(\frac{2}{1+e^{\lambda_j t}} \right)^{\mu_j} x^m \\
&= 2^{-\sum_{j=1}^{r}\mu_j} \sum_{m=0}^{n} S_1(n,m) \frac{\mathrm{Lif}_{k-1}(t) - \mathrm{Lif}_k(t)}{t} E_m(x|\lambda;\mu) \\
&= 2^{-\sum_{j=1}^{r}\mu_j} \sum_{m=0}^{n} S_1(n,m) \big(\mathrm{Lif}_{k-1}(t) - \mathrm{Lif}_k(t) \big) \frac{E_{m+1}(x|\lambda;\mu)}{m+1} \\
&= 2^{-\sum_{j=1}^{r}\mu_j} \sum_{m=0}^{n} \frac{S_1(n,m)}{m+1} \left(\sum_{l=0}^{m+1} \frac{t^l}{l!(l+1)^{k-1}} E_{m+1}(x|\lambda;\mu) - \sum_{l=0}^{m+1} \frac{t^l}{l!(l+1)^k} E_{m+1}(x|\lambda;\mu) \right) \\
&= 2^{-\sum_{j=1}^{r}\mu_j} \sum_{m=0}^{n} \frac{S_1(n,m)}{m+1} \left(\sum_{l=0}^{m+1} \frac{\binom{m+1}{l}}{(l+1)^{k-1}} E_{m+1-l}(x|\lambda;\mu) - \sum_{l=0}^{m+1} \frac{\binom{m+1}{l}}{(l+1)^k} E_{m+1-l}(x|\lambda;\mu) \right) \\
&= 2^{-\sum_{j=1}^{r}\mu_j} \sum_{m=0}^{n} \frac{S_1(n,m)}{m+1} \sum_{l=1}^{m+1} \frac{\binom{m+1}{l} l}{(l+1)^k} E_{m+1-l}(x|\lambda;\mu) \\
&= 2^{-\sum_{j=1}^{r}\mu_j} \sum_{m=0}^{n} S_1(n,m) \sum_{l=0}^{m} \frac{\binom{m}{l}}{(m+2-l)^k} E_l(x|\lambda;\mu) \,.
\end{aligned}
$$

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Thus, we obtain

$$
\begin{aligned}
s_{n+1}^{(k)}(x) &= x s_n^{(k)}(x-1) \\
&- 2^{-1-\sum_{j=1}^{r}\mu_j} \sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{i=1}^{r} S_1(n,m)\frac{\binom{m}{l}}{(l+1)^k}\mu_i\lambda_i E_{m-l}(x+\lambda_i-1|\lambda;\mu+e_i) \\
&+ 2^{-\sum_{j=1}^{r}\mu_j} \sum_{m=0}^{n}\sum_{l=0}^{m} \frac{\binom{m}{l}}{(m+2-l)^k} S_1(n,m)E_l(x-1|\lambda;\mu)\,,
\end{aligned}
$$

which is (22).
On the other hand, by (14) with (22), we have

$$
\begin{aligned}
&\frac{g'(t)}{g(t)} s_n^{(k)}(x) \\
&= \left(\sum_{i=1}^{r}\frac{\mu_i\lambda_i e^{\lambda_i t}}{1+e^{\lambda_i t}} - \frac{\mathrm{Lif}_k'(t)}{\mathrm{Lif}_k(t)}\right) s_n^{(k)}(x) \\
&= \frac{1}{2}\sum_{i=1}^{r}\mu_i\lambda_i e^{\lambda_i t}\frac{2}{1+e^{\lambda_i t}}\sum_{j=0}^{n}\sum_{l=0}^{n-j}\binom{n}{l} S_1(n-l,j)s_l^{(k)}x^j \\
&\quad - 2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}\sum_{l=0}^{m}\frac{\binom{m}{l}}{(m+2-l)^k} S_1(n,m)E_l(x|\lambda;\mu)\,.
\end{aligned}
\tag{26}
$$

The first term in (26) is

$$
\begin{aligned}
&\frac{1}{2}\sum_{j=0}^{n}\sum_{l=0}^{n-j}\binom{n}{l} S_1(n-l,j)s_l^{(k)}\sum_{i=1}^{r}\mu_i\lambda_i e^{\lambda_i t}\frac{2}{1+e^{\lambda_i t}}x^j \\
&= \frac{1}{2}\sum_{j=0}^{n}\sum_{l=0}^{n-j}\binom{n}{l} S_1(n-l,j)s_l^{(k)}\sum_{i=1}^{r}\mu_i\lambda_i e^{\lambda_i t}\lambda_i^j E_j\left(\frac{x}{\lambda_i}\right) \\
&= \frac{1}{2}\sum_{j=0}^{n}\sum_{l=0}^{n-j}\binom{n}{l} S_1(n-l,j)s_l^{(k)}\sum_{i=1}^{r}\mu_i\lambda_i^{j+1} E_j\left(\frac{x+\lambda_i}{\lambda_i}\right)\,.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
s_{n+1}^{(k)}(x) &= x s_n^{(k)}(x-1) \\
&- \frac{1}{2}\sum_{j=0}^{n}\sum_{l=0}^{n-j}\sum_{i=1}^{r}\binom{n}{l}\mu_i\lambda_i^{j+1} S_1(n-l,j)s_l^{(k)}E_j\left(\frac{x+\lambda_i-1}{\lambda_i}\right) \\
&+ 2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}\sum_{l=0}^{m}\frac{\binom{m}{l}}{(m+2-l)^k} S_1(n,m)E_l(x-1|\lambda;\mu)\,.
\end{aligned}
$$

which is (23). ∎

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3.5 Differentiation

Theorem 5

$$\frac{d}{dx}s_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)=n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}s_l^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\,. \quad (27)$$

Proof. We shall use

$$\frac{d}{dx}s_n(x)=\sum_{l=0}^{n-1}\binom{n}{l}\left\langle\bar{f}(t)|x^{n-l}\right\rangle s_l(x)$$

(*Cf.* [9, Theorem 2.3.12]). Since

$$\begin{aligned}\left\langle\bar{f}(t)|x^{n-l}\right\rangle&=\left\langle\ln(1+t)|x^{n-l}\right\rangle\\&=\left\langle\sum_{m=1}^{\infty}\frac{(-1)^{m-1}t^m}{m}\Big|x^{n-l}\right\rangle\\&=\sum_{m=1}^{n-l}\frac{(-1)^{m-1}}{m}\left\langle t^m|x^{n-l}\right\rangle\\&=\sum_{m=1}^{n-l}\frac{(-1)^{m-1}}{m}(n-l)!\delta_{m,n-l}\\&=(-1)^{n-l-1}(n-l-1)!\,,\end{aligned}$$

with (12), we have

$$\begin{aligned}&\frac{d}{dx}s_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\&=\sum_{l=0}^{n-1}\binom{n}{l}(-1)^{n-l-1}(n-l-1)!s_l^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\&=n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}s_l^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\,,\end{aligned}$$

which is the identity (27). ∎

3.6 A more relation

The classical Cauchy numbers c_n are defined by

$$\frac{t}{\ln(1+t)}=\sum_{n=0}^{\infty}c_n\frac{t^n}{n!}$$

(see e.g. [2, 6]).

Theorem 6 *For $n \geq 1$, we have*

$$\begin{aligned}
&s_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\
&= xs_{n-1}^{(k)}(x-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) + \frac{1}{n}\sum_{l=1}^{n}\binom{n}{l}c_{n-l}\big(s_l^{(k-1)}(x-1) - s_l^{(k)}(x-1)\big)\\
&\quad - \sum_{i=1}^{r}\mu_i\lambda_i s_{n-1}^{(k)}(x+\lambda_i-1|\lambda;\mu+e_i)\,. \qquad (28)
\end{aligned}$$

Proof. For $n \geq 1$, we have

$$\begin{aligned}
&s_n^{(k)}(y|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\
&= \left\langle \sum_{l=0}^{\infty} s_l^{(k)}(y|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\frac{t^l}{l!}\Big| x^n \right\rangle\\
&= \left\langle \prod_{j=1}^{r}\big(1+(1+t)^{\lambda_j}\big)^{-\mu_j}\mathrm{Lif}_k\big(\ln(1+t)\big)(1+t)^y \Big| x^n \right\rangle\\
&= \left\langle \partial_t\left(\prod_{j=1}^{r}\big(1+(1+t)^{\lambda_j}\big)^{-\mu_j}\mathrm{Lif}_k\big(\ln(1+t)\big)(1+t)^y\right)\Big| x^{n-1} \right\rangle\\
&= \left\langle \left(\partial_t\prod_{j=1}^{r}\big(1+(1+t)^{\lambda_j}\big)^{-\mu_j}\right)\mathrm{Lif}_k\big(\ln(1+t)\big)(1+t)^y \Big| x^{n-1} \right\rangle\\
&\quad + \left\langle \prod_{j=1}^{r}\big(1+(1+t)^{\lambda_j}\big)^{-\mu_j}\Big(\partial_t\mathrm{Lif}_k\big(\ln(1+t)\big)\Big)(1+t)^y \Big| x^{n-1} \right\rangle\\
&\quad + \left\langle \prod_{j=1}^{r}\big(1+(1+t)^{\lambda_j}\big)^{-\mu_j}\mathrm{Lif}_k\big(\ln(1+t)\big)\left(\partial_t(1+t)^y\right) \Big| x^{n-1} \right\rangle .
\end{aligned}$$

The third term is

$$\begin{aligned}
&y\left\langle \prod_{j=1}^{r}\big(1+(1+t)^{\lambda_j}\big)^{-\mu_j}\mathrm{Lif}_k\big(\ln(1+t)\big)(1+t)^{y-1} \Big| x^{n-1} \right\rangle\\
&= ys_{n-1}^{(k)}(y-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\,.
\end{aligned}$$

Since

$$\mathrm{Lif}_{k-1}\big(\ln(1+t)\big) - \mathrm{Lif}_k\big(\ln(1+t)\big) = \left(\frac{1}{2^{k-1}} - \frac{1}{2^k}\right)t + \cdots\,,$$

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the second term is

$$\left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\frac{\mathrm{Lif}_{k-1}\big(\ln(1+t)\big)-\mathrm{Lif}_k\big(\ln(1+t)\big)}{(1+t)\ln(1+t)}(1+t)^y\middle| x^{n-1}\right\rangle$$

$$=\left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\right.$$

$$\left.\frac{\mathrm{Lif}_{k-1}\big(\ln(1+t)\big)-\mathrm{Lif}_k\big(\ln(1+t)\big)}{t}(1+t)^{y-1}\middle| \frac{t}{\ln(1+t)}x^{n-1}\right\rangle$$

$$=\left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\right.$$

$$\left.\frac{\mathrm{Lif}_{k-1}\big(\ln(1+t)\big)-\mathrm{Lif}_k\big(\ln(1+t)\big)}{t}(1+t)^{y-1}\middle| \sum_{l=0}^{\infty}c_l\frac{t^l}{l!}x^{n-1}\right\rangle$$

$$=\sum_{l=0}^{n-1}\binom{n-1}{l}c_l\left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\right.$$

$$\left.(1+t)^{y-1}\middle|\frac{\mathrm{Lif}_{k-1}\big(\ln(1+t)\big)-\mathrm{Lif}_k\big(\ln(1+t)\big)}{t}x^{n-1-l}\right\rangle$$

$$=\sum_{l=0}^{n-1}\frac{1}{n-l}\binom{n-1}{l}c_l\left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\right.$$

$$\left.(1+t)^{y-1}\middle|\left(\mathrm{Lif}_{k-1}\big(\ln(1+t)\big)-\mathrm{Lif}_k\big(\ln(1+t)\big)\right)x^{n-l}\right\rangle$$

$$=\sum_{l=0}^{n-1}\frac{1}{n-l}\binom{n}{l}c_l\left(\left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\mathrm{Lif}_{k-1}\big(\ln(1+t)\big)(1+t)^{y-1}\middle| x^{n-l}\right\rangle\right.$$

$$\left.-\left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\mathrm{Lif}_{k}\big(\ln(1+t)\big)(1+t)^{y-1}\middle| x^{n-l}\right\rangle\right)$$

$$=\frac{1}{n}\sum_{l=0}^{n-1}\binom{n}{l}c_l\left(s_{n-l}^{(k-1)}(y-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)-s_{n-l}^{(k)}(y-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\right)$$

$$=\frac{1}{n}\sum_{l=1}^{n}\binom{n}{l}c_{n-l}\left(s_{l}^{(k-1)}(y-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)-s_{l}^{(k)}(y-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\right).$$

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Since

$$\begin{aligned}&\partial_t \prod_{j=1}^{r}\bigl(1+(1+t)^{\lambda_j}\bigr)^{-\mu_j}\\&=-\sum_{i=1}^{r}\mu_i\lambda_i(1+t)^{\lambda_i-1}\bigl(1+(1+t)\lambda_i\bigr)^{-1}\prod_{j=1}^{r}\bigl(1+(1+t)^{\lambda_j}\bigr)^{-\mu_j},\end{aligned}$$

the first term is

$$\begin{aligned}&-\sum_{i=1}^{r}\mu_i\lambda_i\left\langle \bigl(1+(1+t)\lambda_i\bigr)^{-1}\prod_{j=1}^{r}\bigl(1+(1+t)^{\lambda_j}\bigr)^{-\mu_j}\mathrm{Lif}_k\bigl(\ln(1+t)\bigr)(1+t)^{y+\lambda_i-1}|x^{n-1}\right\rangle\\&=-\sum_{i=1}^{r}\mu_i\lambda_i s_{n-1}^{(k)}(y+\lambda_i-1|\lambda;\mu+e_i)\,.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}&s_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\&=xs_{n-1}^{(k)}(x-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)+\frac{1}{n}\sum_{l=1}^{n}\binom{n}{l}c_{n-l}\bigl(s_l^{(k-1)}(x-1)-s_l^{(k)}(x-1)\bigr)\\&\quad-\sum_{i=1}^{r}\mu_i\lambda_i s_{n-1}^{(k)}(x+\lambda_i-1|\lambda;\mu+e_i)\,,\end{aligned}$$

which is the identity (28). ∎

3.7 A relation including the Stirling numbers of the first kind

Theorem 7 *For $n-1\ge m\ge 1$, we have*

$$\begin{aligned}&m\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)s_l^{(k)}(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\&=(m-1)\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)s_l^{(k)}(-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\&\quad+\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)s_l^{(k-1)}(-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\&\quad-m\sum_{l=0}^{n-m-1}\sum_{i=1}^{r}\binom{n-1}{l}S_1(n-l-1,m)\mu_i\lambda_i s_l^{(k)}(\lambda_i-1|\lambda;\mu+e_i)\,.\end{aligned}\tag{29}$$

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Proof. We shall compute

$$\left\langle \prod_{j=1}^{r}\bigl(1+(1+t)^{\lambda_j}\bigr)^{-\mu_j}\mathrm{Lif}_k\bigl(\ln(1+t)\bigr)\bigl(\ln(1+t)\bigr)^m \middle| x^n \right\rangle$$

in two different ways. On the one hand, it is equal to

$$\begin{aligned}
&\left\langle \prod_{j=1}^{r}\bigl(1+(1+t)^{\lambda_j}\bigr)^{-\mu_j}\mathrm{Lif}_k\bigl(\ln(1+t)\bigr) \middle| \bigl(\ln(1+t)\bigr)^m x^n \right\rangle\\
&= \left\langle \prod_{j=1}^{r}\bigl(1+(1+t)^{\lambda_j}\bigr)^{-\mu_j}\mathrm{Lif}_k\bigl(\ln(1+t)\bigr) \middle| m!\sum_{l=m}^{\infty} S_1(l,m)\frac{t^l}{l!}x^n \right\rangle\\
&= m!\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)\left\langle \prod_{j=1}^{r}\bigl(1+(1+t)^{\lambda_j}\bigr)^{-\mu_j}\mathrm{Lif}_k\bigl(\ln(1+t)\bigr) \middle| x^{n-l} \right\rangle\\
&= m!\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)\left\langle \sum_{i=0}^{\infty} s_i^{(k)}(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\frac{t^i}{i!} \middle| x^{n-l} \right\rangle\\
&= m!\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)s_{n-l}^{(k)}(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\
&= m!\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)s_l^{(k)}(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\,.
\end{aligned}$$

On the other hand, it is equal to

$$\begin{aligned}
&\left\langle \partial_t\left(\prod_{j=1}^{r}\bigl(1+(1+t)^{\lambda_j}\bigr)^{-\mu_j}\mathrm{Lif}_k\bigl(\ln(1+t)\bigr)\bigl(\ln(1+t)\bigr)^m\right) \middle| x^{n-1} \right\rangle\\
&= \left\langle \left(\partial_t\prod_{j=1}^{r}\bigl(1+(1+t)^{\lambda_j}\bigr)^{-\mu_j}\right)\mathrm{Lif}_k\bigl(\ln(1+t)\bigr)\bigl(\ln(1+t)\bigr)^m \middle| x^{n-1} \right\rangle\\
&\quad + \left\langle \prod_{j=1}^{r}\bigl(1+(1+t)^{\lambda_j}\bigr)^{-\mu_j}\ \bigl(\partial_t\mathrm{Lif}_k\bigl(\ln(1+t)\bigr)\bigr)\ \bigl(\ln(1+t)\bigr)^m \middle| x^{n-1} \right\rangle\\
&\quad + \left\langle \prod_{j=1}^{r}\bigl(1+(1+t)^{\lambda_j}\bigr)^{-\mu_j}\mathrm{Lif}_k\bigl(\ln(1+t)\bigr)\ \bigl(\partial_t\bigl(\ln(1+t)\bigr)^m\bigr) \middle| x^{n-1} \right\rangle\,. \qquad (30)
\end{aligned}$$

The third term of (30) is equal to

$$\begin{aligned}
& m\left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\mathrm{Lif}_k\left(\ln(1+t)\right)(1+t)^{-1}\middle|\left(\ln(1+t)\right)^{m-1}x^{n-1}\right\rangle \\
&= m\left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\mathrm{Lif}_k\left(\ln(1+t)\right)(1+t)^{-1}\middle| \right. \\
&\qquad \left. (m-1)!\sum_{l=m-1}^{\infty}S_1(l,m-1)\frac{t^l}{l!}x^{n-1}\right\rangle \\
&= m!\sum_{l=m-1}^{n-1}\binom{n-1}{l}S_1(l,m-1) \\
&\quad \times\left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\mathrm{Lif}_k\left(\ln(1+t)\right)(1+t)^{-1}\middle|x^{n-1-l}\right\rangle \\
&= m!\sum_{l=m-1}^{n-1}\binom{n-1}{l}S_1(l,m-1)s_{n-1-l}^{(k)}(-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) \\
&= m!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)s_{l}^{(k)}(-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\,.
\end{aligned}$$

The second term of (30) is equal to

$$\begin{aligned}
& \left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\left(\frac{\mathrm{Lif}_{k-1}\left(\ln(1+t)\right)-\mathrm{Lif}_k\left(\ln(1+t)\right)}{(1+t)\ln(1+t)}\right)\left(\ln(1+t)\right)^{m}\middle|x^{n-1}\right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\mathrm{Lif}_{k-1}\left(\ln(1+t)\right)(1+t)^{-1}\middle|\left(\ln(1+t)\right)^{m-1}x^{n-1}\right\rangle \\
&\quad -\left\langle \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\mathrm{Lif}_{k}\left(\ln(1+t)\right)(1+t)^{-1}\middle|\left(\ln(1+t)\right)^{m-1}x^{n-1}\right\rangle \\
&= (m-1)!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)s_{l}^{(k-1)}(-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) \\
&\quad -(m-1)!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)s_{l}^{(k)}(-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\,.
\end{aligned}$$

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The first term of (30) is equal to

$$
\begin{aligned}
&\left\langle \left(\partial_t \prod_{j=1}^{r} \left(1+(1+t)^{\lambda_j}\right)^{-\mu_j} \right) \mathrm{Lif}_k\left(\ln(1+t)\right)\left(\ln(1+t)\right)^m \middle| x^{n-1} \right\rangle \\
&= -\sum_{i=1}^{r} \mu_i\lambda_i \left\langle \left(1+(1+t)^{\lambda_i}\right)^{-1} \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_i}\right)^{-\mu_j} \right. \\
&\qquad\qquad \left. \mathrm{Lif}_k\left(\ln(1+t)\right)(1+t)^{\lambda_i-1} \middle| \left(\ln(1+t)\right)^m x^{n-1} \right\rangle \\
&= -\sum_{i=1}^{r} \mu_i\lambda_i \left\langle \left(1+(1+t)^{\lambda_i}\right)^{-1} \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_i}\right)^{-\mu_j} \right. \\
&\qquad\qquad \left. \mathrm{Lif}_k\left(\ln(1+t)\right)(1+t)^{\lambda_i-1} \middle| m!\sum_{l=m}^{\infty} S_1(l,m)\frac{t^l}{l!} x^{n-1} \right\rangle \\
&= -m!\sum_{i=1}^{r}\mu_i\lambda_i \sum_{l=m}^{n-1}\binom{n-1}{l} S_1(l,m) \\
&\quad \times \left\langle \left(1+(1+t)^{\lambda_i}\right)^{-1} \prod_{j=1}^{r}\left(1+(1+t)^{\lambda_i}\right)^{-\mu_j} \mathrm{Lif}_k\left(\ln(1+t)\right)(1+t)^{\lambda_i-1} \middle| x^{n-1-l} \right\rangle \\
&= -m!\sum_{i=1}^{r}\mu_i\lambda_i \sum_{l=m}^{n-1}\binom{n-1}{l} S_1(l,m) s_{n-1-l}^{(k)}(\lambda_i-1|\lambda;\mu+e_i) \\
&= -m!\sum_{l=0}^{n-m-1}\sum_{i=1}^{r}\binom{n-1}{l} S_1(n-1-l,m)\mu_i\lambda_i s_l^{(k)}(\lambda_i-1|\lambda;\mu+e_i)\,.
\end{aligned}
$$

Therefore, we get, for $n-1 \geq m \geq 1$,

$$
\begin{aligned}
& m!\sum_{l=0}^{n-m}\binom{n}{l} S_1(n-l,m) s_l^{(k)}(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) \\
&= m!\sum_{l=0}^{n-m}\binom{n-1}{l} S_1(n-l-1,m-1) s_l^{(k)}(-1) \\
&\quad + (m-1)!\sum_{l=0}^{n-m}\binom{n-1}{l} S_1(n-l-1,m-1) s_l^{(k-1)}(-1) \\
&\quad - (m-1)!\sum_{l=0}^{n-m}\binom{n-1}{l} S_1(n-l-1,m-1) s_l^{(k)}(-1) \\
&\quad - m!\sum_{l=0}^{n-m-1}\sum_{i=1}^{r}\binom{n-1}{l} S_1(n-l-1,m)\mu_i\lambda_i s_l^{(k)}(\lambda_i-1|\lambda;\mu+e_i)\,.
\end{aligned}
$$

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Dividing both sides by $(m-1)!$, we obtain for $n-1 \geq m \geq 1$

$$\begin{aligned}
& m \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l, m) s_l^{(k)}(\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r) \\
&= (m-1) \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) s_l^{(k)}(-1|\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r) \\
&\quad + \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) s_l^{(k-1)}(-1|\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r) \\
&\quad - m \sum_{l=0}^{n-m-1} \sum_{i=1}^{r} \binom{n-1}{l} S_1(n-l-1, m) \mu_i \lambda_i s_l^{(k)}(\lambda_i - 1|\lambda; \mu + e_i)\,.
\end{aligned}$$

Thus, we get (29). ∎

3.8 A relation with the falling factorials

Theorem 8

$$s_n^{(k)}(x|\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r) = \sum_{m=0}^{n} \binom{n}{m} s_{n-m}^{(k)}(x)_m\,. \tag{31}$$

Proof. For (12) and (19), assume that $s_n^{(k)}(x|\lambda_1, \ldots, \lambda_r; \mu_1, \ldots, \mu_r) = \sum_{m=0}^{n} C_{n,m}(x)_m$. By (11), we have

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!} \left\langle \frac{1}{\prod_{j=1}^{r} \left(1 + e^{\lambda_j \ln(1+t)}\right)^{\mu_j}} \mathrm{Lif}_k\big(\ln(1+t)\big) t^m \Big| x^n \right\rangle \\
&= \frac{1}{m!} \left\langle \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} \mathrm{Lif}_k\big(\ln(1+t)\big) \Big| t^m x^n \right\rangle \\
&= \binom{n}{m} \left\langle \prod_{j=1}^{r} \left(1 + (1+t)^{\lambda_j}\right)^{-\mu_j} \mathrm{Lif}_k\big(\ln(1+t)\big) \Big| x^{n-m} \right\rangle \\
&= \binom{n}{m} s_{n-m}^{(k)}\,.
\end{aligned}$$

Thus, we get the identity (31). ∎

3.9 A relation with higher-order Frobenius-Euler polynomials

For $\alpha \in \mathbb{C}$ with $\alpha \neq 1$, the Frobenius-Euler polynomials of order r, $H_n^{(r)}(x|\alpha)$ are defined by the generating function

$$\left(\frac{1-\alpha}{e^t - \alpha}\right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|\alpha) \frac{t^n}{n!}$$

(see e.g. [4]).

Theorem 9

$$s_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)=\sum_{m=0}^{n}\left(\sum_{j=0}^{n-m}\sum_{l=0}^{n-m-j}\binom{s}{j}\binom{n-j}{l}(n)_j\right.$$
$$\left.\times(1-\alpha)^{-j}S_1(n-j-l,m)s_l^{(k)}\right)H_m^{(s)}(x|\alpha)\,. \tag{32}$$

Proof. For (12) and

$$H_n^{(s)}(x|\alpha)\sim\left(\left(\frac{e^t-\alpha}{1-\alpha}\right)^s,t\right)\,, \tag{33}$$

assume that $s_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)=\sum_{m=0}^n C_{n,m}H_m^{(s)}(x|\alpha)$. By (11), similarly to the proof of (29), we have

$$\begin{aligned}
C_{n,m}&=\frac{1}{m!}\left\langle\frac{\left(\frac{e^{\ln(1+t)}-\alpha}{1-\alpha}\right)^s}{\prod_{j=1}^r\left(1+e^{\lambda_j\ln(1+t)}\right)^{\mu_j}}\mathrm{Lif}_k\big(\ln(1+t)\big)\big(\ln(1+t)\big)^m\Big|x^n\right\rangle\\
&=\frac{1}{m!(1-\alpha)^s}\left\langle\prod_{j=1}^r\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\mathrm{Lif}_k\big(\ln(1+t)\big)\big(\ln(1+t)\big)^m(1-\alpha+t)^s\Big|x^n\right\rangle\\
&=\frac{1}{m!(1-\alpha)^s}\\
&\quad\times\left\langle\prod_{j=1}^r\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\mathrm{Lif}_k\big(\ln(1+t)\big)\big(\ln(1+t)\big)^m\Big|\sum_{i=0}^{\min\{s,n\}}\binom{s}{i}(1-\alpha)^{s-i}t^ix^n\right\rangle\\
&=\frac{1}{m!(1-\alpha)^s}\sum_{i=0}^{n-m}\binom{s}{i}(1-\alpha)^{s-i}(n)_i\\
&\quad\times\left\langle\prod_{j=1}^r\left(1+(1+t)^{\lambda_j}\right)^{-\mu_j}\mathrm{Lif}_k\big(\ln(1+t)\big)\big(\ln(1+t)\big)^m\Big|x^{n-i}\right\rangle\\
&=\frac{1}{m!(1-\alpha)^s}\sum_{i=0}^{n-m}\binom{s}{i}(1-\alpha)^{s-i}(n)_i\sum_{l=0}^{n-m-i}m!\binom{n-i}{l}S_1(n-i-l,m)s_l^{(k)}\\
&=\sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{s}{i}\binom{n-i}{l}(n)_i(1-\alpha)^{-i}S_1(n-i-l,m)s_l^{(k)}\,.
\end{aligned}$$

Thus, we get the identity (32). ∎

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3.10 A relation with higher-order Bernoulli polynomials

Bernoulli polynomials $\mathfrak{B}_n^{(r)}(x)$ of order r are defined by

$$\left(\frac{t}{e^t-1}\right)^r e^{xt}=\sum_{n=0}^\infty\frac{\mathfrak{B}_n^{(r)}(x)}{n!}t^n$$

(see e.g. [9, Section 2.2]). In addition, Cauchy numbers of the first kind $\mathfrak{C}_n^{(r)}$ of order r are defined by

$$\left(\frac{t}{\ln(1+t)}\right)^r=\sum_{n=0}^\infty\frac{\mathfrak{C}_n^{(r)}}{n!}t^n$$

(see e.g. [1, (2.1)], [8, (6)]).

Theorem 10

$$\begin{aligned}&s_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\&=\sum_{m=0}^n\left(\sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{n}{i}\binom{n-i}{l}\mathfrak{C}_i^{(s)}S_1(n-i-l,m)s_l^{(k)}\right)\mathfrak{B}_m^{(s)}(x)\,.\end{aligned}\tag{34}$$

Proof. For (12) and

$$\mathfrak{B}_n^{(s)}(x)\sim\left(\left(\frac{e^t-1}{t}\right)^s,t\right),\tag{35}$$

assume that $s_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)=\sum_{m=0}^n C_{n,m}\mathfrak{B}_m^{(s)}(x)$. By (11), similarly to the proof of (29), we have

$$\begin{aligned}C_{n,m}&=\frac{1}{m!}\left\langle\frac{\left(\frac{e^{\ln(1+t)}-1}{\ln(1+t)}\right)^s}{\prod_{j=1}^r(1+e^{\lambda_j\ln(1+t)})^{\mu_j}}\mathrm{Lif}_k\bigl(\ln(1+t)\bigr)\bigl(\ln(1+t)\bigr)^m\Big|x^n\right\rangle\\
&=\frac{1}{m!}\left\langle\prod_{j=1}^r\bigl(1+(1+t)^{\lambda_j}\bigr)^{-\mu_j}\mathrm{Lif}_k\bigl(\ln(1+t)\bigr)\bigl(\ln(1+t)\bigr)^m\Big|\left(\frac{t}{\ln(1+t)}\right)^s x^n\right\rangle\\
&=\frac{1}{m!}\left\langle\prod_{j=1}^r\bigl(1+(1+t)^{\lambda_j}\bigr)^{-\mu_j}\mathrm{Lif}_k\bigl(\ln(1+t)\bigr)\bigl(\ln(1+t)\bigr)^m\Big|\sum_{i=0}^\infty\mathfrak{C}_i^{(s)}\frac{t^i}{i!}x^n\right\rangle\\
&=\frac{1}{m!}\sum_{i=0}^{n-m}\mathfrak{C}_i^{(s)}\binom{n}{i}\left\langle\prod_{j=1}^r\bigl(1+(1+t)^{\lambda_j}\bigr)^{-\mu_j}\mathrm{Lif}_k\bigl(\ln(1+t)\bigr)\bigl(\ln(1+t)\bigr)^m\Big|x^{n-i}\right\rangle\\
&=\frac{1}{m!}\sum_{i=0}^{n-m}\mathfrak{C}_i^{(s)}\binom{n}{i}\sum_{l=0}^{n-m-i}m!\binom{n-i}{l}S_1(n-i-l,m)s_l^{(k)}\\
&=\sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{n}{i}\binom{n-i}{l}\mathfrak{C}_i^{(s)}S_1(n-i-l,m)s_l^{(k)}\,.\end{aligned}$$

Thus, we get the identity (34). ∎

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On the hyperstablity of a functional equation in commutative groups

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Abstract. Using the fixed point method, we prove the hyperstablity of the functional equation

$$f(ax+by) = \frac{(a+b)}{2}f(x+y) + \frac{(a-b)}{2}f(x-y),$$

where a, b are different integers greater than 1, in the class of functions from a commutative group into a commutative complete metric group.

1. **Introduction and preliminaries**

The stability problem of functional equations originated from a question of Ulam [33] concerning the stability of group homomorphisms. Hyers [19] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [4] for additive mappings and by Rassias [29] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [17] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [6, 12, 13, 14, 15, 16, 20, 21, 25, 27, 28, 30, 31, 34]).

We say a functional equation $\mathfrak{D}$ is *hyperstable* if any function f satisfying the equation $\mathfrak{D}$ approximately is a true solution of $\mathfrak{D}$. It seems that the first hyperstability result was published in [6] and concerned the ring homomorphisms. However, The term *hyperstability* has been used for the first time in [24]. Quite often the hyperstability is confused with superstability, which admits also bounded functions. Numerous papers on this subject have been published and we refer to [1, 2, 3, 5, 7, 8, 9, 11, 18, 24, 26, 32].

Throughout this paper, we will denote the set of natural numbers by $\mathbb{N}$, the set of integers by $\mathbb{Z}$ and the set of real numbers by $\mathbb{R}$. Let $\mathbb{N}_+$ be the set of positive integers. By $\mathbb{N}_m$, $m \in \mathbb{N}_+$, we will denote the set of all integers greater than or equal to m. Let $\mathbb{R}_0 = [0, \infty)$ the set of nonnegative real numbers and $\mathbb{R}_+ = (0, \infty)$ the set of positive real numbers. We write B^A to mean " the family of all functions mapping from a nonempty set A into a nonempty set B ".

Definition 1.1. Let X be a nonempty set, (Y, d) be a metric space, $\varepsilon \in \mathbb{R}_0^{X^n}$ and $\mathcal{F}_1$, $\mathcal{F}_2$ be operators mapping from a nonempty set $\mathcal{D} \subset Y^X$ into Y^{X^n}. We say that the operator equation

$$\mathcal{F}_1\varphi(x_1, \dots, x_n) = \mathcal{F}_2\varphi(x_1, \dots, x_n), \quad (x_1, \dots, x_n \in X) \tag{1.1}$$

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is ε-hyperstable provided that every $\varphi_0 \in \mathcal{D}$ which satisfies

$$d\left(\mathcal{F}_1\varphi_0(x_1,\ldots,x_n),\mathcal{F}_2\varphi_0(x_1,\ldots,x_n)\right) \le \varepsilon(x_1,\ldots,x_n), \quad (x_1,\ldots,x_n \in X)$$

fulfills the equation (1.1).

In 2011, Kenary [22] introduced and proved the Hyers-Ulam stability for the following functional equation

$$f(ax+by) = \frac{(a+b)}{2}f(x+y) + \frac{(a-b)}{2}f(x-y) \tag{1.2}$$

in non-Archimedean normed spaces and in random normed spaces, where m, n are different integers greater than 1. In 2011, Kenary, Jang and Park [23] proved the Hyers-Ulam stability of (1.2) in various normed spaces by using the fixed point method.

In this paper, using the fixed point method derived from [10, Theorem 1], we prove the hyperstability of (1.2) in the class of functions from a commutative group into a commutative complete metric group.

Before proceeding to the main results, we state the following theorem which is useful for our purpose.

Theorem 1.2. ([10, Theorem 1]) *Let X be a nonempty set, (Y,d) a complete metric space, $f_1,\ldots,f_s\colon X \to X$ and $L_1,\ldots,L_s\colon X \to \mathbb{R}_0$ be given mappings. Let $\Lambda\colon \mathbb{R}_0^X \to \mathbb{R}_0^X$ be a linear operator defined by*

$$\Lambda\delta(x) := \sum_{i=1}^{s} L_i(x)\delta(f_i(x)), \tag{1.3}$$

for $\delta \in \mathbb{R}_0^X$ and $x \in X$. If $\mathcal{T}\colon Y^X \to Y^X$ is an operator satisfying the inequality

$$d\left(\mathcal{T}\xi(x),\mathcal{T}\mu(x)\right) \le \sum_{i=1}^{s} L_i(x) d\left(\xi(f_i(x)),\mu(f_i(x))\right), \quad \xi,\mu \in Y^X, x \in X,$$

and a function $\varepsilon\colon X \to \mathbb{R}_0$ and a mapping $\varphi\colon X \to Y$ satisfy

$$d\left(\mathcal{T}\varphi(x),\varphi(x)\right) \le \varepsilon(x), \quad (x \in X),$$

$$\varepsilon^*(x) := \sum_{k=0}^{\infty} \Lambda^k\varepsilon(x) < \infty, \quad (x \in X),$$

then for every $x \in X$, the limit

$$\psi(x) := \lim_{n\to\infty} \mathcal{T}^n\varphi(x),$$

exists and the function $\psi \in Y^X$ so defined is a unique fixed point of $\mathcal{T}$ with

$$d\left(\varphi(x),\psi(x)\right) \le \varepsilon^*(x), \qquad (x \in X).$$

2. Main results

Suppose $(G,+)$ and $(H,+)$ are abelian groups, and d is a metric on H such that

(i): d is invariant with respect to $+$, that is,

$$d(u+w,v+w) = d(u,v), \qquad (u,v,w \in H);$$

(ii): (H,d) is a complete metric space.

We will denote by $Aut(G)$ the family of all automorphisms of G. Moreover, for each $u\colon G \to G$ we write $ux := u(x)$ for $x \in G$ and we define u' by $u'x := x - ux$ for $x \in G$.
Let

$$l(G) := \big\{u \in Aut(G) : (u'-u), (au'+bu) \in Aut(G),$$
$$\alpha_u := \frac{2}{a+b}\lambda(au'+bu) + \frac{|a-b|}{a+b}\lambda(u'-u) < 1\} \neq \varnothing, \tag{2.1}$$

where

$$\lambda(u) := \inf\{t \in \mathbb{R}_0 : \varepsilon(ux, uy) \le t\varepsilon(x,y), \quad \forall x, y \in G\}$$

for $u \in Aut(G)$ and $\varepsilon\colon G^2 \to \mathbb{R}_0$. The following theorem is a result concerning the hyperstability of the functional equation (1.2).

Theorem 2.1. *Let $f : G \to H$ be a mapping satisfying the inequality*

$$d\left(f(x+y), \frac{2}{a+b}f(ax+by) - \frac{a-b}{a+b}f(x-y)\right) \le \varepsilon(x,y) \tag{2.2}$$

for all $x, y \in G$, where $\varepsilon\colon G^2 \to \mathbb{R}_0$ is an arbitrary function. Assume that there exists a nonempty subset $\mathcal{U} \subset l(G)$ such that

$$u \circ v = v \circ u, \quad \forall u, v \in \mathcal{U},$$

and

$$\begin{aligned} &\inf\big\{\varepsilon(u'x, ux) : u \in \mathcal{U}\big\} = 0, \quad \forall x \in G,\\ &\sup\{\alpha_u : u \in \mathcal{U}\} < 1, \end{aligned} \tag{2.3}$$

then f is a solution of (1.2) on G.

Proof. Let us fix $u \in \mathcal{U}$. Replacing x with $u'x$ and y with ux in (2.2), we get

$$d\left(f(x), \frac{2}{a+b}f\Big((au'+bu)x\Big) - \frac{a-b}{a+b}f\Big((u'-u)x\Big)\right) \le \varepsilon(u'x, ux) := \varepsilon_u(x) \tag{2.4}$$

for all $x \in G$. We define the operators $\mathcal{T}_u\colon H^G \to H^G$ and $\Lambda_u\colon \mathbb{R}_0^G \to \mathbb{R}_0^G$ by

$$\mathcal{T}_u\xi(x) := \frac{2}{a+b}\xi\Big((au'+bu)x\Big) - \frac{a-b}{a+b}\xi\Big((u'-u)x\Big), \tag{2.5}$$

$$\Lambda_u\delta(x) := \frac{2}{a+b}\delta\Big((au'+bu)x\Big) + \frac{|a-b|}{a+b}\delta\Big((u'-u)x\Big)$$

for all $x \in G$, $\xi \in H^G$ and $\delta \in \mathbb{R}_0^G$. Then (2.4) becomes

$$d\Big(f(x), \mathcal{T}_u f(x)\Big) \le \varepsilon_u(x)$$

for all $x \in G$.

The operator $\Lambda_u : \mathbb{R}_0^G \to \mathbb{R}_0^G$ has the form given by (1.3) with $s = 2$ and $f_1(x) = (au'+bu)x$, $f_2(x) = (u'-u)x$, $L_1(x) = \frac{2}{a+b}$, $L_2(x) = \frac{|a-b|}{a+b}$ for all $x \in G$.
Further,

$$\begin{aligned} d\left(\mathcal{T}_u\xi(x), \mathcal{T}_u\mu(x)\right) = d\Bigg(&\frac{2}{a+b}\xi\Big((au'+bu)x\Big) - \frac{a-b}{a+b}\xi\Big((u'-u)x\Big), \\ &\frac{2}{a+b}\mu\Big((au'+bu)x\Big) - \frac{a-b}{a+b}\mu\Big((u'-u)x\Big)\Bigg) \\ \leq \frac{2}{a+b} d\Big(\xi\big((au'+bu)x\big), \mu\big((au'+bu)x\big)\Big) &+ \frac{|a-b|}{a+b} d\Big(\xi\big((u'-u)x\big), \mu\big((u'-u)x\big)\Big) \end{aligned}$$

for all $x \in G$ and $\xi, \mu \in H^G$.

Note that, in view of the definition of $\lambda(u)$,

$$\varepsilon(ux, uy) \leq \lambda(u)\varepsilon(x, y), \qquad x, y \in G,$$

So it is easy to show by induction on k that

$$\Lambda_u^k \varepsilon_u(x) \leq \alpha_u^k \varepsilon(u'x, ux),$$

for all $x \in G$, where

$$\alpha_u = \left(\frac{2}{a+b}\lambda(au'+bu) + \frac{|a-b|}{a+b}\lambda(u'-u)\right).$$

Hence

$$\varepsilon^*(x) := \sum_{k=0}^{\infty} \Lambda_u^k \varepsilon_u(x) \leq \varepsilon(u'x, ux) \sum_{k=0}^{\infty} \alpha_u^k = \frac{\varepsilon(u'x, ux)}{1-\alpha_u} < \infty$$

for all $x \in G$. By Theorem 1.2, there exists a unique solution $F_u : G \to H$ of the equation

$$F_u(x) = \frac{2}{a+b} F_u\Big((au'+bu)x\Big) - \frac{a-b}{a+b} F_u\Big((u'-u)x\Big)$$

for all $x \in G$, which is a fixed point of $\mathcal{T}_u$ such that

$$d\Big(F_u(x), f(x)\Big) \leq \frac{\varepsilon(u'x, ux)}{1-\alpha_u}$$

for all $x \in G$. Moreover,

$$F_u(x) = \lim_{k\to\infty} \mathcal{T}_u^k f(x)$$

for all $x \in G$.

To prove that F_u satisfies the functional equation (1.2) on G, just prove the following inequality

$$d\Big(\mathcal{T}_u^n f(x+y), \frac{2}{a+b}\mathcal{T}_u^n f(ax+by) - \frac{a-b}{a+b}\mathcal{T}_u^n f(x-y)\Big) \leq \alpha_u^n \varepsilon(x, y) \tag{2.6}$$

for all $x, y \in G$, and $n \in \mathbb{N}$.

Indeed, if $n = 0$ then (2.6) is simply (2.2). So, take $n \in \mathbb{N}_+$ and suppose that (2.6) holds for n

and $x, y \in G$. Then, by using (2.5) and the triangle inequality, we have

$$\begin{aligned}
& d\left(\mathcal{T}_u^{n+1} f(x+y), \frac{2}{a+b}\mathcal{T}_u^{n+1} f(ax+by) - \frac{a-b}{a+b}\mathcal{T}_u^{n+1} f(x-y)\right) \\
&= d\left(\frac{2}{a+b}\mathcal{T}_u^n f\Big((au'+bu)(x+y)\Big) - \frac{a-b}{a+b}T_u^n f\Big((u'-u)(x+y)\Big),\right. \\
&\quad (\frac{2}{a+b})^2 \mathcal{T}_u^n f\Big((au'+bu)(ax+by)\Big) - (\frac{2}{a+b})(\frac{a-b}{a+b})T_u^n f\Big((u'-u)(ax+by)\Big) \\
&\quad \left. -(\frac{2}{a+b})(\frac{a-b}{a+b})T_u^n f\Big((au'+bu)(x-y)\Big) + (\frac{a-b}{a+b})^2 T_u^n f\Big((u'-u)(x-y)\Big)\right) \\
&\leq \frac{2}{a+b} d\left(\mathcal{T}_u^n f\Big((au'+bu)(x+y)\Big),\right. \\
&\qquad\qquad \left.\frac{2}{a+b}\mathcal{T}_u^n f\Big((au'+bu)(ax+by)\Big) - \frac{a-b}{a+b}T_u^n f\Big((au'+bu)(x-y)\Big)\right) \\
&\quad + \frac{|a-b|}{a+b} d\left(\mathcal{T}_u^n f\Big((u'-u)(x+y)\Big),\right. \\
&\qquad\qquad \left.\frac{2}{a+b}\mathcal{T}_u^n f\Big((u'-u)(ax+by)\Big) - \frac{a-b}{a+b}T_u^n f\Big((u'-u)(x-y)\Big)\right) \\
&\leq \alpha_u^n \left(\frac{2}{a+b}\varepsilon\Big((au'+bu)x, (au'+bu)y\Big) + \frac{|a-b|}{a+b}\varepsilon\Big((u'-u)x, (u'-u)y\Big)\right) \\
&\leq \varepsilon(x,y)\alpha_u^n \left(\frac{2}{a+b}\lambda(au'+bu) + \frac{|a-b|}{a+b}\lambda(u'-u)\right) \\
&= \alpha_u^{n+1}\varepsilon(x,y).
\end{aligned}$$

By induction, we have shown that (2.6) holds for all $x, y \in G$. Letting $n \to \infty$ in (2.6), we get

$$F_u(ax+by) = \frac{a+b}{2}F_u(x+y) + \frac{a-b}{2}F_u(x-y)$$

for all $x, y \in G$. Thus, we have proved that for every $u \in \mathcal{U}$ there exists a function $F_u : G \to H$ which is a solution of the functional equation (1.2) on G and satisfies

$$d\Big(f(x), F_u(x)\Big) \leq \frac{\varepsilon(u'x, ux)}{1-\alpha_u}$$

for all $x \in G$. By (2.3), we get

$$\begin{aligned}
d\Big(f(x), F_u(x)\Big) &\leq \frac{\inf_{u\in\mathcal{U}} \varepsilon(u'x, ux)}{1-\sup_{u\in\mathcal{U}} \alpha_u} \\
&= 0
\end{aligned}$$

for all $x \in G$. This means that $F_u(x) = f(x)$ for all $x \in G$ and $u \in \mathcal{U}$, and hence

$$f(ax+by) = \frac{a+b}{2}f(x+y) + \frac{a-b}{2}f(x-y)$$

for all $x, y \in G$, which implies that f satisfies the functional equation (1.2) on G. □

In the next theorem, we will study the hyperstablity of the functional equation (1.2) on G without 0 the neutral element, because of the reason that one can easily deduce some applications.

Theorem 2.2. *Let $f : G \to H$ be a mapping satisfying the inequality*

$$d\left(f(x+y), \frac{2}{a+b}f(ax+by) - \frac{a-b}{a+b}f(x-y)\right) \le \varepsilon(x,y)$$

for all $x, y \in G\backslash\{0\}$, where 0 is the neutral element of the group $(G,+)$ and $\varepsilon : (G\backslash\{0\})^2 \to \mathbb{R}_0$ is an arbitrary function. Assume that there exists a nonempty subset $\mathcal{U} \subset l(G)$ such that

$$u \circ v = v \circ u \qquad (u, v \in \mathcal{U}),$$

and

$$\begin{aligned} &\inf\left\{\varepsilon(u'x, ux) : u \in \mathcal{U}\right\} = 0, \quad \forall x \in G\backslash\{0\}, \\ &\sup\left\{\alpha_u : u \in \mathcal{U}\right\} < 1, \end{aligned} \tag{2.7}$$

then f is a solution of the functional equation (1.2) on $G\backslash\{0\}$.

Proof. The proof is the same as in the proof of Theorem 2.1. □

3. Some consequences

From Theorem 2.2, we can obtain the following corollaries as natural results.

Corollary 3.1. *Let E and F be a normed space and a Banach space, respectively. Assume that X is a subgroup of the group $(E,+)$, $p < 0$, $q < 0$ and $c \ge 0$. If $f : X \to F$ satisfies*

$$\left\| f(ax+by) - \frac{a+b}{2}f(x+y) - \frac{a-b}{2}f(x-y) \right\| \le c\left(\|x\|^p + \|y\|^q\right) \tag{3.1}$$

for all $x, y \in X\backslash\{0\}$, then f satisfies the functional equation (1.2) on $X\backslash\{0\}$.

Proof. The proof follows from Theorem 2.2 by taking

$$\varepsilon(x,y) = c\left(\|x\|^p + \|y\|^q\right), \quad x, y \in X\backslash\{0\},$$

with some real numbers $c \ge 0$, $p < 0$, $q < 0$ and $d(x,y) = \|x - y\|$. For each $m \in \mathbb{N}_+$ define $u_m \colon X\backslash\{0\} \to X\backslash\{0\}$ by $u_m x := -mx$ and $u'_m \colon X\backslash\{0\} \to X\backslash\{0\}$ by $u'_m x := (1+m)x$. Then

$$\begin{aligned} \varepsilon(u_m x, u_k y) &= \varepsilon(-mx, -ky) = c\left(\|-mx\|^p + \|-ky\|^q\right) \\ &= cm^p\|x\|^p + ck^q\|y\|^q \le (m^p + k^q)\, c\left(\|x\|^p + \|y\|^q\right) \\ &= (m^p + k^q)\, \varepsilon(x,y) \end{aligned}$$

for all $x \in X \setminus \{0\}$, $k, m \in \mathbb{N}_+$. Hence

$$\lim_{m\to\infty} \varepsilon(u'_m x, u_m y) \qquad \le \lim_{m\to\infty} \left((1+m)^p + m^q\right)\varepsilon(x,y) = 0$$

for all $x, y \in X\backslash\{0\}$. Then (2.7) is valid with $\lambda(u_m) = m^p + m^q$ for $m \in \mathbb{N}_+$, and there exists $n_0 \in \mathbb{N}_+$ such that

$$\frac{2}{a+b}\left(\left|a + (a-b)m\right|^p + \left|a + (a-b)m\right|^q\right) + \frac{|a-b|}{a+b}\left((2m+1)^p + (2m+1)^q\right) < 1 \ (m \ge n_0).$$

So it easily seen that (2.1) is fulfilled with

$$\mathcal{U} := \left\{u_m \in Aut\ X : m \in \mathbb{N}_{n_0}\right\}.$$

Therefore, by Theorem 2.2, every $f : X \to F$ satisfying (3.1) is a solution of the functional equation (1.2) on $X\backslash\{0\}$. □

Corollary 3.2. *Let E and F be a normed space and a Banach space, respectively. Assume that X is a subgroup of the group $(E,+)$, $p,q \in \mathbb{R}$, $p+q<0$ and $c \geq 0$. If $f : X \to F$ satisfies*

$$\left\| f(ax+by) - \frac{a+b}{2} f(x+y) - \frac{a-b}{2} f(x-y) \right\| \leq c \, \|x\|^p \, \|y\|^q$$

for all $x, y \in X \backslash \{0\}$, then f satisfies the functional equation (1.2) on $X \backslash \{0\}$.

Proof. It is easily seen that the function ε given by

$$\varepsilon(x,y) = c \, \|x\|^p \, \|y\|^q \quad x, y \in X \backslash \{0\},$$

satisfies (2.7), since

$$\varepsilon(mx,ky) = c \, \|mx\|^p \, \|ky\|^q = c \, |m|^p \, |k|^q \, \|x\|^p \, \|y\|^q = |m|^p \, |k|^q \, \varepsilon(x,y)$$

for all $x, y \in X \backslash \{0\}$, $k, m \in \mathbb{Z}$, and $km \neq 0$.

The remainder of the proof is similar to the proof of Corollary 3.1. □

By an analogous conclusion, the function ε given by

$$\varepsilon(x,y) = c \left(\|x\|^p + \|y\|^q + \|x\|^p \, \|y\|^q \right), \quad x, y \in X \backslash \{0\},$$

satisfies (2.7), since

$$\begin{aligned} \varepsilon(mx,ky) &= c \left(\|mx\|^p + \|ky\|^q + \|mx\|^p \, \|ky\|^q \right) \\ &= c \left(|m|^p \, \|x\|^p + |k|^q \, \|y\|^p + |m|^p \, |k|^q \, \|x\|^p \, \|y\|^q \right) \\ &\leq \left(|m|^p + |k|^q + |m|^p \, |k|^q \right) \varepsilon(x,y) \end{aligned}$$

for all $x, y \in X \backslash \{0\}$, $k, m \in \mathbb{Z}$, and $km \neq 0$. So we have the following corollary.

Corollary 3.3. *Let E and F be a normed space and a Banach space, respectively. Assume that X is a subgroup of the group $(E,+)$, and $p<0$, $q<0$, $p+q<0$ and $c \geq 0$. If $f : X \to F$ satisfies*

$$\left\| f(ax+by) - \frac{a+b}{2} f(x+y) - \frac{a-b}{2} f(x-y) \right\| \leq c \left(\|x\|^p + \|y\|^q + \|x\|^p \, \|y\|^q \right)$$

for all $x, y \in X \backslash \{0\}$, then f satisfies the functional equation (1.2) on $X \backslash \{0\}$.

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A fractional finite difference inclusion

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Abstract. In this manuscript we investigated the fractional finite difference inclusion $\Delta^{\mu}_{\mu-2}x(t) \in F(t, x(t), \Delta x(t))$ via the boundary conditions $\Delta x(b+\mu) = A$ and $x(\mu-2) = B$, where $1 < \mu \leq 2$, $A, B \in \mathbb{R}$ and $F : \mathbb{N}^{b+\mu+2}_{\mu-2} \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is a compact valued multifunction.

Keywords: Fixed point of multifunction, Fractional finite difference inclusion, Hausdorff metric.

1 Introduction

We recall that there are several published works on the existence of solutions for some fractional finite difference equations. In [1, 3, 7] the boundary value problems for discrete fractional equations were explain in detail. In [4] the discrete nabla fractional Taylor formulae were described together with some well known inequalities. In [8] the authors investigated a k-dimensional system of fractional finite difference equations. The papers [11, 12, 13, 14] reported several results on discrete boundary value problem and existence results for fractional difference equations. Further results can be seen in [17, 19, 20, 21] and the references therein. The readers can find more details about elementary notions and definitions of fractional finite difference equations in [5, 6, 10, 15] and [16].

In [2] it was proved the existence of solutions for nonlinear fractional q-difference inclusions involving convex as well as non-convex valued maps with nonlocal Robin (separated) conditions. However,in the of our knowledge there is no research on fractional finite difference inclusions so far. Here, we give a motivation about the importance of studying the fractional finite difference inclusions. For example, consider the fractional finite difference equation $\Delta^{\mu} y(t) = h(t+\mu-2, y(t+\mu-2))$ via the boundary conditions $y(\mu-3) = 0$, $\Delta^{\alpha} y(\mu-1-\alpha) = 0$ and $\Delta^{\beta} y(\mu+b+1-\beta) = 0$, where $t \in \mathbb{N}^{b+3}_0$, $b \in \mathbb{N}_0$, $2 < \mu \leq 3$, $1 < \alpha \leq 2$, $0 < \beta \leq 1$ and $h : \mathbb{N}^{b+\mu+1}_{\mu-2} \times \mathbb{R} \to \mathbb{R}$ is a map. Define the compact valued multifunction $T_h : \mathbb{N}^{b+\mu+1}_{\mu-2} \times \mathbb{R} \to 2^{\mathbb{R}}$ by $T_h(t) = \{h(t+\mu-2, y(t+\mu-2))\}$. It is easy to check that each solution of the fractional finite difference equation $\Delta^{\mu} y(t) = h(t+\mu-2, y(t+\mu-2))$ is a solution for the fractional finite difference inclusion $\Delta^{\mu} y(t) \in T_h(t)$. Thus, studying the fractional finite difference inclusions needs more potential mathematical abilities. In this paper,

we provide some preliminaries to investigate the existence of solution for a fractional finite difference inclusion.

We recall some basic definitions utilized in the rest of the manuscript. The Gamma function is defined by $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$ for the complex numbers z in which the real part of z is positive (see [18]). Now, we define $t^{\underline{\mu}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\mu)}$ for all $t, \mu \in \mathbb{R}$ whenever the right-hand side is defined (see [7] and [16]). In this paper, we use the notations $\mathbb{N}_p = \{p, p+1, p+2, \ldots\}$ for all $p \in \mathbb{R}$ and $\mathbb{N}_p^q = \{p, p+1, p+2, \ldots, q\}$ for all real numbers p and q whenever $q - p$ is a natural number. Let $\mu > 0$ with $m - 1 < \mu < m$ for some natural number m. The μ-th fractional sum of f based at a is defined by $\Delta_a^{-\mu} f(t) = \frac{1}{\Gamma(\mu)} \sum_{k=a}^{t-\mu} (t - \sigma(k))^{\underline{\mu-1}} f(k)$ for all $t \in \mathbb{N}_{a+\mu}$, where $\sigma(k) = k + 1$ is the forward jump operator (see [3, 8]). Similarly, we define $\Delta_a^\mu f(t) = \frac{1}{\Gamma(-\mu)} \sum_{k=a}^{t+\mu} (t - \sigma(k))^{\underline{-\mu-1}} f(k)$ for all $t \in \mathbb{N}_{a+m-\mu}$ (see [7, 16]). Note that, the domain of $\Delta_a^r f$ is $\mathbb{N}_{a+m-r}$ for $r > 0$ and $\mathbb{N}_{a-r}$ for $r < 0$. Also, for the natural number $\mu = m$, we have the known formula $\Delta_a^\mu f(t) = \Delta^m f(t) = \sum_{i=0}^m (-1)^i \binom{m}{i} f(t + m - i)$ (see [7, 16]). Finally, set the trivial sum $\Delta_a^0 f(t) = f(t)$ for all $t \in \mathbb{N}_a$.

Lemma 1.1. *[7] Let $m \geq 1$, $m - 1 < \mu \leq m$ and $h : \mathbb{N}_{\mu-2}^{b+\mu+2} \to \mathbb{R}$ be a map. The general solution of the equation $\Delta_{\mu-2}^\mu y(t) = h(t)$ is given by*

$$y(t) = \sum_{i=1}^m c_i t^{\underline{\mu-i}} + \frac{1}{\Gamma(\mu)} \sum_{s=0}^{t-\mu} (t - \sigma(s))^{\underline{\mu-1}} h(s)$$

for all $t \in \mathbb{N}_{\mu-2}$, where $c_1, \cdots, c_m$ are arbitrary constants.

Let $(\mathcal{X}, d)$ be a metric space. The Hausdorff metric $H_d : 2^{\mathcal{X}} \times 2^{\mathcal{X}} \to [0, \infty)$ is defined by

$$H_d(C, E) = \max\{\sup_{c \in C} d(c, E), \sup_{e \in E} d(C, e)\},$$

where $d(C, e) = \inf_{c \in C} d(c, e)$ ([18]). We denote the set of compact subsets of $\mathcal{X}$ by $P_{cp}(\mathcal{X})$ and the set of closed subsets of $\mathcal{X}$ by $C(\mathcal{X})$. Let $T : \mathcal{X} \to 2^{\mathcal{X}}$ be a multifunction. An element $x \in \mathcal{X}$ is called a fixed point of T whenever $x \in Tx$ ([18]). A multifunction $T : \mathcal{X} \to C(\mathcal{X})$ is called a contraction whenever there exists $\lambda \in (0, 1)$ such that $H_d(T(x), T(y)) \leq \lambda d(x, y)$ for all $x, y \in \mathcal{X}$ ([18]). In 1970, Covitz and Nadler proved next theorem ([9]).

Theorem 1.2. *Each closed valued contraction multifunction on a complete metric space has a fixed point.*

2 Main result

Now, we are ready to investigate the existence of solutions for fractional finite difference inclusion

$$\Delta_{\mu-2}^\mu x(t) \in F(t, x(t), \Delta x(t)) \tag{1}$$

via the boundary conditions $\Delta x(b+\mu) = A$ and $x(\mu-2) = B$, where $x : \mathbb{N}_{\mu-2}^{b+\mu+2} \to \mathbb{R}$ is a map, $1 < \mu \le 2$ and $F : \mathbb{N}_{\mu-2}^{b+\mu+2} \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is a compact valued multifunction.

Lemma 2.1. *Let $1 < \mu \le 2$ and $x : \mathbb{N}_{\mu-2}^{b+\mu+2} \to \mathbb{R}$ and $y : \mathbb{N}_0^{b+2} \to \mathbb{R}$ be two maps. Then x_0 is a solution of the fractional finite difference equation $\Delta_{\mu-2}^{\mu} x(t) = y(t)$ via the boundary conditions $\Delta x(b+\mu) = A$ and $x(\mu-2) = B$ if and only if x_0 is a solution of the fractional sum equation $x(t) = \varphi(t) + \sum_{s=0}^{b+1} G(t,s)y(s)$, where $\varphi(t) = \Big[\frac{A}{(\mu-1)(\mu+b)^{\underline{\mu-2}}} - \frac{B(\mu-2)}{(b+3)\Gamma(\mu)}\Big] t^{\underline{\mu-1}} + \frac{B}{\Gamma(\mu-1)} t^{\underline{\mu-2}}$, $G(t,s) = -\dfrac{t^{\underline{\mu-1}}(b+\mu-\sigma(s))^{\underline{\mu-2}}}{(\mu+b)^{\underline{\mu-2}}} + (t-\sigma(s))^{\underline{\mu-1}}$ whenever $0 \le s \le t-\mu \le b+1$ and $G(t,s) = -\dfrac{t^{\underline{\mu-1}}(b+\mu-\sigma(s))^{\underline{\mu-2}}}{(\mu+b)^{\underline{\mu-2}}}$ whenever $0 \le t-\mu < s \le b+1$.*

Proof. Let x_0 be a solution of the equation $\Delta_{\mu-2}^{\mu} x(t) = y(t)$ via the boundary conditions $\Delta x(b+\mu) = A$ and $x(\mu-2) = B$. By using Lemma 1.1, we get

$$x_0(t) = c_1 t^{\underline{\mu-1}} + c_2 t^{\underline{\mu-2}} + \frac{1}{\Gamma(\mu)} \sum_{s=0}^{t-\mu} (t-\sigma(s))^{\underline{\mu-1}} y(s)$$

where $c_1, c_2 \in \mathbb{R}$. By using the boundary condition $x_0(\mu-2) = B$, we obtain $c_2 = \frac{B}{\Gamma(\mu-1)}$. Since $\Delta x_0(t) = c_1(\mu-1)t^{\underline{\mu-2}} + c_2(\mu-2)t^{\underline{\mu-3}} + \frac{1}{\Gamma(\mu-1)} \sum_{s=0}^{t-\mu+1} (t-\sigma(s))^{\underline{\mu-2}} y(s)$, by using the boundary condition $\Delta x_0(b+\mu) = A$ we get

$$c_1 = \frac{A}{(\mu-1)(\mu+b)^{\underline{\mu-2}}} - \frac{B(\mu-2)}{(b+3)\Gamma(\mu)} - \frac{1}{(\mu+b)^{\underline{\mu-2}}\Gamma(\mu)} \sum_{s=0}^{b+1} (b+\mu-\sigma(s))^{\underline{\mu-2}} y(s).$$

Hence,

$$x_0(t) = \Big[\frac{A}{(\mu-1)(\mu+b)^{\underline{\mu-2}}} - \frac{B(\mu-2)}{(b+3)\Gamma(\mu)}\Big] t^{\underline{\mu-1}} + \frac{B}{\Gamma(\mu-1)} t^{\underline{\mu-2}}$$

$$-\frac{t^{\underline{\mu-1}}}{(\mu+b)^{\underline{\mu-2}}\Gamma(\mu)} \sum_{s=0}^{b+1} (b+\mu-\sigma(s))^{\underline{\mu-2}} y(s) + \frac{1}{\Gamma(\mu)} \sum_{s=0}^{t-\mu} (t-\sigma(s))^{\underline{\mu-1}} y(s) = \varphi(t) + \sum_{s=0}^{b+1} G(t,s)y(s).$$

Now let x_0 be a solution of the equation $x(t) = \varphi(t) + \sum_{s=0}^{b+1} G(s,t)y(s)$. Then, we have

$$x_0(t) = \Big[\frac{A}{(\mu-1)(\mu+b)^{\underline{\mu-2}}} - \frac{B(\mu-2)}{(b+3)\Gamma(\mu)}\Big] t^{\underline{\mu-1}} + \frac{B}{\Gamma(\mu-1)} t^{\underline{\mu-2}}$$

$$-\frac{t^{\underline{\mu-1}}}{(\mu+b)^{\underline{\mu-2}}\Gamma(\mu)} \sum_{s=0}^{b+1} (b+\mu-\sigma(s))^{\underline{\mu-2}} y(s) + \frac{1}{\Gamma(\mu)} \sum_{s=0}^{t-\mu} (t-\sigma(s))^{\underline{\mu-1}} y(s).$$

Since $(\mu-2)^{\underline{\mu-1}} = 0$, $(\mu-2)^{\underline{\mu-2}} = \Gamma(\mu-1)$ and $\sum_{s=0}^{-2} (\mu-2-\sigma(s))^{\underline{\mu-1}} y(s) = 0$, we get $x_0(\mu-2) = B$. A simple calculation shows that $\Delta x_0(\mu+b) = A$. On the other hand, it is easy to check that $x_0(t) = c_1 t^{\underline{\mu-1}} + c_2 t^{\underline{\mu-2}} + \frac{1}{\Gamma(\mu)} \sum_{s=0}^{t-\mu} (t-\sigma(s))^{\underline{\mu-1}} y(s)$ is a solution for the equation $\Delta_{\mu-2}^{\mu} x(t) = y(t)$ and so $\Delta^{\mu} x_0(t) = y(t)$. This completes the proof. ∎

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A function $x : \mathbb{N}_{\mu-2}^{b+\mu+2} \to \mathbb{R}$ is a solution for the problem (1) whenever it satisfies the boundary conditions and there exists a function $y : \mathbb{N}_0^{b+2} \to \mathbb{R}$ such that $y(t) \in F(t, x(t), \Delta x(t))$ for all $t \in \mathbb{N}_0^{b+2}$ and

$$x(t) = \Big[\frac{A}{(\mu-1)(\mu+b)^{\underline{\mu-2}}} - \frac{B(\mu-2)}{(b+3)\Gamma(\mu)}\Big]t^{\underline{\mu-1}} + \frac{B}{\Gamma(\mu-1)}t^{\underline{\mu-2}}$$

$$-\frac{t^{\underline{\mu-1}}}{(\mu+b)^{\underline{\mu-2}}\Gamma(\mu)}\sum_{s=0}^{b+1}(b+\mu-\sigma(s))^{\underline{\mu-2}}y(s) + \frac{1}{\Gamma(\mu)}\sum_{s=0}^{t-\mu}(t-\sigma(s))^{\underline{\mu-1}}y(s).$$

Let $\mathcal{X}$ be the set of all functions $x : \mathbb{N}_{\mu-2}^{b+\mu+2} \to \mathbb{R}$ endowed with the norm

$$\|x\| = \max_{t\in\mathbb{N}_{\mu-2}^{b+\mu+2}} |x(t)| + \max_{t\in\mathbb{N}_{\mu-2}^{b+\mu+2}} |\Delta x(t)|.$$

Suppose that $\{x_n\}$ is a Cauchy sequence in $\mathcal{X}$. For each $\epsilon > 0$, choose a natural number N such that $\|x_n - x_m\| < \epsilon$ for all $m, n > N$. Hence, $\max_{t\in\mathbb{N}_{\mu-2}^{b+\mu+2}} |x_n(t) - x_m(t)| < \epsilon$ and $\max_{t\in\mathbb{N}_{\mu-2}^{b+\mu+2}} |\Delta x_n(t) - \Delta x_m(t)| < \epsilon$. Choose $x(t), z(t) \in \mathbb{R}$ such that $x_n(t) \to x(t)$ and $\Delta x_n(t) \to z(t)$ for all $t \in \mathbb{N}_{\mu-2}^{b+\mu+2}$. Note that, $\Delta x_n(t) = x_n(t+1) - x_n(t)$ for all n and so $\Delta x(t) = x(t+1) - x(t) = z(t)$. This implies that there exists a natural number M such that $|x_n(t) - x(t)| < \frac{\epsilon}{2}$ and $|\Delta x_n(t) - \Delta x(t)| < \frac{\epsilon}{2}$ for all $t \in \mathbb{N}_{\mu-2}^{b+\mu+2}$ and $n > M$. Thus,

$$\|x_n - x\| = \max_{t\in\mathbb{N}_{\mu-2}^{b+\mu+2}} |x_n(t) - x(t)| + \max_{t\in\mathbb{N}_{\mu-2}^{b+\mu+2}} |\Delta x_n(t) - \Delta x(t)| < \epsilon.$$

This shows that $(\mathcal{X}, \|.\|)$ is a Banach space. For each $x \in \mathcal{X}$, put

$$S_{F,x} = \{y : \mathbb{N}_0^{b+2} \to \mathbb{R} :\ y(t) \in F(t, x(t), \Delta x(t)) \text{ for all } t \in \mathbb{N}_0^{b+2}\}.$$

Note that, the selection principle implies that $S_{F,x}$ is nonempty because $F(t, x(t), \Delta x(t)) \neq \emptyset$. Put

$$G_1 = \max_{t\in\mathbb{N}_{\mu-2}^{\mu+b+2}} \frac{1}{(\mu+b)^{\underline{\mu-2}}\Gamma(\mu)}\sum_{s=0}^{b+2}\Big[(b+\mu-\sigma(s))^{\underline{\mu-2}}t^{\underline{\mu-1}} + (\mu+b)^{\underline{\mu-2}}(t-\sigma(s))^{\underline{\mu-1}}\Big]$$

and

$$G_2 = \max_{t\in\mathbb{N}_{\mu-2}^{\mu+b+2}} \frac{1}{(\mu+b)^{\underline{\mu-2}}\Gamma(\mu-1)}\sum_{s=0}^{b+2}\Big[(b+\mu-\sigma(s))^{\underline{\mu-2}}t^{\underline{\mu-2}} + (\mu+b)^{\underline{\mu-2}}(t-\sigma(s))^{\underline{\mu-2}}\Big].$$

Since every Green function is bounded and the summations are finite, G_1 and G_2 are real numbers.

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Theorem 2.2. *Let $g : \mathbb{N}_{\mu-2}^{b+\mu+2} \to \mathbb{R}$ be a map such that $0 < \max_{t\in\mathbb{N}_{\mu-2}^{b+\mu+2}} |g(t)|(G_1 + G_2) < 1$. Suppose that $F : \mathbb{N}_{\mu-2}^{b+\mu+2} \times \mathbb{R} \times \mathbb{R} \to P_{cp}(\mathbb{R})$ is a multifunction such that*

$$H_d(F(t, x_1, x_2), F(t, z_1, z_2)) \leq g(t)(|x_1 - z_1| + |x_2 - z_2|) \tag{2}$$

for all $t \in \mathbb{N}_{\mu-2}^{b+\mu+2}$ and $x_1, x_2, z_1, z_2 \in \mathbb{R}$. Then the inclusion problem (1) has a solution.

Proof. Choose $y \in S_{F,x}$. Define $h \in \mathcal{X}$ by

$$h(t) = \Big[\frac{A}{(\mu-1)(\mu+b)^{\underline{\mu-2}}} - \frac{B(\mu-2)}{(b+3)\Gamma(\mu)}\Big]t^{\underline{\mu-1}} + \frac{B}{\Gamma(\mu-1)}t^{\underline{\mu-2}}$$

$$-\frac{t^{\underline{\mu-1}}}{(\mu+b)^{\underline{\mu-2}}\Gamma(\mu)}\sum_{s=0}^{b+1}(b+\mu-\sigma(s))^{\underline{\mu-2}}y(s) + \frac{1}{\Gamma(\mu)}\sum_{s=0}^{t-\mu}(t-\sigma(s))^{\underline{\mu-1}}y(s)$$

for all $t \in \mathbb{N}_{\mu-2}^{\mu+b+2}$. This shows that the set

$$\Big\{h \in \mathcal{X} : \text{ there exists } y \in S_{F,x} \text{ such that } h(t) = w(t) \text{ for all } t \in \mathbb{N}_{\mu-2}^{\mu+b+2}\Big\}$$

is nonempty, where

$$w(t) = \Big[\frac{A}{(\mu-1)(\mu+b)^{\underline{\mu-2}}} - \frac{B(\mu-2)}{(b+3)\Gamma(\mu)}\Big]t^{\underline{\mu-1}} + \frac{B}{\Gamma(\mu-1)}t^{\underline{\mu-2}}$$

$$-\frac{t^{\underline{\mu-1}}}{(\mu+b)^{\underline{\mu-2}}\Gamma(\mu)}\sum_{s=0}^{b+1}(b+\mu-\sigma(s))^{\underline{\mu-2}}y(s) + \frac{1}{\Gamma(\mu)}\sum_{s=0}^{t-\mu}(t-\sigma(s))^{\underline{\mu-1}}y(s).$$

Now, define the multifunction $T : \mathcal{X} \to 2^{\mathcal{X}}$ by

$$T(x) = \Big\{h \in \mathcal{X} : \text{ there exists } y \in S_{F,x} \text{ such that } h(t) = w(t) \text{ for all } t \in \mathbb{N}_{\mu-2}^{\mu+b+2}\Big\}.$$

First, we show that $T(x)$ is a closed subset of $\mathcal{X}$ for all $x \in \mathcal{X}$. Let $x \in \mathcal{X}$ and $\{u_n\}_{n\geq1}$ be a sequence in $T(x)$ with $u_n \to u$. Choose $y_n \in S_{F,x}$ such that

$$u_n(t) = \Big[\frac{A}{(\mu-1)(\mu+b)^{\underline{\mu-2}}} - \frac{B(\mu-2)}{(b+3)\Gamma(\mu)}\Big]t^{\underline{\mu-1}} + \frac{B}{\Gamma(\mu-1)}t^{\underline{\mu-2}}$$

$$-\frac{t^{\underline{\mu-1}}}{(\mu+b)^{\underline{\mu-2}}\Gamma(\mu)}\sum_{s=0}^{b+1}(b+\mu-\sigma(s))^{\underline{\mu-2}}y_n(s) + \frac{1}{\Gamma(\mu)}\sum_{s=0}^{t-\mu}(t-\sigma(s))^{\underline{\mu-1}}y_n(s)$$

for all $t \in \mathbb{N}_{\mu-2}^{b+\mu+2}$ and $n \geq 1$. Since F has compact values, $\{y_n\}_{n\geq1}$ has a subsequence which converges to some $y : \mathbb{N}_0^{b+2} \to \mathbb{R}$. We denote this subsequence again by $\{y_n\}_{n\geq1}$. It is easy to check that $y \in S_{F,x}$ and

$$u_n(t) \to u(t) = \Big[\frac{A}{(\mu-1)(\mu+b)^{\underline{\mu-2}}} - \frac{B(\mu-2)}{(b+3)\Gamma(\mu)}\Big]t^{\underline{\mu-1}} + \frac{B}{\Gamma(\mu-1)}t^{\underline{\mu-2}}$$

$$-\frac{t^{\underline{\mu-1}}}{(\mu+b)^{\underline{\mu-2}}\Gamma(\mu)}\sum_{s=0}^{b+1}(b+\mu-\sigma(s))^{\underline{\mu-2}}y(s)+\frac{1}{\Gamma(\mu)}\sum_{s=0}^{t-\mu}(t-\sigma(s))^{\underline{\mu-1}}y(s)$$

for all $t \in \mathbb{N}_{\mu-2}^{b+\mu+2}$. This implies that $u \in T(x)$ and so the multifunction T has closed values. Now, we show that T is a contraction multifunction with the constant

$$\lambda = \max_{t\in\mathbb{N}_{\mu-2}^{b+\mu+2}} |g(t)|(G_1+G_2) < 1.$$

Let $x, z \in \mathcal{X}$, $h_1 \in T(x)$ and $h_2 \in T(z)$. Choose $y_1 \in S_{F,x}$ and $y_2 \in S_{F,z}$ such that

$$h_1(t) = \Big[\frac{A}{(\mu-1)(\mu+b)^{\underline{\mu-2}}} - \frac{B(\mu-2)}{(b+3)\Gamma(\mu)}\Big]t^{\underline{\mu-1}} + \frac{B}{\Gamma(\mu-1)}t^{\underline{\mu-2}}$$

$$-\frac{t^{\underline{\mu-1}}}{(\mu+b)^{\underline{\mu-2}}\Gamma(\mu)}\sum_{s=0}^{b+1}(b+\mu-\sigma(s))^{\underline{\mu-2}}y_1(s)+\frac{1}{\Gamma(\mu)}\sum_{s=0}^{t-\mu}(t-\sigma(s))^{\underline{\mu-1}}y_1(s)$$

and

$$h_2(t) = \Big[\frac{A}{(\mu-1)(\mu+b)^{\underline{\mu-2}}} - \frac{B(\mu-2)}{(b+3)\Gamma(\mu)}\Big]t^{\underline{\mu-1}} + \frac{B}{\Gamma(\mu-1)}t^{\underline{\mu-2}}$$

$$-\frac{t^{\underline{\mu-1}}}{(\mu+b)^{\underline{\mu-2}}\Gamma(\mu)}\sum_{s=0}^{b+1}(b+\mu-\sigma(s))^{\underline{\mu-2}}y_2(s)+\frac{1}{\Gamma(\mu)}\sum_{s=0}^{t-\mu}(t-\sigma(s))^{\underline{\mu-1}}y_2(s)$$

for all $t \in \mathbb{N}_{\mu-2}^{b+\mu+2}$. Since

$$H_d(F(t,x(t),\Delta x(t)), F(t,z(t),\Delta z(t))) \le g(t)(|x(t)-z(t)|+|\Delta x(t)-\Delta z(t)|),$$

we get $|y_1(t)-y_2(t)| \le g(t)(|x(t)-z(t)|+|\Delta x(t)-\Delta z(t)|)$ for all $t \in \mathbb{N}_0^{b+2}$. Hence,

$$|h_1(t)-h_2(t)| \le \sum_{s=0}^{b+1}\frac{(b+\mu-\sigma(s))^{\underline{\mu-2}}t^{\underline{\mu-1}}}{\Gamma(\mu)(\mu+b)^{\underline{\mu-2}}}|y_1(s)-y_2(s)| + \sum_{s=0}^{t-\mu}\frac{(t-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)}|y_1(s)-y_2(s)|.$$

Since $\sum_{s=t-\mu+1}^{b+2}\frac{(t-\sigma(s))^{\underline{\mu-1}}}{\Gamma(\mu)} = 0$ and $\frac{(b+\mu-\sigma(b+2))^{\underline{\mu-2}}}{\Gamma(\mu)(\mu+b)^{\underline{\mu-2}}} = 0$, we obtain

$$|h_1(t)-h_2(t)| \le \sum_{s=0}^{b+2}\frac{(b+\mu-\sigma(s))^{\underline{\mu-2}}t^{\underline{\mu-1}} + (\mu+b)^{\underline{\mu-2}}(t-\sigma(s))^{\underline{\mu-1}}}{(\mu+b)^{\underline{\mu-2}}\Gamma(\mu)}|y_1(s)-y_2(s)|$$

and so

$$\max_{t\in\mathbb{N}_{\mu-2}^{\mu+b+2}}|h_1(t)-h_2(t)| \le G_1 \max_{t\in\mathbb{N}_{\mu-2}^{\mu+b+2}}|g(t)|\big(\max_{t\in\mathbb{N}_{\mu-2}^{\mu+b+2}}|x(t)-z(t)| + \max_{t\in\mathbb{N}_{\mu-2}^{\mu+b+2}}|\Delta x(t)-\Delta z(t)|\big)$$

$$= G_1 \max_{t\in\mathbb{N}_{\mu-2}^{\mu+b+2}}|g(t)|\|x-z\|.$$

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Since

$$\Delta h_1(t) = \Big[\frac{A}{(\mu+b)^{\underline{\mu-2}}} - \frac{B(\mu-2)}{(b+3)\Gamma(\mu-1)}\Big] t^{\underline{\mu-2}} + \frac{B}{\Gamma(\mu-2)} t^{\underline{\mu-3}}$$

$$-\frac{t^{\underline{\mu-2}}}{(\mu+b)^{\underline{\mu-2}}\Gamma(\mu-1)} \sum_{s=0}^{b+1} (b+\mu-\sigma(s))^{\underline{\mu-2}} y_1(s) + \frac{1}{\Gamma(\mu-1)} \sum_{s=0}^{t-\mu+1} (t-\sigma(s))^{\underline{\mu-2}} y_1(s),$$

we get

$$|\Delta h_1(t) - \Delta h_2(t)| \le \sum_{s=0}^{b+2} \frac{(b+\mu-\sigma(s))^{\underline{\mu-2}} t^{\underline{\mu-2}} + (\mu+b)^{\underline{\mu-2}} (t-\sigma(s))^{\underline{\mu-2}}}{(\mu+b)^{\underline{\mu-2}}\Gamma(\mu-1)} |y_1(s) - y_2(s)|$$

and so

$$\max_{t\in\mathbb{N}_{\mu-2}^{\mu+b+2}} |\Delta h_1(t) - \Delta h_2(t)| \le G_2 \max_{t\in\mathbb{N}_{\mu-2}^{\mu+b+2}} |g(t)| \big(\max_{t\in\mathbb{N}_{\mu-2}^{\mu+b+2}} |x(t)-z(t)| + \max_{t\in\mathbb{N}_{\mu-2}^{\mu+b+2}} |\Delta x(t) - \Delta z(t)| \big)$$

$$= G_2 \max_{t\in\mathbb{N}_{\mu-2}^{\mu+b+2}} |g(t)| \|x-z\|.$$

Hence, $\|h_1 - h_2\| \le \max_{t\in\mathbb{N}_{\mu-2}^{b+\mu+2}} |g(t)|(G_1+G_2)\|x-z\| = \lambda\|x-z\|$ for all $x, z \in \mathcal{X}$, $h_1 \in T(x)$ and $h_2 \in T(z)$. This implies that $H_d(T(x), T(z)) \le \lambda\|x-z\|$ for all $x, z \in \mathcal{X}$ and so the multifunction T is a contraction with closed values. Now by using Lemma 1.2, T has a fixed point which is a solution for the inclusion problem (1). □

Now, we present an example to illustrate the problem.

Example 2.1. *Consider the fractional finite difference inclusion*

$$\Delta_{-0.5}^{1.5} x(t) \in \Big[0, e^t + \frac{\sin(x(t))}{e^7} + 8t^2 - \frac{|\Delta x(t)|}{e^7 + |\Delta x(t)|}\Big] \tag{3}$$

via the boundary conditions $\Delta x(6.5) = 5$ and $x(-0.5) = -15$. Put $\mu = 1.5$, $b = 5$, $A = 5$, $B = -15$ and $F(t, x_1, x_2) = \Big[0, e^t + \frac{\sin x_1}{e^7} + 8t^2 - \frac{|x_2|}{e^7+|x_2|}\Big]$ for all $t \in \mathbb{N}_{-0.5}^{8.5}$ and $x_1, x_2 \in \mathbb{R}$. Note that, $e^t + \frac{\sin x_1}{e^7} + 8t^2 - \frac{|x_2|}{e^7+|x_2|} > 0$ for all $t \in \mathbb{N}_{-0.5}^{8.5}$ and $x_1, x_2 \in \mathbb{R}$ and so F is a compact valued multifunction on $\mathbb{N}_{-0.5}^{8.5} \times \mathbb{R} \times \mathbb{R}$. Define $g(t) = \frac{1}{165|t|}$. Note that $\max_{t\in\mathbb{N}_{-0.5}^{8.5}} |g(t)| = \frac{2}{165}$,

$$G_1 = \max_{t\in\mathbb{N}_{-0.5}^{8.5}} \frac{1}{(6.5)^{\underline{-0.5}}\Gamma(1.5)} \sum_{s=0}^{7} (5.5-s)^{\underline{-0.5}} t^{\underline{0.5}} + (6.5)^{\underline{-0.5}} (t-\sigma(s))^{\underline{0.5}} \simeq 66.8457$$

and

$$G_2 = \max_{t\in\mathbb{N}_{-0.5}^{8.5}} \frac{1}{(6.5)^{\underline{-0.5}}\Gamma(0.5)} \sum_{s=0}^{7} (5.5-s)^{\underline{-0.5}} t^{\underline{-0.5}} + (6.5)^{\underline{-0.5}} (t-\sigma(s))^{\underline{-0.5}} \simeq 15.3947.$$

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Also, $\lambda = \frac{2}{165}(66.8457 + 15.3947) < 1$. *On the other hand, we have*

$$\sup_{b \in F(t,y_1,y_2)} d(F(t,x_1,x_2),b) = \frac{\sin y_1}{e^7} - \frac{|y_2|}{e^7+|y_2|} - \frac{\sin x_1}{e^7} + \frac{|x_2|}{e^7+|x_2|}$$

whenever $\frac{\sin y_1}{e^7} - \frac{|y_2|}{e^7+|y_2|} > \frac{\sin x_1}{e^7} - \frac{|x_2|}{e^7+|x_2|}$ *and* $\sup_{b \in F(t,y_1,y_2)} d(F(t,x_1,x_2),b) = 0$ *otherwise. This implies that*

$$H_d(F(t,x_1,x_2), F(t,y_1,y_2)) \le \Big|\frac{\sin y_1}{e^7} - \frac{|y_2|}{e^7+|y_2|} - \frac{\sin x_1}{e^7} + \frac{|x_2|}{e^7+|x_2|}\Big|$$

$$\le \frac{1}{e^7}|\sin y_1 - \sin x_1| + \frac{1}{e^7}|y_2 - x_2| \le g(t)(|x_1 - y_1| + |x_2 - y_2|)$$

for all $t \in \mathbb{N}_{-0.5}^{8.5}$ *and* $x_1, x_2, y_1, y_2 \in \mathbb{R}$. *Now by using Theorem 2.2, the problem (3) has at least one solution.*

Remark 2.1. *The values* A *and* B *are arbitrary constants in the problem. In particular, the problem (3) has a solution via the boundary conditions* $\Delta x(6.5) = A$ *and* $x(-0.5) = B$ *for all* $A, B \in \mathbb{R}$.

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Some implicit properties of the second kind Bernoulli polynomials of order α

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Abstract In this paper we define the second kind Bernoulli polynomials of order α in the complex plane and find some interesting properties, symmetric identities of this polynomials. We also derive some relations between the second kind Bernoulli polynomials of order α, Euler polynomials of the second kind of order α, the stirling numbers and other polynomials.

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1. Introduction

The Bernoulli, Euler and Genocchi polynomials have been a subject of investigations and have been used in various branches of mathmatics such as theory of numbers, calculus of finite differences, combinatorial analysis, p-adic analytic number theory etc. These polynomials which are divided the first kind or the second kind have been researched by many mathematicians. Those polynomials have been extended and generalized in various direction, in particular, Bernoulli polynomials have been discussed by C. S. Ryoo, Q.-M. Luo, H. M. Srivastava, and T. Kim, etc(see [1-23]).

In this paper we use the following notations. $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ denotes the ring of integers, $\mathbb{Q}$ denotes the fild of rational numbers, $\mathbb{C}$ denotes the set of complex numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

The second kind Bernoulli numbers $\widetilde{B}_n$ and polynomials $\widetilde{B}_n(x)$ are by means of the

following generating function (see [16-18]):

$$\sum_{n=0}^{\infty} \widetilde{B}_n \frac{t^n}{n!} = \frac{te^t}{e^{2t}-1} \quad (|t| < \pi),$$
$$\sum_{n=0}^{\infty} \widetilde{B}_n(x) \frac{t^n}{n!} = \left(\frac{te^t}{e^{2t}-1}\right) e^{tx} \quad (|t| < \pi), \tag{1.1}$$

respectively. From the above generating function, we can easily see that

$$\widetilde{B}_n := \widetilde{B}_n(0).$$

Also, the Euler polynomials $\widetilde{E}_n(x)$ of second kinds are given by

$$\sum_{n=0}^{\infty} \widetilde{E}_n(x) \frac{t^n}{n!} = \left(\frac{2e^t}{e^{2t}+1}\right) e^{tx} \quad (|t| < \frac{\pi}{2}). \tag{1.2}$$

In [7], they expanded the Euler polynomials using a real or complex parameter α and introduced the Euler numbers and polynomials of the second kind of order α as follows:

$$\sum_{n=0}^{\infty} \widetilde{E}_n^{(\alpha)} \frac{t^n}{n!} = \left(\frac{2e^t}{e^{2t}+1}\right)^{\alpha} \quad (|t| < \frac{\pi}{2}),$$
$$\sum_{n=0}^{\infty} \widetilde{E}_n^{(\alpha)}(x) \frac{t^n}{n!} = \left(\frac{2e^t}{e^{2t}+1}\right)^{\alpha} e^{tx} \quad (|t| < \frac{\pi}{2}). \tag{1.3}$$

We usually define the central factorial numbers by the following expansion formula.

$$\sum_{k=0}^{n} T(n,k)x(x-1^2)(x-2^2)\cdots(x-(k-1)^2) = x^n. \tag{1.4}$$

or by the generating function

$$(2k)! \sum_{n=k}^{\infty} T(n,k) \frac{x^{2n}}{(2n)!} = (e^x + e^{-x} - 2)^k. \tag{1.5}$$

By using (1.4) and (1.5), we can find some properties of the central factorial numbers $T(n,k)$ (see [8,23]).

The stirling numbers of the second kind are the number of ways to partition a set of n objects into k non-empty subsets and are denoted by $S_2(n,k)$. The Stirling numbers of each kind according to the parameters n, k can form mutually inverse triangular matrices. The Stirling numbers of the second kind $S_2(n,k)$ can be defined by means of

$$x^n = \sum_{k=0}^{n} S_2(n,k)x(x-1)(x-2)\cdots(x-n+1), \tag{1.6}$$

or by the generating function

$$(e^x - 1)^k = k! \sum_{n=k}^{\infty} S_2(n,k) \frac{x^n}{n!}. \tag{1.7}$$

It follows from (1.6) or (1.7) that

$$S_2(n,k) = S_2(n-1,k-1) + kS_2(n-1,k),$$

with $S_2(n,0) = 0(n > 0)$, $S_2(n,n) = 1$, $S_2(n,1) = 1(n > 0)$, $S_2(n,k) = 0(k > n$ or $k < 0)$ (see [4,10,22-23]).

Associated Stirling numbers of the second kind $b(n,k)$ are defined by

$$(e^x - 1 - x)^k = k! \sum_{n=2k}^{\infty} b(n,k) \frac{x^n}{n!}. \tag{1.8}$$

It follows from (1.8) that

$$b(n,k) = (n-1)b(n-2,k-1) + kb(n-1,k),$$

with $b(n,0) = 0(n > 0)$, $b(0,0) = 1$, $b(n,1) = 1(n > 1)$, $b(n,k) = 0(2k > n$ or $k < 0)$(see [3-4, 10-11]).

$S_n(x)$ is the stirling polynomials, defined by

$$\sum_{n=0}^{\infty} \frac{S_n(x)}{n!} t^n = \left(\frac{t}{1 - e^{-t}} \right)^{x+1} \tag{1.9}$$

The stirling polynomials are related to the stirling numbers of the first kind $S_1(n,m)$ by

$$S_n(m) = \frac{(-1)^n}{\binom{m}{n}} S_1(m+1, m-n+1),$$

where $\binom{m}{n}$ is a binomial coefficient and m is an integer with $m \geq n$(see [19]).

This paper is organized as follows. In section 2, we construct the second kind Bernoulli polynomials of order α in the complex plane. We also study some interesting and basic properties of their polynomials. In section 3, we find some relations between the second kind Bernoulli polynomials of order α and Euler polynomials of the second kind of order α, the stirling numbers and other polynomials. We also define the power sum polynomials in oder to derive some symmetric identities about the second kind Bernoulli polynomials of order α in the complex plane.

2. Some properties of the second kind Bernoulli polynomials of oder α

In this section, we construct the second kind Bernoulli polynomials of order α in the complex plane. We get some basic propertites of their polynomials.

Definition 2.1. For a real or complex parameter α, the second kind Bernoulli numbers $\widetilde{B}_n^{(\alpha)}$ and the second kind Bernoulli polynomials $\widetilde{B}_n^{(\alpha)}(x)$, each of degree n in x as well as in α, are defined by means of the following generating functions:

$$\sum_{n=0}^{\infty} \widetilde{B}_n^{(\alpha)} \frac{t^n}{n!} = \left(\frac{te^t}{e^{2t}-1}\right)^{\alpha},$$

and

$$\sum_{n=0}^{\infty} \widetilde{B}_n^{(\alpha)}(x) \frac{t^n}{n!} = \left(\frac{te^t}{e^{2t}-1}\right)^{\alpha} e^{tx}, \quad (|t| < \pi;\ 1^{\alpha} := 1),$$

respectively. Note that

$$\widetilde{B}_n^{(\alpha)}(0) := \widetilde{B}_n^{(\alpha)}, \quad \widetilde{B}_n^{(1)} = \widetilde{B}_n(x), \quad \widetilde{B}_n^{(0)}(x) = x^n, \quad (n \in \mathbb{N}_0).$$

From Definition 2,1, we can find the above polynomials by using computer.

n	$\widetilde{B}_n$	$\widetilde{B}_n^{(\alpha)}$	$\widetilde{B}_n^{(\alpha)}(x)$
$n=0$	$\frac{1}{2}$	$\frac{1}{2^\alpha}$	$\frac{1}{2^\alpha}$
$n=1$	0	0	$\frac{1}{2^\alpha}x$
$n=2$	$-\frac{1}{6}$	$\frac{1}{2^\alpha}\left(-\frac{1}{3}\alpha\right)$	$\frac{1}{2^\alpha}\left(-\frac{1}{3}\alpha+x^2\right)$
$n=3$	0	0	$\frac{1}{2^\alpha}(-\alpha x+x^3)$
$n=4$	$\frac{7}{30}$	$\frac{1}{2^\alpha}\left(\frac{1}{270}\alpha+\frac{1}{27}\alpha^2\right)$	$\frac{1}{2^\alpha}\left(\frac{2}{15}\alpha+\frac{1}{3}\alpha^2-2\alpha x^2+x^4\right)$
$n=5$	0	0	$\frac{1}{2^\alpha}\left(\left(\frac{2}{3}\alpha+\frac{5}{3}\alpha^2\right)x-\frac{10}{3}\alpha x^3+x^5\right)$
$n=6$	$-\frac{31}{42}$	$\frac{1}{2^\alpha}\left(-\frac{16}{63}\alpha-\frac{6}{9}\alpha^2-\frac{5}{9}\alpha^3\right)$	$\frac{1}{2^\alpha}\left(-\frac{16}{63}\alpha+\frac{10}{9}\alpha^2-\frac{16}{9}\alpha^2-\frac{5}{9}\alpha^3+(2\alpha+5\alpha^2)x^2-5\alpha x^4+x^6\right)$
...	...	...	...

Table 1: The comparison of $\widetilde{B}_n$, $\widetilde{B}_n^{(\alpha)}$ and $\widetilde{B}_n^{(\alpha)}(x)$

From now on, we will find the basic propertities of the second kind Bernoulli polynomials of order α from Definition 2.1.

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Theorem 2.2. Let $\alpha, \beta \in \mathbb{C}$. Then we find that

$$\widetilde{B}_n^{(\alpha+\beta)}(x+y) = \sum_{k=0}^{n} \binom{n}{k} \widetilde{B}_k^{(\alpha)}(x) \widetilde{B}_{n-k}^{(\beta)}(y).$$

Proof. From Definition 2.1, we obtain as the following equation by the binomial theorem.

$$\begin{aligned}\sum_{n=0}^{\infty} \widetilde{B}_n^{(\alpha+\beta)}(x+y)\frac{t^n}{n!} &= \left(\frac{te^t}{e^{2t}-1}\right)^{\alpha+\beta} e^{t(x+y)}\\ &= \sum_{n=0}^{\infty} \widetilde{B}_n^{(\alpha)}(x)\frac{t^n}{n!} \sum_{n=0}^{\infty} \widetilde{B}_n^{(\beta)}(y)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \widetilde{B}_k^{(\alpha)}(x)\widetilde{B}_{n-k}^{(\beta)}(y)\frac{t^n}{n!}.\end{aligned}$$

Comparing the both sides of $\frac{t^n}{n!}$, we complete the proof.

□

Corollary 2.3. Let $\alpha, \beta \in \mathbb{C}$. From Theorem 2.2, we can see that
(i) If we suppose $\beta = 0$, then

$$\widetilde{B}_n^{(\alpha)}(x+y) = \sum_{k=0}^{n} \binom{n}{k} \widetilde{B}_k^{(\alpha)}(x) y^{n-k} \quad = \sum_{k=0}^{n} \binom{n}{k} \widetilde{B}_k^{(\alpha)}(y) x^{n-k}.$$

(ii) If we suppoes $\beta = 0$ and $y = 0$, then

$$\widetilde{B}_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k} \widetilde{B}_k^{(\alpha)} x^{n-k} \quad = \sum_{k=0}^{n} \binom{n}{k} \widetilde{B}_k^{(\alpha)} x^{k}.$$

Theorem 2.4. Let $\alpha, x \in \mathbb{C}$ and $n \in \mathbb{N}_0$. Then we get that

$$\widetilde{B}_n^{(\alpha)}(-x) = (-1)^n \widetilde{B}_n^{(\alpha)}(x).$$

Proof. Substituting $-x$ instead of x we get the following equation.

$$\begin{aligned}\sum_{n=0}^{\infty} \widetilde{B}_n^{(\alpha)}(-x)\frac{t^n}{n!} &= \left(\frac{te^t}{e^{2t}-1}\right)^{\alpha} e^{-tx}\\ &= \left(\frac{-te^{-t}}{e^{-2t}-1}\right)^{\alpha} e^{-tx}\\ &= \sum_{n=0}^{\infty} (-1)^n \widetilde{B}_n^{(\alpha)}(x)\frac{t^n}{n!}.\end{aligned}$$

Hence the proof of Theorem 2.4. is clear.

□

Theorem 2.5. Let $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}_0$. Then we have

$$\widetilde{B}_n^{(\alpha)}(x+2) - \widetilde{B}^{(\alpha)}(x) = n\widetilde{B}_n^{(\alpha-1)}(x+1) = \sum_{k=0}^{n} \binom{n}{k} k\widetilde{B}_{n-k}^{(\alpha-1)}(x).$$

Proof. By using Definition 2.1, we get that

$$\begin{aligned}\widetilde{B}_n^{(\alpha)}(x+2) - \widetilde{B}^{(\alpha)}(x)\frac{t^n}{n!} &= \left(\frac{te^t}{e^{2t}-1}\right)^{\alpha} e^{t(x+2)} - \left(\frac{te^t}{e^{2t}-1}\right)^{\alpha} e^{tx} \\ &= \left(\frac{te^t}{e^{2t}-1}\right)^{\alpha-1} te^{t(x+1)}.\end{aligned}$$

We can find out the following equation.

$$\left(\frac{te^t}{e^{2t}-1}\right)^{\alpha-1} te^{t(x+1)} = \sum_{n=0}^{\infty} \widetilde{B}_{n-1}^{(\alpha-1)}(x)\frac{t^n}{n!} \sum_{n=0}^{\infty} n\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} k\widetilde{B}_{n-k}^{(\alpha-1)}(x)\frac{t^n}{n!}.$$

We also find the other equation as the follows:

$$\left(\frac{te^t}{e^{2t}-1}\right)^{\alpha-1} te^{t(x+1)} = \sum_{n=0}^{\infty} \widetilde{B}_n^{(\alpha-1)}(x+1)\frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty} n\widetilde{B}_{n-1}^{(\alpha-1)}(x+1)\frac{t^n}{n!}.$$

Hence we terminate the proof of Theorem 2.5.

□

Theorem 2.6. Let $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}_0$. Then we derive that

(i) $\displaystyle\sum_{k=0}^{n-1} \binom{n-1}{k} \widetilde{B}_k^{(\alpha-1)}(x) = \frac{1}{n}\left[\widetilde{B}_n^{(\alpha)}(x+2) - \widetilde{B}_n^{(\alpha)}(x)\right].$

(ii) $\displaystyle\widetilde{B}_n^{(\alpha-1)}(x+1) = \frac{1}{n+1}\left[\sum_{k=0}^{n+1} 2^{n+1-k}\binom{n+1}{k}\widetilde{B}_k^{(\alpha)}(x) - \widetilde{B}_{n+1}^{(\alpha)}(x)\right].$

(iii) $\displaystyle\widetilde{B}_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k}\left[2^{n-k}\widetilde{B}_k^{(\alpha)}(x) - k\widetilde{B}_{n-k}^{(\alpha-1)}(x)\right].$

Proof. (i) Using Corollary 2.3.(i), we can see that

$$\widetilde{B}_n^{(\alpha)}(x+1) = \sum_{k=0}^{n} \binom{n}{k}\widetilde{B}_k^{(\alpha)}(x), \tag{2.1}$$

and

$$\widetilde{B}_n^{(\alpha)}(x+2) = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k}\widetilde{B}_k^{(\alpha)}(x). \tag{2.2}$$

If we combine Theorem 2.5 and (2.1), then we find that

$$\widetilde{B}_n^{(\alpha)}(x+2) - \widetilde{B}_n^{(\alpha)}(x) = n\sum_{k=0}^{n-1}\binom{n-1}{k}\widetilde{B}_k^{(\alpha-1)}(x).$$

Therefore, we can see Theorem 2.6.(i).
(ii) From (2.2) we can transform Theorem 2.5 as following equation:

$$\sum_{k=0}^{n}\binom{n}{k}2^{n-k}\widetilde{B}_k^{(\alpha)}(x) - \widetilde{B}_n^{(\alpha)}(x) = n\widetilde{B}_{n-1}^{(\alpha-1)}(x+1).$$

Hence we have

$$\widetilde{B}_n^{(\alpha-1)}(x+1) = \frac{1}{n+1}\left[\sum_{k=0}^{n+1}2^{n+1-k}\binom{n+1}{k}\widetilde{B}_k^{(\alpha)}(x) - \widetilde{B}_{n+1}^{(\alpha)}(x)\right].$$

(iii) This proof is very similar to the proof of (ii). We find the below equation from Theorem 2.5 and (2.2).

$$\sum_{k=0}^{n}\binom{n}{k}2^{n-k}\widetilde{B}_k^{(\alpha)}(x) - \widetilde{B}_n^{(\alpha)}(x) = \sum_{k=0}^{n}\binom{n}{k}k\widetilde{B}_{n-k}^{(\alpha-1)}(x).$$

From the above equation we know that

$$\widetilde{B}_n^{(\alpha)}(x) = \sum_{k=0}^{n}\binom{n}{k}\left[2^{n-k}\widetilde{B}_k^{(\alpha)}(x) - k\widetilde{B}_{n-k}^{(\alpha-1)}(x)\right].$$

Thus we wind up the proof of Theorem 2.6.

□

Corollary 2.7. Let n be non-negative integer. From Theorem 2.4.(ii), we can see that

$$x^n = \frac{1}{n+1}\sum_{k=0}^{n+1}2^{n+1-k}\binom{n+1}{k}\widetilde{B}_k(x-1).$$

Theorem 2.8. For w_1, w_2 be positive integers and $\alpha \in \mathbb{C}$, we have

$$\begin{aligned}&\sum_{n=0}^{l}\binom{l}{n}\sum_{i=0}^{w_2-1}\sum_{j=0}^{w_1-1}w_1^{l-n}w_2^{n}\widetilde{B}_{l-n}^{(\alpha)}(w_2x+2j\frac{w_2}{w_1})\widetilde{B}_n^{(\alpha)}(w_1y+2i\frac{w_1}{w_2})\\ &=\sum_{n=0}^{l}\binom{l}{n}\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}w_2^{l-n}w_1^{n}\widetilde{B}_{l-n}^{(\alpha)}(w_1x+2j\frac{w_1}{w_2})\widetilde{B}_n^{(\alpha)}(w_2y+2i\frac{w_2}{w_1}).\end{aligned}$$

Proof. We can consider that

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$$A(t) := \frac{(e^{2w_1w_2t}-1)^2(w_1w_2t^2e^{t(w_1+w_2)})^\alpha e^{w_1w_2t(x+y)}}{(e^{2w_1t}-1)^{\alpha+1}(e^{2w_2t}-1)^{\alpha+1}}.$$

Then we are able to express $A(t)$ as follows:

$$\begin{aligned} A(t) &:= \left(\frac{w_1te^{w_1t}}{e^{2w_1t}-1}\right)^\alpha e^{w_1w_2tx}\sum_{i=0}^{w_2-1}e^{2iw_1t}\left(\frac{w_2te^{w_2t}}{e^{2w_2t}-1}\right)^\alpha e^{w_1w_2ty}\sum_{j=0}^{w_1-1}e^{2jw_2t} \\ &= \sum_{l=0}^{\infty}\sum_{n=0}^{l}\binom{l}{n}\sum_{i=0}^{w_2-1}\sum_{j=0}^{w_1-1}w_1^{l-n}w_2^n\widetilde{B}_{l-n}^{(\alpha)}(w_2x+2j\frac{w_2}{w_1})\widetilde{B}_n^{(\alpha)}(w_1y+2i\frac{w_1}{w_2})\frac{t^l}{l!}. \end{aligned} \tag{2.3}$$

We also represent $A(t)$ as the following form.

$$\begin{aligned} A(t) &:= \left(\frac{w_2te^{w_2t}}{e^{2w_2t}-1}\right)^\alpha e^{w_1w_2tx}\sum_{i=0}^{w_1-1}e^{2iw_2t}\left(\frac{w_1te^{w_1t}}{e^{2w_1t}-1}\right)^\alpha e^{w_1w_2ty}\sum_{j=0}^{w_2-1}e^{2jw_1t} \\ &= \sum_{l=0}^{\infty}\sum_{n=0}^{l}\binom{l}{n}\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}w_2^{l-n}w_1^n\widetilde{B}_{l-n}^{(\alpha)}(w_1x+2j\frac{w_1}{w_2})\widetilde{B}_n^{(\alpha)}(w_2y+2i\frac{w_2}{w_1})\frac{t)^l}{l!}. \end{aligned} \tag{2.4}$$

By using the coefficient comparison in (2.3) and (2.4), we proved Theorem 2.8.

□

Corollary 2.9. By substituting $w_1 = 1$ from Theorem 2.8, we easily see that

$$\begin{aligned} &\sum_{n=0}^{l}\binom{l}{n}\sum_{i=0}^{w_2-1}w_2^n\widetilde{B}_{l-n}^{(\alpha)}(w_2x)\widetilde{B}_n^{(\alpha)}(y+\frac{2i}{w_2}) \\ &= \sum_{n=0}^{l}\binom{l}{n}\sum_{j=0}^{w_2-1}w_2^{l-n}\widetilde{B}_n^{(\alpha)}(w_2y)\widetilde{B}_{l-n}^{(\alpha)}(x+\frac{2j}{w_2}). \end{aligned}$$

Theorem 2.10. For w_1, w_2 be positive integers, $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}$, we obtain

$$\begin{aligned} &\sum_{l=0}^{n}\binom{n}{l}lw_1^{n+1-l}w_2^{l-1}\widetilde{B}_{n-l}^{(\alpha-1)}(w_2x)\sum_{i=0}^{w_2-1}\widetilde{B}_{l-1}^{(\alpha)}(w_1y+2i\frac{w_1}{w_2}+\frac{w_1}{w_2}) \\ &= \sum_{l=0}^{n}\binom{n}{l}lw_2^{n+1-l}w_1^{l-1}\widetilde{B}_{n-l}^{(\alpha-1)}(w_1x)\sum_{j=0}^{w_1-1}\widetilde{B}_{l-1}^{(\alpha)}(w_2y+2j\frac{w_2}{w_1}+\frac{w_2}{w_1}). \end{aligned}$$

Proof. Assume that

$$B(t) := \frac{(e^{2w_1w_2t}-1)(w_1w_2t^2e^{t(w_1+w_2)})^\alpha e^{w_1w_2t(x+y)}}{(e^{2w_1t}-1)^\alpha(e^{2w_2t}-1)^\alpha}.$$

Then we express $B(t)$ as follows:

$$
\begin{aligned}
B(t) := &\left(\frac{w_1 t e^{w_1 t}}{e^{2w_1 t}-1}\right)^{\alpha-1} e^{w_1 w_2 t x} w_1 t e^{w_1 t} \sum_{i=0}^{w_2-1} e^{2iw_1 t}\left(\frac{w_2 t e^{w_2 t}}{e^{2w_2 t}-1}\right)^{\alpha} e^{w_1 w_2 t y} \\
&= \sum_{n=1}^{\infty}\sum_{l=0}^{n}\binom{n}{l} l \widetilde{B}_{n-l}^{(\alpha-1)}(w_2 x) w_1^{n+1-l} w_2^{l-1} \sum_{i=0}^{w_2-1} \widetilde{B}_{l-1}^{(\alpha)}(w_1 y + 2i\frac{w_1}{w_2} + \frac{w_1}{w_2})\frac{t^n}{n!}.
\end{aligned}
\tag{2.5}
$$

We can also represent $B(t)$ as the following equation.

$$
\begin{aligned}
B(t) := &\left(\frac{w_2 t e^{w_2 t}}{e^{2w_2 t}-1}\right)^{\alpha-1} e^{w_1 w_2 t x} w_2 t e^{w_2 t} \sum_{j=0}^{w_1-1} e^{2iw_2 t}\left(\frac{w_1 t e^{w_1 t}}{e^{2w_1 t}-1}\right)^{\alpha} e^{w_1 w_2 t y} \\
&= \sum_{n=1}^{\infty}\sum_{l=0}^{n}\binom{n}{l} l \widetilde{B}_{n-l}^{(\alpha-1)}(w_1 x) w_2^{n+1-l} w_1^{l-1} \sum_{j=0}^{w_1-1} \widetilde{B}_{l-1}^{(\alpha)}(w_2 y + 2j\frac{w_2}{w_1} + \frac{w_2}{w_1})\frac{t^n}{n!}.
\end{aligned}
\tag{2.6}
$$

We get the following symmetric equation from (2.5) and (2.6).

$$
\begin{aligned}
&\sum_{l=0}^{n}\binom{n}{l} l w_1^{n+1-l} w_2^{l-1} \widetilde{B}_{n-l}^{(\alpha-1)}(w_2 x) \sum_{i=0}^{w_2-1} \widetilde{B}_{l-1}^{(\alpha)}(w_1 y + 2i\frac{w_1}{w_2} + \frac{w_1}{w_2}) \\
&= \sum_{l=0}^{n}\binom{n}{l} l w_2^{n+1-l} w_1^{l-1} \widetilde{B}_{n-l}^{(\alpha-1)}(w_1 x) \sum_{j=0}^{w_1-1} \widetilde{B}_{l-1}^{(\alpha)}(w_2 y + 2j\frac{w_2}{w_1} + \frac{w_2}{w_1}).
\end{aligned}
$$

Thus, we complete the proof of Theorem 2.10.

□

Corollary 2.11. When $w_1 = 1$ from Theorem 2.10, we can see that

$$
\begin{aligned}
&\sum_{l=0}^{n}\binom{n}{l} l w_2^{l-1} \widetilde{B}_{n-l}^{(\alpha-1)}(w_2 x) \sum_{i=0}^{w_2-1} \widetilde{B}_{l-1}^{(\alpha)}(y + \frac{2i}{w_2} + \frac{1}{w_2}) \\
&= \sum_{l=0}^{n}\binom{n}{l} l w_2^{n+1-l} \widetilde{B}_{n-l}^{(\alpha-1)}(x) \widetilde{B}_{l-1}^{(\alpha)}(w_2 y + w_2).
\end{aligned}
$$

3. Some relations of the second kind Bernoulli polynomials of order α, the various kinds of numbers and polynomials

In this section, we investigate some interesting relations between the second kind Bernoulli polynomials of order α and the Euler polynomials of the second kind of order α. We find some relations between the second kind Bernoulli polynomials of order α, stirling numbers, the associated stirling numbers of the second kind and central factorial numbers. We also derive the analogue of Srivastava-Pintér addition theorem(see [14,20]).

Theorem 3.1. Let $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$. Then one has

$$\widetilde{B}_n^{(\alpha)}(x) = \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} \binom{n}{k} \widetilde{B}_{n-k}^{(\alpha)}(x) \widetilde{E}_k^{(\alpha)}(x) = \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} \binom{n}{k} \widetilde{B}_{n-k}^{(\alpha)} \widetilde{E}_k^{(\alpha)}(2x).$$

Proof. We can find the following equation by using generating function.

$$\sum_{n=0}^{\infty} \widetilde{B}_n^{(\alpha)}(x) \frac{t^n}{n!} = \left(\frac{te^t}{e^{2t}-1}\right)^\alpha e^{tx} = \left(\frac{te^{\frac{1}{2}t}}{e^t-1}\right)^\alpha \left(\frac{e^{\frac{1}{2}t}}{e^t+1}\right)^\alpha e^{tx}.$$

From the above equation we can combine the Euler polynomials of the second kind of oder α, $\widetilde{E}_n^{(\alpha)}$, by using Cauchy product as follows:.

$$\sum_{n=0}^{\infty} \widetilde{B}_n^{(\alpha)}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{2}\right)^n \widetilde{B}_{n-k}^{(\alpha)} \widetilde{E}_k^{(\alpha)}(2x) \frac{t^n}{n!}.$$

Hence, the proof of Theorem 3.1 is clear.

□

From simple transformation, we are able to represent to the following equation from Theorem 3.1.

$$2^n \widetilde{B}_n^{(\alpha)}(\frac{x}{2}) = \sum_{k=0}^{n} \binom{n}{k} \widetilde{B}_{n-k}^{(\alpha)}(\frac{x}{2}) \widetilde{E}_k^{(\alpha)}(\frac{x}{2}) = \sum_{k=0}^{n} \binom{n}{k} \widetilde{B}_{n-k}^{(\alpha)} \widetilde{E}_k^{(\alpha)}(x).$$

By using addition theorem of the Euler polynomials of the second kind of order α, we find the analogue of Srivastava-Pintér addition theorem as the following theorem 3.2.

Theorem 3.2. Let $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}_0$. Then we obtain that

$$\widetilde{B}_n^{(\alpha)}(x+y) = \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} \left[\widetilde{B}_k^{(\alpha)}(y+2) + \widetilde{B}_k^{(\alpha)}(y)\right] \widetilde{E}_{n-k}(x-1).$$

Proof. In [6], we can make the following equation by using simple calculation.

$$x^n = \frac{1}{2}\left[\sum_{k=0}^{n} \binom{n}{k} 2^{n-k} \widetilde{E}_k(x-1) + \widetilde{E}_n(x-1)\right]. \tag{3.1}$$

Substituting (3.1) in Corollary 2.3.(i), inverting the order of summation and using the elementary combinatorial, we get

$$\begin{aligned}
\widetilde{B}_n^{(\alpha)}(x+y) &= \sum_{k=0}^{n} \binom{n}{k} \widetilde{B}_k^{(\alpha)}(y) \left[\frac{1}{2}\left\{\sum_{j=0}^{n-k} \binom{n-k}{j} 2^{n-k-j} \widetilde{E}_j(x-1) + \widetilde{E}_{n-k}(x-1)\right\}\right] \\
&= \frac{1}{2}\left[\sum_{j=0}^{n} \binom{n}{j} \widetilde{E}_j(x-1) \sum_{k=0}^{n-j} \binom{n-j}{k} 2^{n-k-j} \widetilde{B}_k^{(\alpha)}(y) + \sum_{k=0}^{n} \binom{n}{k} \widetilde{B}_k^{(\alpha)}(y) \widetilde{E}_{n-k}(x-1)\right].
\end{aligned}$$

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Note that

$$\widetilde{B}_{n-j}^{(\alpha)}(y+2)=\sum_{k=0}^{n-j}\binom{n-j}{k}2^{n-k-j}\widetilde{B}_k^{(\alpha)}(y).$$

By applying the above form, we derive that

$$\widetilde{B}_n^{(\alpha)}(x+y)=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\left[\widetilde{B}_k^{(\alpha)}(y+2)+\widetilde{B}_k^{(\alpha)}(y)\right]\widetilde{E}_{n-k}(x-1).$$

Hence, we complete the proof of Theorem 3.2.

□

Corollary 3.3. When $\alpha=1$ from Theorem 3.2, we see that

$$\widetilde{B}_n(x+y)=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\left[\widetilde{B}_k(y+2)+\widetilde{B}_k(y)\right]\widetilde{E}_{n-k}(x-1).$$

Corollary 3.4. By setting $y=0$ from Theorem 3.2, we have

$$\widetilde{B}_n^{(\alpha)}(x)=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\left[\widetilde{B}_k^{(\alpha)}(2)+\widetilde{B}_k^{(\alpha)}\right]\widetilde{E}_{n-k}(x-1).$$

Theorem 3.5. Let $n\in\mathbb{N}_0$ and $m\in\mathbb{N}$. Then we get

$$\begin{aligned}\widetilde{B}_n^{(m)}=\frac{n!}{2^m}&\sum_{\substack{v_1,\cdots,v_m=0\\ v_1+\cdots+v_m=n}}\frac{\left(\sum_{k=0}^{v_1}\binom{v_1}{k}\left(\widetilde{B}_{v_1-k}(1)\widetilde{E}_k+\widetilde{B}_{v_1-k}\widetilde{E}_k(-1)\right)\right)\cdots}{v_1\cdots}\\ &\times\frac{\left(\sum_{k=0}^{v_m}\binom{v_m}{k}\left(\widetilde{B}_{v_m-k}(1)\widetilde{E}_k+\widetilde{B}_{v_m-k}\widetilde{E}_k(-1)\right)\right)}{v_m!}.\end{aligned}$$

Proof. Suppose that $\alpha=1$. That is,

$$\begin{aligned}\sum_{n=0}^{\infty}\widetilde{B}_n\frac{t^n}{n!}=\left(\frac{te^t}{e^{2t}-1}\right)&=\frac{te^{3t}}{(e^{2t}-1)(e^{2t}+1)}+\frac{te^t}{(e^{2t}-1)(e^{2t}+1)}\\ &=\sum_{n=0}^{\infty}\widetilde{B}_n(1)\sum_{n=0}^{\infty}\frac{1}{2}\widetilde{E}_n\frac{t^n}{n!}+\sum_{n=0}^{\infty}\widetilde{B}_n\sum_{n=0}^{\infty}\frac{1}{2}\widetilde{E}_n(-1)\frac{t^n}{n!}\\ &=\frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\left(\widetilde{B}_{n-k}(1)\widetilde{E}_k+\widetilde{B}_{n-k}\widetilde{E}_k(-1)\right)\frac{t^n}{n!}.\end{aligned}$$

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Consider that $\alpha = 2$. Then we have

$$\begin{aligned}\sum_{n=0}^{\infty}\widetilde{B}_n^{(2)}\frac{t^n}{n!} &= \left(\frac{1}{2}\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}\left(\widetilde{B}_{n-k}(1)\widetilde{E}_k+\widetilde{B}_{n-k}\widetilde{E}_k(-1)\right)\right\}\frac{t^n}{n!}\right)\\ &\times\left(\frac{1}{2}\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}\left(\widetilde{B}_{n-k}(1)\widetilde{E}_k+\widetilde{B}_{n-k}\widetilde{E}_k(-1)\right)\right\}\frac{t^n}{n!}\right)\\ &= \frac{1}{2^2}\sum_{n=0}^{\infty}\sum_{\substack{v_1,v_2=0\\ v_1+v_2=2}}\frac{\sum_{k=0}^{v_1}\binom{v_1}{k}\left(\widetilde{B}_{v_1-k}(1)\widetilde{E}_k+\widetilde{B}_{v_1-k}\widetilde{E}_k(-1)\right)}{v_1!}\\ &\times\frac{\sum_{k=0}^{v_2}\binom{v_2}{k}\left(\widetilde{B}_{v_2-k}(1)\widetilde{E}_k+\widetilde{B}_{v_2-k}\widetilde{E}_k(-1)\right)}{v_2!}.\end{aligned}$$

Therefore, we can derive the below equation when $\alpha = m$.

$$\begin{aligned}\widetilde{B}_n^{(m)} &= \frac{n!}{2^m}\sum_{\substack{v_1,\cdots,v_m=0\\ v_1+\cdots+v_m=n}}\frac{\left(\sum_{k=0}^{v_1}\binom{v_1}{k}\left(\widetilde{B}_{v_1-k}(1)\widetilde{E}_k+\widetilde{B}_{v_1-k}\widetilde{E}_k(-1)\right)\right)\cdots}{v_1\cdots}\\ &\times\frac{\left(\sum_{k=0}^{v_m}\binom{v_m}{k}\left(\widetilde{B}_{v_m-k}(1)\widetilde{E}_k+\widetilde{B}_{v_m-k}\widetilde{E}_k(-1)\right)\right)}{v_m!}.\end{aligned}$$

□

Theorem 3.6. Let $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}$. Then we have

$$\widetilde{B}_n^{(\alpha)}(x+y) = \sum_{j=0}^{n}\binom{n}{j}\sum_{k=0}^{n-j}\binom{x}{k}k!\widetilde{B}_j^{(\alpha)}(y)S_2(n-j,k),$$

where $S_2(n-j,k)$ is the stirling numbers of the second kind.

Proof. From Definition which is the stirling numbers of the second kind we can find this theorem. Note that

$$x^n = \sum_{k=0}^{n}\binom{x}{k}k!S_2(n,k),$$

where $S_2(n,k)$ are the stirling numbers of the second kind.
By combining the above equation and Corollary 2.3.(i), we can easily see that

$$\widetilde{B}_n^{(\alpha)}(x+y) = \sum_{j=0}^{n}\binom{n}{j}\sum_{k=0}^{n-j}\binom{x}{k}k!\widetilde{B}_j^{(\alpha)}(y)S_2(n-j,k),$$

Therefore, we complete the proof.

□

Theorem 3.7. Let $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$. Then we derive

$$\widetilde{B}_n^{(\alpha)} = \sum_{k=0}^{n} \binom{n}{k} 2^{n-\alpha-k} \sum_{j=0}^{n-k} \frac{(-1)^j}{\binom{n+j-k}{j}} \binom{\alpha+j-1}{j} b(n-k+j,j)\alpha^k,$$

where $b(n-k+j,j)$ is the associated stirling numbers of the second kind.

Proof. By using the associated stirling numbers of the second kind, we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} \widetilde{B}_n^{(\alpha)} \frac{t^n}{n!} &= \left(\frac{te^t}{e^{2t}-1}\right)^{\alpha} = \frac{1}{2^{\alpha}} \left(\frac{1}{1+\frac{1}{2t}(e^{2t}-2t-1)}\right)^{\alpha} e^{\alpha t} \\
&= \frac{1}{2^{\alpha}} \sum_{j=0}^{\infty} \binom{\alpha+j-1}{j} \left(-\frac{1}{2}\right)^j \left(e^{2t}-2t-1\right)^j t^{-j} e^{\alpha t} \\
&= \frac{1}{2^{\alpha}} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha+j-1}{j} j! \sum_{n=j}^{\infty} 2^n b(n+j,j) \frac{t^n}{(n+j)!} e^{\alpha t} \\
&= \frac{1}{2^{\alpha}} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} \sum_{j=0}^{n-k} \frac{(-1)^j}{\binom{n+j-k}{j}} \binom{\alpha+j-1}{j} b(n+j-k,j) \alpha^k \frac{t^n}{n!}.
\end{aligned}$$

Therefore we express $\widetilde{B}_n^{(\alpha)}$ as follows.

$$\widetilde{B}_n^{(\alpha)} = \sum_{k=0}^{n} \binom{n}{k} 2^{n-\alpha-k} \sum_{j=0}^{n-k} \frac{(-1)^j}{\binom{n+j-k}{j}} \binom{\alpha+j-1}{j} b(n-k+j,j)\alpha^k.$$

□

Theorem 3.8. Let $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$. Then we obtain

$$\widetilde{B}_n^{(\alpha)}(x) = \sum_{l=0}^{n} \sum_{k=0}^{n-l} \binom{n-k}{l} 2^{n-\alpha-l-k} \alpha^k \sum_{j=0}^{n-l-k} \binom{n+j}{k} \frac{(-1)^j}{\binom{n+j}{j}} \binom{\alpha+j-1}{j} b(n+j-l-k,j) x^l.$$

Proof. By using the associated stirling numbers of the second kind, we express the polynomials $\widetilde{B}_n^{(\alpha)}(x)$ as follows.

$$\begin{aligned}
\sum_{n=0}^{\infty} \widetilde{B}_n^{(\alpha)}(x) \frac{t^n}{n!} &= \left(\frac{te^t}{e^{2t}-1}\right)^{\alpha} e^{tx} = \sum_{n=0}^{\infty} \widetilde{B}_n^{(\alpha)} \frac{t^n}{n!} \sum_{n=0}^{\infty} x^n \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{l=0}^{n} \sum_{k=0}^{n-l} \binom{n-k}{l} 2^{n-\alpha-l-k} \alpha^k \sum_{j=0}^{n-l-k} \binom{n+j}{k} \frac{(-1)^j}{\binom{n+j}{j}} \binom{\alpha+j-1}{j} \\
&\times b(n+j-l-k,j) x^l \frac{t^n}{n!}.
\end{aligned}$$

From the above equation, we have the Theorem 3.8 as follows:

$$\widetilde{B}_n^{(\alpha)}(x)=\sum_{l=0}^{n}\sum_{k=0}^{n-l}\binom{n-k}{l}2^{n-\alpha-l-k}\alpha^k\sum_{j=0}^{n-l-k}\binom{n+j}{k}\frac{(-1)^j}{\binom{n+j}{j}}\binom{\alpha+j-1}{j}b(n+j-l-k,j)x^l.$$

Thus, we complete the proof of Theorem 3.8.

□

Theorem 3.9. Let $n\in\mathbb{N}_0$ and $\alpha\in\mathbb{C}$. Then we obtain that

$$\left(\widetilde{B}+\widetilde{B}^{(\alpha)}(1+\alpha)\right)^n=\frac{2^nS_n(\alpha)}{2^{\alpha+1}},$$

where $S_n(\alpha)$ is the stirling polynomials.
Proof. Consider that $\widetilde{B}^n=\widetilde{B}_n$. Then we get the following equation from binomial operation and definition of stirling polynomials.

$$\begin{aligned}2^{\alpha+1}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\left(\widetilde{B}+\widetilde{B}^{(\alpha)}(1+\alpha)\right)^n\frac{t^n}{n!}&=2\sum_{n=0}^{\infty}\widetilde{B}_n\frac{\left(\frac{1}{2}t\right)^n}{n!}2^\alpha\sum_{n=0}^{\infty}\widetilde{B}_n^{(\alpha)}(1+\alpha)\frac{\left(\frac{1}{2}t\right)^n}{n!}\\&=\left(\frac{te^t}{e^t-1}\right)^{\alpha+1}=\sum_{n=0}^{\infty}S_n(\alpha)\frac{t^n}{n!}.\end{aligned}$$

Hence we can finish the proof.

□

Theorem 3.10. Let $n\in\mathbb{N}_0$ and $k\in\mathbb{N}$. Then we derive that

$$\binom{n}{2k}\widetilde{B}_{n-2k}^{(-2k)}(x)=\sum_{l=0}^{n-2k}2^{2k+l}T(k+l,k)x^{n-(2k+l)},$$

where $T(n,k)$ is the central factorial numbers.
Proof. Consider that

$$\sum_{n=k}^{\infty}T(n,k)\frac{t^{2n}}{(2n)!}=\frac{(e^t-1)^{2k}}{(2k)!e^{tk}}.$$

In oder to make relation of the second kind Bernoulli polynomials and the central factorial numbers, we can express as the follows:

$$\frac{1}{4^k(2k)!}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\widetilde{B}_n^{(-2k)}(x)\frac{t^{2k+n}}{n!}=\sum_{n=k}^{\infty}T(n,k)\frac{t^{2n}}{(2n)!}\sum_{n=0}^{\infty}\left(\frac{1}{2}x\right)^n\frac{t^n}{n!}.$$

From the above equation, we obtain the below equation by using some calculation and comparing the both sides of $\frac{t^{2k+n}}{(2k+n)!}$.

$$\binom{n}{2k}\left(\frac{1}{2}\right)^n\widetilde{B}_{n-2k}^{(-2k)}(x)=\sum_{l=0}^{n-2k}\frac{T(k+l,k)}{2^{n-2k-l}}x^{n-2k-l}.$$

Therefore the Theorem 3.10 is clear.

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□

Let $l \in \mathbb{N}_0$ and n be non-negative integers. Then we can define $\widetilde{P}_l(n)$ as follows:

$$\widetilde{P}_l(n) = \widetilde{B}_l(2n) - \widetilde{B}_l = l\sum_{i=0}^{n-1}(1+2i)^{l-1}.$$

where $\widetilde{P}_l(n)$ is the symmetric power sum polynomials.

Theorem 3.11. For w_1, w_2 be positive integers and $\alpha \in \mathbb{C}$, we get

$$\begin{aligned}&\sum_{l=0}^{n}\binom{n}{l}\widetilde{B}_{n-l}^{(\alpha-1)}(w_1y)\sum_{s=0}^{l}\binom{l}{s}w_1^{l-s}w_2^{n-l+s}\widetilde{B}_{l-s}^{(\alpha)}(w_2x)\widetilde{P}_s(w_1)\\ &=\sum_{l=0}^{n}\binom{n}{l}\widetilde{B}_{n-l}^{(\alpha-1)}(w_2y)\sum_{s=0}^{l}\binom{l}{s}w_2^{l-s}w_1^{n-l+s}\widetilde{B}_{l-s}^{(\alpha)}(w_1x)\widetilde{P}_s(w_2).\end{aligned}$$

Proof. Where

$$C(t) := \frac{(e^{2w_1w_2t}-1)(w_1w_2t^2e^{t(w_1+w_2)})^{\alpha}e^{w_1w_2t(x+y)}}{(e^{2w_1t}-1)^{\alpha}(e^{2w_2t}-1)^{\alpha}},$$

say that

$$\begin{aligned}C(t) :&= \left(\frac{w_1te^{w_1t}}{e^{2w_1t}-1}\right)^{\alpha}e^{w_1w_2tx}\sum_{i=0}^{w_1-1}w_2te^{(1+2i)w_2t}\left(\frac{w_2te^{w_2t}}{e^{2w_2t}-1}\right)^{\alpha-1}e^{w_1w_2ty}\\ &=\sum_{n=0}^{\infty}\widetilde{B}_n^{(\alpha)}(w_2x)\frac{(w_1t)^n}{n!}\sum_{s=0}^{\infty}\widetilde{P}_s(w_1)\frac{(w_2t)^s}{s!}\sum_{n=0}^{\infty}\widetilde{B}_n^{(\alpha-1)}(w_1y)\frac{(w_2t)^n}{n!}\\ &=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\widetilde{B}_{n-l}^{(\alpha-1)}(w_1y)\sum_{s=0}^{l}\binom{l}{s}w_1^{l-s}w_2^{n-l+s}\widetilde{P}_s(w_1)\widetilde{B}_{l-s}^{(\alpha)}(w_2x)\right)\frac{t^n}{n!},\end{aligned} \tag{3.2}$$

which is true. $A(t)$ get transformed in the other following equation.

$$\begin{aligned}C(t) :&= \left(\frac{w_2te^{w_2t}}{e^{2w_2t}-1}\right)^{\alpha}e^{w_1w_2tx}\sum_{i=0}^{w_2-1}2w_1te^{(1+2i)w_1t}\left(\frac{w_1te^{w_1t}}{e^{2w_1t}-1}\right)^{\alpha-1}e^{w_1w_2ty}\\ &=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\widetilde{B}_{n-l}^{(\alpha-1)}(w_2y)\sum_{s=0}^{l}\binom{l}{s}w_2^{l-s}w_1^{n-l+s}\widetilde{P}_s(w_2)\widetilde{B}_{l-s}^{(\alpha)}(w_1x)\right)\frac{t^n}{n!}.\end{aligned} \tag{3.3}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in (3.2) and (3.3), we get the equation as:

$$\begin{aligned}&\sum_{l=0}^{n}\binom{n}{l}\widetilde{B}_{n-l}^{(\alpha-1)}(w_1y)\sum_{s=0}^{l}\binom{l}{s}w_1^{l-s}w_2^{n-l+s}\widetilde{B}_{l-s}^{(\alpha)}(w_2x)\widetilde{P}_s(w_1)\\ &=\sum_{l=0}^{n}\binom{n}{l}\widetilde{B}_{n-l}^{(\alpha-1)}(w_2y)\sum_{s=0}^{l}\binom{l}{s}w_2^{l-s}w_1^{n-l+s}\widetilde{B}_{l-s}^{(\alpha)}(w_1x)\widetilde{P}_s(w_2).\end{aligned}$$

Therefore we complete the proof.

□

By using Theorem 3.11, we also get symmetric property of the second kind Bernoulli polynomials of order α

Corollary 3.12. By substituting $w_1 = 1$ from Theorem 3.11, we have

$$\begin{aligned}&\sum_{l=0}^{n}\binom{n}{l}\widetilde{B}_{n-l}^{(\alpha-1)}(y)\sum_{s=0}^{l}s\binom{l}{s}w_2^{n-l+s}\widetilde{B}_{l-s}^{(\alpha)}(w_2x)\\&=\sum_{l=0}^{n}\binom{n}{l}\widetilde{B}_{n-l}^{(\alpha-1)}(w_2y)\sum_{s=0}^{l}\binom{l}{s}w_2^{l-s}\widetilde{B}_{l-s}^{(\alpha)}(x)\widetilde{P}_s(w_2).\end{aligned}$$

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An oscillation of the solution for a nonlinear second-order stochastic differential equation

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Abstract. In this paper, we study the oscillatory properties for asymptotic behaviors of solutions of a class of nonlinear second-order stochastic Itô equations. Meanwhile, we investigate existence of zeros of its solutions with probability 1. Sufficient conditions for the oscillation and nonoscillation of solutions are obtained on the half-line $[t_0, \infty)$ for every $t_0 > 0$.

Keywords: Oscillation, Stochastic differential equations, Zeros of solutions, Wiener process, Itô integral

AMS Subject Classification: 60H10; 60H25; 34K11

1 Introduction

During the past few decades, stochastic differential equations (SDEs) are becoming increasingly important as models of stochastic phenomena that play a prominent role in a diverse range of application areas, including mathematical modeling in engineering and physics, geophysical sciences, stochastic control, mechanics, environmental processes, mathematical biology, molecular dynamics for chemistry, epidemiology, economic modeling, industrial mathematics and mathematical finance [1-10]. Indeed, these models can be stochastic for different reasons. Therefore, numerous studies have been performed to understanding their dynamical behaviors, particularly in relation to problems of the specification of the stochastic processes governing the behaviors of an underlying quantity, as well as fundamental microscopic laws generate stochastic behaviors in the case of coarse-graining and modeling error and so on [11-16]. However, a complete understanding of SDEs theory requires familiarity with advanced probability and stochastic processes, whereas solutions of such models are themself stochastic processes.

Further, in particular, second-order differential equations with random coefficients have found wide variety applications in branches of science. Typically, they are mathematical models of objects under the influence of random forces such that the presence of infinite set of zeros of solutions for these equations indicates that the evolution of investigated objects is oscillatory. Recently, research work about oscillation phenomenon occupies an important place in differential stochastic theory due to the sensitivity of stochastic forces and behaviors. Moreover, the stochastic theory for these equations, as well as the theory of oscillatory solutions of deterministic equations have been studied extensively and are well-developed. Oscillation and nonoscillatition conditions for both linear and nonlinear differential equations, difference equations and delay equations have been investigated in [17-22]. The oscillating properties for solutions of difference equations can be found in the excellent monograph of Agarwal et al. [23]. Besides, the authors in the monograph [24] were devoted to the problem of relationship between oscillation behavior of solutions for differential equations and the corresponding difference equations. On the contrary, the theory of the oscillation of stochastic system is not well-developed.

Incidentally, Mao in [25] considered the stochastic equation of the following form

$$\ddot{x} + kx = h\dot{W}(t),$$

where $\dot{W}(t)$ is a Wiener process which is nowhere differentiable. It was proved that the solution with initial values $x(0) = 1$, $\dot{x}(0) = 0$ has infinitely many zeros, all simple, on each half-line $[t_0, \infty)$ for every $t_0 \geq 0$. The first two moments of the first zero were estimated.

In contrast, the more general equation of the form

$$\ddot{x} + k(t, x, \dot{x}) = h\dot{W}(t),$$

was studied in [26]. The author there demonstrated that this equation has infinitely many zeros with probability 1. Consequently, the explicit upper and lower estimates for the expected values of these zeros were obtained. However, the Itô stochastic equations of the form

$$\ddot{x} + (p(t) + q(t)\dot{W}(t))x = 0$$

was considered by method of asymptotic equivalence in [27,28], whereas the oscillation of solutions was analyzed. In the monographs [29,30], the oscillatory properties of solutions for both linear and nonlinear stochastic delay differential equations with multiplicative noise are given. It was shown that such noise induces an oscillation in solutions. Besides, the oscillation of solutions of first order nonlinear stochastic difference equations is investigated in [31].

The purpose of this paper is to study an asymptotic behavior, as $t \to \infty$, of solutions of a second order stochastic Itô equation. Meanwhile, we investigate existence of zeros of its solutions with probability 1. In the sequel, unless otherwise specified, we say that a solution is oscillatory if it has infinitely many zeros with probability 1 on the half-line $[0, \infty)$. A solution which is not oscillatory is called nonoscillatory.

This paper is organized in five sections including the introduction. In the next section, we present some necessary definitions and preliminary results that will be used in this work. In the same time, statement of a second order SDEs is introduced. In Section 3, the discussion of a solution for linear case of second-order SDEs is presented, as well as the conditions of nonoscillatory behavior of its solutions for nonlinear case of SDEs are constructed. Finally, the conclusions are drawn in Section 4.

2 Statement of the problem and auxiliary results

The material in this section is basic in some sense. For the reader's convenience, we present some necessary definitions and auxiliary results related to the SDEs theory that will be used in the remainder of this paper.

Let us consider a nonlinear second-order stochastic equation of the following form

$$\ddot{x} + p(t, x, \dot{x}) + q(t, x, \dot{x})\dot{W}(t) = 0, t \geq 0. \tag{1}$$

While the corresponding system of stochastic Itô equations will be written as

$$\begin{aligned} dx_1 &= x_2 dt, \\ dx_2 &= -p(t, x_1, x_2)dt - q(t, x_1, x_2)dW(t), \end{aligned} \tag{2}$$

where $x \in R^1, t \geq 0, W(t)$ is a standard Wiener process defined on the probability space (Ω, F, P), $\{F_t, t \geq 0\}$ is the family of σ-algebras adapted to $W(t)$, and the functions $p(t, x_1, x_2)$ and $q(t, x_1, x_2)$ are continuous with respect to x_1, $x_2 \in R^1$ for $t \geq 0$, as well as satisfy the Lipschitz condition with respect to x_1, x_2 together with linear growth condition. Without loss of generality, we assume that $p(t, 0, 0) = q(t, 0, 0) = 0$.

It should be noted that the presence of stochastic in equation (1) causes new difficulties in studying the oscillation of solutions. In this regard, we mention here the following remark: Firstly, solutions of equation (1) are random processes, so their zeros are random variables with certain properties. As a consequence, we need to introduce a new definition of zero which is different of the deterministic case ($q = 0$). Secondly, from the Strook-Varadham support theorem, it follows that solutions of equation (1) can be nonoscillatory on finite intervals. Therefore, the oscillatory solutions should be considered only on infinite intervals. Thirdly, since solutions of equation (1) have only first derivative, so we can not use a second derivative to apply the concavity property of the solution between two successive zeros. It is well known that this method is used in the deterministic case.

Nevertheless, system (2) is a particular case of general second-order system, it would seem that this simplifies its investigation, as well it is the system with a degenerate diffusion that completes its investigation by probability methods. Subsequently, under the above assumptions of equation (1) and the corresponding system (2), we assume that the solution $\bar{x}(t) = (x_1(t), x_2(t))$ of system (2) subject to the initial condition $\bar{x}(t_0) = \bar{x}_0$ satisfy all necessary requirements of the existence of a unique solution for $t \geq t_0$, whereas $\bar{x}_0$ is an $F_{t_0}-$measurable random variable. In addition, the process $\bar{x}(t)$ will never reach the origin (0,0), for more details see Lemma 2.3 in [32]. In our notation, let $x_1(t) = x(t)$. Throughout this paper, a solution $x(t)$ of equation (1) is called a nontrivial solution if it satisfy the following condition

$$P\left\{x(t) = 0,\ t > t_0\right\} = 0$$

On the other hand, for any nontrivial solution $x(t)$ of equation (1), where $t \geq t_0 \geq 0$, the random variable τ_1 can be defined as follows

$$\tau_1 = \begin{cases} \inf\left\{t > t_0 \mid x_1(t) = 0\right\}, & \text{if } \left\{t > t_0 \mid x_1(t) = 0\right\} \neq \varnothing, \\ \infty, & \text{otherwise.} \end{cases} \tag{3}$$

Now, we will introduce the definition of zeros of a solution $x(t)$ on the half-line $t > 0$.

Definition 2.1 The random variable τ_1 is called the first zero of a solution $x(t)$ on the interval $t \geq t_0$, if $\tau_1 < \infty$ with probability 1.

In consequence, one can define another random variable τ_2 as follows

$$\tau_2 = \begin{cases} \inf\left\{t > \tau_1 \mid x_1(t) = 0\right\}, & \text{if } \left\{t > \tau_1 \mid x_1(t) = 0\right\} \neq \varnothing, \\ \infty, & \text{otherwise.} \end{cases} \tag{4}$$

Here, the random variable τ_2 is called the second zero of a solution $x(t)$ on the interval $t \geq t_0$, if $\tau_2 < \infty$ with probability 1.

Correspondingly, one can define by induction a sequence of zeros $\{\tau_n\}$ of a solution $x(t)$ on the interval $t \geq t_0$. Particularly, if $t_0 = 0$. Then, we deal with zeros on the half-line $t > 0$.

Definition 2.2 A nontrivial solution $x(t)$ of equation (1) is called oscillatory on the half-line $t > 0$, if it has infinitely many zeros there. Otherwise, it is called nonoscillatory.

3 Main results and behavior solutions of the SDEs

In this section, some definitions and results are briefly reviewed to establish and generalize the results to the main equation in this work. Meanwhile, we study the behavior of the zeros of solutions for a class of second-order SDEs subject to some initial conditions, as well as we detect the conditions of nonoscillatory behavior of its solutions.

3.1 Linear stochastic Itô equation

Consider the following equation

$$\ddot{x} + x = f(t)\dot{W}(t), \tag{5}$$

subject to the initial conditions

$$x(0) = 1, \dot{x}(0) = 0, \tag{6}$$

where $f(t)$ is a nonrandom function defined on $t \geq 0$ such that a stochastic Itô integral

$$\int_0^t f(s)dW(s)$$

is defined for any $t > 0$.

Note that equation (5) is special case of equation (1), so it satisfies all arguments mentioned in the previous part of our work. Further, we give the following theorem regarding to study the behavior of the zeros of solutions for equation (5) with initial conditions (6).

Theorem 3.1 Assume that $f(t)$ satisfies the following conditions:

1. $f(t)$ is differentiable function for $t \geq 0$ such that $f(0) \geq 0$,
2. $(\sin(t-s)f(s))' \leq 0$ for $0 \leq s \leq t \leq \frac{\pi}{2}$.

Then, the solution of equation (5) subject to initial conditions (6) oscillates on the half-line $t \geq 0$. Besides, a mathematical expectation τ_1 of the first zero satisfies the estimation

$$E\tau_1 \geq 2t^\star \Phi(\frac{1}{\sqrt{t^\star}}), \tag{7}$$

where t^* is the solution of the equation

$$f(0) = \cot(t) \tag{8}$$

on $[0, \frac{\pi}{2}]$ and $\Phi(z) = \frac{1}{2\pi}\int_0^z e^{\frac{-u^2}{2}} du$.

Proof. From Itô formula, the representation of the solution of (5) with initial conditions (6) is given by

$$x(t) = \cos(t) + \int_0^t f(s)\sin(t-s)dW(s). \tag{9}$$

Which implies that

$$x(t) = \cos(t) + \sin(t)\int_0^t f(s)\cos(s)dW(s) - \cos(t)\int_0^t f(s)\sin(s)dW(s).$$

Accordingly, the process $x(t)$ can be written as

$$x(t) = \cos(t) + \bar{W}_1(p(t))\sin(t) + \bar{W}_2(q(t))\cos(t),$$

where

$$p(t) = \int_0^t f^2(s)\cos^2(s)ds, \ q(t) = \int_0^t f^2(s)\sin^2(s)ds,$$

and $\bar{W}_1$, $\bar{W}_2$ are Wiener processes.

In contrast, if we consider $x(t)$ at the times $t_m = (2m+\frac{1}{2})\pi$ for $m = 1,2,3,...$, and define a sequence $\{Y_m\}$ by $Y_m = x((2m+\frac{1}{2})\pi) - x((2(m-1)+\frac{1}{2})\pi)$, whereas $x((2m+\frac{1}{2})\pi) = \bar{W}_1\left(\int_0^{(2m+\frac{1}{2})\pi} \cos^2(s)f^2(s)ds\right)$.

Then, $Y_0 = \bar{W}(\frac{\pi}{4})$, $Y_1 = \bar{W}_1(\frac{5\pi}{4}) - \bar{W}_1(\frac{\pi}{4}), ...,$ is a sequence of random variables with mean zero and variance $\int\limits_{(2(m-1)+\frac{1}{2})\pi}^{(2m+\frac{1}{2})\pi} \cos^2(s) f^2(s) ds.$

Here, it is worth to mention that

$$x((2m + \frac{1}{2})\pi) = Y_0 + Y_1 + ... + Y_m. \tag{10}$$

By the familiar theorems on the limits of sums of independent random variables (e.g. the law of the iterated logarithm), it follows that the sequence $\{x((2m+\frac{1}{2})\pi)\}$ has infinitely many switches of sign. Since $x(t)$ is continuous on $[0, \infty)$, so it has infinitely many zeros on $[0, \infty)$. Therefore, it oscillates on $[0, \infty)$.

Now, let us prove the estimation (7) for the first zero of the oscillation. By applying the integration-by-parts formula to (9), we obtain

$$x(t) = \cos(t) - \int\limits_0^t (\sin(t-s)f(s))' dW(s) \geq \cos(t) + \int\limits_0^t (\sin(t-s)f(s))' ds, \tag{11}$$

for $\omega \in \Omega$, where $W(t) \geq -1$ and $0 \leq s \leq t \leq \frac{\pi}{2}$.

From properties of a Wiener process, it follows that

$$P\left\{\omega \mid \max_{t\in[0,T]} W(t) > -1\right\} = 2\Phi(\frac{1}{2\sqrt{T}}), \tag{12}$$

where $\Phi(z) = \frac{1}{2\sqrt{2\pi}} \int\limits_0^z e^{\frac{-u^2}{2}} du.$

As a result, from equation (11), we obtain the estimate

$$x(t) \geq \cos(t) - f(0)\sin(t) > 0, \tag{13}$$

for $t \in [0, t^*)$, where t^* is solution of equation (8).

Hence, from equations (12) and (13), we have

$$P\left\{\tau \geq \cot^{-1}(f(0))\right\} \geq 2\Phi\left(\frac{1}{\sqrt{\cot^{-1}(f(0))}}\right), \tag{14}$$

for the first zero τ_1. By using Chebyshev's inequality, it yields that $E\tau_1 \geq 2t^\star\Phi(\frac{1}{\sqrt{t^\star}})$. The proof is complete. ■

3.2 Nonlinear stochastic Itô equation

Let $P(a, n) = \{x \in R^m \mid (x - a, n) \geq 0\}$, where $a, n \in R^m$, and $(\cdot, \cdot)$ is the usual scalar product. Thus, a polyhedron is any set of the form

$$\bigcap_{\alpha\in I} P(a_\alpha, n_\alpha), \tag{15}$$

where $I = \{1, ..., N\}$ is a finite subset of N.

Now, consider a system of SDEs

$$dx = f(t, x)dt + g(t, x)dW(t) \tag{16}$$

where $x \in R^m$, $f : [0, \infty) \times R^m \to R^m$, $g = [g_{ij}] : [0, \infty] \times R^m \to R^m \times R^r$ are mappings, and $W(t)$ is an $r-$dimensional Wiener process.

Definition 3.1 A set $K \in R^m$ is said to be stochastically invariant for system (16), if for any $x(0) \in K$ and every solution $x(t)$ of equation (1), then $P\{x(t) \in K, t > 0\} = 1$.

The next theorem states conditions of an invariance of the set (15) for system (16).

Theorem 3.2 [33] Let $K = \bigcap_{\alpha \in I} P(a_\alpha, n_\alpha)$ be a polyhedron in R^m. Suppose that the coefficients $f(t, x)$ and $g(t, x)$ of system (16) are defined for $t \geq 0$, $x \in R^m$, and satisfy the following conditions:

1. for each $T > 0$, there exists a constant $Kt_T > 0$ such that for all $x \in K$ and $t \in [0, T)$,

$$\|f(t,x)\|^2 + \|g(t,x)\|^2 \leq K_T(1 + |x|^2);$$

2. for all $T > 0$, $x \in K$, $y \in K$ and $t \in [0, T)$,

$$\|f(t,x) - f(t,y)\| + \|g(t,x) - g(t,y)\| \leq K_T \, |x - y| \, ;$$

3. for each $x \in K$, the functions $f(\cdot, x)$ and $g(\cdot, x)$, defined for $t \geq 0$, are continuous.

Then, the set K is invariant for the system (16) if and only if the following condition holds:

(a) for all $\alpha \in I$ and $x \in K$ such that $(x - a_\alpha, n_\alpha) = 0$, we have

$$(f(t,x), n_\alpha) \geq 0 \text{ and } (g_j(t,x), n_\alpha) = 0,$$

where $t \geq 0$, $j = \overline{1, r}$, and g_j is the $j - th$ column of the matrix $g = [g_{ij}]$.

Now, we use the above theorem to find the conditions of nonoscillatory behavior of the solutions of equation (1). As well, we state the following theorem:

Theorem 3.3 Suppose that the functions p and q in equation (1) satisfy the conditions (1)-(3) of Theorem 3.2. Moreover, if

$$\begin{aligned} &(1)\ p(t, x_1, 0) \geq 0,\ x_1 < 0,\ t \geq 0; \\ &(2)\ p(t, x_1, 0) \leq 0,\ x_1 > 0,\ t \geq 0; \\ &(3)\ q(t, x_1, 0) = 0,\ t \geq 0, x_1 \in R^1. \end{aligned} \tag{17}$$

Then, all solutions of equation (1) with nonrandom initial values such that $x(0) > 0$, $\dot{x}(0) \geq 0$ or $x(0) < 0$, $\dot{x}(0) \leq 0$ are not oscillate on the half-line $[0, \infty)$.

Proof. We consider any solution of equation (1) with initial values $x(0) > 0$, $\dot{x}(0) \geq 0$. It corresponds to the solution (x_1, x_2) of system (2) with initial values $x_1(0) > 0$, $x_2(0) \geq 0$. Obviously, there exists $\epsilon > 0$ such that $0 < \epsilon \leq x_1(0)$.

Let M be a set such that $M = \{(x_1, x_2) \mid x_1 \geq \epsilon,\ x_2 \geq 0\}$. It is a polyhedron, if we set $a_1 = \epsilon$, $n_1 = l_1 = (1, 0)^T$, $a_2 = 0$, $n_2 = l_2 = (0, 1)^T$. Then, $M = \bigcap_{\alpha \in I} P(a_\alpha, n_\alpha)$, where $I = \{1, 2\}$. Consequently, the boundaries of this polyhedron are lines

$$\gamma_1 = \{(x_1, x_2) \mid x_1 = \epsilon,\ x_2 \geq 0\},$$

$$\gamma_2 = \{(x_1, x_2) \mid x_1 \geq \epsilon,\ x_2 = 0\}.$$

Therefore, by using Theorem 3.2, the functions f and q have the form

$$f(t,x_1,x_2) = (x_2, -p(t,x_1,x_2))^T,$$

$$g(t,x_1,x_2) = (0, -q(t,x_1,x_2))^T,$$

Next, we verify the conditions of Theorem 3.2. On the boundary γ_1, we have

$$(f,n_1) = (f,l_1) = x_2 \geq 0, \text{ and } (g,n_1) = (g,l_1) = 0.$$

From condition (3) of equation (17), we have

$$(f,n_2) = (f,l_2) = -p(t,x_1,0) \geq 0,$$

and

$$(g,n_2) = (g,l_2) = -q(t,x_1,0) = 0.$$

on the boundary γ_2.

Again from Theorem 3.2, it follows that the set M is the invariant set for the solutions of system (2). Thus, the curve $(x_1(t), x_2(t))$ does not intersect with probability 1 the line $x_1 = 0$. This means that the solutions of equation (1) with initial values $x(0) > 0$, $\dot{x}(0) \geq 0$ do not oscillate.

It remains to consider the case with initial values $x(0) < 0$, $\dot{x}(0) \leq 0$. We only introduce the polyhedron $M_1 = \{(x_1,x_2) \mid x_1 \leq -\epsilon,\ x_2 \leq 0\}$ instead the set M. Hence, the proof is complete. ■

4 Concluding remarks

The use of SDEs is a natural way to model real-world phenomena under stochastic processes. In this paper, we study the qualitative behavior of nonlinear second order stochastic differential equations. Interest focuses on solutions of such equations which are oscillatory. A nontrivial solution is called oscillatory if it has infinitely many zeros with probability 1 on half-line. Otherwise, it is called nonoscillatory. The sufficient conditions for the oscillation and nonoscillation of solutions are obtained.

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Fixed point theorems and T-stability of Picard iteration for generalized Lipschitz mappings in cone metric spaces over Banach algebras

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Abstract: In this paper, we obtain the existence of non-normal solid cone and some fixed point theorems for generalized Lipschitz contractive mappings in cone metric spaces over Banach algebras. Our results greatly generalize the main work by Xu and Radenović (Fixed Point Theory and Applications, 2014, 2014: 102). Moreover, we verify the P property and T-stability of Picard's iteration. Further, we give an example to illustrate that our works are never a copy of metric results in the literature.

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Keywords: Generalized Lipschitz constant, P property, T stability, Cone metric space over Banach algebra, Solid cone

1 Introduction

Since Huang and Zhang [1] introduced the concept of cone metric space, many scholars have focused on fixed point theorems in such spaces. There are lots of works on fixed point results in the setting of cone metric spaces (see [2-6]). It is said that [1] is well-known as a result of the fact that cone metric spaces generalize metric spaces and expands the famous Banach contraction principle. But recently, it had not yet been a hot topic since some authors appealed to the equivalence of some metric and cone metric fixed point results

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(see [7-12]). Owing to these reasons, people set out to lose interest in studying fixed point theorems in cone metric spaces. However, the present situation has gone better since, very recently, Liu and Xu [13] introduced the concept of cone metric space over Banach algebra and obtained some fixed point theorems in normal cone metric spaces over Banach algebras. Moreover, they gave an example to illustrate that the non-equivalence of versions of fixed point theorems between cone metric spaces over Banach algebras and (general) metric spaces (in usual sense), which shows that it is essentially necessary to investigate fixed points in cone metric spaces over Banach algebras. Lately, Xu and Radenović [15] delete the normality of cones and greatly generalize the main results of [13]. Throughout this paper, we obtain the existence of non-normal solid cone for generalized Lipschitz mappings in cone metric spaces over Banach algebras. Moreover, we present some fixed point theorems for such mappings in such setting by omitting the assumptions of normalities of cones. Our theorems include the main results of [13] and [15]. Furthermore, we consider the mapping's P property and T-stability of Picard's iteration. Our results greatly unite and extend the main work of [13-15] and [17-21]. In addition, we give an example to illustrate our results in cone metric spaces over Banach algebras are never equivalent to the counterpart of metric spaces.

Let $\mathcal{A}$ be a Banach algebra with a unit e, and θ the zero element of $\mathcal{A}$. A nonempty closed convex subset K of $\mathcal{A}$ is called a cone if $\{\theta, e\} \subset K$, $K^2 = KK \subset K$, $K \cap (-K) = \{\theta\}$ and $\lambda K + \mu K \subset K$ for all $\lambda, \mu \geq 0$. On this basis, we define a partial ordering $\preceq$ with respect to K by $x \preceq y$ if and only if $y - x \in K$. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will indicate that $y - x \in \text{int}K$, where $\text{int}K$ stands for the interior of K. If $\text{int}K \neq \emptyset$, then K is said to be a solid cone. Write $\|\cdot\|$ as the norm on $\mathcal{A}$. A cone K is called normal if there is a number $M > 0$ such that for all $x, y \in \mathcal{A}$, $\theta \preceq x \preceq y$ implies $\|x\| \leq M\|y\|$. The least positive number satisfying above is called the normal constant of K.

In the sequel we always suppose that $\mathcal{A}$ is a Banach algebra with a unit e, K is a solid cone in $\mathcal{A}$, and $\preceq$ is a partial ordering with respect to K.

Definition 1.1.([13]) Let X be a nonempty set and $\mathcal{A}$ a Banach algebra. Suppose that the mapping $d : X \times X \to \mathcal{A}$ satisfies:

(i) $\theta \prec d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $d(x, y) = \theta$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x,y) \preceq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space over Banach algebra.

Definition 1.2.([15]) Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}$ a sequence in X. Then

(i) $\{x_n\}$ converges to x whenever for every $c \gg \theta$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ $(n \to \infty)$.

(ii) $\{x_n\}$ is a Cauchy sequence whenever for each $c \gg \theta$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

(iii) (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

Definition 1.3. Let (X, d) be a cone metric space, $\{y_n\}$ a sequence in X and T a self-map of X. Let x_0 be a point of X, $x_{n+1} = Tx_n$ a Picard's iteration in X. The iteration procedure $x_{n+1} = Tx_n$ is said to be T-stable with respect to T if $\{x_n\}$ converges to a fixed point q of T, and for each $c \gg \theta$, there exists a natural number N such that $d(y_{n+1}, Ty_n) \ll c$ for all $n > N$, then $\lim_{n\to\infty} y_n = q$.

Remark 1.4. Comparing Definition 2.1 of [18] and Definition 1.3, we find that, the conditions of the former are stronger than the latter. Actually, if $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = \theta$, then we must have that for each $c \gg \theta$ there exists an integer $N > 0$ such that $d(y_{n+1}, Ty_n) \ll c$ for all $n > N$. But the contrary is not true (see [6]).

Lemma 1.5.([6]) Let $u, v, w \in \mathcal{A}$. If $u \preceq v$ and $v \ll w$, then $u \ll w$.

Lemma 1.6.([6]) Let $\mathcal{A}$ be a Banach algebra and $\{a_n\}$ a sequence in $\mathcal{A}$. If $a_n \to \theta$ $(n \to \infty)$, then for any $c \gg \theta$, there exists N such that for all $n > N$, one has $a_n \ll c$.

Lemma 1.7.([16]) Let $\mathcal{A}$ be a Banach algebra with a unit e, $x \in \mathcal{A}$, then the limit $\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$ exists and the spectral radius $\rho(x)$ satisfies

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf \|x^n\|^{\frac{1}{n}}.$$

If $\rho(x) < |\lambda|$, then $\lambda e - x$ is invertible in $\mathcal{A}$, moreover,

$$(\lambda e - x)^{-1} = \sum_{i=0}^{\infty} \frac{x^i}{\lambda^{i+1}},$$

where λ is a complex constant.

Lemma 1.8.([16]) Let $\mathcal{A}$ be a Banach algebra with a unit e, $a, b \in \mathcal{A}$. If a commutes with b, then

$$\rho(a+b) \leq \rho(a) + \rho(b), \qquad \rho(ab) \leq \rho(a)\rho(b).$$

2 Main results

In this section we give some basic but important properties, which will be used constantly in the sequel. Moreover, we introduce a class of contractive mappings with some generalized Lipschitz constants and prove the existence of non-normal solid cone and several fixed point theorems based on them without the assumption of normalities of cones. In addition, we obtain the fixed point periodic property and T-stability of Picard's iteration. All results greatly generalize the main assertions of [13-15] and [17-21]. Further, we display an example to illustrate the applications. In the end, we give another example to claim that our results in the setting of cone metric spaces over Banach algebras are never equivalent to those in usual metric spaces.

Lemma 2.1. Let $\mathcal{A}$ be a Banach algebra and $k \in \mathcal{A}$. If $\rho(k) < 1$, then $\lim\limits_{n\to\infty} \|k^n\| = 0$.

Proof. Since $\rho(k) = \lim\limits_{n\to\infty} \|k^n\|^{\frac{1}{n}} < 1$, then there exists $\alpha > 0$ such that $\lim\limits_{n\to\infty} \|k^n\|^{\frac{1}{n}} < \alpha < 1$. Letting n be big enough, we obtain $\|k^n\|^{\frac{1}{n}} \leq \alpha$, then $\|k^n\| \leq \alpha^n \to 0$ $(n \to \infty)$. Hence $\|k^n\| \to 0$ $(n \to \infty)$. □

Lemma 2.2. Let $\mathcal{A}$ be a Banach algebra with a unit e, $\{x_n\}$ a sequence in $\mathcal{A}$. If x_n converges to x in $\mathcal{A}$, and for any $n \geq 1$, x_n commutes with x, then $\rho(x_n) \to \rho(x)$ as $n \to \infty$.

Proof. Since x_n commutes with x, then it follows by Lemma 1.8 that

$$\rho(x_n) \leq \rho(x_n - x) + \rho(x) \Rightarrow \rho(x_n) - \rho(x) \leq \rho(x_n - x),$$
$$\rho(x) \leq \rho(x - x_n) + \rho(x_n) \Rightarrow \rho(x) - \rho(x_n) \leq \rho(x - x_n),$$

thus

$$|\rho(x_n) - \rho(x)| \leq \rho(x_n - x) \leq \|x_n - x\| \Rightarrow \rho(x_n) \to \rho(x)(n \to \infty).$$

□

Lemma 2.3. Let $\mathcal{A}$ be a Banach algebra with a unit e and K be a solid cone in $\mathcal{A}$. Let $\{a_n\}$ and $\{c_n\}$ be two sequences in K satisfying the following inequality:

$$a_{n+1} \preceq h a_n + c_n, \tag{2.1}$$

where $h \in K$ and $\rho(h) < 1$. If for each $c \gg \theta$, there exists N such that $c_n \ll c$ for all $n > N$, then $a_n \ll c$ $(n > N)$.

Proof. By virtue of $\rho(h) < 1$, it follows by Lemma 1.7, $e-h$ is invertible and $(e-h)^{-1} = \sum_{i=0}^{\infty} h^i$. Moreover, by Lemma 2.1, it establishes $\|h^n\| \to 0$ $(n \to \infty)$. Assume $c \gg \theta$ be arbitrary. Then there exists N_1 such that for all $n > N_1$, we have

$$c_n \ll \frac{(e-h)c}{2}. \tag{2.2}$$

Since

$$\|h^{n-N_1} a_{N_1+1}\| \leq \|h^{n-N_1}\| \|a_{N_1+1}\| \to 0 \quad (n \to \infty),$$

thus there is N_2 such that for all $n > N_2$, it satisfies

$$h^{n-N_1} a_{N_1+1} \ll \frac{c}{2}. \tag{2.3}$$

Put $N = \max\{N_1, N_2\}$, then for all $n > N$, both (2.2) and (2.3) are satisfied. Taking advantage of (2.1), we speculate that

$$\begin{gathered} a_{n+1} - h a_n \preceq c_n, \\ h a_n - h^2 a_{n-1} \preceq h c_{n-1}, \\ h^2 a_{n-1} - h^3 a_{n-2} \preceq h^2 c_{n-2}, \\ \cdots\cdots\cdots \\ h^{n-N_1-1} a_{N_1+2} - h^{n-N_1} a_{N_1+1} \preceq h^{n-N_1-1} c_{N_1+1}. \end{gathered}$$

Conbine with the above terms, for all $n > N$, it follows that

$$\begin{aligned} a_{n+1} &\preceq h^{n-N_1} a_{N_1+1} + c_n + h c_{n-1} + h^2 c_{n-2} + \cdots + h^{n-N_1-1} c_{N_1+1} \\ &\ll \frac{c}{2} + (e + h + h^2 + \cdots + h^{n-N_1-1}) \cdot \frac{(e-h)c}{2} \\ &\preceq \frac{c}{2} + (e-h)^{-1} \cdot \frac{(e-h)c}{2} = c. \end{aligned}$$

□

Remark 2.4. Lemma 2.3 greatly generalizes Lemma 1 of [17] and Lemma 1.5 of [18]. Virtually, we delete the normality of K. Moreover, our conditions are weaker than them. Indeed, if $a_n \to \theta$ $(n \to \infty)$, then $a_n \ll c$ $(n > N)$. But the converse is not true (see [6]). Further, if $\|k\| < 1$, it is natural that $\rho(k) < 1$. Yet, the converse is not true.

Theorem 2.5. Let (X, d) be a cone metric space over Banach algebra $\mathcal{A}$ and K be a solid cone in $\mathcal{A}$. Suppose that the mapping $T : X \to X$ satisfies the following contractive condition:

$$d(Tx, Ty) \preceq k_1 d(x, y) + k_2 d(x, Tx) + k_3 d(y, Ty) + k_4 d(x, Ty) + k_5 d(y, Tx),$$

for all $x, y \in X$, where $k_i \in K(i = 1, \ldots, 5)$ are generalized Lipschitz constants with $\rho(k_1) + \rho(k_2 + k_3 + k_4 + k_5) < 1$. If k_1 commutes with $k_2 + k_3 + k_4 + k_5$, then there exists a sequence $\{x_n\}$ in X such that it is a Cauchy sequence. Moreover, if $\{d(x_n, y_n)\}$ converges to some non-zero element in $\mathcal{A}$ for any two different Cauchy sequence $\{x_n\}$ and $\{y_n\}$, then K is a non-normal cone.

Proof. Fix $x_0 \in X$ and set $x_{n+1} = Tx_n = T^{n+1}x_0$. Then we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\preceq k_1 d(x_n, x_{n-1}) + k_2 d(x_n, Tx_n) + k_3 d(x_{n-1}, Tx_{n-1}) \\ &\quad + k_4 d(x_n, Tx_{n-1}) + k_5 d(x_{n-1}, Tx_n) \\ &= k_1 d(x_n, x_{n-1}) + k_2 d(x_n, x_{n+1}) + k_3 d(x_{n-1}, x_n) \\ &\quad + k_5 d(x_{n-1}, x_{n+1}) \\ &\preceq (k_1 + k_3 + k_5) d(x_n, x_{n-1}) + (k_2 + k_5) d(x_{n+1}, x_n). \end{aligned} \tag{2.4}$$

We also have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) = d(Tx_{n-1}, Tx_n) \\ &\preceq k_1 d(x_{n-1}, x_n) + k_2 d(x_{n-1}, Tx_{n-1}) + k_3 d(x_n, Tx_n) \\ &\quad + k_4 d(x_{n-1}, Tx_n) + k_5 d(x_n, Tx_{n-1}) \\ &= k_1 d(x_n, x_{n-1}) + k_2 d(x_{n-1}, x_n) + k_3 d(x_n, x_{n+1}) \\ &\quad + k_4 d(x_{n-1}, x_{n+1}) \\ &\preceq (k_1 + k_2 + k_4) d(x_n, x_{n-1}) + (k_3 + k_4) d(x_{n+1}, x_n). \end{aligned} \tag{2.5}$$

Add up (2.4) and (2.5) yields that

$$\begin{aligned} 2d(x_{n+1}, x_n) &\preceq (2k_1 + k_2 + k_3 + k_4 + k_5) d(x_n, x_{n-1}) \\ &\quad + (k_2 + k_3 + k_4 + k_5) d(x_{n+1}, x_n), \end{aligned}$$

which establishes that

$$(2e - k_2 - k_3 - k_4 - k_5) d(x_{n+1}, x_n) \preceq (2k_1 + k_2 + k_3 + k_4 + k_5) d(x_n, x_{n-1}).$$

Put $k = k_2 + k_3 + k_4 + k_5$, then

$$(2e - k) d(x_{n+1}, x_n) \preceq (2k_1 + k) d(x_n, x_{n-1}). \tag{2.6}$$

Since $\rho(k) \leq \rho(k_1) + \rho(k) < 1 < 2$, then by Lemma 1.7 it follows that $2e - k$ is invertible. Furthermore,

$$(2e-k)^{-1} = \sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}}.$$

By multiplying in both sides of (2.6) by $(2e-k)^{-1}$, we arrive at

$$d(x_{n+1}, x_n) \preceq (2e-k)^{-1}(2k_1 + k)d(x_n, x_{n-1}). \tag{2.7}$$

Denote $h = (2e-k)^{-1}(2k_1+k)$, then by (2.7) we get

$$d(x_{n+1}, x_n) \preceq hd(x_n, x_{n-1}) \preceq \cdots \preceq h^n d(x_1, x_0).$$

By Lemma 1.8 we conclude that

$$\rho\Big(\sum_{i=0}^{n} \tfrac{k^i}{2^{i+1}}\Big) \leqslant \sum_{i=0}^{n} \rho\big(\tfrac{k^i}{2^{i+1}}\big) \leqslant \sum_{i=0}^{n} \tfrac{[\rho(k)]^i}{2^{i+1}},$$

which implies by Lemma 2.2 that

$$\rho\Big(\sum_{i=0}^{\infty} \tfrac{k^i}{2^{i+1}}\Big) \leqslant \sum_{i=0}^{\infty} \tfrac{[\rho(k)]^i}{2^{i+1}}.$$

Since k_1 commutes with k, it follows that

$$\begin{aligned}(2e-k)^{-1}(2k_1+k) &= \Big(\sum_{i=0}^{\infty} \tfrac{k^i}{2^{i+1}}\Big)(2k_1+k) = 2\Big(\sum_{i=0}^{\infty} \tfrac{k^i}{2^{i+1}}\Big)k_1 + \sum_{i=0}^{\infty} \tfrac{k^{i+1}}{2^{i+1}} \\ &= 2k_1\Big(\sum_{i=0}^{\infty} \tfrac{k^i}{2^{i+1}}\Big) + k\sum_{i=0}^{\infty} \tfrac{k^i}{2^{i+1}} = (2k_1+k)\Big(\sum_{i=0}^{\infty} \tfrac{k^i}{2^{i+1}}\Big) = (2k_1+k)(2e-k)^{-1},\end{aligned}$$

that is to say, $(2e-k)^{-1}$ commutes with $2k_1+k$. Then by Lemma 1.8 we gain

$$\begin{aligned}\rho(h) = \rho\big((2e-k)^{-1}(2k_1+k)\big) &\leq \rho\big((2e-k)^{-1}\big)\rho(2k_1+k) \\ &\leq \rho\Big(\sum_{i=0}^{\infty} \tfrac{k^i}{2^{i+1}}\Big)[2\rho(k_1)+\rho(k)] \leq \Big(\sum_{i=0}^{\infty} \tfrac{[\rho(k)]^i}{2^{i+1}}\Big)[2\rho(k_1)+\rho(k)] \\ &= \frac{1}{2-\rho(k)}[2\rho(k_1)+\rho(k)] < 1,\end{aligned}$$

which establishes that $e-h$ is invertible and $\|h^m\| \to 0$ $(m \to \infty)$. Thus for all $n > m$,

$$\begin{aligned}d(x_n, x_m) &\preceq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\ &\preceq (h^{n-1} + h^{n-2} + \cdots + h^m)d(x_1, x_0) \\ &= (h^{n-m-1} + h^{n-m-2} + \cdots + h + e)h^m d(x_1, x_0) \\ &\preceq \Big(\sum_{i=0}^{\infty} h^i\Big)h^m d(x_1, x_0) \\ &= (e-h)^{-1}h^m d(x_1, x_0).\end{aligned}$$

Owing to

$$\|(e-h)^{-1}h^m d(x_1,x_0)\| \le \|(e-h)^{-1}\|\|h^m\|\|d(x_1,x_0)\| \to 0\ (m\to\infty),$$

we have $(e-h)^{-1}h^m d(x_1,x_0) \to \theta$ $(m\to\infty)$. So by using Lemma 1.5 and 1.6, we easily see that $\{x_n\}$ is a Cauchy sequence in X.

Now we take $y_0 \in X$ such that $y_0 \ne x_0$. Using the same method as the above mentioned, we can show that $\{y_n\}$ is also a Cauchy sequence if $y_{n+1} = Ty_n = T^{n+1}y_0$. In the following we suppose the contrary, that is, K is normal. We shall prove that $\{d(x_n,y_n)\}$ is convergent in $(\mathcal{A},\|\cdot\|)$ if K is a normal cone with normal constant M. In fact, in view of the completeness of $(\mathcal{A},\|\cdot\|)$, it will be enough to show that the sequence $\{d(x_n,y_n)\}$ is a Cauchy sequence. To this end, let $\varepsilon > 0$ and choose $c \gg \theta$ and $\|c\| < \frac{\varepsilon}{4M+2}$. Since $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences, there is N such that $d(x_n,x_m) \ll c$ and $d(y_n,y_m) \ll c$ for all $n,m > N$. It is clear that

$$d(x_n,y_n) \preceq d(x_n,x_m) + d(x_m,y_m) + d(y_m,y_n) \preceq d(x_m,y_m) + 2c, \tag{2.8}$$

$$d(x_m,y_m) \preceq d(x_m,x_n) + d(x_n,y_n) + d(y_n,y_m) \preceq d(x_n,y_n) + 2c. \tag{2.9}$$

It follows immediately from (2.8) and (2.9) that

$$\theta \preceq d(x_m,y_m) + 2c - d(x_n,y_n) \preceq d(x_n,y_n) + 2c + 2c - d(x_n,y_n) = 4c. \tag{2.10}$$

By virtue of the normality of K, (2.10) means that

$$\|d(x_m,y_m) + 2c - d(x_n,y_n)\| \le 4M\|c\|.$$

Hence, it ensures us that

$$\|d(x_m,y_m) - d(x_n,y_n)\| \le \|d(x_m,y_m) + 2c - d(x_n,y_n)\| + \|2c\| \le (4M+2)\|c\| < \varepsilon,$$

which implies that $\{d(x_n,y_n)\}$ is Cauchy and hence convergent.

Next, put $\lim\limits_{n\to\infty} d(x_n,y_n) = a$, it is evident that $\theta \preceq a$. Finally, we claim that $a = \theta$. Actually, if there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = y_{n_0}$, the claim is clear. Without loss of

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generality, we suppose that $x_n \neq y_n$ for all $n \in \mathbb{N}$. Notice that

$$\begin{aligned} d(x_{n+1}, y_{n+1}) &= d(Tx_n, Ty_n) \\ &\preceq k_1 d(x_n, y_n) + k_2 d(x_n, Tx_n) + k_3 d(y_n, Ty_n) \\ &\quad + k_4 d(x_n, Ty_n) + k_5 d(y_n, Tx_n) \\ &= k_1 d(x_n, y_n) + k_2 d(x_n, x_{n+1}) + k_3 d(y_n, y_{n+1}) \\ &\quad + k_4 d(x_n, y_{n+1}) + k_5 d(y_n, x_{n+1}) \\ &\preceq (k_1 + k_4 + k_5) d(x_n, y_n) + (k_2 + k_5) d(x_n, x_{n+1}) + (k_3 + k_4) d(y_n, y_{n+1}). \end{aligned}$$

Taking the limit as $n \to \infty$, we obtain that

$$a \preceq (k_1 + k_4 + k_5) a.$$

Set $\lambda = k_1 + k_4 + k_5$, then it follows that

$$a \preceq \lambda a \preceq \cdots \preceq \lambda^n a.$$

Because $\lambda \preceq k_1 + k$ leads to $\lambda^n \preceq (k_1 + k)^n$, moreover, by Lemma 2.1, $\rho(k_1 + k) \leq \rho(k_1) + \rho(k) < 1$ leads to $(k_1 + k)^n \to \theta$ $(n \to \infty)$, we claim that, for each $c \gg \theta$, there exists $n_0(c)$ such that $\lambda^n \ll c$ such that for all $n > n_0(c)$. Consequently, $a = \theta$, a contradiction. □

It is clear that if T is a map which has a fixed point u, then u is also a fixed point of T^n for each $n \in \mathbb{N}$. It is well known that the converse is not true. If a map T satisfies $F(T) = F(T^n)$ for each $n \in \mathbb{N}$, where $F(T)$ stands for the set of all fixed points of T, then it is said to have a property P (see [19-21]). The following results are generalizations of the corresponding results in metric and cone metric spaces (see [20-21]). It will be deduced also without using normality of the cone.

Theorem 2.6. Let (X, d) be a complete cone metric space over Banach algebra $\mathcal{A}$ and K be a solid cone in $\mathcal{A}$. Let $T : X \to X$ be a mapping such that $F(T) \neq \emptyset$ and that

$$d(Tx, T^2 x) \preceq k d(x, Tx) \tag{2.11}$$

for all $x \in X$, where $k \in K$ is a generalized Lipschitz constant with $\rho(k) < 1$. Then T has a property P.

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Proof. We will always assume that $n > 1$, since the statement for $n = 1$ is trivial. Let $z \in F(T^n)$. By the hypotheses, it is clear that

$$\begin{aligned} d(z, Tz) &= d(TT^{n-1}z, T^2T^{n-1}z) \preceq kd(T^{n-1}z, T^n z) = kd(TT^{n-2}z, T^2T^{n-2}z) \\ &\preceq k^2 d(T^{n-2}z, T^{n-1}z) \preceq \cdots \preceq k^n d(z, Tz). \end{aligned}$$

On account of $\rho(k) < 1$, it follows by Lemma 2.1 that $\|k^n\| \to 0$ $(n \to \infty)$. Thus $\|k^n d(z, Tz)\| \leq \|k^n\| \|d(z, Tz)\| \to 0$ $(n \to \infty)$. Hence $d(z, Tz) = \theta$, that is., $Tz = z$. □

Theorem 2.7. Let (X, d) be a complete cone metric space over Banach algebra $\mathcal{A}$ and K be a solid cone in $\mathcal{A}$. Suppose that the mapping $T : X \to X$ satisfies the following contractive condition:

$$d(Tx, Ty) \preceq k_1 d(x, y) + k_2 d(x, Tx) + k_3 d(y, Ty) + k_4 d(x, Ty) + k_5 d(y, Tx),$$

for all $x, y \in X$, where $k_i \in K (i = 1, \ldots, 5)$ are generalized Lipschitz constants with $\rho(k_1) + \rho(k_2 + k_3 + k_4 + k_5) < 1$. If k_1 commutes with $k_2 + k_3 + k_4 + k_5$, then T has a unique fixed point in X. Moreover, for arbitrary $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point. Further, T has a property P.

Proof. By using Theorem 2.5, we obtain $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. We shall prove x^* is the fixed point of T. To this end, on the one hand, we have

$$\begin{aligned} d(Tx^*, x^*) &\preceq d(Tx^*, Tx_n) + d(Tx_n, x^*) \\ &\preceq k_1 d(x^*, x_n) + k_2 d(x^*, Tx^*) + k_3 d(x_n, Tx_n) \\ &\quad + k_4 d(x^*, Tx_n) + k_5 d(x_n, Tx^*) + d(Tx_n, x^*) \\ &= k_1 d(x^*, x_n) + k_2 d(x^*, Tx^*) + k_3 d(x_n, x_{n+1}) \\ &\quad + (e + k_4) d(x^*, x_{n+1}) + k_5 d(x_n, Tx^*) \\ &\preceq (k_1 + k_3 + k_5) d(x_n, x^*) + (k_2 + k_5) d(Tx^*, x^*) \\ &\quad + (e + k_3 + k_4) d(x_{n+1}, x^*), \end{aligned}$$

which implies that

$$(e - k_2 - k_5) d(Tx^*, x^*) \preceq (k_1 + k_3 + k_5) d(x_n, x^*) + (e + k_3 + k_4) d(x_{n+1}, x^*). \tag{2.12}$$

On the other hand, we obtain

$$\begin{aligned}
d(Tx^*, x^*) &\preceq d(Tx_n, Tx^*) + d(Tx_n, x^*) \\
&\preceq k_1 d(x_n, x^*) + k_2 d(x_n, Tx_n) + k_3 d(x^*, Tx^*) \\
&\quad + k_4 d(x_n, Tx^*) + k_5 d(x^*, Tx_n) + d(Tx_n, x^*) \\
&= k_1 d(x_n, x^*) + k_2 d(x_n, x_{n+1}) + k_3 d(x^*, Tx^*) \\
&\quad + (e + k_5) d(x^*, x_{n+1}) + k_4 d(x_n, Tx^*) \\
&\preceq (k_1 + k_2 + k_4) d(x_n, x^*) + (k_3 + k_4) d(Tx^*, x^*) \\
&\quad + (e + k_2 + k_5) d(x_{n+1}, x^*),
\end{aligned}$$

which means that

$$(e - k_3 - k_4) d(Tx^*, x^*) \preceq (k_1 + k_2 + k_4) d(x_n, x^*) + (e + k_2 + k_5) d(x_{n+1}, x^*). \tag{2.13}$$

Combining (2.12) and (2.13) yields that

$$\begin{aligned}
&(2e - k_2 - k_3 - k_4 - k_5) d(Tx^*, x^*) \\
&\preceq (2k_1 + k_2 + k_3 + k_4 + k_5) d(x_n, x^*) \\
&\quad + (2e + k_2 + k_3 + k_4 + k_5) d(x_{n+1}, x^*).
\end{aligned}$$

Denote $k = k_2 + k_3 + k_4 + k_5$, then

$$(2e - k) d(Tx^*, x^*) \preceq (2k_1 + k) d(x_n, x^*) + (2e + k) d(x_{n+1}, x^*).$$

Consequently, we deduce that

$$\begin{aligned}
d(Tx^*, x^*) &\preceq (2e - k)^{-1} (2k_1 + k) d(x_n, x^*) \\
&\quad + (2e - k)^{-1} (2e + k) d(x_{n+1}, x^*).
\end{aligned}$$

In view of $x_n \to x^*$ $(n \to \infty)$, then for each $\frac{c}{m} \gg \theta$ $(m = 1, 2, \ldots)$, there exists N_m such that for all $n > N_m$, one has $d(x_n, x^*) \ll \frac{c}{m}$. Hence we speculate that

$$\begin{aligned}
d(Tx^*, x^*) &\preceq (2e - k)^{-1} (2k_1 + k) d(x_n, x^*) + (2e - k)^{-1} (2e + k) d(x_{n+1}, x^*) \\
&\ll [(2e - k)^{-1} (2k_1 + k) + (2e - k)^{-1} (2e + k)] \frac{c}{m} \to \theta \ (m \to \infty),
\end{aligned}$$

which follows that $Tx^* = x^*$.

In the following we shall show the fixed point is unique. Actually, if there is another fixed point y^*, then

$$\begin{aligned}d(x^*,y^*) &\preceq k_1 d(x^*,y^*) + k_2 d(x^*,Tx^*) + k_3 d(y^*,Ty^*)\\ &\quad + k_4 d(x^*,Ty^*) + k_5 d(y^*,Tx^*)\\ &= (k_1+k_4+k_5) d(x^*,y^*).\end{aligned}$$

Set $\lambda = k_1+k_4+k_5$, then it follows that

$$d(x^*,y^*) \preceq \lambda d(x^*,y^*) \preceq \cdots \preceq \lambda^n d(x^*,y^*).$$

By the proof of Theorem 2.5, we claim that, for each $c \gg \theta$, there exists $n_0(c)$ such that $\lambda^n \ll c$ such that for all $n > n_0(c)$. Consequently, $d(x^*,y^*) = \theta$, that is, $x^* = y^*$.

Finally, we shall prove that T has a property P. We have to show that the mapping T satisfies the condition (2.11). Indeed, firstly we have

$$\begin{aligned}d(Tx,T^2x) &= d(Tx,TTx)\\ &\preceq k_1 d(x,Tx) + k_2 d(x,Tx) + k_3 d(Tx,T^2x)\\ &\quad + k_4 d(x,T^2x) + k_5 d(Tx,Tx)\\ &\preceq k_1 d(x,Tx) + k_2 d(x,Tx) + k_3 d(Tx,T^2x)\\ &\quad + k_4 d(x,Tx) + k_4 d(Tx,T^2x),\end{aligned}$$

that is,

$$(e-k_3-k_4) d(Tx,T^2x) \preceq (k_1+k_2+k_4) d(x,Tx). \tag{2.14}$$

Also, we have

$$\begin{aligned}d(Tx,T^2x) &= d(TTx,Tx)\\ &\preceq k_1 d(Tx,x) + k_2 d(Tx,T^2x) + k_3 d(x,Tx)\\ &\quad + k_4 d(Tx,Tx) + k_5 d(x,T^2x)\\ &\preceq k_1 d(x,Tx) + k_2 d(Tx,T^2x) + k_3 d(x,Tx)\\ &\quad + k_5 d(x,Tx) + k_5 d(Tx,T^2x),\end{aligned}$$

that is,

$$(e-k_2-k_5) d(Tx,T^2x) \preceq (k_1+k_3+k_5) d(x,Tx). \tag{2.15}$$

Adding (2.14) and (2.15) we obtain

$$(2e-k)d(Tx,T^2x) \preceq (2k_1+k)d(x,Tx).$$

According to the above proof, we demonstrate that

$$d(Tx,T^2x) \preceq hd(x,Tx),$$

where $h=(2e-k)^{-1}(2k_1+k)$ and $\rho(h)<1$. Therefore, by Theorem 2.6, T has a property P. □

Theorem 2.8. Let (X,d) be a complete cone metric space over Banach algebra $\mathcal{A}$ and K be a solid cone in $\mathcal{A}$. Suppose that the mapping $T:X\to X$ satisfies the following contractive condition:

$$d(Tx,Ty) \preceq k_1d(x,y)+k_2d(x,Tx)+k_3d(y,Ty)+k_4d(x,Ty)+k_5d(y,Tx),$$

for all $x,y\in X$, where $k_i\in K(i=1,\ldots,5)$ are generalized Lipschitz constants with $\rho(k_1)+\rho(k_2+k_3+k_4+k_5)<1$. If k_1 commutes with $k_2+k_3+k_4+k_5$, then Picard's iteration is T-stable.

Proof. By utilizing Theorem 2.7, we obtain that T has a unique fixed point q in X. Assume that $\{y_n\}\subset X$ satisfies the following condition: for each $c\gg\theta$, there exists N such that for all $n>N$, $d(y_{n+1},Ty_n)\ll c$. Firstly we have

$$\begin{aligned}
d(Ty_n,q) &= d(Ty_n,Tq)\\
&\preceq k_1d(y_n,q)+k_2d(y_n,Ty_n)+k_3d(q,Tq)\\
&\quad +k_4d(y_n,Tq)+k_5d(q,Ty_n)\\
&= k_1d(y_n,q)+k_2d(y_n,Ty_n)+k_4d(y_n,q)+k_5d(q,Ty_n)\\
&\preceq (k_1+k_4)d(y_n,q)+k_2[d(y_n,q)+d(q,Ty_n)]+k_5d(q,Ty_n)\\
&= (k_1+k_2+k_4)d(y_n,q)+(k_2+k_5)d(q,Ty_n). \qquad (2.16)
\end{aligned}$$

Secondly, we arrive at

$$\begin{aligned}
d(Ty_n,q) &= d(q,Ty_n)=d(Tq,Ty_n)\\
&\preceq k_1d(q,y_n)+k_2d(q,Tq)+k_3d(y_n,Ty_n)\\
&\quad +k_4d(q,Ty_n)+k_5d(y_n,Tq)\\
&= k_1d(y_n,q)+k_3d(y_n,Ty_n)+k_4d(q,Ty_n)+k_5d(y_n,q)\\
&\preceq (k_1+k_5)d(y_n,q)+k_3[d(y_n,q)+d(q,Ty_n)]+k_4d(q,Ty_n)\\
&= (k_1+k_3+k_5)d(y_n,q)+(k_3+k_4)d(q,Ty_n). \qquad (2.17)
\end{aligned}$$

Adding up (2.16) and (2.17) yields that

$$2d(Ty_n, q) \preceq (2k_1 + k_2 + k_3 + k_4 + k_5)d(y_n, q) + (k_2 + k_3 + k_4 + k_5)d(q, Ty_n),$$

Denote $k = k_2 + k_3 + k_4 + k_5$, then we get

$$(2e - k)d(Ty_n, q) \preceq (2k_1 + k)d(y_n, q).$$

Based on the proof of Theorem 2.5, it is not hard to verify that

$$d(Ty_n, q) \preceq hd(y_n, q),$$

where $h = (2e - k)^{-1}(2k_1 + k)$ and $\rho(h) < 1$.

Setting $a_n = d(y_n, q)$ and $c_n = d(y_{n+1}, Ty_n)$, we claim that

$$a_{n+1} = d(y_{n+1}, q) \preceq d(y_{n+1}, Ty_n) + d(Ty_n, q) \preceq c_n + ha_n.$$

If for each $c \gg \theta$, there exists N such that for all $n > N$, $c_n = d(y_{n+1}, Ty_n) \ll c$. Then, making full use of Lemma 2.3, we get $a_n = d(y_n, q) \ll c$, which leads to $y_n \to q$ as $n \to \infty$. That is to say, the Picard's iteration is T-stable.

Corollary 2.9. Let (X, d) be a complete cone metric space over Banach algebra $\mathcal{A}$ and K be a solid cone in $\mathcal{A}$. Suppose that the mapping $T : X \to X$ satisfies the following contractive condition:

$$d(Tx, Ty) \preceq kd(x, y),$$

for all $x, y \in X$, where $k \in K$ is a generalized Lipschitz constant with $\rho(k) < 1$. Then T has a unique fixed point in X. Moreover, for any $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point. Further, T has a property P and Picard's iteration is T-stable.

Corollary 2.10. Let (X, d) be a complete cone metric space over Banach algebra $\mathcal{A}$ and K be a solid cone in $\mathcal{A}$. Suppose that the mapping $T : X \to X$ satisfies the following contractive condition:

$$d(Tx, Ty) \preceq \frac{k}{2}[d(x, Tx) + d(y, Ty)],$$

for all $x, y \in X$, where $k \in K$ is a generalized Lipschitz constant with $\rho(k) < 1$. Then T has a unique fixed point in X. Moreover, for every $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point. Further, T has a property P and Picard's iteration is T-stable.

Corollary 2.11. Let (X, d) be a complete cone metric space over Banach algebra $\mathcal{A}$ and K be a solid cone in $\mathcal{A}$. Suppose that the mapping $T : X \to X$ satisfies the following contractive condition:

$$d(Tx, Ty) \preceq \frac{k}{2}[d(x, Ty) + d(y, Tx)],$$

for all $x, y \in X$, where $k \in K$ is a generalized Lipschitz constant with $\rho(k) < 1$. Then T has a unique fixed point in X. Moreover, for each $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point. Further, T has a property P and Picard's iteration is T-stable.

Remark 2.12. Throughout the conclusions above, we focus on fixed point theorems in cone metric spaces over Banach algebras instead of the theorems only in cone metric spaces. All the coefficients are vector elements and the multiplications such as $kd(x, y)$ are vector multiplications instead of usual scalar ones, which may bring us more convenience in applications.

Remark 2.13. In our results such as Corollary 2.9, we only suppose the spectral radius of k is less than 1, while $\|k\| < 1$ is not assumed. Generally speaking, it is meaningful since by Remark 2.4, the condition $\rho(k) < 1$ is weaker than that $\|k\| < 1$.

Remark 2.14. Compared with the main results of [13] and [15], our main results in this paper deal not only with the fixed point theorems for generalized Lipschitz mappings, but also with P property and T-stability of Picard's iteration, all in the setting of cone metric spaces under the condition that the underlying cones are solid without assumption of normality. These results may be more valuable to put into use since the cones discussed are not necessarily normal under ordinary conditions. Therefore, it is an interesting thing to discuss the fixed point results in cone metric spaces over Banach algebras without the assumption that the underlying cones are normal. The following examples show that our main results will be very useful.

Example 2.15. Let $\mathcal{A} = C^1_{\mathbb{R}}[0, 1]$ and define a norm on $\mathcal{A}$ by $\|x\| = \|x\|_\infty + \|x'\|_\infty$. Define multiplication in $\mathcal{A}$ as just pointwise multiplication. Then $\mathcal{A}$ is a real Banach algebra with a unit $e = 1$ ($e(t) = 1$ for all $t \in [0, 1]$). The set $K = \{x \in \mathcal{A} : x(t) \geq 0 \text{ for all } t \in [0, 1]\}$ is a cone in $\mathcal{A}$. Moreover, K is a non-normal solid cone (see [6]). Let $X = \{a, b, c\}$. Define $d : X \times X$ by $d(a, b)(t) = d(b, a)(t) = e^t$, $d(b, c)(t) = d(c, b)(t) = 2e^t$, $d(c, a)(t) = d(a, c)(t) = 3e^t$ and $d(x, x)(t) = \theta$ for all $t \in [0, 1]$ and each $x \in X$. We have that (X, d) is a solid cone metric space over Banach algebra $\mathcal{A}$. Further, let $T : X \to X$ be a mapping defined with $Ta = Tb = b$, $Tc = a$ and let $k_1, k_2, k_3, k_4, k_5 \in K$ defined with $k_1(t) = \frac{1}{3}t + \frac{1}{2}$, $k_2(t) = k_3(t) = k_4(t) = k_5(t) = \frac{1}{25}$ for all $t \in [0, 1]$. By careful calculations one can get

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that T is not a Banach contraction and all the conditions of Theorems 2.7 are fulfilled. The point $x = b$ is the unique fixed point of T. By using Theorem 2.7 and Theorem 2.8, we can also conclude that T has a P property and Picard's iteration is T-stable.

Example 2.16. Let $\mathcal{A} = \mathbb{R}^2$ and the norm be $\|(x_1, x_2)\| = |x_1| + |x_2|$. Define the multiplication by

$$xy = (x_1, x_2)(y_1, y_2) = (x_1y_1, x_1y_2 + x_2y_1),$$

where $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{A}$. Then $\mathcal{A}$ is a Banach algebra with a unit $e = (1, 0)$. Taking $X = [0, 0.55] \times (-\infty, +\infty)$, $K = \{(x_1, x_2) \in \mathcal{A} : x_1, x_2 \geq 0\}$ and

$d(x, y) = (|x_1 - y_1|, |x_2 - y_2|)$ for all $x = (x_1, x_2), y = (y_1, y_2) \in X$,

we claim that (X, d) is a cone metric space over $\mathcal{A}$ and K is a normal solid cone with normal constant $M = 1$.

Define a mapping $T : X \to X$ as

$$Tx = T(x_1, x_2) = \Big(\frac{1}{2}\big(\cos\frac{x_1}{2} - |x_1 - \frac{1}{2}|\big), \arctan(1 + |x_2|) + \ln(x_1 + 1)\Big).$$

By using mean value theorem of differentials, it follows that

$$\begin{aligned}
d(Tx, Ty) &= d(T(x_1, x_2), T(y_1, y_2)) \\
&= \Big(\big|\frac{1}{2}\big(\cos\frac{x_1}{2} - \cos\frac{y_1}{2} - |x_1 - \frac{1}{2}| + |y_1 - \frac{1}{2}|\big)\big|, \\
&\qquad |\arctan(1 + |x_2|) - \arctan(1 + |y_2|) + \ln(x_1 + 1) - \ln(y_1 + 1)|\Big) \\
&\preceq \Big(\big|\frac{x_1 + y_1}{4}\big|\big|\frac{x_1 - y_1}{4}\big| + \frac{1}{2}|x_1 - y_1|, \frac{1}{2}|x_2 - y_2| + |x_1 - y_1|\Big) \\
&\preceq \Big(\frac{5}{8}, 1\Big)\big(|x_1 - y_1|, |x_2 - y_2|\big) \\
&= \Big(\frac{5}{8}, 1\Big)d(x, y)
\end{aligned}$$

for all $x, y \in X$. Put $k = (\frac{5}{8}, 1)$. Simple calculations show that all conditions of Corollary 2.9 are satisfied. Thus by Corollary 2.9, T has a unique fixed point in X. Further, T has a property P and Picard's iteration is T-stable.

The following statement indicates our fixed point results in cone metric space over Banach algebra $\mathcal{A}$ are not equivalent to those in metric spaces. In order to end this, put

$$d_1(x, y) = \inf_{\{u \in P : d(x,y) \preceq u\}} \|u\|, \qquad d_2(x, y) = \inf\{r \in \mathbb{R} : d(x, y) \preceq re\},$$

where $x, y \in X$ and $e = (e_1, e_2) \in \mathrm{int}K$. Then by Theorem 2.2 of [10], d_1 and d_2 are both equivalent metrics. Hence we need to consider only one of them. Let us refer to the

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metric d_2. We shall prove our conclusions are not equivalent to the well-known Banach contraction principle, which means Theorem 2.4 of [8] does not hold in the setting of cone metric spaces over Banach algebras. As a matter of fact, taking $x' = (\frac{1}{2}, 0)$, $y' = (0, 0)$, $e = (1, \frac{1}{2})$, we have

$$\begin{aligned} d_2(Tx', Ty') &= \inf\Big\{r \in \mathbb{R} : \Big(\frac{1}{2}\cos\frac{1}{4} - \frac{1}{4}, \ln\frac{3}{2}\Big) \preceq r\Big(1, \frac{1}{2}\Big)\Big\} \\ &= \max\Big\{\frac{1}{2}\cos\frac{1}{4} - \frac{1}{4}, 2\ln\frac{3}{2}\Big\} = 2\ln\frac{3}{2} \\ &\geq \frac{1}{2} = d_2(x', y'), \end{aligned}$$

which implies that there does not exist $\lambda \in [0, 1)$ such that

$$d_2(Tx, Ty) \leq \lambda d_2(x, y)$$

for all $x, y \in X$. Thus it does not satisfy the contractive condition of Banach contraction principle. That is to say, Theorem 2.4 of [8] is unsuitable for cone metric spaces over Banach algebras.

Remark 2.17. Since the contractive mapping in Example 2.15 is generalized Lipschitz mapping, we are easy to make a conclusion that Corollary 2.1 in [5] cannot cope with Example 2.15, which infers that the main results in the setting of cone metric spaces over Banach algebras are very meaningful.

Remark 2.18. In Example 2.16, we are not hard to see that the main results in this paper are indeed more different from the standard results of cone metric spaces presented in the literature. Also, Example 2.16 shows that cone metric space over Banach algebra do be a real generalization of metric space even if some works with the assumption of normal cone.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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Int-soft filters of MTL-algebras

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Abstract. The notions of (Boolean) int-soft filters in MTL-algebras are introduced, and several properties are investigated. Characterizations of (Boolean) int-soft filters are discussed, and a condition for an int-soft filter to be Boolean is provided. The extension property for a Boolean int-soft filter is constructed, and the least int-soft filter containing a given soft set is established.

1. Introduction

The logic MTL, Monoidal t-norm based logic, was introduced by Esteva and Godo in [3]. This logic is very interesting from many points of view. From the logic point of view, it can be regarded as a weak system of Fuzzy Logic. Indeed, it arises from Hájek's Basic Logic BL [4] by replacing the axiom

$$(A\hat{\wedge}(A \to B)) \leftrightarrow (A \wedge B)$$

by the weaker axiom

$$(A\hat{\wedge}(A \to B)) \to (A \wedge B).$$

In connection with the logic MTL, Esteva and Godo [3] introduced a new algebra, called a *MTL-algebra*, and studied several basic properties. They also introduced the notion of (prime) filters in MTL-algebras. Vetterlein [8] studied MTL-algebras arising from partially ordered groups. Borzooei, Khosravi Shoar and Americ [1] discussed some types of filters in MTL-algebras. Morton and van Alten [6] considered the algebraic semantics of the monoidal t-norm logic(MTL) with unary operations (modalities).

In this paper, we introduce the notion of (Boolean) int-soft filters in MTL-algebras, and investigate several properties. We discuss characterizations of (Boolean) int-soft filters, and provide a condition for an int-soft filter to be Boolean. We establish the extension property for a Boolean int-soft filter. We also construct the least int-soft filter containing a given soft set.

2. Preliminaries

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By a *residuated lattice* we shall mean a lattice $L = (L, \leq, \wedge, \vee, \odot, \to, 0, 1)$ containing the least element 0 and the largest element 1, and endowed with two binary operations $\odot$ (called *product*) and $\to$ (called *residuum*) such that

- $\odot$ is associative, commutative and isotone.
- $(\forall x \in L)\ (x \odot 1 = x)$.
- The Galois correspondence holds, that is,

$$(\forall x, y, z \in L)\,(x \odot y \leq z \iff x \leq y \to z).$$

In a residuated lattice, the following are true (see [7]):

(a1) $x \leq y \iff x \to y = 1$.
(a2) $0 \to x = 1$, $1 \to x = x$, $x \to (y \to x) = 1$.
(a3) $y \leq (y \to x) \to x$.
(a4) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$.
(a5) $x \to y \leq (z \to x) \to (z \to y), \quad x \to y \leq (y \to z) \to (x \to z)$.
(a6) $y \leq x \Rightarrow x \to z \leq y \to z,\ z \to y \leq z \to x$.
(a7) $\left(\bigvee_{i\in\Gamma} y_i\right) \to x = \bigwedge_{i\in\Gamma} (y_i \to x)$.

We define $x^* = \bigvee\{y \in L \mid x \odot y = 0\}$, equivalently, $x^* = x \to 0$. Then

(a8) $0^* = 1,\quad 1^* = 0,\quad x \leq x^{**}$, and $x^* = x^{***}$.

Based on the Hájek's results [4], Axioms of MTL and Formulas which are provable in MTL, Esteva and Godo [3] defined the algebras, so called MTL-algebras, corresponding to the MTL-logic in the following way.

Definition 2.1. An *MTL-algebra* is a residuated lattice $L = (L, \leq, \wedge, \vee, \odot, \to, 0, 1)$ satisfying the pre-linearity equation:

$$(x \to y) \vee (y \to x) = 1.$$

In an MTL-algebra, the following are true:

(a9) $x \to (y \vee z) = (x \to y) \vee (x \to z)$.
(a10) $x \odot y \leq x \wedge y$.

Definition 2.2 ([3])**.** Let L be an MTL-algebra. A nonempty subset F of L is called a *filter* of L if it satisfies

(b1) $(\forall x, y \in F)\ (x \odot y \in F)$.
(b2) $(\forall x \in F)\ (\forall y \in L)\ (x \leq y \Rightarrow y \in F)$.

Since $\wedge$ is not definable from $\odot$ and $\to$ in a MTL-algebra, one could consider that the further condition

(b3) $(\forall x, y \in F)\ (x \wedge y \in F)$

should be also required for a filter. However the condition (b3) is indeed redundant because it is a consequence of conditions (b1) and (b2). Namely, since $x \odot y \leq x \wedge y$, if $x, y \in F$ then $x \odot y \in F$ and thus $x \wedge y \in F$ as well.

Proposition 2.3. *A nonempty subset F of an MTL-algebra L is a filter of L if and only if it satisfies:*

(b4) $1 \in F$.

(b5) $(\forall x \in F)\ (\forall y \in L)\ (x \rightarrow y \in F \ \Rightarrow\ y \in F)$.

A soft set theory is introduced by Molodtsov [5], and Çağman et al. [2] provided new definitions and various results on soft set theory.

Let $\mathscr{P}(U)$ denote the power set of an initial universe set U and $A, B, C, \cdots \subseteq E$ where E is a set of parameters.

Definition 2.4 ([2, 5])**.** A *soft set* $\left(\tilde{f}, A\right)$ over U is defined to be the set of ordered pairs

$$\left(\tilde{f}, A\right) := \left\{\left(x, \tilde{f}(x)\right) : x \in E,\ \tilde{f}(x) \in \mathscr{P}(U)\right\},$$

where $\tilde{f} : E \rightarrow \mathscr{P}(U)$ such that $\tilde{f}(x) = \emptyset$ if $x \notin A$.

For a soft set $\left(\tilde{f}, L\right)$ over U, the set

$$i_L\left(\tilde{f}; \gamma\right) = \left\{x \in L \mid \gamma \subseteq \tilde{f}(x)\right\}$$

is called the γ-*inclusive set* of $\left(\tilde{f}, L\right)$.

3. **Int-soft filters**

In what follows, we take an MTL-algebra L as a set of parameters.

Definition 3.1. A soft set $\left(\tilde{f}, L\right)$ over U is called an *int-soft filter* of L if it satisfies:

$$(\forall x, y \in L)\left(\tilde{f}(x \odot y) \supseteq \tilde{f}(x) \cap \tilde{f}(y)\right). \tag{3.1}$$

$$(\forall x, y \in L)\left(x \leq y \ \Rightarrow\ \tilde{f}(x) \subseteq \tilde{f}(y)\right). \tag{3.2}$$

Example 3.2. Let $L = [0, 1]$ and define a product $\odot$ and a residuum $\rightarrow$ on L as follows:

$$x \odot y := \begin{cases} x \wedge y & \text{if } x + y > \frac{1}{2}, \\ 0 & \text{otherwise} \end{cases} \quad x \rightarrow y := \begin{cases} 1 & \text{if } x \leq y, \\ (0.5 - x) \vee y & \text{if } x > y \end{cases}$$

for all $x, y \in L$. Then L is an MTL-algebra. Let $\left(\tilde{f}, L\right)$ be a soft set over U in which

$$\tilde{f}(x) := \begin{cases} \alpha & \text{if } x \in (0.5, 1], \\ \beta & \text{otherwise,} \end{cases}$$

where $\alpha \supseteq \beta$ in $\mathscr{P}(U)$. Then it is routine to verify that $\left(\tilde{f}, L\right)$ is an int-soft filter of L.

We provide characterizations of an int-soft filter.

Theorem 3.3. *A soft set* $\left(\tilde{f}, L\right)$ *over* U *is an int-soft filter of* L *if and only if it satisfies:*

$$(\forall x \in L)\left(\tilde{f}(1) \supseteq \tilde{f}(x)\right), \tag{3.3}$$

$$(\forall x, y \in L)\left(\tilde{f}(y) \supseteq \tilde{f}(x) \cap \tilde{f}(x \to y)\right). \tag{3.4}$$

Proof. Assume that $\left(\tilde{f}, L\right)$ is an int-soft filter of L. Since $x \leq 1$ for all $x \in L$, it follows from (3.2) that $\tilde{f}(x) \subseteq \tilde{f}(1)$ for all $x \in L$. Since $x \leq (x \to y) \to y$, we have $x \odot (x \to y) \leq y$ for all $x, y \in L$ by the Galois correspondence. It follows from (3.2) and (3.1) that

$$\tilde{f}(y) \supseteq \tilde{f}(x \odot (x \to y)) \supseteq \tilde{f}(x) \cap \tilde{f}(x \to y)$$

for all $x, y \in L$.

Conversely, let $\left(\tilde{f}, L\right)$ be a soft set over U which satisfy two conditions (3.3) and (3.4). Let $x, y \in L$ be such that $x \leq y$. Then $x \to y = 1$, and so

$$\tilde{f}(y) \supseteq \tilde{f}(x) \cap \tilde{f}(x \to y) = \tilde{f}(x) \cap \tilde{f}(1) = \tilde{f}(x),$$

for all $x \in L$. This proves (3.2). Using (a4), we know that

$$x \to (y \to (x \odot y)) = (x \odot y) \to (x \odot y) = 1.$$

Using (3.3) and (3.4), we have

$$\begin{aligned}
\tilde{f}(x \odot y) &\supseteq \tilde{f}(y) \cap \tilde{f}(y \to (x \odot y)) \\
&\supseteq \tilde{f}(y) \cap \left(\tilde{f}(x) \cap \tilde{f}(x \to (y \to (x \odot y)))\right) \\
&= \tilde{f}(y) \cap \left(\tilde{f}(x) \cap \tilde{f}(1)\right) \\
&= \tilde{f}(x) \cap \tilde{f}(y)
\end{aligned}$$

for all $x, y \in L$. Therefore $\left(\tilde{f}, L\right)$ is an int-soft filter of L. □

Theorem 3.4. *A soft set* $\left(\tilde{f}, L\right)$ *over* U *is an int-soft filter of* L *if and only if it satisfies:*

$$(\forall a, b, c \in L)\left(a \leq b \to c \;\Rightarrow\; \tilde{f}(c) \supseteq \tilde{f}(a) \cap \tilde{f}(b)\right). \tag{3.5}$$

Proof. Assume that $\left(\tilde{f}, L\right)$ is an int-soft filter of L. Let $a, b, c \in L$ be such that $a \leq b \to c$. Then $\tilde{f}(a) \subseteq \tilde{f}(b \to c)$ by (3.2), and so

$$\tilde{f}(c) \supseteq \tilde{f}(b) \cap \tilde{f}(b \to c) \supseteq \tilde{f}(b) \cap \tilde{f}(a).$$

Conversely, let $\left(\tilde{f}, L\right)$ be a soft set over U satisfying (3.5). Since $x \leq x \to 1$ for all $x \in L$, it follows from (3.5) that

$$\tilde{f}(1) \supseteq \tilde{f}(x) \cap \tilde{f}(x) = \tilde{f}(x)$$

for all $x \in L$. Since $x \to y \leq x \to y$ for all $x, y \in L$, we have

$$\tilde{f}(y) \supseteq \tilde{f}(x) \cap \tilde{f}(x \to y)$$

for all $x, y \in L$. Therefore $\left(\tilde{f}, L\right)$ is an int-soft filter of L. □

Corollary 3.5. *A soft set $\left(\tilde{f}, L\right)$ over U is an int-soft filter of L if and only if it satisfies the following assertion:*

$$\tilde{f}(x) \supseteq \bigcap_{k=1}^{n} \tilde{f}(a_k) \tag{3.6}$$

whenever $a_n \to (\cdots \to (a_2 \to (a_1 \to x))\cdots) = 1$ for every $a_1, a_2, \cdots, a_n \in L$.

Proof. It is by induction. □

Theorem 3.6. *For a filter F of L and $a \in L$, let $\left(\tilde{f}, L\right)$ be a soft set over U defined by*

$$\tilde{f}(x) := \begin{cases} \gamma_1 & \text{if } x \in \{z \in L \mid a \vee z \in F\}, \\ \gamma_2 & \text{otherwise,} \end{cases}$$

for all $x \in L$ where $\gamma_2 \subsetneq \gamma_1$ in $\mathscr{P}(U)$. Then $\left(\tilde{f}, L\right)$ is an int-soft filter of L.

Proof. Since $a \vee 1 \in F$, we have $1 \in \{z \in L \mid a \vee z \in F\}$ and so $\tilde{f}(1) = \gamma_1 \supseteq \tilde{f}(x)$ for all $x \in L$. Now if $y \in \{z \in L \mid a \vee z \in F\}$, then clearly

$$\tilde{f}(y) = \gamma_1 \supseteq \tilde{f}(x) \cap \tilde{f}(x \to y).$$

Suppose that $y \notin \{z \in L \mid a \vee z \in F\}$. Then at least one of x and $x \to y$ does not belong to $\{z \in L \mid a \vee z \in F\}$. Hence

$$\tilde{f}(y) = \gamma_2 = \tilde{f}(x) \cap \tilde{f}(x \to y),$$

and therefore $\left(\tilde{f}, L\right)$ is an int-soft filter of L. □

Theorem 3.7. *A soft set $\left(\tilde{f}, L\right)$ over U is an int-soft filter of L if and only if the nonempty γ-inclusive set $i_L\left(\tilde{f}; \gamma\right)$ is a filter of L for all $\gamma \in \mathscr{P}(U)$.*

Proof. Assume that the nonempty γ-inclusive set $i_L\left(\tilde{f}; \gamma\right)$ is a filter of L for all $\gamma \in \mathscr{P}(U)$. For any $x \in L$, let $\tilde{f}(x) = \gamma$. Then $x \in i_L\left(\tilde{f}; \gamma\right)$. Since $i_L\left(\tilde{f}; \gamma\right)$ is a filter of L, we have $1 \in i_L\left(\tilde{f}; \gamma\right)$ and so $\tilde{f}(x) = \gamma \subseteq \tilde{f}(1)$. For any $x, y \in L$, let $\tilde{f}(x \to y) \cap \tilde{f}(x) = \gamma$. Then $x \to y \in i_L\left(\tilde{f}; \gamma\right)$

and $x \in i_L\left(\tilde{f};\gamma\right)$. It follows from (b5) that $y \in i_L\left(\tilde{f};\gamma\right)$. Hence $\tilde{f}(y) \supseteq \gamma = \tilde{f}(x \to y) \cap \tilde{f}(x)$. Therefore $\left(\tilde{f}, L\right)$ is an int-soft filter of L by Theorem 3.3.

Conversely, suppose that $\left(\tilde{f}, L\right)$ is an int-soft filter of L. Let $\gamma \in \mathscr{P}(U)$ be such that $i_L\left(\tilde{f};\gamma\right) \neq \emptyset$. Then there exists $a \in i_L\left(\tilde{f};\gamma\right)$, and so $\gamma \subseteq \tilde{f}(a)$. It follows from (3.3) that $\gamma \subseteq \tilde{f}(a) \subseteq \tilde{f}(1)$. Thus $1 \in i_L\left(\tilde{f};\gamma\right)$. Let $x, y \in L$ be such that $x \to y \in i_L\left(\tilde{f};\gamma\right)$ and $x \in i_L\left(\tilde{f};\gamma\right)$. Then $\gamma \subseteq \tilde{f}(x \to y)$ and $\gamma \subseteq \tilde{f}(x)$. It follows from (3.4) that

$$\gamma \subseteq \tilde{f}(x \to y) \cap \tilde{f}(x) \subseteq \tilde{f}(y),$$

that is, $y \in i_L\left(\tilde{f};\gamma\right)$. Thus $i_L\left(\tilde{f};\gamma\right)$ $(\neq \emptyset)$ is a filter of L by Proposition 2.3. □

Theorem 3.8. *If $\left(\tilde{f}, L\right)$ is an int-soft filter of L, then the set*

$$\Omega_a := \{x \in L \mid \tilde{f}(x) \supseteq \tilde{f}(a)\}$$

is a filter of L for every $a \in L$.

Proof. Since $\tilde{f}(1) \supseteq \tilde{f}(x)$ for all $x \in L$. we have $1 \in \Omega_a$. Let $x, y \in L$ be such that $x \in \Omega_a$ and $x \to y \in \Omega_a$. Then $\tilde{f}(x) \supseteq \tilde{f}(a)$ and $\tilde{f}(x \to y) \supseteq \tilde{f}(a)$. Since $\tilde{f}$ is an int-soft filter of L, it follows from (3.4) that

$$\tilde{f}(y) \supseteq \tilde{f}(x) \cap \tilde{f}(x \to y) \supseteq \tilde{f}(a)$$

so that $y \in \Omega_a$. Hence Ω_a is a filter of L. □

Theorem 3.9. *Let $a \in L$ and let $\left(\tilde{f}, L\right)$ be a soft set over U. Then*

(1) *If Ω_a is a filter of L, then $\left(\tilde{f}, L\right)$ satisfies the following implication:*

$$(\forall x, y \in L)\,(\tilde{f}(a) \subseteq \tilde{f}(x \to y) \cap \tilde{f}(x) \;\Rightarrow\; \tilde{f}(a) \subseteq \tilde{f}(y)). \tag{3.7}$$

(2) *If $\left(\tilde{f}, L\right)$ satisfies (3.3) and (3.7), then Ω_a is a filter of L.*

Proof. (1) Assume that Ω_a is a filter of L. Let $x, y \in L$ be such that

$$\tilde{f}(a) \subseteq \tilde{f}(x \to y) \cap \tilde{f}(x).$$

Then $x \to y \in \Omega_a$ and $x \in \Omega_a$. Using (b5), we have $y \in \Omega_a$ and so $\tilde{f}(y) \supseteq \tilde{f}(a)$.
(2) Suppose that $\tilde{f}$ satisfies (3.3) and (3.7). From (3.3) it follows that $1 \in \Omega_a$. Let $x, y \in L$ be such that $x \in \Omega_a$ and $x \to y \in \Omega_a$. Then $\tilde{f}(a) \subseteq \tilde{f}(x)$ and $\tilde{f}(a) \subseteq \tilde{f}(x \to y)$, which imply that $\tilde{f}(a) \subseteq \tilde{f}(x) \cap \tilde{f}(x \to y)$. Thus $\tilde{f}(a) \subseteq \tilde{f}(y)$ by (3.7), and so $y \in \Omega_a$. Therefore Ω_a is a filter of L. □

Proposition 3.10. *Let $\left(\tilde{f}, L\right)$ be an int-soft filter of L. Then the following are equivalent:*

(1) $(\forall x, y, z \in L)$ $\left(\tilde{f}(x \to z) \supseteq \tilde{f}(x \to (y \to z)) \cap \tilde{f}(x \to y)\right)$.

(2) $(\forall x, y \in L)\ \left(\tilde{f}(x \to y) \supseteq \tilde{f}(x \to (x \to y))\right)$.

(3) $(\forall x, y, z \in L)\ \left(\tilde{f}((x \to y) \to (x \to z)) \supseteq \tilde{f}(x \to (y \to z))\right)$.

Proof. (1) $\Rightarrow$ (2). Suppose that $\left(\tilde{f}, L\right)$ satisfies the condition (1). Taking $z = y$ and $y = x$ in (1) and using (3.3), we have

$$\begin{aligned}\tilde{f}(x \to y) &\supseteq \tilde{f}(x \to (x \to y)) \cap \tilde{f}(x \to x)\\ &= \tilde{f}(x \to (x \to y)) \cap \tilde{f}(1)\\ &= \tilde{f}(x \to (x \to y))\end{aligned}$$

for all $x, y, z \in L$.

(2) $\Rightarrow$ (3). Suppose that $\left(\tilde{f}, L\right)$ satisfies the condition (2) and let $x, y, z \in L$. Since

$$x \to (y \to z) \leq x \to ((x \to y) \to (x \to z)),$$

it follows that

$$\begin{aligned}\tilde{f}((x \to y) \to (x \to z)) &= \tilde{f}(x \to ((x \to y) \to z))\\ &\supseteq \tilde{f}(x \to (x \to ((x \to y) \to z)))\\ &= \tilde{f}(x \to ((x \to y) \to (x \to z)))\\ &\supseteq \tilde{f}(x \to (y \to z)).\end{aligned}$$

(3) $\Rightarrow$ (1). If $\left(\tilde{f}, L\right)$ satisfies the condition (3), then

$$\begin{aligned}\tilde{f}(x \to y) &\supseteq \tilde{f}((x \to y) \to (x \to z)) \cap \tilde{f}(x \to y)\\ &\supseteq \tilde{f}(x \to (y \to z)) \cap \tilde{f}(x \to y).\end{aligned}$$

This completes the proof. $\square$

Theorem 3.11. *For a fixed element $a \in L$, let $\left(\tilde{f}_a, L\right)$ be a soft set over U defined by*

$$\tilde{f}_a(x) := \begin{cases} \gamma_1 & \text{if } a \leq x,\\ \gamma_2 & \text{otherwise,}\end{cases}$$

where $\gamma_1 \supsetneq \gamma_2$ in $\mathscr{P}(U)$. Then $\left(\tilde{f}_a, L\right)$ is an int-soft filter of L if and only if it satisfies the following implication:

$$(\forall x, y \in L)\,(a \leq y \to x,\ a \leq y \ \Rightarrow\ a \leq x). \tag{3.8}$$

Proof. Assume that $\left(\tilde{f}_a, L\right)$ is an int-soft filter of L and let $x, y \in L$ be such that $a \leq y \to x$ and $a \leq y$. Then $\tilde{f}_a(y \to x) = \gamma_1 = \tilde{f}_a(y)$, and thus

$$\tilde{f}_a(x) \supseteq \tilde{f}_a(y \to x) \cap \tilde{f}_a(y) = \gamma_1$$

which implies that $\tilde{f}_a(x) = \gamma_1$ and so $a \leq x$.

Conversely, suppose that (3.8) is valid. Note that $i_L\left(\tilde{f}_a;\gamma_2\right) = L$ and $i_L\left(\tilde{f}_a;\gamma_1\right) = \{x \in L \mid a \leq x\}$. Obviously $1 \in i_L\left(\tilde{f}_a;\gamma_1\right)$. Let $x, y \in L$ be such that $x \in i_L\left(\tilde{f}_a;\gamma_1\right)$ and $x \to y \in i_L\left(\tilde{f}_a;\gamma_1\right)$. Then $a \leq x$ and $a \leq x \to y$, which imply from (3.8) that $a \leq y$, that is, $y \in i_L\left(\tilde{f}_a;\gamma_1\right)$. Hence $i_L\left(\tilde{f}_a;\gamma_1\right)$ is a filter of L. Using Theorem 3.7, we know that $\left(\tilde{f}_a, L\right)$ is an int-soft filter of L. □

Definition 3.12. An int-soft filter $\left(\tilde{f}, L\right)$ of L is said to be *Boolean* if it satisfies the following identity

$$(\forall x \in L)\left(\tilde{f}(x \vee x^*) = \tilde{f}(1)\right). \tag{3.9}$$

Proposition 3.13. *Every Boolean int-soft filter $\left(\tilde{f}, L\right)$ of L satisfies the following inclusion:*

$$(\forall x, y, z \in L)\left(\tilde{f}(x \to z) \supseteq \tilde{f}(x \to (z^* \to y)) \cap \tilde{f}(y \to z)\right). \tag{3.10}$$

Proof. Using (a5), we have

$$y \to z \leq (z^* \to y) \to (z^* \to z) \leq (x \to (z^* \to y)) \to (x \to (z^* \to z)).$$

It follows from (3.2) that

$$\tilde{f}(y \to z) \subseteq \tilde{f}((x \to (z^* \to y)) \to (x \to (z^* \to z)))$$

so from (3.4) that

$$\begin{aligned}\tilde{f}(x \to (z^* \to z)) &\supseteq \tilde{f}(x \to (z^* \to y)) \cap \tilde{f}((x \to (z^* \to y)) \to (x \to (z^* \to z)))\\ &\supseteq \tilde{f}(x \to (z^* \to y)) \cap \tilde{f}(y \to z).\end{aligned}$$

Since

$$z^* \vee z = ((z^* \to z) \to z) \wedge ((z \to z^*) \to z^*) \leq (z^* \to z) \to z,$$

we have $\tilde{f}((z^* \to z) \to z) \supseteq \tilde{f}(z^* \vee z) = \tilde{f}(1)$. Since

$$x \to (z^* \to z) \leq ((z^* \to z) \to z) \to (x \to z),$$

it follows from (3.2) that

$$\tilde{f}(x \to (z^* \to z)) \subseteq \tilde{f}(((z^* \to z) \to z) \to (x \to z)).$$

Thus

$$\begin{aligned}\tilde{f}(x \to z) &\supseteq \tilde{f}((z^* \to z) \to z) \cap \tilde{f}(((z^* \to z) \to z) \to (x \to z))\\ &\supseteq \tilde{f}(1) \cap \tilde{f}(x \to (z^* \to z))\\ &= \tilde{f}(x \to (z^* \to z))\\ &\supseteq \tilde{f}(x \to (z^* \to y)) \cap \tilde{f}(y \to z).\end{aligned}$$

This completes the proof. □

We provide a condition for an int-soft filter to be Boolean.

Proposition 3.14. *If an int-soft filter* $\left(\tilde{f}, L\right)$ *of* L *satisfies the following inclusion*

$$(\forall x, y \in L)\left(\tilde{f}(x) \supseteq \tilde{f}((x \to y) \to x)\right), \tag{3.11}$$

then it is Boolean.

Proof. Using (a2), (a4) and (a5), we have

$$\begin{aligned} 1 &= x \to ((x^* \to x) \to x) \\ &\leq ((x^* \to x) \to x)^* \to x^* \\ &\leq (x^* \to x) \to (((x^* \to x) \to x)^* \to x) \\ &= ((x^* \to x) \to x)^* \to ((x^* \to x) \to x) \\ &= (((x^* \to x) \to x) \to 0) \to ((x^* \to x) \to x). \end{aligned}$$

It follows from (3.2), (3.3) and (3.11) that

$$\tilde{f}((x^* \to x) \to x) \supseteq \tilde{f}((((x^* \to x) \to x) \to 0) \to ((x^* \to x) \to x)) = \tilde{f}(1).$$

Using (a7) and (a9), since

$$\begin{aligned} (x^* \to x) \to x &\leq ((x^* \to x) \to x) \vee ((x^* \to x) \to x^*) \\ &= (x^* \to x) \to (x \vee x^*) \\ &= (1 \wedge (x^* \to x)) \to (x \vee x^*) \\ &= ((x \to x) \wedge (x^* \to x)) \to (x \vee x^*) \\ &= ((x \vee x^*) \to x) \to (x \vee x^*), \end{aligned}$$

we get

$$\begin{aligned} \tilde{f}(1) &= \tilde{f}((x^* \to x) \to x) \\ &\subseteq \tilde{f}(((x \vee x^*) \to x) \to (x \vee x^*)) \\ &\subseteq \tilde{f}(x \vee x^*), \end{aligned}$$

and so $\tilde{f}(x \vee x^*) = \tilde{f}(1)$. Therefore $\left(\tilde{f}, L\right)$ is Boolean. $\square$

Proposition 3.15. *If an int-soft filter* $\left(\tilde{f}, L\right)$ *of* L *satisfies the condition* (3.10)*, then it satisfies the condition* (3.11)*.*

Proof. Since $(x \to y) \to x \leq x^* \to x$, it follows from (3.2) that

$$\begin{aligned} \tilde{f}(x) &= \tilde{f}(1 \to x) \\ &\supseteq \tilde{f}(1 \to (x^* \to x^*)) \cap \tilde{f}(x^* \to x) \\ &\supseteq \tilde{f}(1) \cap \tilde{f}((x \to y) \to x) \\ &= \tilde{f}((x \to y) \to x). \end{aligned}$$

Hence $\left(\tilde{f}, L\right)$ satisfies the condition (3.11). □

Proposition 3.16. *If an int-soft filter $\left(\tilde{f}, L\right)$ of L satisfies* (3.11)*, then it satisfies the following inclusion:*

$$(\forall x, y, z \in L)\left(\tilde{f}(x \to z) \supseteq \tilde{f}(x \to (y \to z)) \cap \tilde{f}(x \to y)\right). \tag{3.12}$$

Proof. Since $x \to (y \to z) = y \to (x \to z) \leq (x \to y) \to (x \to (x \to z))$, it follows from (3.2) that

$$\tilde{f}(x \to (y \to z)) \subseteq \tilde{f}((x \to y) \to (x \to (x \to z)))$$

so from (3.4) that

$$\begin{aligned}\tilde{f}(x \to (x \to z)) &\supseteq \tilde{f}(x \to y) \cap \tilde{f}((x \to y) \to (x \to (x \to z)))\\ &\supseteq \tilde{f}(x \to y) \cap \tilde{f}(x \to (y \to z)).\end{aligned}$$

Since

$$x \to (x \to z) \leq x \to (((x \to z) \to z) \to z) = ((x \to z) \to z) \to (x \to z),$$

we have

$$\begin{aligned}\tilde{f}(x \to z) &\supseteq \tilde{f}(((x \to z) \to z) \to (x \to z))\\ &\supseteq \tilde{f}(x \to (x \to z))\\ &\supseteq \tilde{f}(x \to y) \cap \tilde{f}(x \to (y \to z))\end{aligned}$$

by using (3.2) and (3.10). This completes the proof. □

Proposition 3.17. *Every Boolean int-soft filter $\left(\tilde{f}, L\right)$ of L satisfies the following inclusion:*

$$(\forall x, y, z \in L)\left(\tilde{f}(x \to z) \supseteq \tilde{f}(x \to (y \to z)) \cap \tilde{f}(x \to y)\right).$$

Proof. Note that $x \to (y \to z) = y \to (x \to z) \leq (x \to y) \to (x \to (x \to z))$ and

$$x \to (x \to z) \leq x \to (((x \to z) \to z) \to z) = ((x \to z) \to z) \to (x \to z)$$

for all $x, y, z \in L$. It follows from (3.2), (3.4), and Propositions 3.13 and 3.14 that

$$\begin{aligned}\tilde{f}(x \to z) &\supseteq \tilde{f}(((x \to z) \to z) \to (x \to z))\\ &\supseteq \tilde{f}(x \to (x \to z))\\ &\supseteq \tilde{f}(x \to y) \cap \tilde{f}((x \to y) \to (x \to (x \to z)))\\ &\supseteq \tilde{f}(x \to y) \cap \tilde{f}(x \to (y \to z)).\end{aligned}$$

This completes the proof. □

Combining Propositions 3.13, 3.14 and 3.15, we have a characterization of a Boolean int-soft filter.

Theorem 3.18. *Let $\left(\tilde{f}, L\right)$ be an int-soft filter of L. Then the following assertions are equivalent:*

(1) $\left(\tilde{f}, L\right)$ *is Boolean.*

(2) $\left(\tilde{f}, L\right)$ *satisfies the condition* (3.10).

(3) $\left(\tilde{f}, L\right)$ *satisfies the condition* (3.11).

Proposition 3.19. *Every Boolean int-soft filter* $\left(\tilde{f}, L\right)$ *of* L *satisfies:*

$$(\forall x, y \in L)\left(\tilde{f}(x \to y) \subseteq \tilde{f}(((y \to x) \to x) \to y)\right). \tag{3.13}$$

Proof. Let $\left(\tilde{f}, L\right)$ be a Boolean int-soft filter of L. Since $y \leq ((y \to x) \to x) \to y$, we have

$$(((y \to x) \to x) \to y) \to x \leq y \to x \tag{3.14}$$

by (a6). Using (a4), (a5), (a6) and (3.14), we get

$$\begin{aligned} x \to y &\leq ((y \to x) \to x) \to ((y \to x) \to y) \\ &= (y \to x) \to (((y \to x) \to x) \to y) \\ &\leq ((((y \to x) \to x) \to y) \to x) \to (((y \to x) \to x) \to y) \end{aligned}$$

and so

$$\begin{aligned} \tilde{f}(((y \to x) \to x) \to y) &\supseteq \tilde{f}(((((y \to x) \to x) \to y) \to x) \to (((y \to x) \to x) \to y)) \\ &\supseteq \tilde{f}(x \to y) \end{aligned}$$

by Theorem 3.18(3) and (3.2). □

Theorem 3.20. (Extension Property for Boolean int-soft filter) *Let* $\left(\tilde{f}, L\right)$ *and* $(\tilde{g}, L)$ *be two int-soft filters of* L *such that* $\tilde{f}(1) = \tilde{g}(1)$ *and* $\tilde{f}(x) \subseteq \tilde{g}(x)$ *for all* $x \in L$. *If* $\left(\tilde{f}, L\right)$ *is Boolean, then so is* $(\tilde{g}, L)$.

Proof. Assume that $\left(\tilde{f}, L\right)$ is a Boolean int-soft filter of L. Then $\tilde{f}(x \vee x^*) = \tilde{f}(1)$ for all $x \in L$. It follows from the hypothesis that

$$\tilde{g}(x \vee x^*) \supseteq \tilde{f}(x \vee x^*) = \tilde{f}(1) = \tilde{g}(1). \tag{3.15}$$

Combining (3.15) and (3.3), we have $\tilde{g}(x \vee x^*) = \tilde{g}(1)$ for all $x \in L$. Hence $(\tilde{g}, L)$ is a Boolean int-soft filter of L. □

For any soft set $\left(\tilde{f}, L\right)$ over U, let $(\tilde{g}, L)$ be a soft set over U in which

$$\tilde{g}(x) := \bigcup \left\{ \bigcap_{k=1}^{n} \tilde{f}(a_n) \;\middle|\; \begin{array}{c} a_n \to (\cdots \to (a_2 \to (a_1 \to x)) \cdots) = 1, \\ a_1, a_2, \cdots, a_n \in L \end{array} \right\} \tag{3.16}$$

for all $x \in L$. Let $a, b, x \in L$ be such that $a \leq b \to x$. Take $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_m \in L$ such that

$$\begin{aligned}
&a_n \to (\cdots \to (a_2 \to (a_1 \to a))\cdots) = 1,\\
&b_m \to (\cdots \to (b_2 \to (b_1 \to b))\cdots) = 1,\\
&\tilde{g}(a) = \bigcap_{k=1}^{n} \tilde{f}(a_k),\\
&\tilde{g}(b) = \bigcap_{j=1}^{m} \tilde{f}(b_j).
\end{aligned}$$

Then

$$b_m \to (\cdots \to (b_1 \to (a_n \to (\cdots \to (a_2 \to (a_1 \to x))\cdots)))\cdots) = 1,$$

and so

$$\begin{aligned}
\tilde{g}(x) &\supseteq \tilde{f}(a_1) \cap \tilde{f}(a_2) \cap \cdots \cap \tilde{f}(a_n) \cap \tilde{f}(b_1) \cap \tilde{f}(b_2) \cap \cdots \tilde{f}(b_m)\\
&= \left(\bigcap_{k=1}^{n} \tilde{f}(a_k)\right) \cap \left(\bigcap_{j=1}^{m} \tilde{f}(b_j)\right)\\
&= \tilde{g}(a) \cap \tilde{g}(b).
\end{aligned}$$

Hence $(\tilde{g}, L)$ is an int-soft filter of L by Theorem 3.4. Since $x \to x = 1$ for all $x \in L$, we have $\tilde{f}(x) \subseteq \tilde{g}(x)$ for all $x \in L$. Thus $(\tilde{g}, L)$ contains $\left(\tilde{f}, L\right)$. Let $\left(\tilde{h}, L\right)$ be an int-soft filter of L that contains $\left(\tilde{f}, L\right)$. Then

$$\begin{aligned}
\tilde{g}(x) &= \bigcup \left\{ \bigcap_{k=1}^{n} \tilde{f}(a_n) \;\middle|\; \begin{array}{c} a_n \to (\cdots \to (a_2 \to (a_1 \to x))\cdots) = 1, \\ a_1, a_2, \cdots, a_n \in L \end{array} \right\}\\
&\subseteq \bigcup \left\{ \bigcap_{k=1}^{n} \tilde{h}(a_n) \;\middle|\; \begin{array}{c} a_n \to (\cdots \to (a_2 \to (a_1 \to x))\cdots) = 1, \\ a_1, a_2, \cdots, a_n \in L \end{array} \right\}\\
&\subseteq \bigcup \tilde{h}(x) = \tilde{h}(x)
\end{aligned}$$

by Corollary 3.5, that is, $(\tilde{g}, L)$ is contained in $\left(\tilde{h}, L\right)$.

We summarize this as follows:

Theorem 3.21. *For any soft set* $\left(\tilde{f}, L\right)$ *over* U, *the soft set* $(\tilde{g}, L)$ *over* U *in which*

$$\tilde{g}(x) := \bigcup \left\{ \bigcap_{k=1}^{n} \tilde{f}(a_n) \;\middle|\; \begin{array}{c} a_n \to (\cdots \to (a_2 \to (a_1 \to x))\cdots) = 1, \\ a_1, a_2, \cdots, a_n \in L \end{array} \right\}$$

for all $x \in L$ *is the least int-soft filter of* L *that contains* $\left(\tilde{f}, L\right)$.

Conclusion

Based on the soft set theory, we have introduced the notion of (Boolean) int-soft filters in MTL-algebras, and have investigated several properties. We have discussed characterizations of (Boolean) int-soft filters, and have provided a condition for an int-soft filter to be Boolean. We have established the extension property for a Boolean int-soft filter. We have also constructed the least int-soft filter containing a given soft set. Future research will focus on applying the notions and contents to other types of filters in related algebraic structures, and on studying it again by using Boolean algebra instead of $\mathcal{P}(U)$.

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Convergence Analysis of New Iterative Approximating Schemes with Errors for Total Asymptotically Nonexpansive Mappings in Hyperbolic Spaces

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Abstract. The purpose of this paper is to introduce the concept of total asymptotically nonexpansive mappings and to prove some Δ-convergence theorems of iteration processes with errors to approximating a common fixed point for four total asymptotically nonexpansive mappings in hyperbolic spaces. The results presented in the paper extend and improve some recent results announced in the current literature.

Key Words and Phrases. New iterative approximations with errors, asymptotically nonexpansive mapping, total asymptotically nonexpansive mapping, common fixed point, convergence analysis.

AMS Subject Classification. 47H09, 47H10, 54E70.

1 Introduction and preliminaries

Most of the problems in various disciplines of science are nonlinear in nature, whereas fixed point theory proposed in the setting of normed linear spaces or Banach spaces majorly depends on the linear structure of the underlying spaces. A nonlinear framework for fixed point theory is a metric space embedded with a 'convex structure'. The class of hyperbolic spaces, nonlinear in nature, is a general abstract theoretic setting with rich geometrical structure for metric fixed point theory. The study of hyperbolic spaces has been largely motivated and dominated by questions about hyperbolic groups, one of the main objects of study in geometric group theory.

Throughout this paper, we work in the setting of hyperbolic spaces due to Kohlenbach [1], defined below, which is more restrictive than the hyperbolic type introduced in [2] and more general than the concept of hyperbolic space in [3].

A hyperbolic spaces is a metric space (X, d) together with a mapping $W : X^2 \times [0, 1] \to X$ satisfying

(i) $d(u, W(x, y, \alpha)) \le \alpha d(u, x) + (1 - \alpha)d(u, y)$;
(ii) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta|d(x, y)$;
(iii) $W(x, y, \alpha) = W(y, x, (1 - \alpha))$;
(iv) $d(W(x, z, \alpha), W(y, w, \alpha)) \le \alpha d(x, y) + (1 - \alpha)d(z, w)$

for all $u, x, y, z, w \in X$ and $\alpha, \beta \in [0, 1]$ (see also [4]). A nonempty subset K of a hyperbolic space X is convex if $W(x, y, \alpha) \in K$ for all $x, y \in K$ and $\alpha \in [0, 1]$. The class of hyperbolic spaces contains normed spaces and convex subsets thereof, the Hilbert open unit ball equipped with the hyperbolic metric [5], Hadamard manifolds as well as CAT(0) spaces in the sense of Gromov (see [6]).

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A hyperbolic space is uniformly convex [7, 8] if for any $r > 0$ and $\epsilon \in (0, 2]$ there exists $\delta \in (0, 1]$ such that for all $u, x, y \in X$, we have

$$d(W(x, y, \frac{1}{2}), u) \le (1 - \delta)r,$$

provided $d(x, u) \le r$, $d(y, u) \le r$ and $d(x, y) \ge \epsilon r$.

A map $\eta : (0, +\infty) \times (0, 2] \to (0, 1]$, which provides such $\delta = \eta(r, \epsilon)$ for given $r > 0$ and $\epsilon \in (0, 2]$, is known as a modulus of uniform convexity of X. We call η monotone if it decreases with r (for fixed ϵ), i.e., $\forall \epsilon > 0$, $\forall r_2 \ge r_1 > 0$ $(\eta(r_2, \epsilon) \le \eta(r_1, \epsilon))$.

In the sequel, let (X, d) be a metric space, and let K be a nonempty subset of X. We shall denote the fixed point set of a mapping T by $F(T) = \{x \in K : Tx = x\}$.

A mapping $T : K \to K$ is said to be nonexpansive if

$$d(Tx, Ty) \le d(x, y), \quad \forall x, y \in K.$$

A mapping $T : K \to K$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [0, +\infty)$ with $k_n \to 0$ such that

$$d(T^n x, T^n y) \le (1 + k_n)d(x, y), \quad \forall n \ge 1, x, y \in K.$$

A mapping $T : K \to K$ is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$d(T^n x, T^n y) \le Ld(x, y), \quad \forall n \ge 1, x, y \in K.$$

Definition 1.1 A mapping $T : K \to K$ is said to be $(\{\mu_n\}, \{\xi_n\}, \rho)$-total asymptotically nonexpansive if there exist nonnegative sequences $\{\mu_n\}$, $\{\xi_n\}$ with $\mu_n \to 0$, $\xi_n \to 0$ and a strictly increasing continuous function $\rho : [0, +\infty) \to [0, +\infty)$ with $\rho(0) = 0$ such that

$$d(T^n x, T^n y) \le d(x, y) + \mu_n \rho\big(d(x, y)\big) + \xi_n, \quad \forall n \ge 1, x, y \in K.$$

Remark 1.1 From the definitions, it is to know that each nonexpansive mapping is an asymptotically nonexpansive mapping with a sequence $\{k_n = 0\}$, and each asymptotically nonexpansive mapping is a $(\{\mu_n\}, \{\xi_n\}, \rho)$-total asymptotically nonexpansive mapping with $\xi_n = 0$, and $\rho(t) = t$, $t \ge 0$.

The existence of fixed points of various nonlinear mappings has relevant applications in many branches of nonlinear analysis and topology. On the other hand, there are certain situations where it is difficult to derive conditions for the existence of fixed points for certain types of nonlinear mappings. It is worth to mention that fixed point theory for nonexpansive mappings, a limit case of a contraction mapping when the Lipschitz constant is allowed to be 1, requires tools far beyond metric fixed point theory. Iteration schemas are the only main tool for analysis of generalized nonexpansive mappings. Fixed point theory has a computational flavor as one can define effective iteration schemas for the computation of fixed points of various nonlinear mappings. The problem of finding a common fixed point of some nonlinear mappings acting on a nonempty convex domain often arises in applied mathematics.

On the other hand, Zhao et al. [9] introduced a mixed type iteration for total asymptotically nonexpansive mappings in hyperbolic spaces, and prove some Δ-convergence theorems for the iteration process approximating to a common fixed point; Zhao et al. [10] consider convergence theorems for total asymptotically nonexpansive mappings in hyperbolic spaces. Furthermore, Fukhar-ud-din and Kalsoom [11] extended iterative process with errors to asymptotically nonexpansive mappings in hyperbolic spaces, and obtained some convergence results.

Motivated and inspired by the above works, the purpose of this paper is to introduce the concepts of total asymptotically nonexpansive mappings and to prove some Δ-convergence theorems of iteration process with errors for approximating a common fixed point of four total asymptotically nonexpansive mappings in hyperbolic spaces. The results presented in the paper extend and improve some recent results given in [9-25].

2 Preliminaries

In order to define the concept of Δ-convergence in the general setup of hyperbolic spaces, we first collect some basic concepts.

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space X. For $x \in X$, we define a continuous functional $r(\cdot, \{x_n\}) : X \to [0, +\infty)$ by

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}.$$

The asymptotic center $A_K(\{x_n\})$ of a bounded sequence $\{x_n\}$ with respect to $K \subset X$ is the set

$$A_K(\{x_n\}) = \{x \in X : r(x, \{x_n\}) \le r(y, \{x_n\}), \forall y \in K\}.$$

This is the set of minimizers of the functional $r(\cdot, \{x_n\})$. If the asymptotic center is taken with respect to X, then it is simply denoted by $A(\{x_n\})$. It is known that uniformly convex Banach spaces and CAT(0) spaces enjoy the property that 'bounded sequences have unique asymptotic centers with respect to closed convex subsets'. The following lemma is due to Leustean [26] and ensures that this property also holds in a complete uniformly convex hyperbolic space.

Lemma 2.1 ([26]) Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset K of X.

Recall that a sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and call x the Δ-limit of $\{x_n\}$.

Lemma 2.2 ([27]) Let $\{a_n\}$, $\{b_n\}$ and $\{\omega_n\}$ be sequences of nonnegative real numbers satisfying

$$a_{n+1} \le (1 + \omega_n)a_n + b_n, \quad \forall n \ge 1.$$

If $\sum_{n=1}^{\infty} \omega_n < +\infty$ and $\sum_{n=1}^{\infty} b_n < +\infty$, then $\lim_{n\to\infty} a_n$ exists. If there exists a subsequence $\{a_{n_i}\} \subset \{a_n\}$ such that $a_{n_i} \to 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3 ([17]) Let (X, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\limsup_{n\to\infty} d(x_n, x) \le c, \limsup_{n\to\infty} d(y_n, x) \le c,$$
$$\lim_{n\to\infty} d(W(x_n, y_n, \alpha_n), x) = c$$

for some $c \ge 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Lemma 2.4 [17] Let K be a nonempty closed convex subset of uniformly convex hyperbolic space and $\{x_n\}$ be a bounded sequence in K such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \zeta$. If $\{y_m\}$ is another sequence in K such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = \zeta$, then $\lim_{m\to\infty} y_m = y$.

3 Main results

In this section, we give our main results.

Theorem 3.1 Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $T_i : K \to K, i = 1, 2$, be a uniformly L_i-Lipschitzian and $(\{\mu_n^i\}, \{\xi_n^i\}, \rho^i)$-total asymptotically nonexpansive mapping with $\{\mu_n^i\}$ and $\{\xi_n^i\}$ satisfying $\lim_{n\to\infty} \mu_n^i = 0$, $\lim_{n\to\infty} \xi_n^i = 0$ and a strictly increasing continuous function $\rho^i : [0, +\infty) \to [0, +\infty)$ with $\rho^i(0) = 0, i = 1, 2$, let $S_i : K \to K, i = 1, 2$, be a uniformly $\hat{L}_i$-Lipschitzian and $(\{\hat{\mu}_n^i\}, \{\hat{\xi}_n^i\}, \hat{\rho}^i)$-total asymptotically nonexpansive mapping with $\{\hat{\mu}_n^i\}$

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and $\{\hat{\xi}_n^i\}$ satisfying $\lim_{n\to\infty}\hat{\mu}_n^i = 0$, $\lim_{n\to\infty}\hat{\xi}_n^i = 0$ and a strictly increasing continuous function $\hat{\rho}^i : [0, +\infty) \to [0, +\infty)$ with $\hat{\rho}^i(0) = 0, i = 1, 2$. Assume that $\mathcal{F} := \bigcap_{i=1}^2 (F(T_i) \cap F(S_i)) \neq \phi$, and for arbitrarily chosen $x_1 \in K$, a new iterative approximating scheme $\{x_n\}$ with errors is defined as follows:

$$\begin{aligned} x_{n+1} &= W(S_1^n x_n, W(T_1^n y_n, u_n, \theta_{n_1}), \alpha_n), \\ y_n &= W(S_2^n x_n, W(T_2^n x_n, v_n, \theta_{n_2}), \beta_n), \end{aligned} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{\zeta_n\}$, $\{\lambda_n\}$ are sequences in $[0, 1]$ and $\{u_n\}$, $\{v_n\}$ are bounded sequences in K and $\theta_{n_1} = 1 - \frac{\gamma_n}{1-\alpha_n} = \frac{\delta_n}{1-\alpha_n}$, $\theta_{n_2} = 1 - \frac{\lambda_n}{1-\beta_n} = \frac{\zeta_n}{1-\beta_n}$. Let $\{\mu_n^i\}$, $\{\xi_n^i\}$, ρ^i, $\{\hat{\mu}_n^i\}$, $\{\hat{\xi}_n^i\}$, $\hat{\rho}^i$, $i = 1, 2$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{\zeta_n\}$ and $\{\lambda_n\}$ satisfy the following conditions:

(i) $\sum_{n=1}^\infty \mu_n^i < +\infty$, $\sum_{n=1}^\infty \hat{\mu}_n^i < +\infty$, $\sum_{n=1}^\infty \xi_n^i < +\infty$, $\sum_{n=1}^\infty \hat{\xi}_n^i < +\infty$, $\sum_{n=1}^\infty \gamma_n < +\infty$, $\sum_{n=1}^\infty \lambda_n < +\infty$, $i = 1, 2$;

(ii) There exist constants $a, b \in (0, 1)$ such that $\{\alpha_n\} \subset [a, b]$, $\{\beta_n\} \subset [a, b]$, $\{\delta_n\} \subset [a, b]$, $\{\zeta_n\} \subset [a, b]$ and $\lim_{n\to\infty} \alpha_n = \alpha \in [a, b]$;

(iii) There exist a constant $M^* > 0$ such that $\rho^i(r) \leq M^* r$ and $\hat{\rho}^i(r) \leq M^* r$, $r > 0, i = 1, 2$;

(iv) $d(x, y) \leq d(S_i x, y)$ for all $x, y \in K$ and $i = 1, 2$.

Then the iterative sequence $\{x_n\}$ defined by (3.1) Δ-converges to a common fixed point of $\mathcal{F} := \bigcap_{i=1}^2 (F(T_i) \cap F(S_i))$.

Proof. Set $L = \max\{L_i, \hat{L}_i, i = 1, 2\}$, $\mu_n = \max\{\mu_n^i, \hat{\mu}_n^i, i = 1, 2\}$, and $\xi_n = \max\{\xi_n^i, \hat{\xi}_n^i, i = 1, 2\}$, $\rho = \max\{\rho^i, \hat{\rho}^i, i = 1, 2\}$. By condition (i), we know that $\sum_{n=1}^\infty \mu_n < +\infty$, $\sum_{n=1}^\infty \xi_n < +\infty$. The proof of Theorem 3.1 is divided into three steps.

Step 1. We first prove that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in \mathcal{F}$.

For any given $p \in \mathcal{F}$, since T_i and $S_i, i = 1, 2$, are total asymptotically nonexpansive mappings, by condition (iii) and (3.1), we have

$$\begin{aligned} & d(x_{n+1}, p) \\ &= d(W(S_1^n x_n, W(T_1^n y_n, u_n, \theta_{n_1}), \alpha_n), p) \\ &\leq \alpha_n d(S_1^n x_n, p) + (1 - \alpha_n) d(W(T_1^n y_n, u_n, \theta_{n_1}), p) \\ &\leq \alpha_n d(S_1^n x_n, p) + (1 - \alpha_n)\big(\theta_{n_1} d(T_1^n y_n, p) + (1 - \theta_{n_1}) d(u_n, p)\big) \\ &\leq \alpha_n d(S_1^n x_n, p) + \delta_n d(T_1^n y_n, p) + \gamma_n d(u_n, p) \\ &\leq \alpha_n [d(x_n, p) + \mu_n \rho\big(d(x_n, p)\big) + \xi_n] \\ &\quad + \delta_n [d(y_n, p) + \mu_n \rho\big(d(y_n, p)\big) + \xi_n] + \gamma_n d(u_n, p) \\ &\leq \alpha_n [(1 + \mu_n M^*) d(x_n, p) + \xi_n] \\ &\quad + \delta_n [(1 + \mu_n M^*) d(y_n, p) + \xi_n] + \gamma_n d(u_n, p) \\ &\leq \alpha_n (1 + \mu_n M^*) d(x_n, p) + \delta_n (1 + \mu_n M^*) d(y_n, p) \\ &\quad + \gamma_n d(u_n, p) + (\alpha_n + \delta_n) \xi_n, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} d(y_n, p) \quad &= d(W(S_2^n x_n, W(T_2^n x_n, v_n, \theta_{n_2}), \beta_n), p) \\ &\leq \beta_n d(S_2^n x_n, p) + (1 - \beta_n) d(W(T_2^n x_n, v_n, \theta_{n_2}), p) \\ &\leq \beta_n d(S_2^n x_n, p) + (1 - \beta_n)[\theta_{n_2} d(T_2^n x_n, p) + (1 - \theta_{n_2}) d(v_n, p)] \\ &\leq \beta_n d(S_2^n x_n, p) + \zeta_n d(T_2^n x_n, p) + \lambda_n d(v_n, p) \\ &\leq \beta_n [d(x_n, p) + \mu_n \rho\big(d(x_n, p)\big) + \xi_n] \\ &\quad + \zeta_n [d(x_n, p) + \mu_n \rho\big(d(x_n, p)\big) + \xi_n] + \lambda_n d(v_n, p) \\ &\leq (\beta_n + \zeta_n)[(1 + \mu_n M^*) d(x_n, p) + \xi_n] + \lambda_n d(v_n, p) \\ &= (\beta_n + \zeta_n)(1 + \mu_n M^*) d(x_n, p) + \lambda_n d(v_n, p) + (\beta_n + \zeta_n) \xi_n \\ &\leq (1 + \mu_n M^*) d(x_n, p) + \lambda_n d(v_n, p) + \xi_n. \end{aligned} \tag{3.3}$$

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Substituting (3.3) into (3.2) and simplifying it, we have

$$\begin{aligned} d(x_{n+1},p) \quad & \le \delta_n(1+\mu_nM^*)[(1+\mu_nM^*)d(x_n,p)+\lambda_n d(v_n,p)+\xi_n] \\ & \quad +\alpha_n(1+\mu_nM^*)d(x_n,p)+\gamma_n d(u_n,p)+(\alpha_n+\delta_n)\xi_n \\ & = (\alpha_n+\delta_n+\alpha_n\mu_nM^*+2\delta_n\mu_nM^*+\delta_n\mu_n^2{M^*}^2)d(x_n,p) \\ & \quad +\gamma_n d(u_n,p)+\lambda_n\delta_n(1+\mu_nM^*)d(v_n,p) \\ & \quad +[\alpha_n+\delta_n+\delta_n(1+\mu_nM^*)]\xi_n \\ & \le [1+\mu_nM^*(\alpha_n+2\delta_n+\delta_n\mu_nM^*)]d(x_n,p) \\ & \quad +\gamma_n d(u_n,p)+\lambda_n\delta_n(1+\mu_nM^*)d(v_n,p) \\ & \quad +[1+\delta_n(1+\mu_nM^*)]\xi_n \\ & = (1+\omega_n)d(x_n,p)+b_n, \end{aligned} \tag{3.4}$$

where $\omega_n=\mu_nM^*(\alpha_n+2\delta_n+\delta_n\mu_nM^*)$, $b_n=\gamma_n d(u_n,p)+\lambda_n\delta_n(1+\mu_nM^*)d(v_n,p)+[1+\delta_n(1+\mu_nM^*)]\xi_n$. Since $\sum_{n=1}^{\infty}\mu_n<+\infty$, $\sum_{n=1}^{\infty}\xi_n<+\infty$ and condition (i),(ii), and $\{u_n\}$, $\{v_n\}$ are bounded sequences in K, it follows from Lemma 2.2 that $\lim_{n\to\infty} d(x_n,p)$ exists for each $p\in\mathcal{F}$.

Step 2. We show that

$$\lim_{n\to\infty} d(x_n,T_ix_n)=0, \qquad \lim_{n\to\infty} d(x_n,S_ix_n)=0, \quad i=1,2. \tag{3.5}$$

For each $p\in\mathcal{F}$, from the proof of Step 1, we know that $\lim_{n\to\infty} d(x_n,p)$ exists. We may assume that $\lim_{n\to\infty} d(x_n,p)=c\ge 0$. If $c=0$, then the conclusion is trivial. Next, we deal with the case $c>0$. From (3.3), we have

$$d(y_n,p)\le(1+\mu_nM^*)d(x_n,p)+\lambda_n d(v_n,p)+\xi_n. \tag{3.6}$$

Taking lim sup on both sides in (3.6), we have

$$\limsup_{n\to\infty} d(y_n,p)\le c. \tag{3.7}$$

In addition, since

$$d(T_1^n y_n,p)\le d(y_n,p)+\mu_n\rho\big(d(y_n,p)\big)+\xi_n\le(1+\mu_nM^*)d(y_n,p)+\xi_n$$

and

$$d(S_1^n x_n,p)\le(1+\mu_nM^*)d(x_n,p)+\xi_n,$$

we have

$$\limsup_{n\to\infty} d(T_1^n y_n,p)\le c \tag{3.8}$$

and

$$\limsup_{n\to\infty} d(S_1^n x_n,p)\le c. \tag{3.9}$$

Also

$$\begin{aligned} & d(W(T_1^n y_n,u_n,\theta_{n_1}),p) \\ & \le \theta_{n_1} d(T_1^n y_n,p)+(1-\theta_{n_1})d(u_n,p) \\ & = \frac{\delta_n}{1-\alpha_n}d(T_1^n y_n,p)+\frac{\gamma_n}{1-\alpha_n}d(u_n,p) \\ & \le d(T_1^n y_n,p)+\frac{\gamma_n}{1-b}d(u_n,p). \end{aligned} \tag{3.10}$$

Since $\sum_{n=1}^{\infty}\gamma_n<+\infty$, $\lim_{n\to\infty}\gamma_n=0$, by boundedness of $\{u_n\}$ in K and (3.8), taking lim sup on both sides in (3.10), we have

$$\limsup_{n\to\infty} d(W(T_1^n y_n,u_n,\theta_{n_1}),p)\le c. \tag{3.11}$$

By $\lim_{n\to\infty} d(x_{n+1}, p) = c$, it is easy to prove that

$$\lim_{n\to\infty} d(W(S_1^n x_n, W(T_1^n y_n, u_n, \theta_{n_1}), \alpha_n), p) = c. \tag{3.12}$$

It follows from (3.9), (3.11), (3.12) and Lemma 2.3 that

$$\lim_{n\to\infty} d(S_1^n x_n, W(T_1^n y_n, u_n, \theta_{n_1})) = 0. \tag{3.13}$$

Since

$$\begin{aligned} d(x_{n+1}, p) \quad &= d(W(S_1^n x_n, W(T_1^n y_n, u_n, \theta_{n_1}), \alpha_n), p) \\ &\le d(S_1^n x_n, p) + d(S_1^n x_n, x_{n+1}) \\ &\le d(S_1^n x_n, p) + (1-\alpha_n) d(S_1^n x_n, W(T_1^n y_n, u_n, \theta_{n_1})), \end{aligned}$$

with the help of (3.13), we have

$$\liminf_{n\to\infty} d(S_1^n x_n, p) \ge c.$$

Combined with (3.9), it yields that

$$\lim_{n\to\infty} d(S_1^n x_n, p) = c. \tag{3.14}$$

Since

$$\begin{aligned} d(x_{n+1}, p) \quad &\le \alpha_n d(S_1^n x_n, p) + \delta_n d(T_1^n y_n, p) + \gamma_n d(u_n, p) \\ &\le \alpha_n d(S_1^n x_n, p) + \delta_n [d(y_n, p) + \mu_n \rho\big(d(y_n, p)\big) + \xi_n] \\ &\quad + \gamma_n d(u_n, p) \\ &\le \alpha_n d(S_1^n x_n, p) + \delta_n (1 + \mu_n M^*) d(y_n, p) + \gamma_n d(u_n, p) + \delta_n \xi_n \\ &\le \alpha_n d(S_1^n x_n, p) + (1-\alpha_n)(1 + \mu_n M^*) d(y_n, p) \\ &\quad + \gamma_n d(u_n, p) + (1-\alpha_n)\xi_n, \end{aligned}$$

we get

$$\begin{aligned} &\frac{d(x_{n+1}, p) - \alpha_n d(S_1^n x_n, p)}{1-\alpha_n} \\ &\le (1 + \mu_n M^*) d(y_n, p) + \frac{\gamma_n}{1-\alpha_n} d(u_n, p) + \xi_n. \end{aligned}$$

By condition (ii), (3.12) and (3.14), we have

$$\liminf_{n\to\infty} d(y_n, p) \ge c.$$

Combined with (3.7), it yields that

$$\lim_{n\to\infty} d(y_n, p) = c. \tag{3.15}$$

By the same method and (3.15), we can also prove that

$$\lim_{n\to\infty} d(S_2^n x_n, W(T_2^n x_n, v_n, \theta_{n_2})) = 0. \tag{3.16}$$

It follows from virtue of condition (iv), (3.13), and (3.16) that

$$\lim_{n\to\infty} d(x_n, W(T_1^n y_n, u_n, \theta_{n_1})) \le \lim_{n\to\infty} d(S_1^n x_n, W(T_1^n y_n, u_n, \theta_{n_1})) = 0, \tag{3.17}$$

and

$$\lim_{n\to\infty} d(x_n, W(T_2^n x_n, v_n, \theta_{n_2})) \le \lim_{n\to\infty} d(S_2^n x_n, W(T_2^n x_n, v_n, \theta_{n_2})) = 0. \tag{3.18}$$

From (3.1) and (3.16), we have

$$\begin{aligned} d(y_n, S_2^n x_n) &= d(W(S_2^n x_n, W(T_2^n x_n, v_n, \theta_{n_2}), \beta_n), S_2^n x_n) \\ &\leq (1-\beta_n) d(S_2^n x_n, W(T_2^n x_n, v_n, \theta_{n_2})) \to 0 \\ &\quad (\text{as} \quad n \to \infty) \end{aligned} \tag{3.19}$$

and

$$d(x_n, y_n) = d(x_n, W(T_2^n x_n, v_n, \theta_{n_2})) + d(S_2^n x_n, W(T_2^n x_n, v_n, \theta_{n_2})) + d(S_2^n x_n, y_n).$$

It follows from (3.16), (3.18) and (3.19) that

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \tag{3.20}$$

This together with (3.17) implies that

$$\begin{aligned} & d(x_n, W(T_1^n x_n, u_n, \theta_{n_1})) \\ & \leq d(x_n, W(T_1^n y_n, u_n, \theta_{n_1})) + d(W(T_1^n y_n, u_n, \theta_{n_1}), W(T_1^n x_n, u_n, \theta_{n_1})) \\ & \leq d(x_n, W(T_1^n y_n, u_n, \theta_{n_1})) + \theta_{n_1} d(T_1^n y_n, T_1^n x_n) \\ & \leq d(x_n, W(T_1^n y_n, u_n, \theta_{n_1})) + \frac{\delta_n}{1-\alpha_n} L d(y_n, x_n) \to 0 \quad (n \to \infty). \end{aligned} \tag{3.21}$$

On the other hand, from (3.13) and (3.20), we have

$$\begin{aligned} & d(S_1^n x_n, W(T_1^n x_n, u_n, \theta_{n_1})) \\ & \leq d(S_1^n x_n, W(T_1^n y_n, u_n, \theta_{n_1})) + d(W(T_1^n y_n, u_n, \theta_{n_1}), W(T_1^n x_n, u_n, \theta_{n_1})) \\ & \leq d(S_1^n x_n, W(T_1^n y_n, u_n, \theta_{n_1})) + \theta_{n_1} d(T_1^n y_n, T_1^n x_n) \\ & \leq d(S_1^n x_n, W(T_1^n y_n, u_n, \theta_{n_1})) + \frac{\delta_n}{1-\alpha_n} L d(y_n, x_n) \to 0 \quad (n \to \infty). \end{aligned} \tag{3.22}$$

From (3.21) and (3.22), we have that

$$\begin{aligned} d(S_1^n x_n, x_n) &\leq d(S_1^n x_n, W(T_1^n x_n, u_n, \theta_{n_1})) \\ &\quad + d(W(T_1^n x_n, u_n, \theta_{n_1}), x_n) \to 0 \quad (n \to \infty). \end{aligned} \tag{3.23}$$

In addition, since

$$\begin{aligned} d(x_{n+1}, x_n) &= d(W(S_1^n x_n, W(T_1^n y_n, u_n, \theta_{n_1}), \alpha_n), x_n) \\ &\leq \alpha_n d(S_1^n x_n, x_n) + (1-\alpha_n) d(W(T_1^n y_n, u_n, \theta_{n_1}), x_n), \end{aligned}$$

it follows from (3.17) and (3.23) that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.24}$$

Observe that

$$\begin{aligned} d(x_n, T_1^n x_n) &\leq d(x_n, W(T_1^n x_n, u_n, \theta_{n_1})) + d(W(T_1^n x_n, u_n, \theta_{n_1}), T_1^n x_n) \\ &\leq d(x_n, W(T_1^n x_n, u_n, \theta_{n_1})) + (1-\theta_{n_1}) d(T_1^n x_n, u_n) \\ &\leq d(x_n, W(T_1^n x_n, u_n, \theta_{n_1})) \\ &\quad + \frac{\gamma_n}{1-\alpha_n} [d(T_1^n x_n, x_n) + d(x_n, p) + d(u_n, p)], \end{aligned}$$

then

$$\begin{aligned} d(x_n, T_1^n x_n) &\leq \frac{1-\alpha_n}{1-\alpha_n-\gamma_n} d(x_n, W(T_1^n x_n, u_n, \theta_{n_1})) \\ &\quad + \frac{\gamma_n}{1-\alpha_n-\gamma_n} [d(x_n, p) + d(u_n, p)]. \end{aligned} \tag{3.25}$$

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By boundedness of $\{u_n\}$ in K and condition (i), (ii) and $\lim_{n\to\infty} d(x_n, p)$ exists and (3.21), (3.25), we have

$$\lim_{n\to\infty} d(x_n, T_1^n x_n) = 0. \tag{3.26}$$

Similarly, we can also prove that

$$\lim_{n\to\infty} d(x_n, T_2^n x_n) = 0. \tag{3.27}$$

For all $i = 1, 2$, now we know

$$\begin{aligned} d(x_n, T_i x_n) \quad &\le d(x_n, x_{n+1}) + d(x_{n+1}, T_i^{n+1} x_{n+1}) \\ &\quad + d(T_i^{n+1} x_{n+1}, T_i^{n+1} x_n) + d(T_i^{n+1} x_n, T_i x_n) \\ &\le (1+L) d(x_n, x_{n+1}) + d(x_{n+1}, T_i^{n+1} x_{n+1}) + L d(T_i^n x_n, x_n). \end{aligned}$$

It follows from (3.24), (3.26) and (3.27) that

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0, \quad i = 1, 2.$$

By virtue of condition (iv), i.e.,

$$d(S_1 x_n, , W(T_1^n x_n, u_n, \theta_{n_1})) \le d(S_1^n x_n, , W(T_1^n x_n, u_n, \theta_{n_1})),$$

we have

$$\begin{aligned} d(x_n, S_1 x_n) \quad &\le d(x_n, , W(T_1^n x_n, u_n, \theta_{n_1})) + d(S_1 x_n, , W(T_1^n x_n, u_n, \theta_{n_1})) \\ &\le d(x_n, , W(T_1^n x_n, u_n, \theta_{n_1})) + d(S_1^n x_n, , W(T_1^n x_n, u_n, \theta_{n_1})), \end{aligned}$$

from (3.21) and (3.22), which implies that

$$\lim_{n\to\infty} d(x_n, S_1 x_n) = 0.$$

By the same method, we can also prove that

$$\lim_{n\to\infty} d(x_n, S_2 x_n) = 0.$$

Step 3. We shall prove that the sequence $\{x_n\}$ Δ-converges to a common fixed point of $\mathcal{F} := \bigcap_{i=1}^2 (F(T_i) \cap F(S_i))$.

In fact, for each $p \in F$, $\lim_{n\to\infty} d(x_n, p)$ exist. This implies that the sequence $\{d(x_n, p)\}$ is bounded, so is the sequence $\{x_n\}$. Hence, by virtue of Lemma 2.1, $\{x_n\}$ has a unique asymptotic center $A_K(\{x_n\}) = \{x\}$.

Let $\{u_n\}$ be any subsequence of $\{x_n\}$ with $A_K(\{u_n\}) = \{u\}$. It follows from (3.5) that

$$\lim_{n\to\infty} d(u_n, T_i u_n) = 0. \tag{3.28}$$

Next, we show that $u \in F(T_i)$, for all $i = 1, 2$. For this, we define a sequence $\{z_n^i\}$ in K by $z_m^i = T_i^m u$, for all $i = 1, 2$. So we calculate

$$\begin{aligned} & d(z_m^i, u_n) \\ & \le d(T_i^m u, T_i^m u_n) + d(T_i^m u_n, T_i^{m-1} u_n) + \cdots + d(T_i u_n, u_n) \\ & \le d(u, u_n) + \mu_m \rho\big(d(u, u_n)\big) + \xi_m + \sum_{k=1}^m d(T_i^k u_n, T_i^{k-1} u_n) \\ & \le (1 + \mu_m M^*) d(u, u_n) + \xi_m + \sum_{k=1}^m d(T_i^k u_n, T_i^{k-1} u_n). \end{aligned} \tag{3.29}$$

Since T_i is uniformly L-Lipschitzian, it follows from (3.29) that

$$d(z_m^i, u_n) \leq (1 + \mu_m M^*)d(u, u_n) + \xi_m + mLd(T_i u_n, u_n).$$

Taking lim sup on both sides of the above estimate and using (3.28), we have

$$\begin{aligned} r(z_m^i, \{u_n\}) &= \limsup_{n\to\infty} d(z_m^i, u_n) \\ &\leq (1 + \mu_m M^*) \limsup_{n\to\infty} d(u, u_n) + \xi_m \\ &= (1 + \mu_m M^*) r(u, \{u_n\}) + \xi_m, \end{aligned}$$

and so

$$\limsup_{m\to\infty} r(z_m^i, \{u_n\}) \leq r(u, \{u_n\}).$$

Based on $A_K(\{u_n\}) = \{u\}$ and the definition of asymptotic center $A_K(\{u_n\})$ of a bounded sequence $\{u_n\}$ with respect to $K \subset X$, we have

$$r(u, \{u_n\}) \leq r(y, \{u_n\}), \quad \forall y \in K.$$

This implies that

$$\liminf_{m\to\infty} r(z_m^i, \{u_n\}) \geq r(u, \{u_n\}).$$

Hence, we have

$$\lim_{m\to\infty} r(z_m^i, \{u_n\}) = r(u, \{u_n\}).$$

It follows from Lemma 2.4 that $\lim_{m\to\infty} z_m^i = u$, namely, $\lim_{m\to\infty} T_i^m u = u$. As T_i is uniformly continuous, so that $T_i u = T_i(\lim_{m\to\infty} T_i^m u) = \lim_{m\to\infty} T_i^{m+1} u = u$. That is, $u \in F(T_i)$. Similarly, we also can show that $u \in F(S_i)$, for all $i = 1, 2$. Hence, u is the common fixed point of T_i and S_i, for all $i = 1, 2$. And we want to show $x = u$, suppose $x \neq u$, by the uniqueness of asymptotic centers,

$$\begin{aligned} \limsup_{n\to\infty} d(u_n, u) &< \limsup_{n\to\infty} d(u_n, x) \\ &\leq \limsup_{n\to\infty} d(x_n, x) \\ &< \limsup_{n\to\infty} d(x_n, u) \\ &= \limsup_{n\to\infty} d(u_n, u), \end{aligned}$$

a contradiction. Thus we have $x = u$. Since $\{u_n\}$ is an arbitrary subsequence of $\{x_n\}$, $A(\{u_n\}) = \{x\}$ for all subsequence $\{u_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$ Δ-converges to a common fixed point of $\mathcal{F} := \bigcap_{i=1}^2 (F(T_i) \cap F(S_i))$. This completes the proof. □

The following theorem can be obtained from Theorem 3.1 immediately.

Theorem 3.2 Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $T_i : K \to K, i = 1, 2$, be a uniformly L_i-Lipschitzian and $(\{\mu_n^i\}, \{\xi_n^i\}, \rho^i)$-total asymptotically nonexpansive mapping with $\{\mu_n^i\}$ and $\{\xi_n^i\}$ satisfying $\lim_{n\to\infty} \mu_n^i = 0$, $\lim_{n\to\infty} \xi_n^i = 0$ and a strictly increasing continuous function $\rho^i : [0, +\infty) \to [0, +\infty)$ with $\rho^i(0) = 0, i = 1, 2$, let $S_i : K \to K, i = 1, 2$, be a uniformly $\hat{L}_i$-Lipschitzian and asymptotically nonexpansive mapping with $\{k_n^i\} \subset [0, +\infty)$ satisfying $\lim_{n\to\infty} k_n^i = 0$. Assume that $\mathcal{F} := \bigcap_{i=1}^2 (F(T_i) \cap F(S_i)) \neq \phi$, and for arbitrarily chosen $x_1 \in K$, $\{x_n\}$ is defined as follows:

$$\begin{aligned} x_{n+1} &= W(S_1^n x_n, W(T_1^n y_n, u_n, \theta_{n_1}), \alpha_n), \\ y_n &= W(S_2^n x_n, W(T_2^n x_n, v_n, \theta_{n_2}), \beta_n), \end{aligned} \tag{3.30}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{\zeta_n\}$, $\{\lambda_n\}$ are sequences in $[0,1]$ and $\{u_n\}$, $\{v_n\}$ are bounded sequences in K and $\theta_{n_1} = 1 - \frac{\gamma_n}{1-\alpha_n} = \frac{\delta_n}{1-\alpha_n}$, $\theta_{n_2} = 1 - \frac{\lambda_n}{1-\beta_n} = \frac{\zeta_n}{1-\beta_n}$. Let $\{\mu_n^i\}$, $\{\xi_n^i\}$, ρ^i, $\{k_n^i\}$, $i = 1,2$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{\zeta_n\}$ and $\{\lambda_n\}$ satisfy the following conditions:

(i) $\sum_{n=1}^{\infty} \mu_n^i < +\infty$, $\sum_{n=1}^{\infty} \xi_n^i < +\infty$, $\sum_{n=1}^{\infty} k_n^i < +\infty$, $\sum_{n=1}^{\infty} \gamma_n < +\infty$, $\sum_{n=1}^{\infty} \lambda_n < +\infty$, $i = 1,2$;

(ii) There exist constants $a, b \in (0,1)$ such that $\{\alpha_n\} \subset [a,b]$, $\{\beta_n\} \subset [a,b]$, $\{\delta_n\} \subset [a,b]$, $\{\zeta_n\} \subset [a,b]$ and $\lim_{n\to\infty} \alpha_n = \alpha \in [a,b]$;

(iii) There exist a constant $M^* > 0$ such that $\rho^i(r) \le M^* r$, $r > 0, i = 1,2$;

(iv) $d(x,y) \le d(S_i x, y)$ for all $x, y \in K$ and $i = 1,2$.

Then the sequence $\{x_n\}$ defined by (3.30) Δ-converges to a common fixed point of $\mathcal{F} := \bigcap_{i=1}^{2}(F(T_i) \cap F(S_i))$.

Proof. Take $\hat{\rho}^i(t) = t$, $t \ge 0$, $\hat{\xi}_n^i = 0$, $\hat{\mu}_n^i = k_n^i$, $i = 1,2$, in Theorem 3.1. Since all the conditions in Theorem 3.1 are satisfied, it follows from Theorem 3.1 that the sequence $\{x_n\}$ Δ-converges to a common fixed point of $\mathcal{F} := \bigcap_{i=1}^{2}(F(T_i) \cap F(S_i))$. This completes the proof of Theorem 3.2. □

Theorem 3.3 Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $T_i : K \to K, i = 1,2$, be a uniformly L_i-Lipschitzian and $(\{\mu_n^i\}, \{\xi_n^i\}, \rho^i)$-total asymptotically nonexpansive mapping with $\{\mu_n^i\}$ and $\{\xi_n^i\}$ satisfying $\lim_{n\to\infty} \mu_n^i = 0$, $\lim_{n\to\infty} \xi_n^i = 0$ and a strictly increasing continuous function $\rho^i : [0,+\infty) \to [0,+\infty)$ with $\rho^i(0) = 0, i = 1,2$, let $S_i : K \to K, i = 1,2$, be a uniformly $\hat{L}_i$-Lipschitzian and asymptotically nonexpansive mapping with $\{k_n^i\} \subset [0,+\infty)$ satisfying $\lim_{n\to\infty} k_n^i = 0$. Assume that $\mathcal{F} := \bigcap_{i=1}^{2}(F(T_i) \cap F(S_i)) \ne \phi$, and for arbitrarily chosen $x_1 \in K$, $\{x_n\}$ is defined as follows:

$$
\begin{aligned}
x_{n+1} &= W(S_1^n x_n, T_1^n y_n, \alpha_n), \\
y_n &= W(S_2^n x_n, T_2^n x_n, \beta_n),
\end{aligned} \tag{3.31}
$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. Let $\{\mu_n^i\}$, $\{\xi_n^i\}$, ρ^i, $\{k_n^i\}$, $i = 1,2$, $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the following conditions:

(i) $\sum_{n=1}^{\infty} \mu_n^i < +\infty$, $\sum_{n=1}^{\infty} \xi_n^i < +\infty$, $\sum_{n=1}^{\infty} k_n^i < +\infty$;

(ii) There exist constants $a, b \in (0,1)$ such that $\{\alpha_n\} \subset [a,b]$, $\{\beta_n\} \subset [a,b]$, $\lim_{n\to\infty} \alpha_n = \alpha \in [a,b]$;

(iii) There exist a constant $M^* > 0$ such that $\rho^i(r) \le M^* r$, $r > 0, i = 1,2$;

(iv) $d(x,y) \le d(S_i x, y)$ for all $x, y \in K$ and $i = 1,2$.

Then the sequence $\{x_n\}$ defined by (3.31) Δ-converges to a common fixed point of $\mathcal{F} := \bigcap_{i=1}^{2}(F(T_i) \cap F(S_i))$.

Proof. Take $\hat{\rho}^i(t) = t$, $t \ge 0$, $\hat{\xi}_n^i = 0$, $\hat{\mu}_n^i = k_n^i$, $i = 1,2$ and $\gamma_n \equiv \lambda_n \equiv 0$ in Theorem 3.1. Since all the conditions in Theorem 3.1 are satisfied, it follows from Theorem 3.1 that the sequence $\{x_n\}$ Δ-converges to a common fixed point of $\mathcal{F} := \bigcap_{i=1}^{2}(F(T_i) \cap F(S_i))$. This completes the proof. □

Theorem 3.4 Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $T_i : K \to K, i = 1,2$, be a uniformly L_i-Lipschitzian and $(\{\mu_n^i\}, \{\xi_n^i\}, \rho^i)$-total asymptotically nonexpansive mapping with $\{\mu_n^i\}$ and $\{\xi_n^i\}$ satisfying $\lim_{n\to\infty} \mu_n^i = 0$, $\lim_{n\to\infty} \xi_n^i = 0$ and a strictly increasing continuous function $\rho^i : [0,+\infty) \to [0,+\infty)$ with $\rho^i(0) = 0, i = 1,2$, Suppose that $\mathcal{F} := \bigcap_{i=1}^{2} F(T_i) \ne \phi$, and for arbitrarily chosen $x_1 \in K$, $\{x_n\}$ is defined as follows:

$$
\begin{aligned}
x_{n+1} &= W(x_n, T_1^n y_n, \alpha_n), \\
y_n &= W(x_n, T_2^n x_n, \beta_n),
\end{aligned} \tag{3.32}
$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. Let $\{\mu_n^i\}$, $\{\xi_n^i\}$, ρ^i, $i = 1,2$, $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the following conditions:

(i) $\sum_{n=1}^{\infty} \mu_n^i < +\infty$, $\sum_{n=1}^{\infty} \xi_n^i < +\infty$;

(ii) There exist constants $a, b \in (0,1)$ such that $\{\alpha_n\} \subset [a,b]$, $\{\beta_n\} \subset [a,b]$;

(iii) There exist a constant $M^* > 0$ such that $\rho^i(r) \le M^* r$, $r > 0, i = 1,2$.

Then the sequence $\{x_n\}$ defined by (3.32) Δ-converges to a common fixed point of $\mathcal{F} := \bigcap_{i=1}^{2} F(T_i)$.

Proof. Take $\gamma_n \equiv \lambda_n \equiv 0$ and $S_i = I, i = 1, 2$ in Theorem 3.1. Since all the conditions in Theorem 3.1 are satisfied, it follows from Theorem 3.1 that the sequence $\{x_n\}$ Δ-converges to a common fixed point of $\mathcal{F} := \bigcap_{i=1}^{2} F(T_i)$. □

Remark 3.1 The results of Theorems 3.3 and 3.4 improve the corresponding results in Theorem 2.1 of [9] and Theorem 7 of [10], respectively.

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SOME OSTROWSKI TYPE INTEGRAL INEQUALITIES FOR DOUBLE INTEGRAL ON TIME SCALES

WAJEEHA IRSHAD, MUHAMMAD IQBAL BHATTI, AND MUHAMMAD MUDDASSAR*

ABSTRACT. Weighted montgomery identity on time scales for functions of two variables is established. Corresponding discrete and continuous versions of montgomery identities for functions of two variables are obtained. By using the obtained weighted montgomery identity on time scales, an Ostrowski type inequality for double integrals on time scales is pointed out as well.

1. INTRODUCTION AND PRELIMINARY RESULTS

The Ostrowski type inequality, which was originally presented by Ostrowski in [14], can be used to estimate the absolute deviation of a function from its integral mean.
In [6], Bohner and Matthews derived the Montgomery identity on time scales and established the following Ostrowski inequality on time scales, which unifies and extends corresponding discrete [7], continuous [13] and other cases.

Theorem 1. *Let $a, b, s, t \in \mathbb{T}$ with $a < b$ $f : [a, b] \to \mathbb{R}$ be a differentiable function with the property that, $M = \sup_{a<t<b} |f\Delta(t)| < \infty$, induces*

$$\left| f(t) - \frac{1}{(b-a)} \int_a^b f(\sigma(s))\Delta s \right| \leq \frac{M}{b-a} + (h_2(t,a) + h_2(t,b)) \tag{1.1}$$

is the best possible in the sense that rightside cannot be replaced by a smaller quantity. Where $h(.,.)$ is defined by definition (6) below.

2. TIME SCALE ESSENTIALS

During the past decades, with the development of the theory of differential and integral equations as well as difference equations, a lot of integral and difference inequalities have

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been discovered e.g., and the references therein, which play an important role in the research of boundedness, global existence, stability of solutions of differential and integral equations as well as difference equations. On the other hand, Hilger initiated the theory of time scales as a theory capable to contain both difference and differential calculus in a consistent way. Since then many authors have expounded on various aspects of the theory of dynamic equations on time scales including various inequalities on time scales. The existence of a derivative at a point in a time scale depends on the type of the point itself, because time scale may not be connected. Points are classified according to two major operators, the forward jump operators and the backward jump operators.

Definition 1. *For an arbitrary time scale T, the forward jump operator, $\sigma : \mathbb{T} \to \mathbb{T}$, is defined to be*

$$\sigma(t) := inf\{s \in \mathbb{T} : s > t\}$$

If the set of all points in T that are larger than t is empty, then $inf\phi = sup\mathbb{T}$.
If $\mathbb{T}$ has a maximum t, then $\sigma(t) = t$.
The backward jump operator, $\rho : \mathbb{T} \to \mathbb{T}$, is

$$\rho(t) := sup\{s \in \mathbb{T} : s < t\}$$

If the set of all points in $\mathbb{T}$ that are less than t is empty, then $sup\phi = inf\mathbb{T}$.
If T has a minimum t, then $\rho(t) = t$.

Definition 2. *Using these two operators for $t \in \mathbb{T}$, the point can be classified in the following manner, t is right-scattered if $\sigma(t) > t$, t is right-dense if $\sigma(t) = t$, t is left-scattered if $\rho(t) < t$, t is left-dense if $\rho(t) = t$, t is isolated if it is left- and right-scattered: $\rho(t) < t < \sigma(t)$and t is dense if it is both left- and right-dense $\rho(t) = t = \sigma(t)$ for $t \in \mathbb{T}$.*

Definition 3. *The graininess function, $\mu : T \to [0, \infty)$, is defined to be*

$$\mu(t) = \sigma(t) - t$$

The graininess function essentially describes the step size between two consecutive points in $\mathbb{T}$. Oftentimes the differences in results obtained from discrete and continuous calculus stem from the different value of the graininess function evaluated at a given point t.

Definition 4. *The derivative in time scale calculus, called the delta derivative, determines the rates of forward change over a time scale. For a function $f : \mathbb{T} \to R$ and $t \in \mathbb{T}^k$, the*

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delta derivative of f at t, $f^{\Delta}(t)$, is defined to be the number, when it exists, where for any given $\epsilon > 0$, there exists a neighborhood U of t such that

$$\left|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]\right| \leq \epsilon|\sigma(t) - s| \tag{2.2}$$

is true for all $s \in U$. Here, $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$ when $\mathbb{T}$ has a left-scattered maximum m; otherwise, $\mathbb{T}^k = \mathbb{T}$[1]. Since the delta derivative definition involves the forward jump operator, if the time scale has a left scattered maximum m, then one cannot jump past this point. Therefore, this point is removed from the set of points used to determine the delta derivative. However, if the time scale does not contain such a left-scattered maximum, then $\mathbb{T}^k$ is equivalent to the time scale.
Take $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$, $\mu(t) = 0, f^{\Delta} = f'$ is the derivative used in standard calculus. If $\mathbb{T} = \mathbb{Z}$, $\sigma(t) = t + 1$, $\mu(t) = 1$, $f^{\Delta} = \Delta f$ is the forward difference operator used in difference equations.

Theorem 2 (Properties of the Delta Derivative). *$f : \mathbb{T} \to R$ be a function and $t \in \mathbb{T}^k$ as defined above. For such a function, the following properties hold:*

(1) *If f is delta differentiable at t, then f is continuous at t.*

(2) *If t is right-scattered and f is continuous at t, then the delta derivative of f, f^{Δ}, is defined as follows*

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$$

(3) *If t is right-dense, then the delta derivative at t is as follows (if and only if the limit exists as a finite number)*

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

(4) *If f is delta differentiable at t, then*

$$f(\sigma(t)) = f(t) + \mu(t)\, f^{\Delta}(t)$$

(5) *If $\mathbb{T} = \mathbb{R}$, then the delta derivative is $f'(t)$ from continuous calculus.*

(6) *If $\mathbb{T} = \mathbb{Z}$, then the delta derivative is the forward difference, Δf, from discrete calculus.*

Definition 5. *If $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^k$, then $F(t)$ is said to be anti-derivative of $f(t)$ and $f(t)$ is said to be delta integrable provided that $f(t)$ is rd-continuous. The cauchy integral of $f(t)$ is defined by $\int_r^s f(t)\Delta(t) = F(s) - F(r)$.*

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Theorem 3. *Let f, g be rd-continuous, $a, b, c \in \mathbb{T}$ and $\alpha, \beta \in \mathbb{R}$, then*

(1) $\int_a^b (\alpha f(t) + \beta g(t))\triangle t = \alpha \int_a^b f(t)\triangle t + \beta \int_a^b g(t)\triangle t$
(2) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$
(3) $\int_a^c f(t)\Delta t = \int_a^b f(t)\Delta t + \int_b^c f(t)\Delta t$
(4) $\int_a^b f(t)\, g^{\Delta}(t)\, \Delta t = f(b)\, g(b) - f(a)\, g(a) - \int_a^b f^{\Delta}(t)\, g(\sigma(t))\Delta t$

Theorem 4. *If f is Δ-integrable on $[a, b]$, then so is $|f|$, and*

$$\left|\int_a^b f(t)\Delta t\right| \leq \int_a^b |f(t)|\Delta t$$

Let $\mathbb{T}_1$, $\mathbb{T}_2$ be two time scales. Let σ_i, ρ_i and Δ_i be the forward jump operator, the backward jump operator and the delta differentiation, respectively on $\mathbb{T}_i$, for $i = 1, 2$. Let $a, b \in \mathbb{T}_1$, $c, d \in \mathbb{T}_2$, with $a < b$, $c < d$. $[a, b)$ and $[c, d)$ are the half-closed bounded intervals in $\mathbb{T}_1$ and $\mathbb{T}_2$ respectively, and a "rectangle" in $\mathbb{T}_1 \times \mathbb{T}_2$ by

$$\mathbb{R} = [a, b) \times [c, d) = \{(t_1, t_2) : t_1 \in [a, b), t_2 \in [c, d)\}$$

Let f be a real valued function on $\mathbb{T}_1 \times \mathbb{T}_2$. This function f is said to be rd-continuous in t_2 if $a_1 \in \mathbb{T}_1$, then function f is real valued function on $\mathbb{T}_1 \times \mathbb{T}_2$, this function f is said to be rd-continuous in t_2 if $a_1 \in \mathbb{T}_1$, then $f(a_1, t_2)$ is rd-continuous on $\mathbb{T}_2$. CC_{rd} denotes the set of functions $f(a_1, t_2)$ on $\mathbb{T}_1 \times \mathbb{T}_2$, having the properties:

(1) f is rd-continuous in t_1 and t_2.
(2) If $(x_1, x_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ with x_1 right dense and x_2 right dense, then f is continuous at (x_1, x_2).

Definition 6. *Let $g_k, h_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$ be defined by $g_0(t, s) = h_0(t, s) = 1$ for all $s, t \in \mathbb{T}$ and then recursively by*

$$g_{k+1}(t, s) = \int_s^t g_k\,(\sigma(\tau), s)\, \Delta\tau \forall s, t \in \mathbb{T} \tag{2.3}$$

$$h_{k+1}(t, s) = \int_s^t h_k(\sigma(\tau), s)\Delta\tau \forall s, t \in \mathbb{T} \tag{2.4}$$

3. Main Results

Lemma 1 (Weighted Montgomery Identity on Time Scales)**.** *Let $g : [a, b] \to [0, \infty)$, $G : [c, d] \to [0, \infty)$ be rd-continuous and positive and $h : [a, b] \to \mathbb{R}$, $H : [c, d] \to \mathbb{R}$ be invertible and differentiable, such that $g(t_1) = h^{\Delta_1}(t_1)$ on $[a, b]$ and $G(t_2) = H^{\Delta_2}(t_2)$. Let $a, b, s_1, t_1 \in \mathbb{T}_1$, $c, d, s_2, t_2 \in \mathbb{T}_2$ with $a < b, c < d$,*

$A = h^{-1}\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right)$, $B = h^{-1}\left(\left(1-\frac{\alpha}{2}\right)h(b)+\frac{\alpha}{2}h(a)\right)$,
$C = H^{-1}\left(\left(1-\frac{\beta}{2}\right)G(c)+\frac{\beta}{2}H(d)\right)$, $D = H^{-1}\left(\left(1-\frac{\beta}{2}\right)G(d)+\frac{\beta}{2}H(c)\right)$
and

$f:[a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2}\to\mathbb{R}$ *is* $\Delta_1\Delta_2$ *differentiable. Then for all* $s_1\in[A,B]$,$s_2\in[C,D]$, $0\le\alpha,\beta\le 1$, *we have*

$$
\begin{aligned}
&\int_a^b\!\!\int_c^d W(t_1,t_2,s_1,s_2)\frac{\partial^2 f(s_1,s_2)}{\Delta_1 s_1\Delta_2 s_2}\Delta_1 s_1\Delta_2 s_2\\
&=\!\left(\frac{\beta}{2}H(d)-\frac{\beta}{2}H(c)\right)\left\{h(b)-\left(\left(1-\frac{\alpha}{2}\right)h(b)+\frac{\alpha}{2}h(a)\right)\right\}(f(b,c)-f(a,d)+f(b,d))\\
&+\left(\frac{\alpha}{2}h(a)+\frac{\alpha}{2}h(b)\right)\{(1-\beta)H(c)+(\beta-1)H(d)\}\left(f(a,t_2)+f(b,t_2)\right)\\
&+\left(\frac{\beta}{2}H(c)+\frac{\beta}{2}H(d)\right)\{(1-\alpha)h(b)+(\alpha-1)h(a)\}\left(f(t_1,c)-f(t_1,d)\right)\\
&+\left((1-\alpha)h(b)+(\alpha-1)h(a)\right)\left((1-\beta)H(d)+(\beta-1)H(c)\right)f(t_1,t_2)\\
&+\left[H(c)-\left(\left(1-\frac{\beta}{2}\right)H(c)+\frac{\beta}{2}H(d)\right)\right]\int_a^b h'(s_1)f(\sigma(s_1),c)\Delta_1 s_1\\
&-\left[H(t_2)-\left(\left(1-\frac{\beta}{2}\right)H(c)+\frac{\beta}{2}H(d)\right)\right]\int_a^b h'(s_1)f(\sigma(s_1),t_2)\Delta_1 s_1\\
&-\int_c^d H'(s_2)\left((1-\alpha)h(a)+(\alpha-1)h(b)\right)f(t_1,\sigma(s_2))\Delta_2 s_2\\
&-\int_c^d H'(s_2)\left\{\left(h(a)-\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right)\right)\right\}f(a,\sigma(s_2))\Delta_2 s_2\\
&-\int_c^d H'(s_2)\left\{\left(h(b)-\left(\left(1-\frac{\alpha}{2}\right)h(b)+\frac{\alpha}{2}h(a)\right)\right)\right\}f(b,\sigma(s_2))\Delta_2 s_2\\
&+\int_c^d H'(s_2)\left\{\int_a^b h'(s_1)f(\sigma(s_1),\sigma(s_2))\Delta_1 s_1\right\}\Delta_2(t_2)
\end{aligned}
$$

where

$$
W_1(t_1,t_2,s_1,s_2)=\begin{cases}
h(s_1)-\left((1-\frac{\alpha}{2})h(a)+\frac{\alpha}{2}h(b)\right), & s_1\in[a,t_1)\\
h(s_1)-\left(\frac{\alpha}{2}h(a)+(1-\frac{\alpha}{2})h(b)\right), & s_1\in[t_1,b]\\
H(s_2)-\left((1-\frac{\beta}{2})H(c)+\frac{\beta}{2}H(d)\right), & s_2\in[c,t_2)\\
H(s_2)-\left(\frac{\beta}{2}H(c)+(1-\frac{\beta}{2})H(d)\right), & s_2\in[t_2,d]
\end{cases}
$$

Proof. Let's start with

$$\int_a^b\int_c^d W(t_1,t_2,s_1,s_2)\frac{\partial^2 f(s_1,s_2)}{\Delta_1 s_1\Delta_2 s_2}\Delta_1 s_1\Delta_2 s_2$$

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$$=\int_a^{t_1}\int_c^{t_2}\left\{h(s_1)-\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right)\right\}\left\{H(s_2)-\left(\left(1-\frac{\beta}{2}\right)H(c)+\frac{\beta}{2}H(d)\right)\right\}$$
$$\frac{\partial^2 f(s_1,s_2)}{\Delta_1 s_1\Delta_2 s_2}\Delta_1 s_1\Delta_2 s_2+\int_a^{t_1}\int_{t_2}^{d}\left\{h(s_1)-\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right)\right\}$$
$$\left\{H(s_2)-\left(\left(1-\frac{\beta}{2}\right)H(d)+\frac{\beta}{2}H(c)\right)\right\}\frac{\partial^2 f(s_1,s_2)}{\Delta_1 s_1\Delta_2 s_2}\Delta_1 s_1\Delta_2 s_2$$
$$+\int_{t_1}^{b}\int_c^{t_2}\left\{h(s_1)-\left(\left(1-\frac{\alpha}{2}\right)h(b)+\frac{\alpha}{2}h(a)\right)\right\}\left\{H(s_2)-\left(\left(1-\frac{\beta}{2}\right)H(c)+\frac{\beta}{2}H(d)\right)\right\}$$
$$\frac{\partial^2 f(s_1,s_2)}{\Delta_1 s_1\Delta_2 s_2}\Delta_1 s_1\Delta_2 s_2+\int_{t_1}^{b}\int_{t_2}^{d}\left\{h(s_1)-\left(\left(1-\frac{\alpha}{2}\right)h(b)+\frac{\alpha}{2}h(a)\right)\right\}\{H(s_2)-$$
$$\left(\left(1-\frac{\beta}{2}\right)H(d)+\frac{\beta}{2}H(c)\right)\Big\}\frac{\partial^2 f(s_1,s_2)}{\Delta_1 s_1\Delta_2 s_2}\Delta_1 s_1\Delta_2 s_2.$$

Now

$$=\int_a^{t_1}\left\{h(s_1)-\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right)\right\}\left[\left\{H(t_2)-\left(\left(1-\frac{\beta}{2}\right)H(c)+\frac{\beta}{2}H(d)\right)\right\}\right.$$
$$\left.\frac{\partial f(s_1,t_2)}{\Delta_1 s_1}-\left\{H(c)-\left(\left(1-\frac{\beta}{2}\right)H(c)+\frac{\beta}{2}H(d)\right)\right\}\frac{\partial f(s_1,c)}{\Delta_1 s_1}-\int_c^{t_2}H'(s_2)\frac{\partial f(s_1,\sigma(s_2))}{\Delta_1 s_1}\right]\Delta_2 s_2$$
$$+\int_a^{t_1}\left\{h(s_1)-\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right)\right\}\left[\left\{H(d)-\left(\left(1-\frac{\beta}{2}\right)H(d)+\frac{\beta}{2}H(c)\right)\right\}\right.$$
$$\left.\frac{\partial f(s_1,d)}{\Delta_1 s_1}-\left\{H(t_2)-\left(\left(1-\frac{\beta}{2}\right)H(d)+\frac{\beta}{2}H(c)\right)\right\}\frac{\partial f(s_1,t_2)}{\Delta_1 s_1}-\int_{t_2}^{d}H'(s_2)\frac{\partial f(s_1,\sigma(s_2))}{\Delta_1 s_1}\right]\Delta_2 s_2$$
$$+\int_{t_1}^{b}\left\{h(s_1)-\left(\left(1-\frac{\alpha}{2}\right)h(b)+\frac{\alpha}{2}h(a)\right)\right\}\left[\left\{H(t_2)-\left(\left(1-\frac{\beta}{2}\right)H(c)+\frac{\beta}{2}H(d)\right)\right\}\right.$$
$$\left.\frac{\partial f(s_1,t_2)}{\Delta_1 s_1}-\left\{H(c)-\left(\left(1-\frac{\beta}{2}\right)H(c)+\frac{\beta}{2}H(d)\right)\right\}\frac{\partial f(s_1,c)}{\Delta_1 s_1}-\int_c^{t_2}H'(s_2)\frac{\partial f(s_1,\sigma(s_2))}{\Delta_1 s_1}\right]\Delta_2 s_2$$
$$+\int_{t_1}^{b}\left\{h(s_1)-\left(\left(1-\frac{\alpha}{2}\right)h(b)+\frac{\alpha}{2}h(a)\right)\right\}\left[\left\{H(d)-\left(\left(1-\frac{\beta}{2}\right)H(d)+\frac{\beta}{2}H(c)\right)\right\}\right.$$
$$\left.\frac{\partial f(s_1,t_2)}{\Delta_1 s_1}-\left\{H(t_2)-\left(\left(1-\frac{\beta}{2}\right)H(d)+\frac{\beta}{2}H(c)\right)\right\}\frac{\partial f(s_1,c)}{\Delta_1 s_1}-\int_c^{t_2}H'(s_2)\frac{\partial f(s_1,\sigma(s_2))}{\Delta_1 s_1}\right]\Delta_2 s_2$$

and

$$=\left[H(t_2)-\left(\left(1-\frac{\beta}{2}\right)H(c)+\frac{\beta}{2}H(d)\right)\right]\left\{\left[h(t_1)-\left(1-\frac{\alpha}{2}\right)h(a)-\frac{\alpha}{2}h(b)\right]f(t_1,t_2)\right.$$
$$\left.-\left[h(a)-\left(1-\frac{\alpha}{2}\right)h(a)-\frac{\alpha}{2}h(b)\right]f(a,t_2)-\int_a^{t_1}h'(s_1)f(\sigma(s_1),t_2)\Delta_1 s_1\right\}$$

$$
\begin{aligned}
&-\left[H(c)-\left(\left(1-\frac{\beta}{2}\right)H(c)+\frac{\beta}{2}H(d)\right)\right]\left\{\left[h(t_1)-\left(1-\frac{\alpha}{2}\right)h(a)-\frac{\alpha}{2}h(b)\right]f(t_1,c)\right.\\
&\left.-\left[h(a)-\left(1-\frac{\alpha}{2}\right)h(a)-\frac{\alpha}{2}h(b)\right]f(a,c)-\int_a^{t_1}h'(s_1)\,f\left(\sigma(s_1),c\right)\Delta_1 s_1\right\}\\
&-\int_c^{t_2}H'(s_2)\left\{\left[h(t_1)-\left(1-\frac{\alpha}{2}\right)h(a)-\frac{\alpha}{2}h(b)\right]f\left(t_1,\sigma(s_2)\right)-\left[h(a)-\left(1-\frac{\alpha}{2}\right)\right.\right.\\
&\qquad\left.\left.h(a)-\frac{\alpha}{2}h(b)\right]f\left(a,\sigma(s_2)\right)-\int_a^{t_1}h'(s_1)f\left(\sigma(s_1),\sigma(s_2)\right)\Delta_1 s_1\right\}\Delta_2(s_2)
\end{aligned}
$$

$$
\begin{aligned}
&\left[H(d)-\left(1-\frac{\beta}{2}\right)H(d)+\frac{\beta}{2}H(c)\right]\left\{\left[h(t_1)-\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right)\right]f(t_1,d)\right.\\
&\left.-\left[h(a)-\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right)\right]f(a,d)-\int_a^{t_1}h'(s_1)f\left(\sigma(s_1),d\right)\Delta_1 s_1\right\}\\
&-\left[H(t_2)-\left(\left(1-\frac{\beta}{2}\right)H(d)+\frac{\beta}{2}H(c)\right)\right]\left\{\left[h(s_1)-\left(1-\frac{\alpha}{2}\right)h(a)-\frac{\alpha}{2}h(b)\right]f(t_1,t_2)\right.\\
&\left.-\left[h(a)-\left(1-\frac{\alpha}{2}\right)h(a)-\frac{\alpha}{2}h(b)\right]f(a,t_2)-\int_a^{t_1}h'(s_1)f\left(\sigma(s_1),d\right)\Delta_1 s_1\right\}\\
&-\int_{t_2}^{d}H'(s_2)\left\{\left[h(t_1)-\left(1-\frac{\alpha}{2}\right)h(a)-\frac{\alpha}{2}h(b)\right]f\left(t_1,\sigma(s_2)\right)-\left[h(a)-\left(1-\frac{\alpha}{2}\right)h(a)\right.\right.\\
&\qquad\left.\left.-\frac{\alpha}{2}h(b)\right]f\left(a,\sigma(s_2)\right)-\int_a^{t_1}h'(s_1)f\left(\sigma(s_1),\sigma(s_2)\right)\Delta_1 s_1\right\}\Delta_2(s_2)+
\end{aligned}
$$

$$
\begin{aligned}
&\left[H(t_2)-\left(\left(1-\frac{\beta}{2}\right)H(c)+\frac{\beta}{2}H(d)\right)\right]\left\{\left[h(b)-\left(1-\frac{\alpha}{2}\right)h(b)-\frac{\alpha}{2}h(a)\right]f(b,t_2)\right.\\
&\left.-\left[h(t_1)-\left(1-\frac{\alpha}{2}\right)h(b)-\frac{\alpha}{2}h(a)\right]f(t_1,t_2)-\int_{t_1}^{b}h'(s_1)f\left(\sigma s_1,t_2\right)\Delta_1 s_1\right\}\\
&-\left[H(c)-\left(\left(1-\frac{\beta}{2}\right)H(c)+\frac{\beta}{2}H(d)\right)\right]\left\{\left[h(t_1)-\left(1-\frac{\alpha}{2}\right)h(b)-\frac{\alpha}{2}h(a)\right]f(b,c)\right.\\
&\left.-\left[h(t_1)-\left(1-\frac{\alpha}{2}\right)h(b)-\frac{\alpha}{2}h(a)\right]f(t_1,c)-\int_{t_1}^{b}h'(s_1)f\left(\sigma(s_1),c\right)\Delta_1 s_1\right\}\\
&-\int_c^{y}h'(t)\left\{\left(h(b)-\left(\left(1-\frac{\alpha}{2}\right)h(b)+\frac{\alpha}{2}h(a)\right)\right)f\left(b,\sigma(t)\right)-\left[h(t_1)-\left(1-\frac{\alpha}{2}\right)h(b)\right.\right.\\
&\qquad\left.\left.+\frac{\alpha}{2}h(a)\right]f\left(t_1,\sigma(s_2)\right)-\int_{t_1}^{b}h'(s_1)f\left(\sigma(s_1),\sigma(s_2)\right)\Delta_1 s\right\}\Delta_2(s_2)+
\end{aligned}
$$

$$\left[H(d)-\left(\left(1-\frac{\beta}{2}\right)H(d)+\frac{\beta}{2}H(c)\right)\right]\left\{\left[h(b)-\left(1-\frac{\alpha}{2}\right)h(b)-\frac{\alpha}{2}h(a)\right]f(b,d)\right.$$
$$\left.-\left[h(t_1)-\left(1-\frac{\alpha}{2}\right)h(b)-\frac{\alpha}{2}h(a)\right]f(t_1,d)-\int_{t_1}^{b}h'(s_1)f\left(\sigma(s_1),\sigma(S_2)\right)\Delta_1 s_1\right\}$$
$$-\left[h(y)-\left(\left(1-\frac{\beta}{2}\right)H(d)+\frac{\beta}{2}H(c)\right)\right]\left\{\left[h(b)-\left(1-\frac{\alpha}{2}\right)h(b)-\frac{\alpha}{2}h(a)\right]f(b,t_2)\right.$$
$$\left.-\left[h(t_1)-\left(1-\frac{\alpha}{2}\right)h(b)+\frac{\alpha}{2}h(a)\right]f(t_1,t_2)-\int_{t_1}^{b}H'(s_2)f\left(\sigma(s_1),t_2\right)\Delta_1 s\right\}$$
$$-\int_{t_2}^{d}H'(s_2)\left\{\left(h(t_1)-\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right)\right)f\left(t_1,\sigma(s_2)\right)-\left[h(a)-\left(1-\frac{\alpha}{2}\right)h(a)\right.\right.$$
$$\left.\left.+\frac{\alpha}{2}h(b)\right]f\left(a,\sigma(t)\right)-\int_{a}^{t_1}h'(s_1)f\left(\sigma(s_1),\sigma(s_2)\right)\Delta_1 s_1\right\}\Delta_2(s_2)$$

$$=\left(\frac{\beta}{2}H(d)-\frac{\beta}{2}H(c)\right)\left(h(b)-\left(\left(1-\frac{\alpha}{2}\right)h(b)+\frac{\alpha}{2}h(a)\right)\right)\left(f(b,c)-f(a,d)+f(b,d)\right)$$
$$+\left(\frac{\alpha}{2}h(a)+\frac{\alpha}{2}h(b)\right)\left(\left(1-\beta\right)H(c)+\left(\beta-1\right)H(d)\right)\left(f(a,t_2)+f(b,t_2)\right)$$
$$+\left(\frac{\beta}{2}H(c)+\frac{\beta}{2}H(d)\right)\left(\left(1-\alpha\right)h(b)+\left(\alpha-1\right)h(a)\right)\left(f(t_1,c)-f(t_1,d)\right)$$
$$+\left(\left(1-\alpha\right)h(b)+\left(\alpha-1\right)h(a)\right)\left(\left(1-\beta\right)H(d)+\left(\beta-1\right)H(c)\right)f(t_1,t_2)$$
$$+\left[H(c)-\left(\left(1-\frac{\beta}{2}\right)H(c)+\frac{\beta}{2}H(d)\right)\right]\int_{a}^{b}h'(s_1)f\left(\sigma(s_1),c\right)\Delta_1 s_1$$
$$-\left[h(t_2)-\left(\left(1-\frac{\beta}{2}\right)H(c)+\frac{\beta}{2}H(d)\right)\right]\int_{a}^{b}h'(s_1)f\left(\sigma(s_1),t_2\right)\Delta_1 s_1$$
$$-\int_{c}^{d}H'(s_2)\left(\left(1-\alpha\right)h(a)+\left(\alpha-1\right)h(b)\right)f\left(t_1,\sigma(s_2)\right)\Delta_2 s_2$$
$$+\int_{c}^{d}H'(s_2)\left\{\left(h(a)-\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right)\right)\right\}f\left(a,\sigma(s_2)\right)\Delta_2 s_2$$
$$+\int_{c}^{d}H'(s_2)\left\{\int_{a}^{b}h'(s_1)f\left(\sigma(s_1),\sigma(s_2)\right)\Delta_1 s_1\right\}\Delta_2(s_2)$$

□

Remark 1. *When* $\mathbb{T}=\mathbb{R}$

$$\int_a^b\int_c^d W(t_1,t_2,s_1,s_2)\frac{d^2f(s_1,s_2)}{d_1s_1d_2s_2}d_1s_1d_2s_2$$
$$=\left(\frac{\beta}{2}H(d)-\frac{\beta}{2}H(c)\right)\left\{h(b)-\left(\left(1-\frac{\alpha}{2}\right)h(b)+\frac{\alpha}{2}h(a)\right)\right\}\left(f(b,c)-f(a,d)+f(b,d)\right)$$

$$
\begin{aligned}
&+\left(\frac{\alpha}{2}h(a)+\frac{\alpha}{2}h(b)\right)\{(1-\beta)\,H(c)+(\beta-1)\,H(d)\}\,(f(a,t_2)+f(b,t_2))\\
&+\left(\frac{\beta}{2}H(c)+\frac{\beta}{2}H(d)\right)\{(1-\alpha)h(b)+(\alpha-1)\,h(a)\}\,(f(t_1,c)-f(t_1,d))\\
&+((1-\alpha)\,h(b)+(\alpha-1)\,h(a))\,((1-\beta)\,H(d)+(\beta-1)\,H(c))\,f(t_1,t_2)\\
&+\left[H(c)-\left(\left(1-\frac{\beta}{2}\right)H(c)+\frac{\beta}{2}H(d)\right)\right]\int_a^b h'(s_1)f(s_1,c)d_1s_1\\
&-\left[H(t_2)-\left(\left(1-\frac{\beta}{2}\right)H(c)+\frac{\beta}{2}H(d)\right)\right]\int_a^b h'(s_1)f(s_1,t_2)d_1s_1\\
&-\int_c^d H'(s_2)\,((1-\alpha)\,h(a)+(\alpha-1)\,h(b))\,f(t_1,s_2)d_2s_2\\
&+\int_c^d H'(s_2)\left\{\left(h(a)-\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right)\right)\right\}f(a,s_2)d_2s_2\\
&+\int_c^d H'(s_2)\left\{\int_a^b h'(s_1)f(s_1,s_2)d_1s_1\right\}d_2(t_2)
\end{aligned}
$$

Remark 2. $\mathbb{T}=\mathbb{Z}$

$$
\begin{aligned}
&\sum_{s_1=a}^{b-1}\sum_{s_2=c}^{d-1}W(t_1,t_2,s_1,s_2)\\
&=\left(\frac{\beta}{2}H(d)-\frac{\beta}{2}H(c)\right)\left\{h(b)-\left(\left(1-\frac{\alpha}{2}\right)h(b)+\frac{\alpha}{2}h(a)\right)\right\}(f(b,c)-f(a,d)+f(b,d))\\
&+\left(\frac{\alpha}{2}h(a)+\frac{\alpha}{2}h(b)\right)\{(1-\beta)\,H(c)+(\beta-1)\,H(d)\}\,(f(a,t_2)+f(b,t_2))\\
&+\left(\frac{\beta}{2}H(c)+\frac{\beta}{2}H(d)\right)\{(1-\alpha)\,h(b)+(\alpha-1)\,h(a)\}\,(f(t_1,c)-f(t_1,d))\\
&+((1-\alpha)\,h(b)+(\alpha-1)\,h(a))\,((1-\beta)\,H(d)+(\beta-1)\,H(c))\,f(t_1,t_2)\\
&+\left[H(c)-\left(\left(1-\frac{\beta}{2}\right)H(c)+\frac{\beta}{2}H(d)\right)\right]\sum_{s_1=a}^{b-1}h'(s_1)f(t_1+1,c)\\
&-\left[H(t_2)-\left(\left(1-\frac{\beta}{2}\right)H(c)+\frac{\beta}{2}H(d)\right)\right]\sum_{s_1=a}^{b-1}h'(s_1)f(t_1+1,t_2)\\
&-\sum_{s_2=c}^{d-1}H'(s_2)\,((1-\alpha)\,h(a)+(\alpha-1)\,h(b))\,f(t_1,t_2+1)\\
&+\sum_{s_2=c}^{d-1}H'(s_2)\left\{\left(h(a)-\left(\left(1-\frac{\alpha}{2}\right)h(a)+\frac{\alpha}{2}h(b)\right)\right)\right\}f(a,t_2+1)\\
&+\sum_{s_1=a}^{b-1}\sum_{s_2=c}^{d-1}H'(s_2)\left\{\int_a^b h'(s_1)f(t_1+1,t_2+1)\right\}
\end{aligned}
$$

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Remark 3. *By taking $h(s_1) = s_1$, $H(s_2) = s_2$, we obtain*

$$\int_a^b\int_c^d W(t_1,t_2,s_1,s_2)\frac{\partial^2 f(s_1,s_2)}{\Delta_1 s_1\Delta_2 s_2}\Delta_1 s_1\Delta_2 s_2 = (1-\alpha)(1-\beta)(b-a)(d-c)f(t_1,t_2)$$
$$+(b-a)(d-c)\left\{(1-\beta)\frac{\alpha}{2}\left[f(a,t_2)+f(b,t_2)\right]+\frac{\beta}{2}(1-\alpha)\left[f(t_1,c)+f(t_1,d)\right]\right\}$$
$$+\frac{\alpha\beta}{4}(b-a)(d-c)\left\{f(b,c)+f(a,d)+f(b,d)\right\}-\frac{\beta}{2}(d-c)\int_a^b\{f(\sigma(s_1),c)$$
$$+f(\sigma(s_1),d)\}\Delta_1 s_1-\frac{\alpha}{2}(b-a)\int_c^d\{f(a,\sigma(t))+f(b,\sigma(t))\}\Delta_2 t_2$$
$$+\int_a^b\int_c^d f(\sigma(s_1),\sigma(s_2))\Delta_1 s_1\Delta_2 s_2$$

Theorem 5. *Under the conditions of Lemma, if $f^{\Delta_1\Delta_2}\in L_2\left((a,b)_{\mathbb{T}_1}\times(c,d)_{\mathbb{T}_2}\right)$, with $h(s_1)=s_1$, $H(s_2)=s_2$, then we have*

$$(1-\alpha)(1-\beta)(b-a)(d-c)f(t_1,t_2)+(b-a)(d-c)\Big[(1-\beta)\frac{\alpha}{2}\left[f(a,t_2)+f(b,t_2)\right]$$
$$+\frac{\beta}{2}(1-\alpha)\{f(t_1,c)+f(t_1,d)\}\Big]+\frac{\alpha}{2}\frac{\beta}{2}(b-a)(d-c)\{f(b,c)+f(a,d)+f(b,d)\}$$
$$-\frac{\beta}{2}(d-c)\int_a^b\{f(\sigma(s_1),c)+f(\sigma(s_1),d)\}\Delta_1 s_1-\frac{\alpha}{2}(b-a)\int_c^d\{f(a,\sigma(t))$$
$$+f(b,\sigma(t))\}\Delta_2 t_2+\int_a^b\int_c^d f(\sigma(s_1),\sigma(s_2))\Delta_1 s_1\Delta_2 s_2-\frac{[f(b,d)-f(a,d)-f(b,c)+f(a,c)]}{(b-a)^2(d-c)^2}$$
$$\times\left[h_2\left(t_1,a+\alpha\frac{b-a}{2}\right)-h_2\left(a,a+\alpha\frac{b-a}{2}\right)+h_2\left(b,b-\alpha\frac{b-a}{2}\right)-h_2\left(t_1,b-\alpha\frac{b-a}{2}\right)\right]$$
$$\times\left[h_2\left(t_2,c+\alpha\frac{d-c}{2}\right)-h_2\left(c,c+\alpha\frac{d-c}{2}\right)+h_2\left(d,d-\alpha\frac{d-c}{2}\right)-h_2\left(t_2,d-\alpha\frac{d-c}{2}\right)\right]$$
$$\leq\left[\left\{\frac{b^3-a^3}{3}-2\left(a+\alpha\frac{b-a}{2}\right)\left(h_2\left(t_1,a+\alpha\frac{b-a}{2}\right)\right)-h_2\left(a,a+\alpha\frac{b-a}{2}\right)-\left(a+\alpha\frac{b-a}{2}\right)^2(t_1-a)\right\}\right]$$
$$-2\left(b-\alpha\frac{b-a}{2}\right)\left(\left(h_2\left(b,b-\alpha\frac{b-a}{2}\right)-h_2\left(t_1,b-\alpha\frac{b-a}{2}\right)\right)-\left(b-\alpha\frac{b-a}{2}\right)^2(b-t_1)\right)$$
$$\times\left[\left\{\frac{d^3-c^3}{3}-2\left(c+\alpha\frac{d-c}{2}\right)\left(h_2\left(t_2,c+\alpha\frac{d-c}{2}\right)\right)-h_2\left(c,c+\alpha\frac{d-c}{2}\right)-\left(c+\alpha\frac{d-c}{2}\right)^2(t_2-c)\right\}\right]$$
$$-2\left(d-\alpha\frac{d-c}{2}\right)\left(\left(h_2\left(d,d-\alpha\frac{d-c}{2}\right)-h_2\left(t_2,d-\alpha\frac{d-c}{2}\right)\right)-\left(d-\alpha\frac{d-c}{2}\right)^2(d-t_2)\right)$$
$$-\frac{1}{(b-a)(d-c)}\left[\left(\left(h_2\left(t_1,a+\alpha\frac{b-a}{2}\right)\right)-h_2\left(a,a+\alpha\frac{b-a}{2}\right)+\left(\left(h_2\left(b,b-\alpha\frac{b-a}{2}\right)\right)\right.\right.\right.$$

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$$-h_2\left(t_1, b-\alpha\frac{b-a}{2}\right)\Big)\Big]\Big)\left[\left(\left(h_2\left(t_2, c+\alpha\frac{d-c}{2}\right)\right)-h_2\left(c, c+\alpha\frac{d-c}{2}\right)\right.\right.$$
$$\left.\left.+\left(\left(h_2\left(d, d-\alpha\frac{d-c}{2}\right)\right)-h_2\left(t_2, d-\alpha\frac{d-c}{2}\right)\right)\right)\right]\sqrt{T\left(f^{\Delta_1\Delta_2}\right)}.$$

where

$$T(f)=\int_a^b\int_c^d f^2(s_1,s_2)\Delta_2 s_2\Delta_1 s_1-\frac{1}{(b-a)(d-c)}\left(\int_a^b\int_c^d f(s_1,s_2)\Delta_2 s_2\Delta_1 s_1\right)^2$$

Proof. From the definition of $W(t_1,t_2,s_1,s_2)$, and taking $h(s_1)=s_1$, $H(s_2)=s_2$, we obtain

$$\int_a^b\int_c^d W(t_1,t_2,s_1,s_2)\Delta_2 s_2\Delta_1 s_1=\int_a^b W_1(t_1,s_1)\Delta_1 s_1\int_c^d W_2(t_2,s_2)\Delta_2 s_2$$

$$\left[\int_a^b\left\{s_1-\left(a+\alpha\frac{b-a}{2}\right)\right\}\Delta_1 s_1+\int_{t_1}^b\left\{s_1-\left(b-\alpha\frac{b-a}{2}\right)\right\}\Delta_1 s_1\right]$$
$$\times\left[\int_c^{t_2}\left\{s_2-\left(c+\alpha\frac{d-c}{2}\right)\right\}\Delta_2 s_2+\int_{t_2}^d\left\{s_2-\left(d-\alpha\frac{d-c}{2}\right)\right\}\Delta_2 s_2\right]$$

$$=\left[h_2\left(t_1, a+\alpha\frac{b-a}{2}\right)-h_2\left(a, a+\alpha\frac{b-a}{2}\right)+h_2\left(b, b-\alpha\frac{b-a}{2}\right)-h_2\left(t_1, b-\alpha\frac{b-a}{2}\right)\right]$$
$$\times\left[h_2\left(t_2, c+\alpha\frac{d-c}{2}\right)-h_2\left(c, c+\alpha\frac{d-c}{2}\right)+h_2\left(d, d-\alpha\frac{d-c}{2}\right)-h_2\left(t_2, d-\alpha\frac{d-c}{2}\right)\right]$$

and

$$\int_a^b\int_c^d W^2(t_1,t_2,s_1,s_2)\Delta_2 s_2\Delta_1 s_1=\int_a^b W_1^2(t_1,s_1)\Delta_1 s\int_c^d W_2^2(t_2,s_2)\Delta_2 s_2$$
$$=\left\{\int_a^{t_1}\left[s_1-\left(a+\alpha\frac{b-a}{2}\right)\right]^2\Delta_1 s_1+\int_{t_1}^b\left[s_1-\left(b-\alpha\frac{b-a}{2}\right)\right]^2\right\}\Delta_1 s_1$$
$$\times\left\{\int_c^{t_2}\left[s_2-\left(c+\alpha\frac{d-c}{2}\right)\right]^2\Delta_2 s_2+\int_{t_2}^d\left[s_2-\left(d-\alpha\frac{d-c}{2}\right)\right]^2\right\}\Delta_2 s_2$$
$$=\left[\int_a^{t_1}\left\{s_1^2-2\left(a+\alpha\frac{b-a}{2}\right)\left(s_1-\left(a+\alpha\frac{b-a}{2}\right)-\left(a+\alpha\frac{b-a}{2}\right)^2\right)\right\}\Delta_1 s_1\right.$$
$$\left.+\int_{t_1}^b\left\{s_1^2-2\left(b-\alpha\frac{b-a}{2}\right)\left(s_1-\left(b-\alpha\frac{b-a}{2}\right)-\left(b-\alpha\frac{b-a}{2}\right)^2\right)\right\}\Delta_1 s_1\right]$$
$$\times\left[\int_c^{t_2}\left\{s_2^2-2\left(c+\alpha\frac{d-c}{2}\right)\left(s_2-\left(c+\alpha\frac{d-c}{2}\right)-\left(c+\alpha\frac{d-c}{2}\right)^2\right)\right\}\Delta_2 s_2\right.$$
$$\left.+\int_{t_2}^d\left\{s_2^2-2\left(d-\alpha\frac{d-c}{2}\right)\left(s_2-\left(d-\alpha\frac{d-c}{2}\right)-\left(d-\alpha\frac{d-c}{2}\right)^2\right)\right\}\Delta_2 s_2\right]$$

$$\leq\left[\int_a^{t_1}\left\{\frac{s_1^2+s_1\sigma(s_1)+(\sigma(s_1))^2}{3}-2\left(a+\alpha\frac{b-a}{2}\right)\left(s_1-\left(a+\alpha\frac{b-a}{2}\right)-\left(a+\alpha\frac{b-a}{2}\right)^2\right)\Delta_1 s_1\right\}\right]$$
$$\left[\int_{t_1}^{b}\left\{\frac{s^2+s_1\sigma(s_1)+(\sigma(s_1))^2}{3}-2\left(b-\alpha\frac{b-a}{2}\right)\left(s_1-\left(b-\alpha\frac{b-a}{2}\right)-\left(b-\alpha\frac{b-a}{2}\right)^2\right)\Delta_1 s_1\right\}\right]$$
$$\left[\int_c^{t_2}\left\{\frac{s_2^2+s_2\sigma(s_2)+(\sigma(s_2))^2}{3}-2\left(c+\alpha\frac{d-c}{2}\right)\left(s_2-\left(c+\alpha\frac{d-c}{2}\right)-\left(c+\alpha\frac{d-c}{2}\right)^2\right)\Delta_2 s_2\right\}\right]$$
$$\left[\int_y^{d}\left\{\frac{s_2^2+s_2\sigma(s_2)+(\sigma(s_2))^2}{3}-2\left(d-\alpha\frac{d-c}{2}\right)\left(s_2-\left(d-\alpha\frac{d-c}{2}\right)-\left(d-\alpha\frac{d-c}{2}\right)^2\right)\Delta_2 s_2\right\}\right]$$
$$=\frac{{t_1}^3-a^3}{3}-2\left(a+\alpha\frac{b-a}{2}\right)\left[h_2\left(t_1,a+\alpha\frac{b-a}{2}\right)-h_2\left(a,a+\alpha\frac{b-a}{2}\right)\right]-\left(a+\alpha\frac{b-a}{2}\right)^2(t_1-a)$$
$$+\frac{b^3-{t_1}^3}{3}-2\left(b-\alpha\frac{b-a}{2}\right)\left[h_2\left(b,b-\alpha\frac{b-a}{2}\right)-h_2\left(t_1,b-\alpha\frac{b-a}{2}\right)\right]-\left(b-\alpha\frac{b-a}{2}\right)^2(b-t_1)$$
$$+\frac{{t_2}^3-c^3}{3}-2\left(c+\alpha\frac{d-c}{2}\right)\left[h_2\left(t_2,c+\alpha\frac{d-c}{2}\right)-h_2\left(c,c+\alpha\frac{d-c}{2}\right)\right]-\left(c+\alpha\frac{d-c}{2}\right)^2(t_2-c)$$
$$+\frac{d^3-t_2^3}{3}-2\left(d-\alpha\frac{d-c}{2}\right)\left[h_2\left(d,d-\alpha\frac{d-c}{2}\right)-h_2\left(t_2,d-\alpha\frac{d-c}{2}\right)\right]-\left(d-\alpha\frac{d-c}{2}\right)^2(d-t_2)$$

Furthermore, we have

$$\int_a^b\int_c^d\left[W(t_1,t_2,s_1,s_2)-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d W(t_1,t_2,s_1,s_2)\Delta_2 s_2\Delta_1 s_1\right]\Delta_2 s_2\Delta_1 s_1$$
$$\times\left[\frac{\partial^2 f(s_1,s_2)}{\Delta_1 s_1\Delta_2 s_2}-\frac{(1}{(b-a)(d-c)}\int_a^b\int_c^d\frac{\partial^2 f(s_1,s_2)}{\Delta_1 s_1\Delta_2 s_2}\right]\Delta_2 s_2\Delta_1 s_1$$
$$\int_a^b\int_c^d W(t_1,t_2,s_1,s_2)\frac{\partial^2 f(s_1,s_2)}{\Delta_1 s_1\Delta_2 s_2}\Delta_2 s_2\Delta_1 s_1-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d W(t_1,t_2,s_1,s_2)\Delta_2 s_2\Delta_1 s_1$$
$$\times\int_a^b\int_c^d\frac{\partial^2 f(s_1,s_2)}{\Delta_1 s_1\Delta_2 s_2}\Delta_2 s_2\Delta_1 s_1$$
$$=\int_a^b\int_c^d W(t_1,t_2,s_1,s_2)\frac{\partial^2 f(s_1,s_2)}{\Delta_1 s_1\Delta_2 s_2}\Delta_2 s_2\Delta_1 s_1-\frac{[f(b,d)-f(a,d)-f(b,c)+f(a,c)]}{(b-a)(d-c)}$$
$$\times\int_a^b\int_c^d W(t_1,t_2,s_1,s_2)\Delta_2 s_2\Delta_1 s_1$$

On the other hand

$$\int_a^b\int_c^d\left[W(t_1,t_2,s_1,s_2)-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d W(t_1,t_2,s_1,s_2)\Delta_2 s_2\Delta_1 s_1\right]\Delta_2 s_2\Delta_1 s_1$$
$$\times\left[\frac{\partial^2 f(s_1,s_2)}{\Delta_1 s_1\Delta_2 s_2}-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d\frac{\partial^2 f(s_1,s_2)}{\Delta_1 s_1\Delta_2 s_2}\right]\Delta_2 s_2\Delta_1 s_1$$

$$\leq \left\| W(t_1,t_2,...) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d W(t_1,t_2,s_1,s_2)\Delta_2 s_2 \Delta_1 s_1 \right\|_2$$

$$\times \left\| \frac{\partial^2 f(.,.)}{\Delta_1 s \Delta_2 t} - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \frac{\partial^2 f(s_1,s_2)}{\Delta_1 s_1 \Delta_2 s_2} \right\|_2$$

$$= \left[\int_a^b\int_c^d W^2(t_1,t_2,s_1,s_2)\Delta_2 s_2 \Delta_1 s_1 - \frac{1}{(b-a)(d-c)} \left(\int_a^b\int_c^d W(t_1,t_2,s_1,s_2)\Delta_2 s_2 \Delta_1 s_1 \right)^2 \right]^{\frac{1}{2}}$$

$$\times \left[\int_a^b \int_c^d \left(\frac{\partial^2 f(s_1,s_2)}{\Delta_1 s_1 \Delta_2 s_2} \right)^2 - \frac{1}{(b-a)(d-c)} \left(\int_a^b \int_c^d \frac{\partial^2 f(s_1,s_2)}{\Delta_1 s \Delta_2 s_2} \right)^2 \right]^{\frac{1}{2}}$$

$$\times \left[\int_a^b\int_c^d W^2(t_1,t_2,s_1,s_2)\Delta_2 s_2 \Delta_1 s_1 - \frac{1}{(b-a)(d-c)} \left(\int_a^b\int_c^d W(t_1,t_2,s_1,s_2)\Delta_2 s_2 \Delta_1 s_1 \right)^2 \right]^{\frac{1}{2}}$$

$$\times \sqrt{T(f^{\Delta_1 \Delta_2})}.$$

$\square$

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The Henstock-Stieltjes integral for fuzzy-number-valued functions on a infinite interval†

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Abstract: In this paper, the Henstock-Stieltjes integral for fuzzy-number-valued functions on a infinite interval which is an extension of the usual fuzzy Riemann-Stieltjes integral on infinite interval is firstly defined and discussed. Several necessary and sufficient conditions of the integrability for fuzzy-number-valued functions are given by means of the Henstock-Stieltjes integral of real-valued functions on infinite interval and Henstock integral of fuzzy-number-valued functions on infinite interval.

Keywords: Fuzzy numbers; Fuzzy Henstock integral; Stieltjes integral

AMS subject classifications. 26E50; 28E10.

1 Introduction

Recently, in order to complete the theory of fuzzy integrals and to meet the solving need of the fuzzy differential equations [1-3], fuzzy integrals of fuzzy-number-valued functions have been studied by many authors from different points of views, including Nanda [4], Wu et al. [5] and other authors [6-9]. As an extension for Riemann integral and Lebesgue integral, the Stieltjes integral plays an important role in probability theory, stochastic processes, physics, econometrics, biometrics and numerical analysis[10-13] in the Mathematics analysis. In fact, the establishment of the Stieltjes integral was related to the moment of inertia in physics [14]. Until 1909, Riesz presented a general expression for the linear functional of the space of the continuous functions in a finite interval by Stieltjes integral [15]. After Riesz' work, people find that the Stieltjes integral is a powerful tool in several branches of mathematics. In the fuzzy analysis, in 1968, Zadeh defined the probability measure of a fuzzy event by using the Lebesgue-Stieltjes integral of the membership function [16]. It is well known that the notion of the Stieltjes integral for fuzzy-number-valued functions was originally proposed by Nanda [4] in 1989. In 1998, Wu [17] discussed and defined the concept of fuzzy Riemann-Stieltjes integral by means of the representation theorem of fuzzy-number-valued functions, whose membership function could be obtained by solving a nonlinear programming problem, but it is difficult to calculate and extend to the higher-dimensional space. In 2006, Ren et al. introduced the concept of two kinds of fuzzy Riemann-Stieltjes integral for fuzzy-number-valued functions [18,19] and showed that a continuous fuzzy-number-valued function was fuzzy Riemann-Stieltjes integrable with respect to a real-valued increasing function. To overcome the limitations of the existing studies and to characterize continuous linear functionals on the space of Henstock integrable fuzzy-number-valued functions, in 2014, the concept of the Henstock-Stieltjes integral for fuzzy-number-valued functions is defined and discussed, and some useful results for this integral are shown [20].

The expectations of fuzzy random variables were investigated by M. L. Puri and D. A. Ralescu in 1986 [21]. It well known that the notion of a fuzzy random variable as a fuzzy-number-valued function and the expectation $E(X)$ of a fuzzy random variable X was defined by a fuzzy integral $E(X) = \int X$ or set-valued integral of X_λ [21]. In 2007, the concept of the fuzzy Henstock integral on infinite interval is proposed and discussed in order to solve the expectation $E(X)$ of a fuzzy random variable X which distribution function has some kinds of discontinuity or non-integrability [7]. In this paper, the Henstock-Stieltjes integral for fuzzy-number-valued functions on infinite interval which is an extension of the usual fuzzy Riemann-Stieltjes integral on infinite interval is firstly defined and discussed. Several necessary and sufficient conditions of the integrability for fuzzy-number-valued functions are given by means of the

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Henstock-Stieltjes integral of real-valued functions on infinite interval and Henstock integral of fuzzy-number-valued functions on infinite interval.

2 Preliminaries

Fuzzy set $\tilde{u} \in E^1$ is called a fuzzy number if $\tilde{u}$ is a normal, convex fuzzy set, upper semi-continuous and supp $u = \overline{\{x \in \mathbb{R} \mid u(x) > 0\}}$ is compact. Here $\bar{A}$ denotes the closure of A. We use E^1 to denote the fuzzy number space [22].

Let $\tilde{u}, \tilde{v} \in E^1, k \in \mathbb{R}$, the addition and scalar multiplication are defined by

$$[\tilde{u} + \tilde{v}]_\lambda = [\tilde{u}]_\lambda + [\tilde{v}]_\lambda, \qquad [k\tilde{u}]_\lambda = k[\tilde{u}]_\lambda,$$

respectively, where $[\tilde{u}]_\lambda = \{x : u(x) \geqslant \lambda\} = [u_\lambda^-, u_\lambda^+]$, for any $\lambda \in [0, 1]$.

We use the Hausdorff distance between fuzzy numbers given by $D : E^1 \times E^1 \to [0, +\infty)$ as follows [22]:

$$D(\tilde{u}, \tilde{v}) = \sup_{\lambda\in[0,1]} d([\tilde{u}]_\lambda, [\tilde{v}]_\lambda) = \sup_{\lambda\in[0,1]} \max\{|u_\lambda^- - v_\lambda^-|, |u_\lambda^+ - v_\lambda^+|\},$$

where d is the Hausdorff metric. $D(\tilde{u}, \tilde{v})$ is called the distance between $\tilde{u}$ and $\tilde{v}$.

Lemma 2.1 [22]. If $\tilde{u} \in E^1$, then

(1) $[\tilde{u}]_\lambda$ is non-empty bounded closed interval for all $\lambda \in [0, 1]$;

(2) $[\tilde{u}]_{\lambda_1} \supset [\tilde{u}]_{\lambda_2}$ for any $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant 1$;

(3) for any $\{\lambda_n\}$ converging increasingly to $\lambda \in (0, 1]$,

$$\bigcap_{n=1}^{\infty} [\tilde{u}]_{\lambda_n} = [\tilde{u}]_\lambda.$$

Conversely, if for all $\lambda \in [0, 1]$, there exists $A_\lambda \subset \mathbb{R}$ satisfying (1) $\sim$ (3), then there exists a unique $\tilde{u} \in E^1$ such that $[\tilde{u}]_\lambda = A_\lambda, \lambda \in (0, 1]$, and $[\tilde{u}]_0 = \overline{\bigcup_{\lambda\in(0,1]}[\tilde{u}]_\lambda} \subset A_0$.

Definition 2.1 [7, 20, 23]. $\overline{R}$ denote the generalized real line, for $\tilde{f}$ defined on $[a, +\infty]$, we define $\tilde{f}(+\infty) = \tilde{0}$, and $\tilde{0} \cdot (+\infty) = \tilde{0}$.

Let $\delta : [a, +\infty] \to R^+$ be a positive real function. A division $P = \{[x_{i-1}, x_i]; \xi_i\}$ is said to be δ-fine, if the following conditions are satisfied:

(1)$a = x_0 < x_1 < ... < x_{n-1} = b < x_n = +\infty$;

(2)$\xi_i \in [x_{i-1}, x_i] \subset O(\xi_i), i = 1, 2, ..., n$;

where $O(\xi_i) = (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for $i = 1, 2, ..., n - 1$, and $O(\xi_n) = [b, +\infty)$.

For brevity, we write $T = \{[u, v]; \xi\}$, where $[u, v]$ denotes a typical interval in T and ξ is the associated point of $[u, v]$.

Definition 2.2. Let $\alpha : [a, +\infty] \to \mathbb{R}$ be an increasing function. A function $f : [a, +\infty] \to \mathbb{R}$ is Henstock-Stieltjes integrable with respect to α on $[a, +\infty]$ if there exists a real number I such that for every $\varepsilon > 0$, there is a function $\delta(x) > 0$ on $[a, +\infty]$ such that for any δ-fine division $T = \{[x_{i-1}, x_i]; \xi_i\}_{i=1}^n$, we have

$$|\sum_{i=1}^{n} f(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})] - I| < \varepsilon.$$

As usual, we write $(HS)\int_a^{+\infty} f(x)\mathrm{d}\alpha = I$ and $(f, \alpha) \in HS[a, +\infty]$.

Recall, also, that a function $\tilde{f} : [a, b] \to E^1$ is said to be bounded if there exists $M \in \mathbb{R}$ such that $\|\tilde{f}(x)\| = D(\tilde{f}(x), \tilde{0}) \leqslant M$ for any $x \in [a, b]$. Notice that here $\|\tilde{f}(x_0)\|$ does not stand for the norm of E^1.

3 The fuzzy Henstock-Stieltjes integral on infinite interval and its properties

In this section we shall give the definition of the Henstock-Stieltjes integral for fuzzy-number-valued functions on a infinite interval.

Definition 3.1. Let $\alpha : [a, +\infty] \to \mathbb{R}$ be an increasing function. A fuzzy-number-valued function $\tilde{f}(x)$ is said to be fuzzy Henstock-Stieltjes integrable with respect to α on $[a, +\infty]$ if there exists a fuzzy

number $\tilde{H} \in E^1$ such that for every $\varepsilon > 0$, there is a function $\delta(x) > 0$ on $[a, +\infty]$ such that for any δ-fine division $T = \{[x_{i-1}, x_i]; \xi_i\}_{i=1}^n$, we have

$$D(\sum_{i=1}^{n} \tilde{f}(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})], \tilde{H}) < \varepsilon.$$

We write $(FHS) \int_a^{+\infty} \tilde{f}(x)\mathrm{d}\alpha = \tilde{H}$ and $(\tilde{f}, \alpha) \in FHS[a, +\infty]$.

The definition of $\tilde{f} \in FHS(-\infty, a]$ is similar. Naturally, we define $\tilde{f} \in FHS(-\infty, +\infty)$ iff $\tilde{f} \in FHS(-\infty, a]$ and $\tilde{f} \in FHS[a, +\infty)$, and furthermore

$$(FHS) \int_{-\infty}^{+\infty} \tilde{f}(x)d\mathrm{d}\alpha = (FHS) \int_{-\infty}^{a} \tilde{f}(x)d\mathrm{d}\alpha + (FHS) \int_{a}^{+\infty} \tilde{f}(x)d\mathrm{d}\alpha.$$

Remark 3.1. It is clear, if $\tilde{f}(x)$ is a real-valued function then Definition 3.1 implies the definition of (HS) integral on infinite interval introduced by [20]; if $\alpha(x) = x$, then Definition 3.1 implies the definition of (FH) integral introduced by Gong et al. [7].

Remark 3.2. From the definition of the fuzzy Henstock-Stieltjes integral and the fact that (E^1, D) is a complete metric space, we can easily obtain the following conclusions.

Theorem 3.1. Let $\alpha : [a, +\infty] \to \mathbb{R}$ be an increasing function. A fuzzy-number-valued function $\tilde{f}$ is fuzzy Henstock-Stieltjes integrable with respect to α on $[a, +\infty]$ if and only if for every $\epsilon > 0$, there is a function $\delta(x) > 0$ on $[a, +\infty]$ such that for any δ-fine division $T = \{[u, v]; \xi\}$ and $T^{'} = \{[u^{'}, v^{'}]; \xi^{'}\}$, we have

$$D(\sum_{T} \tilde{f}(\xi)[\alpha(v) - \alpha(u)], \sum_{T'} \tilde{f}(\xi^{'})[\alpha(v^{'}) - \alpha(u^{'})]) < \varepsilon.$$

Theorem 3.2. Let $\alpha : [a, +\infty] \to \mathbb{R}$ be an increasing function and let $\tilde{f} : [a, +\infty] \to E^1$. Then the following statements are equivalent:

(1) $(\tilde{f}, \alpha) \in FHS[a, +\infty]$ and $(FHS) \int_a^{+\infty} \tilde{f}(x)\mathrm{d}\alpha = \tilde{A}$;

(2) for any $\lambda \in [0, 1]$, f_λ^- and f_λ^+ are Henstock-Stieltjes integrable with respect to α on $[a, +\infty]$ for any $\lambda \in [0, 1]$ uniformly ($\delta(x)$ is independent of $\lambda \in [0, 1]$), and

$$[(FHS) \int_a^{+\infty} \tilde{f}(x)\mathrm{d}\alpha]_\lambda = [(HS) \int_a^{+\infty} f_\lambda^-(x)\mathrm{d}\alpha, (HS) \int_0^{+\infty} f_\lambda^+(x)\mathrm{d}\alpha].$$

(3) For any $b > a$, $\tilde{f} \in FH[a, b]$, $\lim\limits_{b\to+\infty} \int_a^b \tilde{f}(x)\mathrm{d}\alpha$ as a fuzzy number exists and

$$\lim_{b\to+\infty} \int_a^b \tilde{f}(x)\mathrm{d}\alpha = \int_a^{+\infty} \tilde{f}(x)\mathrm{d}\alpha.$$

Proof. First, we prove that (1) is equivalent to (2).

(1) *implies* (2): If $\int_a^{+\infty} \tilde{f}(x)\mathrm{d}\alpha = \tilde{A}$, then given $\varepsilon > 0$, there exists a positive-valued function $\delta(x)$ on $[a, +\infty]$ such that for any $\delta-$fine division of $[a, +\infty] : T = \{[x_{i-1}, x_i]; \xi_i\}$, we have

$$D(\sum_{i} \tilde{f}(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})), \tilde{A}) < \varepsilon,$$

i.e.

$$\sup_{\lambda\in[0,1]} \max\{|[\sum_{i} \tilde{f}(\xi_i)(\alpha(x_i) - \alpha(x_{i-1}))]_\lambda^- - A_\lambda^-|, |[\sum_{i} \tilde{f}(\xi_i)(\alpha(x_i) - \alpha(x_{i-1}))]_\lambda^+ - A_\lambda^+|\} < \varepsilon,$$

so for any $\lambda \in [0, 1]$ and any $\delta-$fine division $T = \{[x_{i-1}, x_i]; \xi_i\}$, we have

$$|[\sum_{i} \tilde{f}(\xi_i)(\alpha(x_i) - \alpha(x_{i-1}))]_\lambda^- - A_\lambda^-| = |\sum_{i} f_\lambda^-(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) - A_\lambda^-| < \varepsilon,$$

$$|[\sum_i \tilde{f}(\xi_i)(\alpha(x_i)-\alpha(x_{i-1}))]^+_\lambda - A^+_\lambda| = |\sum_i f^+_\lambda(\xi_i)(\alpha(x_i)-\alpha(x_{i-1})) - A^+_\lambda| < \varepsilon.$$

Thus, f^-_λ and f^+_λ are Henstock integrable uniformly with respect to $\lambda \in [0,1]$ on $[a,+\infty)$, and

$$A_\lambda = [\int_a^{+\infty} f^-_\lambda(x)\mathrm{d}\alpha, \int_a^{+\infty} f^+_\lambda(x)\mathrm{d}\alpha].$$

Conversely, since f^-_λ and f^+_λ are Henstock integrable uniformly with respect to $\lambda \in [0,1]$ on $[a,+\infty)$, then given $\varepsilon > 0$, there exists a positive-valued function $\delta(x)$ on $[a,+\infty]$ such that for any $\delta-$fine division $T = \{[x_{i-1}, x_i]; \xi_i\}$, and for any $\lambda \in [0,1]$, we have

$$|\sum_i f^-_\lambda(\xi_i)(\alpha(x_i)-\alpha(x_{i-1})) - A^-_\lambda| < \varepsilon, \tag{0.1}$$

$$|\sum_i f^+_\lambda(\xi_i)(\alpha(x_i)-\alpha(x_{i-1})) - A^+_\lambda| < \varepsilon. \tag{0.2}$$

We can prove that the class of closed intervals $\{[A^-_\lambda, A^+_\lambda] : \lambda \in [0,1]\}$ determines a fuzzy number. In fact, $[A^-_\lambda, A^+_\lambda]$ satisfies the conditions of lemma 2.1.

(1) Since $f^-_\lambda(x) \le f^+_\lambda(x), \lambda \in [0,1]$, then $A^-_\lambda \le A^+_\lambda$, i.e. $[A^-_\lambda, A^+_\lambda]$ is a closed interval, $\lambda \in [0,1]$,

(2) For any $0 \le \lambda_1 \le \lambda_2 \le 1$,

$$f^-_{\lambda_1}(x) \le f^-_{\lambda_2}(x) \le f^+_{\lambda_2}(x) \le f^+_{\lambda_1}(x).$$

This implies

$$\int_a^{+\infty} f^-_{\lambda_1}(x)\mathrm{d}\alpha \le \int_a^{+\infty} f^-_{\lambda_2}(x)\mathrm{d}\alpha \le \int_a^{+\infty} f^+_{\lambda_2}(x)\mathrm{d}\alpha \le \int_a^{+\infty} f^+_{\lambda_1}(x)\mathrm{d}\alpha.$$

That is, $[A^-_{\lambda_1}, A^+_{\lambda_1}] \supset [A^-_{\lambda_2}, A^+_{\lambda_2}]$.

(3) For any $\{\lambda_n\}$ increasingly converging to $\lambda \in (0,1]$,

$$\bigcap_{n=1}^{\infty} [\tilde{f}(x)]_{\lambda_n} = [\tilde{f}(x)]_\lambda,$$

i.e.

$$\bigcap_{n=1}^{\infty} [f^-_{\lambda_n}(x), f^+_{\lambda_n}(x)] = [f^-_\lambda(x), f^+_\lambda(x)].$$

That is

$$\lim_{n\to\infty} f^-_{\lambda_n}(x) = f^-_\lambda(x), \lim_{n\to\infty} f^+_{\lambda_n}(x) = f^+_\lambda(x).$$

Note that

$$f^-_0(x) \le f^-_{\lambda_n}(x) \le f^-_1(x), f^+_1(x) \le f^+_{\lambda_n}(x) \le f^+_0(x).$$

This implies

$$0 \le f^-_{\lambda_n}(x) - f^-_0(x) \le f^-_1(x) - f^-_0(x), 0 \le f^+_{\lambda_n}(x) - f^+_1(x) \le f^+_0(x) - f^+_1(x).$$

By the non-negativeness and Henstock integrability of $f^-_1 - f^-_0, f^+_0 - f^+_1$, we know that $f^-_1 - f^-_0, f^+_0 - f^+_1$ are Lebesgue integrable (refer to [9]). Hence $f^-_{\lambda_n}(x) - f^-_0(x), f^+_{\lambda_n}(x) - f^+_1(x)$ are Lebesgue integrable, and

$$\lim_{n\to\infty} \int_a^{+\infty} (f^-_{\lambda_n}(x) - f^-_0(x))\mathrm{d}\alpha = \int_a^{+\infty} (f^-_\lambda(x) - f^-_0(x))\mathrm{d}\alpha,$$

$$\lim_{n\to\infty} \int_a^{+\infty} (f^+_{\lambda_n}(x) - f^+_1(x))\mathrm{d}\alpha = \int_a^{+\infty} (f^+_\lambda(x) - f^+_1(x))\mathrm{d}\alpha.$$

That is

$$\lim_{n\to\infty} \int_a^{+\infty} f^-_{\lambda_n}(x)\mathrm{d}\alpha = \int_a^{+\infty} f^-_\lambda(x)\mathrm{d}\alpha, \lim_{n\to\infty} \int_a^{+\infty} f^+_{\lambda_n}(x)\mathrm{d}\alpha = \int_a^{+\infty} f^+_\lambda(x).$$

Thus, $\bigcap_{n=1}^{\infty}[A^-_{\lambda_n}, A^+_{\lambda_n}] = [A^-_{\lambda}, A^+_{\lambda}]$.

Combining the inequality (1) and (2) we obtain

$$D(\sum_i(\alpha(x_i) - \alpha(x_{i-1}))\tilde{f}(\xi_i), \tilde{A}) < \varepsilon,$$

i.e.

$$\tilde{f} \in FH[a, +\infty), \int_a^{+\infty} \tilde{f}(x)\mathrm{d}\alpha = \tilde{A}.$$

We'll prove that (i) is equivalent to (iii) as follows.

(1) *implies* (3): Let $\varepsilon > 0$. Suppose $\tilde{f} \in FH[a, +\infty)$. There exists a positive-valued function δ on $[a, +\infty]$ such that

$$D(\sum_i \tilde{f}(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})), \int_a^{+\infty} \tilde{f}) < \varepsilon.$$

for any $\delta-$fine division of $[a, +\infty] : T = \{[x_{i-1}, x_i]; \xi_i\}_{i=1}^n$. On the other hand, by the Cuachy Rule about $\tilde{f} \in FH[a, b]$ (refer to Th 2.3 of [24]), then $\tilde{f} \in FH[a, b]$ for any $b > a$. There is a positive-valued function δ_1 on $[a, b]$ such that for any δ_1-fine division of $[a, b] : T = \{[x_{i-1}, x_i]; \xi_i\}_{i=1}^n$, we have

$$D(\sum_i \tilde{f}(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})), \int_a^{b} \tilde{f}) < \varepsilon.$$

We may assume that $\delta_1 \leq \delta$ for any $\xi \in [a, b]$. Then

$$\begin{aligned} &D(\int_a^{+\infty} \tilde{f}, \int_a^b \tilde{f}) \\ &\leq D(\sum_i \tilde{f}(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})), \int_a^{+\infty} \tilde{f}) + D(\sum_i \tilde{f}(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})), \int_a^{b} \tilde{f}) + D(\tilde{f}(+\infty)\mu([b, +\infty)) \\ &< \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

Hence

$$\lim_{b\to+\infty} \int_a^b \tilde{f}(x)\mathrm{d}\alpha = \int_a^{+\infty} \tilde{f}(x)\mathrm{d}\alpha.$$

(3) *implies* (1): Let $\varepsilon > 0$. Choose a sequence $a = b_0 < b_1 < b_2 < ...., b_k \uparrow +\infty$. Since $\tilde{f} \in FH[b_{k-1}, b_k], k = 1, 2, 3, ...$, there exist δ_k such that

$$D(\sum_{[b_{k-1}, b_k]} \tilde{f}(\xi)(v - u), \int_{b_{k-1}}^{b_k} \tilde{f}) < \varepsilon/2^{k+2}.$$

for any δ_k-fine division on $[b_{k-1}, b_k], k = 1, 2, 3, ...$ Suppose $\lim\limits_{b\to+\infty} \int_a^b \tilde{f}(x)\mathrm{d}\alpha = \tilde{A}$. Choose N such that $b > b_N$ which implies $D(\int_a^b \tilde{f}(x)\mathrm{d}\alpha, \tilde{A}) < \varepsilon/2$.

Define

$$\delta(\xi) = \begin{cases} \delta_1(\xi), & \xi \in [b_0, b_1), \\ \delta_k(\xi), & \xi \in (b_{k-1}, b_k), k = 1, 2, 3, ..., \\ min(\delta_k(b_k), \delta_{k+1}(b_k) & \xi = b_k, k = 1, 2, 3, ... \end{cases}$$

For any $\delta-$fine division $P=\{[x_{i-1},x_i];\xi_i\}$ satisfies $i=1,2,...,n$, $O(\xi_n)=[b,+\infty)$ and $b>b_N$, we have

$$
\begin{aligned}
&D(\sum_i \tilde{f}(\xi_i)(\alpha(x_i)-\alpha(x_{i-1})),\tilde{A})\\
&\le D(\int_a^b \tilde{f},\tilde{A})+D(\int_a^b \tilde{f},\sum_i \tilde{f}(\xi_i)(\alpha(x_i)-\alpha(x_{i-1})))\\
&<\varepsilon/2+D(\int_a^b \tilde{f},\sum_{i=1}^{n-1} \tilde{f}(\xi_i)(\alpha(x_i)-\alpha(x_{i-1}))+\tilde{f}(+\infty)\mu([b,+\infty))\\
&\le \varepsilon/2+\sum_{k=1}^{+\infty}\varepsilon/2^{k+2}=2\varepsilon
\end{aligned}
$$

Hence, $\tilde{f}\in FH[a,+\infty)$ and

$$\lim_{b\to+\infty}\int_a^b \tilde{f}(x)\mathrm{d}\alpha=\int_a^{+\infty}\tilde{f}(x)dx.$$

The proof is complete.

Theorem 3.3. Let $\alpha:[a,+\infty)\to\mathbb{R}$ be an increasing function such that $\alpha\in C^1[a,+\infty)$ and $\tilde{f}:[a,+\infty)\to E^1$ be a bounded fuzzy-number-valued function. Then $\tilde{f}$ is fuzzy Henstock-Stieltjes integrable with respect to α on $[a,+\infty)$ if and only if $\tilde{f}\alpha^{'}$ is fuzzy Henstock integrable on $[a,+\infty)$. Furthermore, we have

$$(FHS)\int_a^{+\infty}\tilde{f}(x)\mathrm{d}\alpha=(FH)\int_a^{+\infty}\tilde{f}(x)\alpha^{'}(x)\mathrm{d}x,$$

where (FH) integral denotes the fuzzy Henstock integral introduced by Wu et al. [5].

Proof. Since $\tilde{f}:[a,+\infty)\to E^1$ is bounded on $[a,+\infty)$, $\sup_{x\in[a,+\infty)}D(\tilde{f}(x),\tilde{0})$ exists. Continuity of $\alpha^{'}$ on $[a,b]$ implies uniform continuity on $[a,b]$ for any $b>a$. Hence, for each $\epsilon>0$, there exists $\eta>0$ such that

$$|\alpha^{'}(x)-\alpha^{'}(y)|<\frac{\epsilon}{3\sup_{x\in[a,+\infty)}D(\tilde{f}(x),\tilde{0})\cdot(b-a)}$$

for any $x,y\in[a,b]$ satisfying $|x-y|<\eta$. Choose a positive-valued function $\delta_1(x)$ on $[a,b]$ with $\delta_1(x)<\eta$ for all $x\in[a,b]$. Let $T=\{[x_{i-1},x_i];\xi_i\}_{i=1}^n$ be a δ_1-fine division on $[a,b]$, then by Lagrange mean value theorem, there exists $\bar{x}_i\in[x_{i-1},x_i]$ such that

$$\alpha(x_i)-\alpha(x_{i-1})=\alpha^{'}(\bar{x}_i)(\alpha(x_i)-\alpha(x_{i-1})),\qquad(1\leqslant i\leqslant n).$$

Since $|\bar{x}_i-x_i|\leqslant\delta_1(x_i)<\eta$ for $1\leqslant i\leqslant n$, we have

$$|\alpha^{'}(\bar{x}_i)-\alpha^{'}(x_i)|<\frac{\epsilon}{3\sup_{x\in[a,+\infty)}D(\tilde{f}(x),\tilde{0})\cdot(b-a)}$$

for $1\leqslant i\leqslant n$. Hence, for any δ_1-fine division $T=\{[x_{i-1},x_i];\xi_i\}_{i=1}^n$ on $[a,b]$, we have

$$
\begin{aligned}
&D(\sum_{i=1}^n \tilde{f}(\xi_i)[\alpha(x_i)-\alpha(x_{i-1})],\sum_{i=1}^n \tilde{f}(\xi_i)\alpha^{'}(\xi_i)(\alpha(x_i)-\alpha(x_{i-1})))\\
&=D(\sum_{i=1}^n \tilde{f}(\xi_i)\alpha^{'}(\bar{x}_i)(\alpha(x_i)-\alpha(x_{i-1})),\sum_{i=1}^n \tilde{f}(\xi_i)\alpha^{'}(\xi_i)(\alpha(x_i)-\alpha(x_{i-1})))\\
&\leqslant\sum_{i=1}^n D(\tilde{f}(\xi_i)\alpha^{'}(\bar{x}_i)(\alpha(x_i)-\alpha(x_{i-1})),\tilde{f}(\xi_i)\alpha^{'}(\xi_i)(\alpha(x_i)-\alpha(x_{i-1})))
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \sup_{\lambda\in[0,1]} \max\{|f_\lambda^-(\xi_i)[\alpha'(\bar{x}_i) - \alpha'(\xi_i)]|, |f_\lambda^+(\xi_i)[\alpha'(\bar{x}_i) - \alpha'(\xi_i)]|\}(\alpha(x_i) - \alpha(x_{i-1})) \\
&\leqslant \sum_{i=1}^{n} |\alpha'(\bar{x}_i) - \alpha'(\xi_i)|(\alpha(x_i) - \alpha(x_{i-1})) \sup_{\lambda\in[0,1]} \max\{|f_\lambda^-(\xi_i)|, |f_\lambda^+(\xi_i)|\} \\
&\leqslant (b-a) \cdot \frac{\epsilon}{3\sup_{x\in[a,+\infty)} D(\tilde{f}(x),\tilde{0}) \cdot (b-a)} \cdot \sup_{x\in[a,+\infty)} D(\tilde{f}(x),\tilde{0}) \\
&< \frac{\epsilon}{3}. \qquad (*)
\end{aligned}
$$

On the other hand, since $\tilde{f}$ is fuzzy Henstock-Stieltjes integrable with respect to α on $[a,+\infty)$, by Theorem 3.1, there is a function $\delta_2(x) > 0$ such that for any δ_2-fine division $T = \{[u,v];\xi\}$ and $T' = \{[u',v'];\xi'\}$ on $[a,b]$, we have

$$D(\sum_T \tilde{f}(\xi)[\alpha(v) - \alpha(u)], \sum_{T'} \tilde{f}(\xi')[\alpha(v') - \alpha(u')]) < \frac{\varepsilon}{3}.$$

Define $\delta(x)$ on $[a,b]$ by $\delta(x) = \min\{\delta_1(x), \delta_2(x)\}$. Then for any δ-fine division $T = \{[u,v];\xi\}$ and $T' = \{[u',v'];\xi'\}$ on $[a,b]$, we have

$$
\begin{aligned}
&D(\sum_T \tilde{f}(\xi)\alpha'(\xi)(v-u), \sum_{T'} \tilde{f}(\xi')\alpha'(\xi')(v'-u')) \\
&\leqslant D(\sum_T \tilde{f}(\xi)\alpha'(\xi)(v-u), \sum_T \tilde{f}(\xi)[\alpha(v) - \alpha(u)]) \\
&+ D(\sum_T \tilde{f}(\xi)[\alpha(v) - \alpha(u)], \sum_{T'} \tilde{f}(\xi')[\alpha(v') - \alpha(u')]) \\
&+ D(\sum_{T'} \tilde{f}(\xi')[\alpha(v') - \alpha(u')], \sum_{T'} \tilde{f}(\xi')\alpha'(\xi')(v'-u')) \\
&< \epsilon.
\end{aligned}
$$

Hence, $\tilde{f}\alpha'$ is Henstock integrable on $[a,b]$ for any $[a,b]$ by Theorem 2.3 of [5], and by above formula (*), we know that

$$(FHS)\int_a^b \tilde{f}(x)\mathrm{d}\alpha = (FH)\int_a^b \tilde{f}(x)\alpha'(x)\mathrm{d}x.$$

Applied Theorem 3.1, $\tilde{f}\alpha'$ is Henstock integrable on $[a,+\infty)$.

Conversely, if $\tilde{f}\alpha'$ is Henstock integrable on $[a,+\infty)$, then by Theorem 2.3 of [5], for each $\epsilon > 0$, there is a function $\delta_3(x) > 0$ such that for any δ_3-fine division $T = \{[u,v];\xi\}$ and $T' = \{[u',v'];\xi'\}$, we have

$$D(\sum_T \tilde{f}(\xi)\alpha'(\xi)(v-u), \sum_{T'} \tilde{f}(\xi')\alpha'(\xi')(v'-u')) < \frac{\varepsilon}{3}.$$

Define $\delta(x)$ on $[a,+\infty)$ by $\delta(x) = \min\{\delta_1(x), \delta_3(x)\}$. Then for any δ-fine division $T = \{[u,v];\xi\}$ and $T' = \{[u',v'];\xi'\}$, we have

$$
\begin{aligned}
&D(\sum_T \tilde{f}(\xi)[\alpha(v) - \alpha(u)], \sum_{T'} \tilde{f}(\xi')[\alpha(v') - \alpha(u')]) \\
&\leqslant D(\sum_T \tilde{f}(\xi)[\alpha(v) - \alpha(u)], \sum_T \tilde{f}(\xi)\alpha'(\xi)(v-u)) \\
&+ D(\sum_T \tilde{f}(\xi)\alpha'(\xi)(v-u), \sum_{T'} \tilde{f}(\xi')\alpha'(\xi')(v'-u')) \\
&+ D(\sum_{T'} \tilde{f}(\xi')\alpha'(\xi')(v'-u'), \sum_{T'} \tilde{f}(\xi')[\alpha(v') - \alpha(u')]) \\
&< \epsilon.
\end{aligned}
$$

Hence, $\tilde{f}$ is fuzzy Henstock-Stieltjes integrable with respect to α on $[a, +\infty)$.

In the following part, we will prove the equation $(FHS)\int_a^{+\infty}\tilde{f}\mathrm{d}\alpha = (FH)\int_a^{+\infty}\tilde{f}\alpha^{'}\mathrm{d}x$. For any division $T : a = x_0 < x_1 < x_2 <, \cdots, < x_n = b$, according to the Lagrange mean value theorem, we have

$$\alpha(x_i) - \alpha(x_{i-1}) = \alpha^{'}(\bar{x}_i)(\alpha(x_i) - \alpha(x_{i-1})), \qquad (x_{i-1} < \bar{x}_i < x_i).$$

This implies

$$\sum_{i=1}^{n}\tilde{f}(\bar{x}_i)[\alpha(x_i) - \alpha(x_{i-1})] = \sum_{i=1}^{n}\tilde{f}(\bar{x}_i)\alpha^{'}(\bar{x}_i)(\alpha(x_i) - \alpha(x_{i-1})).$$

That is

$$(FHS)\int_a^{+\infty}\tilde{f}(x)\mathrm{d}\alpha = (FH)\int_a^{+\infty}\tilde{f}(x)\alpha^{'}(x)\mathrm{d}x.$$

The proof is complete.

Theorem 3.4. Let $\alpha : [a, +\infty) \to \mathbb{R}$ be an increasing function such that $\alpha \in C^1[a, +\infty)$, $|\alpha^{'}(x)| \leqslant M$ and $\tilde{f} = \tilde{0}$ *a.e.* on $[a, +\infty)$ (i.e. $\tilde{f}(x) = \tilde{0}$ on $[a, +\infty)$ except a Lebesgue zero measure set). Then $(\tilde{f}, \alpha) \in FHS[a, +\infty)$ and $(FHS)\int_a^{+\infty}\tilde{f}(x)\mathrm{d}\alpha = \tilde{0}$.

Proof. Since $\alpha(x) \in C^1[a, +\infty)$, and $|\alpha^{'}(x)| \leqslant M$ for all $x \in [a, +\infty)$. By Lagrange mean value theorem, there exists $\xi \in [x_{i-1}, x_i]$ such that

$$\alpha(x_i) - \alpha(x_{i-1}) = \alpha^{'}(\xi)(\alpha(x_i) - \alpha(x_{i-1})) \leqslant M(\alpha(x_i) - \alpha(x_{i-1})).$$

Let $S = \{x|\tilde{f}(x) \neq \tilde{0}\}$ and for each positive integer n, set $S = \bigcup S_n \subset [a, +\infty)$, where $S_n = \{x|n-1 < D(\tilde{f}(x), \tilde{0}) \leqslant n\}, n = 1, 2, 3, \cdots$. For every $\epsilon > 0$ and a positive integer n, choose an open set G_n such that $S_n \subset G_n$ and $\mu(G_n) < \frac{\epsilon}{nM2^n}$. Define $\delta(x)$ on $[a, +\infty)$ by

$$\delta(x) = \begin{cases} 1, & x \in [a, +\infty)\backslash S, \\ \delta(x), & \text{such that } (x - \delta(x), x + \delta(x)) \subset G_n, x \in S_n, n = 1, 2, \cdots. \end{cases}$$

For any δ-fine division $T = \{[x_{i-1}, x_i]; \xi_i\}$, we have

$$\begin{aligned} D(\int_a^{+\infty}\tilde{f}(x)\mathrm{d}\alpha, \tilde{0}) &= D(\sum_{\xi_i \in S}\tilde{f}(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})] + \sum_{\xi_i \in [a,+\infty)\backslash S}\tilde{f}(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})], \tilde{0}) \\ &= D(\sum_{\xi_i \in S}\tilde{f}(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})], \tilde{0}) \leqslant \sum_{\xi_i \in S} D(\tilde{f}(\xi_i)[\alpha(x_i) - \alpha(x_{i-1})], \tilde{0}) \\ &\leqslant M\sum_{i=1}^{\infty}\sum_{\xi_i \in S_i} D(\tilde{f}(\xi_i), \tilde{0})(\alpha(x_i) - \alpha(x_{i-1})) < M\sum_{i=1}^{\infty} i \cdot \frac{\varepsilon}{iM2^i} \\ &= \varepsilon. \end{aligned}$$

The proof is complete.

Remark 3.3. Let $\alpha : [a, +\infty) \to \mathbb{R}$ be an increasing function and $\alpha \in C^1[a, +\infty)$, and $|\alpha^{'}(x)| \leqslant M$. If $\tilde{f}(x) = \tilde{g}(x)$ a.e. on $[a, +\infty)$ and $(\tilde{f}, \alpha) \in FHS[a, +\infty)$, then $(\tilde{g}, \alpha) \in FHS[a, +\infty)$ and

$$(FHS)\int_a^{+\infty}\tilde{f}(x)\mathrm{d}\alpha = (FHS)\int_a^{+\infty}\tilde{g}(x)\mathrm{d}\alpha.$$

Using Theorem 3.4, naturally, we have the following conclusion.

Theorem 3.5. Let $\alpha : [a, +\infty) \to \mathbb{R}$ be an increasing function. If $\tilde{f}(x) = \tilde{0}$ a.e.s. on $[a, +\infty)$ (i.e. $\tilde{f}(x) = \tilde{0}$ on $[a, +\infty)$ except a α-Lebesgue-Stieltjes zero measure set), then $(\tilde{f}, \alpha) \in FHS[a, +\infty)$ and $(FHS)\int_a^{+\infty}\tilde{f}(x)\mathrm{d}\alpha = \tilde{0}$.

4 Conclusion

The aim of this paper is attempt to extend the theory of the fuzzy Henstock-Stieltjes integral on a infinite interval, we firstly define and discuss the Henstock-Stieltjes integral for fuzzy-number-valued functions on a infinite interval. On the other hand, the integrability of the fuzzy Henstock-Stieltjes integral on a infinite interval are also shown and discussed. In the future, we shall consider the continuity and the differentiability of the primitive for the fuzzy Henstock-Stieltjes integral on a infinite interval, the quadrature rules for the fuzzy Henstock-Stieltjes integral on a infinite interval, the convergence theorems for sequences of the fuzzy Henstock-Stieltjes integrable functions on a infinite interval, and so on.

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New weighted q-Čebyšev-Grüss type inequalities for double integrals

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Abstract: In this paper, we establish the weighted double q-integrals Montgomery identity for functions of two independent variables, then obtain weighted q-Čebyšev-Grüss type inequalities for double integrals. Furthermore, weighted q-Ostrowski type inequalities for double integrals are also given.

Keywords: Čebyšev-Grüss type inequalities; Ostrowski type inequalities; Montgomery identity; double q-integrals

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1 Introduction and preliminaries

1882, P.L. Čebyšev [7] prove that, if $f', g' \in L_\infty[a,b]$, then

$$|T(f,g)| \leq \frac{1}{12}(b-a)^2\|f'\|_\infty\|g'\|_\infty, \tag{1.1}$$

where for two functions $f,g:[a,b]\to\mathbb{R}$, the functional

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right), \tag{1.2}$$

and $\|\cdot\|_\infty$ denotes the norm in $L_\infty[a,b]$ defined as $\|f\|_\infty = ess \sup\limits_{t\in[a,b]}|f(t)|$.

In 1935, G. Grüss [13] showed that

$$|T(f,g)| \leq \frac{1}{4}(M-m)(N-n), \tag{1.3}$$

provided m, M, n and N are real numbers satisfying the conditions,

$$m \leq f(x) \leq M, \quad n \leq g(x) \leq N, \tag{1.4}$$

for all $x\in[a,b]$, where $T(f,g)$ is as defined by (1.2).

In 1938, Ostrowski [19] proved the following integral inequality:

Let $f: I\to\mathbb{R}$, where $I\subset\mathbb{R}$ is an interval, be a mapping that is differentiable in the interior of I ($IntI$), and let $a,b\in IntI$, $a<b$. If $|f'(t)|\leq M$, $\forall t\in(a,b)$, then,

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)M, \tag{1.5}$$

for all $x\in[a,b]$.

During the past few years, many researchers have given considerable attention to the above results and various generalizations, extensions and variants of these inequalities (1.1), (1.3) and (1.5) have appeared in the literature, see [1, 2, 3, 6, 8, 9, 11, 12, 16, 17, 18, 20, 21, 22] and the references cited therein. Find new

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inequalities in the multidimensional cases still an interesting problem. In [4, 10], the authors proved the double integrals Montgomery identity:

$$f(x,y) = \frac{1}{b-a}\int_a^b f(t,y)dt + \frac{1}{d-c}\int_c^d f(x,s)ds - \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dtds + \int_a^b\int_c^d P(x,t)Q(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dtds, \quad (1.6)$$

where $f : [a,b]\times[c,d] \to \mathbb{R}$ is differentiable, the derivative $\frac{\partial^2 f(t,s)}{\partial t\partial s}$ is integrable on $[a,b]\times[c,d]$, and the Peano kernels $P(x,t)$ and $Q(y,x)$ are defined by

$$P(x,t) = \begin{cases} \frac{t-a}{b-a}, & a \le t \le x, \\ \frac{t-b}{b-a}, & x \le t \le b \end{cases} \quad \text{and} \quad Q(y,s) = \begin{cases} \frac{s-c}{d-c}, & c \le s \le y, \\ \frac{s-d}{d-c}, & y \le s \le d. \end{cases}$$

Furthermore, Guezane-Lakoud and Aissaoui [14] established new extension of the weighted Montgomery identity (1.6) for functions of two independent variables, then obtained new Čebyšev type inequalities.

For the sake of convenience, some definitions and propositions are cited on q-integral as follows. Some details see [5, 15].

In what follows, q is a real number satisfying $0 < q < 1$.

Definition 1.1 ([5]). For an arbitrary function $f(x)$, the q-differential is defined by $(d_q f)(x) = f(qx) - f(x)$. In particular, $d_q x = (q-1)x$. q-derivative is defined by

$$(D_q f)(x) = \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{(q-1)x}, \quad (D_q f)(0) = \lim_{x\to 0}(D_q f)(x).$$

Clearly, if $f(x)$ is differentiable, then $\lim_{q\to 1^-}(D_q f)(x) = \frac{df(x)}{dx}$. And q-derivatives of higher order by

$$(D_q^0 f)(x) = f(x) \quad \text{and} \quad (D_q^n f)(x) = D_q(D_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

Definition 1.2 ([5]). Suppose $0 < a < b$. The definite q-integral is defined as

$$(I_q f)(x) = \int_0^x f(t)d_q t = x(1-q)\sum_{n=0}^{\infty} f(xq^n)q^n, \quad x \in [0,b]. \quad (1.7)$$

and

$$\int_a^b f(x)d_q x = \int_0^b f(x)d_q x - \int_0^a f(x)d_q x. \quad (1.8)$$

Similarly as done for derivatives, an operator I_q^n can be defined, namely,

$$(I_q^0 f)(x) = f(x) \quad \text{and} \quad (I_q^n f)(x) = I_q(I_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

The definite q-integral defined above is too general for our purpose of studying inequalities. For example, if $f(x) \ge 0$, it is not necessarily true that $\int_a^b f(x)d_q x \ge 0$.

From now on, we will use a special type of the definite q-integral, which we will call the restricted definite q-integral. Throughout all the paper, we will use the following notations:

$$c_j = bq^j, \quad \text{for} \quad j \in \{0, 1, \cdots, n\}, \quad a = c_n = bq^n.$$

Definition 1.3 ([5]). Let $0 < q < 1$, $b > 0$, and $n \in \mathbb{Z}^+$. The restricted q-integral is defined as $\int_{bq^n}^b f(x)d_q x$.

The following formula readily follows from (1.7) and (1.8):

$$\int_a^b f(x)d_qx = \int_{bq^n}^b f(x)d_qx = (1-q)b\sum_{j=0}^{n-1} q^j f(bq^j) = (1-q)\sum_{j=0}^{n-1} c_j f(c_j).$$

Note that the restricted integral $\int_a^b f(x)d_qx$ is just a finite sum, so no questions about convergency arise. It is easy to check that

$$\int_a^b D_q f d_qx = f(b) - f(a).$$

Obviously, if $f(x) \geq g(x)$ on $[a,b]$, then $\int_a^b f(x)d_qx \geq \int_a^b g(x)d_qx$. If $0 < k < n$, then

$$\int_a^b f(x)d_qx = \int_a^{c_k} f(x)d_qx + \int_{c_k}^b f(x)d_qx.$$

The following is the formula for the q-integration by parts:

$$\int_a^b f(x)(D_qg)(x)d_qx = [f(x)g(x)]_a^b - \int_a^b g(qx)(D_qf)(x)d_qx.$$

Cauciman [15] gave q-integral Grüss's inequality as follows: *Assume that* (1.4) *holds, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)d_qx - \left(\frac{1}{b-a}\int_a^b f(x)d_qx\right)\left(\frac{1}{b-a}\int_a^b g(x)d_qx\right)\right| \leq \frac{1}{4}(M-m)(N-n).$$

Assume that $w : [a,b] \to [0,\infty)$ satisfying $\int_a^b w(x)d_qx = 1$. Set $W(t) = \int_a^t w(x)d_qx$ for $t \in [a,b]$, $W(t) = 0$ for $t < a$, and $W(t) = 1$ for $t > b$. We give weighted q-integral Peano kernel $P_w(x,t)$ defined by

$$P_w(x,t) = \begin{cases} W(t), & a \leq t \leq x, \\ W(t)-1, & x \leq t \leq b. \end{cases}$$

Then the following weighted q-integral Montgomery identity holds: (see [23])

$$f(x) = \int_a^b w(t)f(qt)d_qt + \int_a^b P_w(x,t)(D_qf)(t)d_qt.$$

In 2011, Yang [23] obtained the following inequalities:

$$|T(w,f,g)| \leq \|D_qf\|\|D_qg\|\int_a^b w(x)H^2(qx)d_qx,$$

and

$$|T(w,f,g)| \leq \frac{1}{2}\int_a^b w(x)[|g(qx)|\|D_qf\| + |f(qx)|\|D_qg\|]H(qx)d_qx,$$

where $\|\cdot\|$ as $\|h\| = \sup_{t\in[a,b]}|h(t)|$ for $h \in C[a,b]$,

$$T(w,f,g) = \int_a^b w(x)f(qx)g(qx)d_qx - \left(\int_a^b w(x)f(qx)d_qx\right)\left(\int_a^b w(x)g(qx)d_qx\right),$$

and

$$H(x) = \int_a^b |P_w(x,t)|d_qt$$

for all $x \in [a,b]$.

Motivated by the results mentioned above, by using weighted q-integral Montgomery identity for functions of two independent variables, we establish some new weighted q-Čebyšev type inequalities for double integrals. Furthermore, weighted q-Ostrowski type inequalities for double integrals are also given.

2 Weighted q-Čebyšev type inequalities for double integrals

Assume that $w : [a, b] \to \mathbb{R}_0 = [0, \infty)$ and $u : [c, d] \to \mathbb{R}_0$ satisfying $\int_a^b w(x)d_{q_1}x = \int_c^d u(y)d_{q_2}y = 1$, where $0 < q_1, q_2 < 1$. Set $W(t) = \int_a^t w(x)d_{q_1}x$ for $t \in [a, b]$ and $U(s) = \int_a^s u(y)d_{q_2}y$ for $s \in [c, d]$, so we have $W(a) = U(c) = 0$ and $W(b) = U(b) = 1$. We give the following weighted q-integral Peano kernels $P_w(x, t)$ and $Q_u(y, t)$ defined by

$$P_w(x,t) = \begin{cases} W(t), & a \le t \le x, \\ W(t) - 1, & x \le t \le b \end{cases} \quad \text{and} \quad Q_u(y,s) = \begin{cases} U(s), & c \le s \le y, \\ U(s) - 1, & y \le s \le d. \end{cases} \tag{2.1}$$

We use the following notations to simplify details of the presentation. Let $\frac{\partial f(t,s)}{d_{q_1}t}$ and $\frac{\partial f(t,s)}{d_{q_2}s}$ be partial q-derivative on t and s, respectively. For some suitable functions $w : [a, b] \to \mathbb{R}_0$, $u : [c, d] \to \mathbb{R}_0$ and $f, g : \Omega = [a, b] \times [c, d] \to \mathbb{R}$, we set

$$\begin{aligned} T(w,u,f,g) = &\int_a^b \int_c^d w(x)u(y)f(q_1x,q_2y)g(q_1x,q_2y)d_{q_1}xd_{q_2}y \\ &- \int_a^b \int_c^d w(x)u(y)g(q_1x,q_2y)\left(\int_a^b w(t)f(q_1t,q_2y)d_{q_1}t\right)d_{q_1}xd_{q_2}y \\ &- \int_a^b \int_c^d w(x)u(y)g(q_1x,q_2y)\left(\int_c^d u(s)f(q_1x,q_2s)d_{q_2}s\right)d_{q_1}xd_{q_2}y \\ &+ \left(\int_a^b \int_c^d w(x)u(y)f(q_1x,q_2y)d_{q_1}xd_{q_2}y\right)\left(\int_a^b \int_c^d w(x)u(y)g(q_1x,q_2y)d_{q_1}xd_{q_2}y\right) \end{aligned}$$

and define $\|\cdot\|$ as $\|h\| = \sup_{(t,s)\in\Omega} |h(t, s)|$ for $h \in C(\Omega, \mathbb{R})$.

Theorem 2.1. *Let $f : \Omega \to \mathbb{R}$, $w : [a, b] \to \mathbb{R}_0$ and $u : [c, d] \to \mathbb{R}_0$ satisfying $\int_a^b w(x)d_{q_1}x = \int_c^d u(y)d_{q_2}y = 1$, then*

$$\begin{aligned} f(x,y) = \int_a^b w(t)f(q_1t,y)d_{q_1}t + \int_c^d u(s)f(x,q_2s)d_{q_2}s - \int_a^b \int_c^d w(t)u(s)f(q_1t,q_2s)d_{q_1}td_{q_2}s \\ + \int_a^b \int_c^d P_w(x,t)Q_u(y,s)\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s, \end{aligned} \tag{2.2}$$

for $(x, y) \in \Omega$, where the weighted q-integral Peano kernels $P_w(x, t)$ and $Q_u(s, y)$ are defined by (2.1).

Proof. According to the weighted q-integral Peano kernels $P_w(x, t)$ and $Q_u(s, y)$ and the proof of Theorem 1 in [23], we obtain

$$\begin{aligned} &\int_a^b \int_c^d P_w(x,t)Q_u(y,s)\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s = \int_a^b P_w(x,t)\left(\int_c^d Q_u(y,s)\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}d_{q_2}s\right)d_{q_1}t \\ &= \int_a^b P_w(x,t)\left(\frac{\partial f(t,y)}{d_{q_1}t} - \int_c^d u(s)\frac{\partial f(t,q_2s)}{d_{q_1}t}d_{q_2}s\right)d_{q_1}t \\ &= \int_a^b P_w(x,t)\frac{\partial f(t,y)}{d_{q_1}t}d_{q_1}t - \int_c^d u(s)\left(\int_a^b P_w(x,t)\frac{\partial f(t,q_2s)}{d_{q_1}t}d_{q_1}t\right)d_{q_2}s \\ &= \left(f(x,y) - \int_a^b w(t)f(q_1t,y)d_{q_1}t\right) - \int_c^d u(s)\left(f(x,q_2s) - \int_a^b w(t)f(q_1t,q_2s)d_{q_1}t\right)d_{q_2}s \\ &= f(x,y) - \int_a^b w(t)f(q_1t,y)d_{q_1}t - \int_c^d u(s)f(x,q_2s)d_{q_2}s + \int_a^b \int_c^d w(t)u(s)f(q_1t,q_2s)d_{q_1}td_{q_2}s. \end{aligned}$$

Thus we have

$$f(x,y)=\int_a^b w(t)f(q_1t,y)d_{q_1}t+\int_c^d u(s)f(t,q_2s)d_{q_2}s-\int_a^b\int_c^d w(t)u(s)f(q_1t,q_2s)d_{q_1}td_{q_2}s$$
$$+\int_a^b\int_c^d P_w(x,t)Q_u(y,s)\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s,$$

and this completes the proof. □

Theorem 2.2. *Let $f,g:\Omega\to\mathbb{R}$, $w:[a,b]\to\mathbb{R}_0$ and $u:[c,d]\to\mathbb{R}_0$ satisfying $\int_a^b w(x)d_{q_1}x=\int_c^d u(y)d_{q_2}y=1$, then*

$$|T(w,u,f,g)|\le\left\|\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}\right\|\left\|\frac{\partial g(t,s)}{d_{q_1}td_{q_2}s}\right\|\int_a^b\int_c^d w(x)u(y)H^2(q_1x,q_2y)d_{q_1}xd_{q_2}y, \tag{2.3}$$

where

$$H(x,y)=\int_a^b\int_c^d|P_w(x,t)Q_u(y,s)|d_{q_1}td_{q_2}s.$$

Proof. Since the functions f and g satisfy the hypothesis of Theorem 2.1, the following identities hold:

$$f(x,y)=\int_a^b w(t)f(q_1t,y)d_{q_1}t+\int_c^d u(s)f(t,q_2s)d_{q_2}s-\int_a^b\int_c^d w(t)u(s)f(q_1t,q_2s)d_{q_1}td_{q_2}s$$
$$+\int_a^b\int_c^d P_w(x,t)Q_u(y,s)\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s, \tag{2.4}$$

and

$$g(x,y)=\int_a^b w(t)g(q_1t,y)d_{q_1}t+\int_c^d u(s)g(x,q_2s)d_{q_2}s-\int_a^b\int_c^d w(t)u(s)g(q_1t,q_2s)d_{q_1}td_{q_2}s$$
$$+\int_a^b\int_c^d P_w(x,t)Q_u(y,s)\frac{\partial g(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s. \tag{2.5}$$

Due to the above two inequalities (2.4) and (2.5), we have

$$f(q_1x,q_2y)=\int_a^b w(t)f(q_1t,q_2y)d_{q_1}t+\int_c^d u(s)f(q_1x,q_2s)d_{q_2}s-\int_a^b\int_c^d w(t)u(s)f(q_1t,q_2s)d_{q_1}td_{q_2}s$$
$$+\int_a^b\int_c^d P_w(q_1x,t)Q_u(q_2y,s)\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s, \tag{2.6}$$

and

$$g(q_1x,q_2y)=\int_a^b w(t)g(q_1t,q_2y)d_{q_1}t+\int_c^d u(s)g(q_1x,q_2s)d_{q_2}s-\int_a^b\int_c^d w(t)u(s)g(q_1t,q_2s)d_{q_1}td_{q_2}s$$
$$+\int_a^b\int_c^d P_w(q_1x,t)Q_u(q_2y,s)\frac{\partial g(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s. \tag{2.7}$$

Multiplying (2.6) by (2.7), we obtain

$$\left(f(q_1x,q_2y)-\int_a^b w(t)f(q_1t,q_2y)d_{q_1}t-\int_c^d u(s)f(q_1x,q_2s)d_{q_2}s+\int_a^b\int_c^d w(t)u(s)f(q_1t,q_2s)d_{q_1}td_{q_2}s\right)$$
$$\times\left(g(q_1x,q_2y)-\int_a^b w(t)g(q_1t,q_2y)d_{q_1}t-\int_c^d u(s)g(q_1x,q_2s)d_{q_2}s+\int_a^b\int_c^d w(t)u(s)g(q_1t,q_2s)d_{q_1}td_{q_2}s\right)$$
$$=\left(\int_a^b\int_c^d P_w(q_1x,t)Q_u(q_2y,s)\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s\right)\left(\int_a^b\int_c^d P_w(q_1x,t)Q_u(q_2y,s)\frac{\partial g(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s\right).$$

Consequently,

$$
\begin{aligned}
&f(q_1x,q_2y)g(q_1x,q_2y) - f(q_1x,q_2y)\int_a^b w(t)g(q_1t,q_2y)d_{q_1}t - f(q_1x,q_2y)\int_c^d u(s)g(q_1x,q_2s)d_{q_2}s \\
&\quad + f(q_1x,q_2y)\int_a^b\int_c^d w(t)u(s)g(q_1t,q_2s)d_{q_1}td_{q_2}s - g(q_1x,q_2y)\int_a^b w(t)f(q_1t,q_2y)d_{q_1}t \\
&\quad + \int_a^b w(t)f(q_1t,q_2y)d_{q_1}t\int_a^b w(t)g(q_1t,q_2y)d_{q_1}t + \int_a^b w(t)f(q_1t,q_2y)d_{q_1}t\int_c^d u(s)g(q_1x,q_2s)d_{q_2}s \\
&\quad - \int_a^b w(t)f(q_1t,q_2y)d_{q_1}t\int_a^b\int_c^d w(t)u(s)g(q_1t,q_2s)d_{q_1}td_{q_2}s - g(q_1x,q_2y)\int_c^d u(s)f(q_1x,q_2s)d_{q_2}s \\
&\quad + \int_c^d u(s)f(q_1x,q_2s)d_{q_2}s\int_a^b w(t)g(q_1t,q_2y)d_{q_1}t + \int_c^d u(s)f(q_1x,q_2s)d_{q_2}s\int_c^d u(s)g(q_1x,q_2s)d_{q_2}s \\
&- \int_c^d u(s)f(q_1x,q_2s)d_{q_2}s\int_a^b\int_c^d w(t)u(s)g(q_1t,q_2s)d_{q_1}td_{q_2}s + g(q_1x,q_2y)\int_a^b\int_c^d w(t)u(s)f(q_1t,q_2s)d_{q_1}td_{q_2}s \\
&\quad - \int_a^b w(t)g(q_1t,q_2y)d_{q_1}t\int_a^b\int_c^d w(t)u(s)f(q_1t,q_2s)d_{q_1}td_{q_2}s - \int_a^b\int_c^d w(t)u(s)f(q_1t,q_2s)d_{q_1}td_{q_2}s \\
&\quad \times \int_c^d u(s)g(q_1x,q_2s)d_{q_2}s + \int_a^b\int_c^d w(t)u(s)f(q_1t,q_2s)d_{q_1}td_{q_2}s\int_a^b\int_c^d w(t)u(s)g(q_1t,q_2s)d_{q_1}td_{q_2}s \\
&= \left(\int_a^b\int_c^d P_w(q_1x,t)Q_u(q_2y,s)\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s\right)\left(\int_a^b\int_c^d P_w(q_1x,t)Q_u(q_2y,s)\frac{\partial g(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s\right) \quad (2.8)
\end{aligned}
$$

Multiplying both sides of (2.8) by $w(x)u(y)$, then q-integrating the resultant identity over Ω, we get

$$
\begin{aligned}
T(w,u,f,g) = \int_a^b\int_c^d w(x)u(y)&\left[\left(\int_a^b\int_c^d P_w(q_1x,t)Q_u(q_2y,s)\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s\right)\right. \\
&\left.\times\left(\int_a^b\int_c^d P_w(q_1x,t)Q_u(q_2y,s)\frac{\partial g(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s\right)\right]d_{q_1}xd_{q_2}y.
\end{aligned}
$$

Finally, using the properties of modulus we observe that

$$
\begin{aligned}
|T(w,u,f,g)| &\le \left\|\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}\right\|\left\|\frac{\partial g(t,s)}{d_{q_1}td_{q_2}s}\right\|\int_a^b\int_c^d w(x)u(y)\left[\left(\int_a^b\int_c^d |P_w(q_1x,t)Q_u(q_2y,s)|d_{q_1}td_{q_2}s\right)\right. \\
&\quad\left.\times\left(\int_a^b\int_c^d |P_w(q_1x,t)Q_u(q_2y,s)|d_{q_1}td_{q_2}s\right)\right]d_{q_1}xd_{q_2}y \\
&= \left\|\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}\right\|\left\|\frac{\partial g(t,s)}{d_{q_1}td_{q_2}s}\right\|\int_a^b\int_c^d w(x)u(y)H^2(q_1x,q_2y)d_{q_1}xd_{q_2}y.
\end{aligned}
$$

This completes the proof of Theorem 2.2. □

Theorem 2.3. *Let $f,g:\Omega\to\mathbb{R}$, $w:[a,b]\to\mathbb{R}_0$ and $u:[c,d]\to\mathbb{R}_0$ satisfying $\int_a^b w(x)d_{q_1}x = \int_c^d u(y)d_{q_2}y = 1$, then*

$$
\begin{aligned}
|T(w,u,f,g)| &\le \frac{1}{2}\int_a^b\int_c^d w(x)u(y) \\
&\quad\times\left(|g(q_1x,q_2y)|\left\|\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}\right\| + |f(q_1x,q_2y)|\left\|\frac{\partial g(t,s)}{d_{q_1}td_{q_2}s}\right\|\right)H(q_1x,q_2y)d_{q_1}xd_{q_2}y, \quad (2.9)
\end{aligned}
$$

where $H(x,y)$ is defined in Theorem 2.2.

Proof. Multiplying both sides of (2.6) and (2.7) by $w(x)u(y)g(q_1x,q_2y)$ and $w(x)u(y)f(q_1x,q_2y)$, adding the resulting identities and rewriting, we have

$$\begin{aligned}
&w(x)u(y)f(q_1x,q_2y)g(q_1x,q_2y)\\
&\quad=\frac{1}{2}\Bigg(w(x)u(y)g(q_1x,q_2y)\int_a^b w(t)f(q_1t,q_2y)d_{q_1}t+w(x)u(y)g(q_1x,q_2y)\int_c^d u(s)f(q_1x,q_2s)d_{q_2}s\\
&-w(x)u(y)g(q_1x,q_2y)\int_a^b\int_c^d w(t)u(s)f(q_1t,q_2s)d_{q_1}td_{q_2}s+w(x)u(y)f(q_1x,q_2y)\int_a^b w(t)g(q_1t,q_2y)d_{q_1}t\\
&+w(x)u(y)f(q_1x,q_2y)\int_c^d u(s)g(q_1x,q_2s)d_{q_2}s-w(x)u(y)f(q_1x,q_2y)\int_a^b\int_c^d w(t)u(s)g(q_1t,q_2s)d_{q_1}td_{q_2}s\Bigg)\\
&\qquad+\frac{1}{2}\Bigg(w(x)u(y)g(q_1x,q_2y)\int_a^b\int_c^d P_w(q_1x,t)Q_u(q_2y,s)\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s\\
&\qquad\quad+w(x)u(y)f(q_1x,q_2y)\int_a^b\int_c^d P_w(q_1x,t)Q_u(q_2y,s)\frac{\partial g(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s\Bigg).
\end{aligned}\tag{2.10}$$

Q-integrating both sides of (2.10) with respect to x from a to b and y from c to d and rewriting we have

$$\begin{aligned}
T(w,uf,g)=&\frac{1}{2}\Bigg(\int_a^b\int_c^d w(x)u(y)g(q_1x,q_2y)\left(\int_a^b\int_c^d P_w(q_1x,t)Q_u(q_2y,s)\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s\right)d_{q_1}xd_{q_2}y\\
&+\int_a^b\int_c^d w(x)u(y)f(q_1x,q_2y)\left(\int_a^b\int_c^d P_w(q_1x,t)Q_u(q_2y,s)\frac{\partial g(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s\right)d_{q_1}xd_{q_2}y\Bigg).
\end{aligned}\tag{2.11}$$

Finally, from (2.11) and using the properties of modulus we observe that

$$\begin{aligned}
T(w,uf,g)\le&\frac{1}{2}\Bigg(\int_a^b\int_c^d w(x)u(y)|g(q_1x,q_2y)|\left(\int_a^b\int_c^d |P_w(q_1x,t)Q_u(q_2y,s)|\left|\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}\right|d_{q_1}td_{q_2}s\right)d_{q_1}xd_{q_2}y\\
&+\int_a^b\int_c^d w(x)u(y)|f(q_1x,q_2y)|\left(\int_a^b\int_c^d |P_w(q_1x,t)Q_u(q_2y,s)|\left|\frac{\partial g(t,s)}{d_{q_1}td_{q_2}s}\right|d_{q_1}td_{q_2}s\right)d_{q_1}xd_{q_2}y\Bigg)\\
\le&\frac{1}{2}\int_a^b\int_c^d w(x)u(y)\left(|g(q_1x,q_2y)|\left\|\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}\right\|+|f(q_1x,q_2y)|\left\|\frac{\partial g(t,s)}{d_{q_1}td_{q_2}s}\right\|\right)\\
&\times\left(\int_a^b\int_c^d |P_w(q_1x,t)Q_u(q_2y,s)d_{q_1}td_{q_2}s\right)d_{q_1}xd_{q_2}y\\
=&\frac{1}{2}\int_a^b\int_c^d w(x)u(y)\left(|g(q_1x,q_2y)|\left\|\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}\right\|+|f(q_1x,q_2y)|\left\|\frac{\partial g(t,s)}{d_{q_1}td_{q_2}s}\right\|\right)H(q_1x,q_2y)d_{q_1}xd_{q_2}y.
\end{aligned}$$

This completes the proof of Theorem 2.3. □

Remark 2.4. If $q_1,q_2\to 1^-$, by Definitions 1.2 and 1.2, the partial q-derivative and double q-integrals are the usual partial derivative and double integrals, so Theorems 2.2 and 2.3 are reduced to Theorems 3 and 4 in [14].

Theorem 2.5. *Let $f,g:\Omega\to\mathbb{R}$, $w:[a,b]\to\mathbb{R}_0$ and $u:[c,d]\to\mathbb{R}_0$ satisfying $\int_a^b w(x)d_{q_1}x=\int_c^d u(y)d_{q_2}y=1$, then*

$$|T(w,u,f,g)|\le\|g(q_1x,q_2y)\|\left\|\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}\right\|\int_a^b\int_c^d w(x)u(y)H(q_1x,q_2y)d_{q_1}xd_{q_2}y,\tag{2.12}$$

and

$$|T(w,u,f,g)| \leq \|f(q_1x,q_2y)\| \left\| \frac{\partial g(t,s)}{d_{q_1}td_{q_2}s} \right\| \int_a^b \int_c^d w(x)u(y)H(q_1x,q_2y)d_{q_1}xd_{q_2}y, \tag{2.13}$$

where $H(x,y)$ is defined in Theorem 2.2.

Proof. We prove only (2.12), since the proof of (2.13) is similar. The identity (2.6) shows that

$$f(q_1x,q_2y) = \int_a^b w(t)f(q_1t,q_2y)d_{q_1}t + \int_c^d u(s)f(q_1x,q_2s)d_{q_2}s - \int_a^b \int_c^d w(t)u(s)f(q_1t,q_2s)d_{q_1}td_{q_2}s + \int_a^b \int_c^d P_w(q_1x,t)Q_u(q_2y,s)\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s, \tag{2.14}$$

for $(x,y) \in \Omega$.

Now, if we multiply (2.14) by $w(x)u(y)g(q_1x,q_2y)$ and q-integrate over $(x,y) \in \Omega$, we deduce

$$\begin{aligned}
&\int_a^b \int_c^d w(x)u(y)f(q_1x,q_2y)g(q_1x,q_2y)d_{q_1}xd_{q_2}y \\
=&\int_a^b \int_c^d w(x)u(y)g(q_1x,q_2y)\left(\int_a^b w(t)f(q_1t,q_2y)d_{q_1}t\right)d_{q_1}xd_{q_2}y \\
&+\int_a^b \int_c^d w(x)u(y)g(q_1x,q_2y)\left(\int_c^d u(s)f(q_1x,q_2s)d_{q_2}s\right)d_{q_1}xd_{q_2}y \\
&-\int_a^b \int_c^d w(x)u(y)g(q_1x,q_2y)d_{q_1}xd_{q_2}y \int_a^b \int_c^d w(t)u(s)f(q_1t,q_2s)d_{q_1}td_{q_2}s \\
&+\int_a^b \int_c^d w(x)u(y)g(q_1x,q_2y)\left(\int_a^b \int_c^d P_w(q_1x,t)Q_u(q_2y,s)\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s\right)d_{q_1}xd_{q_2}y,
\end{aligned}$$

which provides another representation for the functional $T(w,u,f,g)$ namely,

$$T(w,u,f,g) = \int_a^b \int_c^d w(x)u(y)g(q_1x,q_2y)\left(\int_a^b \int_c^d P_w(q_1x,t)Q_u(q_2y,s)\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s\right)d_{q_1}xd_{q_2}y, \tag{2.15}$$

From (2.15) and using modules properties, it yields

$$\begin{aligned}
|T(w,u,f,g)| \leq &\int_a^b \int_c^d w(x)u(y)|g(q_1x,q_2y)|\left(\int_a^b \int_c^d |P_w(q_1x,t)Q_u(q_2y,s)|\left|\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}\right|d_{q_1}td_{q_2}s\right)d_{q_1}xd_{q_2}y \\
\leq &\|g(q_1x,q_2y)\|\left\|\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}\right\|\int_a^b \int_c^d w(x)u(y)\left(\int_a^b \int_c^d |P_w(q_1x,t)Q_u(q_2y,s)|d_{q_1}td_{q_2}s\right)d_{q_1}xd_{q_2}y \\
= &\|g(q_1x,q_2y)\|\left\|\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}\right\|\int_a^b \int_c^d w(x)u(y)H(q_1x,q_2y)d_{q_1}xd_{q_2}y.
\end{aligned}$$

This completes the proof of Theorem 2.5. □

3 Weighted q-Ostrowski type inequalities for double integrals

For some given functions $f, g : \Omega \to \mathbb{R}$ and $w : [a,b] \to \mathbb{R}_0$ and $u : [c,d] \to \mathbb{R}_0$ satisfying $\int_a^b w(x) d_{q_1} x = \int_c^d u(y) d_{q_2} y = 1$,

$$\begin{aligned} S(w,u,f,g) = f(q_1x,q_2y)g(q_1x,q_2y) - \frac{1}{2}\Bigg(& f(q_1x,q_2y)\int_a^b w(t)g(q_1t,q_2y)d_{q_1}t + \int_c^d u(s)g(q_1x,q_2s)d_{q_2}s \\ & \times f(q_1x,q_2y) + g(q_1x,q_2y)\int_a^b w(t)f(q_1t,q_2y)d_{q_1}t + g(q_1x,q_2y)\int_c^d u(s)f(q_1x,q_2s)d_{q_2}s \\ & - f(q_1x,q_2y)\int_a^b\int_c^d w(t)u(s)g(q_1t,q_2s)d_{q_1}td_{q_2}s - g(q_1x,q_2y)\int_a^b\int_c^d w(t)u(s)f(q_1t,q_2s)d_{q_1}td_{q_2}s\Bigg). \end{aligned}$$

Theorem 3.1. *Let $f, g : \Omega \to \mathbb{R}$, $w : [a,b] \to \mathbb{R}_0$ and $u : [c,d] \to \mathbb{R}_0$ satisfying $\int_a^b w(x)d_{q_1}x = \int_c^d u(y)d_{q_2}y = 1$, then*

$$|S(w,u,f,g)| \le \frac{1}{2}\left(|f(q_1x,q_2y)|\left\|\frac{\partial g(t,s)}{d_{q_1}td_{q_2}s}\right\| + |g(q_1x,q_2y)|\left\|\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}\right\|\right)H(q_1x,q_2y), \tag{3.1}$$

and

$$|T(w,u,f,g)| \le \frac{1}{2}\left(|f(q_1x,q_2y)|\left\|\frac{\partial g(t,s)}{d_{q_1}td_{q_2}s}\right\| + |g(q_1x,q_2y)|\left\|\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}\right\|\right)\int_a^b\int_c^d w(x)u(y)H(q_1x,q_2y)d_{q_1}xd_{q_2}y. \tag{3.2}$$

where $H(x,y)$ is defined in Theorem 2.2.

Proof. Multiplying both sides of (2.6) and (2.7) by $g(q_1x,q_2y)$ and $f(q_1x,q_2y)$, adding the resulting identities and rewriting we have

$$\begin{aligned} f(q_1x,q_2y)g(q_1x,q_2y) = \frac{1}{2}\Bigg(& f(q_1x,q_2y)\int_a^b w(t)g(q_1t,q_2y)d_{q_1}t + f(q_1x,q_2y)\int_c^d u(s)g(q_1x,q_2s)d_{q_2}s \\ & + g(q_1x,q_2y)\int_a^b w(t)f(q_1t,q_2y)d_{q_1}t + g(q_1x,q_2y)\int_c^d u(s)f(q_1x,q_2s)d_{q_2}s \\ & - f(q_1x,q_2y)\int_a^b\int_c^d w(t)u(s)g(q_1t,q_2s)d_{q_1}td_{q_2}s - g(q_1x,q_2y)\int_a^b\int_c^d w(t)u(s)f(q_1t,q_2s)d_{q_1}td_{q_2}s\Bigg) \\ & + \frac{1}{2}\Bigg(f(q_1x,q_2y)\int_a^b\int_c^d P_w(q_1x,t)Q_u(q_2y,s)\frac{\partial g(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s \\ & + g(q_1x,q_2y)\int_a^b\int_c^d P_w(q_1x,t)Q_u(q_2y,s)\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s\Bigg), \end{aligned} \tag{3.3}$$

which implies

$$\begin{aligned} S(w,u,f,g) = \frac{1}{2}\Bigg(& f(q_1x,q_2y)\int_a^b\int_c^d P_w(q_1x,t)Q_u(q_2y,s)\frac{\partial g(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s \\ & + g(q_1x,q_2y)\int_a^b\int_c^d P_w(q_1x,t)Q_u(q_2y,s)\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s\Bigg). \end{aligned}$$

We observe

$$|S(w,u,f,g)| \le \frac{1}{2}\left(|f(q_1x,q_2y)|\left\|\frac{\partial g(t,s)}{d_{q_1}td_{q_2}s}\right\| + |g(q_1x,q_2y)|\left\|\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}\right\|\right)H(q_1x,q_2y).$$

Multiplying both sides of (3.3) by $w(x)u(y)$ and q-integrate over $(x,y)\in\Omega$, we deduce

$$T(w,u,f,g)=\frac{1}{2}\left(\int_a^b\int_c^d w(x)u(y)f(q_1x,q_2y)\left(\int_a^b\int_c^d P_w(q_1x,t)Q_u(q_2y,s)\frac{\partial g(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s\right)d_{q_1}xd_{q_2}y\right.$$
$$\left.+\int_a^b\int_c^d w(x)u(y)g(q_1x,q_2y)\left(\int_a^b\int_c^d P_w(q_1x,t)Q_u(q_2y,s)\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}d_{q_1}td_{q_2}s\right)d_{q_1}xd_{q_2}y\right).$$

We observe

$$|T(w,u,f,g)|\le\frac{1}{2}\left(|f(q_1x,q_2y)|\left\|\frac{\partial g(t,s)}{d_{q_1}td_{q_2}s}\right\|+|g(q_1x,q_2y)|\left\|\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}\right\|\right)\int_a^b\int_c^d w(x)u(y)H(q_1x,q_2y)d_{q_1}xd_{q_2}y.$$

This completes the proof of Theorem 3.1. □

Let $g(x,y)=1$, we have the following corollary.

Corollary 3.2. *Let $f:\Omega\to\mathbb{R}$, $w:[a,b]\to\mathbb{R}_0$ and $u:[c,d]\to\mathbb{R}_0$ satisfying $\int_a^b w(x)d_{q_1}x=\int_c^d u(y)d_{q_2}y=1$, then*

$$\left|f(x,y)-\int_a^b w(t)f(q_1t,y)d_{q_1}t-\int_c^d u(s)f(x,q_2s)d_{q_2}s+\int_a^b\int_c^d w(t)u(s)f(q_1t,q_2s)d_{q_1}td_{q_2}s\right|$$
$$\le H(x,y)\left\|\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}\right\|,$$

where $H(x,y)$ is defined in Theorem 2.2, and especially, let $w(x)=\frac{1}{b-a}$ and $u(y)=\frac{1}{d-c}$, we get

$$\left|f(x,y)-\int_a^b w(t)f(q_1t,y)d_{q_1}t-\int_c^d u(s)f(x,q_2s)d_{q_2}s+\int_a^b\int_c^d w(t)u(s)f(q_1t,q_2s)d_{q_1}td_{q_2}s\right|$$
$$\le\frac{1}{4(q_1+1)(q_2+1)}\left((b-a)^2+4\left(x-\frac{a+b}{2}\right)^2\right)\left((d-c)^2+4\left(y-\frac{c+d}{2}\right)^2\right)\left\|\frac{\partial f(t,s)}{d_{q_1}td_{q_2}s}\right\|. \quad (3.4)$$

Remark 3.3. If $q_1,q_2\to 1^-$, the inequality (3.4) are reduced to the main result in [4].

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QUADRATIC ρ-FUNCTIONAL INEQUALITIES IN NORMED SPACES

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Abstract. In this paper, we solve the quadratic ρ-functional inequalities

$$\begin{aligned}&\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\ &\leq \left\|\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\right\|,\end{aligned} \tag{0.1}$$

where ρ is a number with $|\rho|<1$ and

$$\begin{aligned}&\left\|2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right\| \\ &\leq \|\rho(f(x+y)+f(x-y)-2f(x)-2f(y))\|,\end{aligned} \tag{0.2}$$

where ρ is a number with $|\rho|<\frac{1}{2}$. Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequalities (0.1) and (0.2) in normed spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [29] concerning the stability of group homomorphisms.

The functional equation

$$f(x+y)=f(x)+f(y)$$

is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [14] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [20] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [11] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [23] for mappings $f: X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [6] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [7] proved the Hyers-Ulam stability of the quadratic functional equation. The functional equation $f\left(\frac{x+y}{2}\right)=\frac{1}{2}f(x)+\frac{1}{2}f(y)$ is called the *Jensen type quadratic functional equation*. The stability problems of several functional equations

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have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 3, 16, 17, 21, 22, 25, 26, 27, 28, 30, 31]).

In [12], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\| \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}).$$

See also [10, 24]. Gilányi [13] and Fechner [9] proved the Hyers-Ulam stability of the functional inequality (1.1). Park, Cho and Han [18] proved the Hyers-Ulam stability of additive functional inequalities.

Lemma 1.1. (Banach fixed-point theorem) *Let (S, d) be a complete metric space and let $T : S \to S$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in S$, there exists a positive integer n_0 such that*

(1) $d(T^n x, T^{n+1}x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence $\{T^n x\}$ converges to a fixed point y^* of T;*
(3) *y^* is the unique fixed point of T in the set $Y = \{y \in S \mid d(T^{n_0}x, y) < \infty\}$;*
(4) $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Ty)$ *for all $y \in Y$.*

Since we defined the metric d as generalized metric in order to use this lemma in the proof of the problem we extend the lemma.

Lemma 1.2. ([8]) *Let (S, d) be a complete generalized metric space and let $J : S \to S$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in S$, either*

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in S \mid d(J^{n_0}x, y) < \infty\}$;*
(4) $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ *for all $y \in Y$.*

In 1996, Isac and Rassias [15] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [4, 5, 19]).

In Section 2, we solve the quadratic ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.1) in normed spaces.

In Section 3, we solve the quadratic ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.2) in normed spaces.

Throughout this paper, assume that X is a normed space and Y is a Banach space.

2. Hyers-Ulam stability of the quadratic ρ-functional inequality (0.1): a fixed point approach

In this section, assume that $|\rho| < 1$.

We solve the quadratic ρ-functional inequality (0.1) in normed spaces.

Lemma 2.1. *A mapping $f : X \to Y$ satisfies*

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\leq \left\| \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right) \right\| \end{aligned} \tag{2.1}$$

for all $x, y \in X$ if and only if $f : X \to Y$ is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $\|2f(0)\| \leq |\rho|\|2f(0)\|$. So $f(0) = 0$.

Letting $y = x$ in (2.1), we get $\|f(2x) - 4f(x)\| \leq 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4} f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\leq \left\| \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right) \right\| \\ &= \frac{|\rho|}{2} \|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$.

The converse is obviously true. □

We prove the Hyers-Ulam stability of the quadratic ρ-functional inequalty (0.1) in Banach spaces.

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(a, b) \leq 4\alpha \varphi\left(\frac{a}{2}, \frac{b}{2}\right) \tag{2.3}$$

for all $a, b \in X$. Let $f : X \to Y$ be a mapping satisfying

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\leq \left\| \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right) \right\| + \varphi(x, y) \end{aligned} \tag{2.4}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{4 - 4\alpha} \varphi(x, x)$$

for all $x \in X$.

Proof. Consider the set

$$S := \{h : X \to Y\}$$

and let d be the generalized metric on S:

$$d(g,h) := \inf\{\mu \in \mathbb{R}_+ : \|g-h\| \le \mu\varphi(x,x), x \in S\}$$

It is easy to show that (S,d) is complete. Let J be the linear mapping from S to S such that

$$Jg(x) := \frac{1}{4}g(2x) \tag{2.5}$$

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then from (2.3) and (2.5), we get

$$\|Jg(x) - Jh(x)\| = \left\|\frac{1}{4}g(2x) - \frac{1}{4}h(2x)\right\| \le \frac{1}{4}\varepsilon\varphi(2x,2x) \le \alpha\varepsilon\varphi(x,x)$$

This means $d(Jg, Jh) \le \alpha d(g,h)$.

So the function $J : S \to S$ is a contractive mapping such that

$$d(Jg, Jh) \le \alpha d(g,h)$$

for $0 \le \alpha < 1$.

Letting $y = x$ in (2.4), we get

$$\|f(2x) - 4f(x)\| \le \varphi(x,x)$$

and so

$$\|f(x) - Jf(x)\| \le \frac{1}{4}\varphi(x,x)$$

for all $x \in X$. Thus we get $d(f, Jf) \le \frac{1}{4}$.

By Lemma 1.2, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(2a\right) = 4Q(a) \tag{2.6}$$

for all $a \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|f(a) - Q(a)\| \quad \le \quad \mu\varphi\left(a,a\right)$$

for all $a \in X$;

(2) $d(J^l f, Q) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} \frac{1}{4^l} f\left(2^l a\right) = Q(a)$$

for all $a \in X$;

(3) $d(f,Q) \le \frac{1}{1-\alpha}d(f,Jf)$, which implies the inequality

$$d(f,Q) \le \frac{1}{4-4\alpha}.$$

So

$$\|f(a) - Q(a)\| \leq \frac{1}{4-4\alpha}\varphi(a,a)$$

for all $a \in X$.

Then

$$\begin{aligned}
&\|Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y)\| \\
&\quad = \lim_{l\to\infty} \left\| \frac{1}{4^l}(f(2^l(x+y)) + f(2^l(x-y)) - 2f(2^l x) - 2f(2^l y)) \right\| \\
&\quad \leq \lim_{l\to\infty} \left\| \frac{\rho}{4^l}(2f(2^{l-1}(x+y)) + 2f(2^{l-1}(x-y)) - f(2^l x) - f(2^l y)) \right\| \\
&\qquad + \lim_{l\to\infty} \frac{1}{4^l}\varphi(2^l x, 2^l y) \\
&\quad = \left\| \rho \left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y) \right) \right\|
\end{aligned}$$

for all $x, y \in X$. Hence

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$$

for all x, y. So $Q : X \to Y$ is quadratic. □

Remark 2.3. We could prove the same statement with the same manner in spite of replacing the condition $\varphi(a,b) \leq 4\alpha\varphi\left(\frac{a}{2}, \frac{b}{2}\right)$ into $\varphi(a,b) \leq \frac{1}{4}\alpha\varphi(2a, 2b)$ by defining J such that $Jg(x) = 4g(\frac{x}{2})$ instead of $Jg(x) = \frac{1}{4}g(2x)$. It could be also applied to Theorem 3.2.

Corollary 2.4. *Let $r \neq 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned}
&\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\
&\quad \leq \left\| \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right) \right\| + \theta(\|x\|^r + \|y\|^r)
\end{aligned}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{2\theta}{|4-2^r|}\|x\|^r$$

for all $x \in X$.

3. Hyers-Ulam stability of the quadratic ρ-functional inequality (0.2): a fixed point approach

In this section, assume that $|\rho| < \frac{1}{2}$.

We solve the quadratic ρ-functional inequality (0.2) in normed spaces.

Lemma 3.1. *A mapping $f : X \to Y$ satisfies*

$$\begin{aligned}
&\left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right\| \\
&\quad \leq \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\|
\end{aligned} \tag{3.1}$$

for all $x, y \in X$ if and only if $f : X \to Y$ is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $x = y = 0$ in (3.1), we get $\|2f(0)\| \le |\rho|\|2f(0)\|$. So $f(0) = 0$.

Letting $y = 0$ in (3.1), we get

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \le 0 \tag{3.2}$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{4}f(x)$ for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned}&\frac{1}{2}\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\quad = \left\|2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right\| \\ &\quad \le |\rho|\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\|\end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$.

The converse is obviously true. □

We prove the Hyers-Ulam stability of the quadratic ρ-functional inequalty (0.2) in Banach spaces.

Theorem 3.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(a, b) \le 4\alpha\varphi\left(\frac{a}{2}, \frac{b}{2}\right)$$

for all $a, b \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\begin{aligned}&\left\|2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right\| \\ &\quad \le \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\| + \varphi(x, y)\end{aligned}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{4 - 4\alpha}\varphi(x, 0)$$

for all $x \in X$.

Proof. Consider the set

$$S := \{h : X \to Y\}$$

and let d be the generalized metric on S:

$$d(g, h) := \inf\{\mu \in \mathbb{R}_+ : \|g - h\| \le \mu\varphi(x, 0), x \in S\}$$

It is easy to show that (S, d) is complete. Let J be the linear mapping from S to S such that

$$Jg(x) := \frac{1}{4}g(2x)$$

Let $Q : X \to Y$ be defined as in the proof of Theorem 2.2. Then

$$\begin{aligned}
&\left\|2Q\left(\frac{x+y}{2}\right)+2Q\left(\frac{x-y}{2}\right)-Q(x)-Q(y)\right\| \\
&\quad= \lim_{l\to\infty}\left\|\frac{1}{4^l}(2f(2^{l-1}(x+y))+2f(2^{l-1}(x-y))-f(2^lx)-f(2^ly))\right\| \\
&\quad\le \lim_{l\to\infty}\left\|\frac{\rho}{4^l}(f(2^l(x+y))+f(2^l(x-y))-2f(2^lx)-2f(2^ly))\right\| \\
&\qquad+\lim_{l\to\infty}\frac{1}{4^l}\varphi(2^lx,2^ly) \\
&\quad= \|\rho(Q(x+y)+Q(x-y)-2Q(x)-2Q(y))\|
\end{aligned}$$

for all $x, y \in X$. Hence

$$Q(x+y)+Q(x-y)=2Q(x)+2Q(y)$$

for all x, y. So $Q : X \to Y$ is quadratic. $\square$

Corollary 3.3. *Let $r \neq 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned}
&\left\|2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right\| \\
&\quad\le \|\rho(f(x+y)+f(x-y)-2f(x)-2f(y))\|+\theta(\|x\|^r+\|y\|^r)
\end{aligned}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x)-Q(x)\| \le \frac{\theta}{|4-2^r|}\|x\|^r$$

for all $x \in X$.

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Solution of the Ulam stability problem for quartic (a, b)-functional equations

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Abstract. The "oldest quartic" functional equation was introduced and solved by the author of this paper (see: Glas. Mat. Ser. III 34 (54) (1999), no. 2, 243-252) which is of the form:

$$f(x+2y)+f(x-2y)=4[f(x+y)+f(x-y)]-6f(x)+24f(y).$$

Interesting results have been achieved by S.A. Mohiuddine et al., since 2009. In this paper, we are introducing new quartic functional equations, and establish fundamental formulas for the general solution of such functional equations and for "Ulam stability" of pertinent quartic functional inequalities.

Keywords and phrases: Quartic functional equations and inequalities; Various normed spaces; Ulam stability.
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1. INTRODUCTION

In 1940 S. M. Ulam [1] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following famous "*stability Ulam question*":

We are given a group G and a metric group G' with metric $\rho(.,.)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all x, y in G, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?

By now an affirmative answer has been given in several cases, and some interesting variations of the problem have also been investigated. We shall call such an $f : G \to G'$ an *approximate homomorphism*.

In 1941 D. H. Hyers [2] considered the case of approximately additive mappings $f : E \to E'$ where E and E' are Banach spaces and f satisfies the following *Hyers' inequality*

$$\|f(x+y)-f(x)-f(y)\| \le \epsilon$$

for all $x, y \in E$. It was shown that the limit

$$L(x)=\lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying $\|f(x) - L(x)\| \le \epsilon$.

No continuity conditions are required for this result, but if $f(tx)$ is continuous in the real variable t for each fixed x, then L is linear, and if f is continuous at a single point of E then $L : E \to E'$ is also continuous.

In 1982-1994, a generalization of this result was proved by the author J. M. Rassias [3–7], as follows. He introduced the following *weaker condition* (*or weaker inequality or the generalized Cauchy inequality*)

$$\|f(x+y)-[f(x)+f(y)]\| \le \theta \|x\|^p \|y\|^q$$

for all x, y in E, controlled by (or involving) a product of different powers of norms, where $\theta \ge 0$ and real $p, q : r = p + q \ne 1$, and retained the condition of continuity of $f(tx)$ in t for fixed x. Besides he investigated that it is possible to replace ϵ in the above *Hyers' inequality*, by a non-negative real-valued function such that the pertinent series converges and other conditions hold and still obtain stability results. In all the cases investigated in these results, the approach to the existence question was to prove

asymptotic type formulas: $L(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$; $L(x) = \lim_{n\to\infty} 2^n f(2^{-n} x)$.

Theorem (J. M. Rassias:1982-1994). Let X be a real normed linear space and let Y be a real complete normed linear space. Assume in addition that $f : X \to Y$ is an approximately additive mapping for which there exist constants $\theta \geq 0$ and $p, q \in \mathbb{R}$ such that $r = p + q \neq 1$ and f satisfies the "*generalized Cauchy inequality*"

$$\left\|f(x+y) - [f(x) + f(y)]\right\| \leq \theta \left\|x\right\|^p \left\|y\right\|^q$$

for all $x, y \in X$. Then there exists a unique additive mapping $L : X \to Y$ satisfying

$$\left\|f(x) - L(x)\right\| \leq \frac{\theta}{|2^r - 2|}\left\|x\right\|^r$$

for all $x \in X$. If in addition $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is an $\mathbb{R}$-linear mapping.

In 1940, Ulam, S. M. [1] proposed the "Ulam stability problem": *When does a linear transformation near an "approximately linear" transformation exist*? Since then, many specialists on this "famous Ulam problem", have investigated interesting functional equations, for instance: D. H. Hyers [2], in 1941; T. Aoki [8], in 1950; T. M. Rassias [9], in 1978; Z. Gajda [10], in 1991; T. M. Rassias and P. Šemrl [11], in 1992; P. Găvruta [12], in 1994; S.-M. Jung [13], in 1998; K. W. Jun and H. M. Kim [14], in 2002; R. P. Agarwal et al. [15], in 2003, and others. Interesting Ulam-Hyers stability results have been established by S. A. Mohiuddine et al. ([16–19]). The "oldest quartic" functional equation was introduced and solved by the author of this paper, [20], which is of the form:

$$f(x + 2y) + f(x - 2y) = 4[f(x + y) + f(x - y)] - 6f(x) + 24f(y).$$

Since then various quartic equations have been proposed and solved by a number of experts in the area of functional equations and inequalities. For more details on these concepts, one can be referred to [21–30]. For further research in various normed spaces, we are introducing new quartic functional equations, and establish fundamental formulas for the general solution of such functional equations and for "Ulam stability" of pertinent quartic functional inequalities.

2. ON (a, b)-QUARTIC FUNCTIONAL EQUATIONS

1. Stability of General a-Quartic Functional Equation

$$2\big[f(ax + y) + f(x + ay)\big] + a(a - 1)^2 f(x - y) = 2(a^2 - 1)^2\big[f(x) + f(y)\big] + a(a + 1)^2 f(x + y) \tag{1.1}$$

where $a \neq 0, a \neq \pm 1$.

Replacing $x = y = 0$ in (1.1) one gets $4a^2(1 - a^2)f(0) = 0$, or

$$f(0) = 0. \tag{1.2}$$

Similarly, substituting $x = x, y = 0$ in (1.1), we obtain

$$f(ax) = a^4 f(x) + (a^2 - 1)^2 f(0) = a^4 f(x) \tag{1.3}$$

Also assuming $f(2x) = 16f(x)$, replacing $x = x, y = x$ in (1.1), and setting $k = a+1 \neq 0, \pm 1$, one obtains

$$4f(kx) + a(a - 1)^2 f(0) = 4(a^2 - 1)^2 f(x) + a(a + 1)^2 f(2x), \tag{1.4}$$

or

$$f(kx) + \frac{1}{4}a(a - 1)^2 f(0) = (a^2 - 1)^2 f(x) + 4a(a + 1)^2 f(x) = k^4 f(x), \text{ or}$$

$$f(kx) = k^4 f(x) - \frac{1}{4}a(a - 1)^2 f(0) = k^4 f(x). \tag{1.5}$$

Without assuming $f(2x) = 16f(x)$, replacing $x = x, y = -x$ in (1.1), and setting $l = a - 1 \neq 0, \pm 1$, one obtains

$$2\big[f(lx) + f(-lx)\big] + a(a-1)^2 f(2x) = 2(a^2-1)^2\big[f(x) + f(-x)\big] + a(a+1)^2 f(0) \tag{1.6}$$

Placing $-x$ on x in (1.6), and then subtracting the new equation from (1.6), we get $a(a-1)^2[f(2x) - f(-2x)] = 0$. Letting $x/2$ on x, we find that f is an "*even function*", such that $f(-x) = f(x)$. Thus from (1.6), we obtain

$$4f(lx) + a(a-1)^2 f(2x) = 4(a^2-1)^2 f(x) + a(a+1)^2 f(0). \tag{1.7}$$

Assuming $f(2x) = 16f(x)$, we get

$$f(lx) = l^4 f(x) + \frac{1}{4}a(a+1)^2 f(0) = l^4 f(x).$$

Without assuming $f(2x) = 16f(x)$, subtracting (1.7) from (1.4), we obtain from (1.2) that

$$f(kx) - f(lx) = \frac{1}{2}a(a^2+1)f(2x) = \left[\left(\frac{k}{2}\right)^4 - \left(\frac{l}{2}\right)^4\right] f(2x).$$

Replacing $x \to x/2$, we get that

$$\text{“}f\left(\frac{k}{2}x\right) = \left(\frac{k}{2}\right)^4 f(x) \quad \text{if and only if} \quad f\left(\frac{l}{2}x\right) = \left(\frac{l}{2}\right)^4 f(x)\text{”}. \tag{1.8}$$

Employing the "quartic mean", we have equivalently that

$$\text{“}\overline{f}_{\frac{k}{2}}(x) = \frac{f\left(\frac{k}{2}x\right)}{\left(\frac{k}{2}\right)^4} = f(x) \ \text{ iff } \ \overline{f}_{\frac{l}{2}}(x) = \frac{f\left(\frac{l}{2}x\right)}{\left(\frac{l}{2}\right)^4} = f(x)\text{”}.$$

Let X be a real normed linear space and let Y be a real complete normed linear space. Assume $f : X \to Y$, satisfying the following general a-quartic functional inequality

$$\big\|2\big[f(ax+y) + f(x+ay)\big] + a(a-1)^2 f(x-y) - 2(a^2-1)^2\big[f(x) + f(y)\big] - a(a+1)^2 f(x+y)\big\| \leq c \tag{1.9}$$

where $a \neq 0, a \neq \pm 1$. Replacing $x = y = 0$ in (1.9), one gets

$$\big\|f(0)\big\| \leq c/4a^2|a^2-1|. \tag{1.10}$$

Substituting $x = x, y = 0$ in (1.9), and employing the triangle inequality, we obtain:

$$\big\|f(ax) - a^4 f(x)\big\| \leq c_2 = \frac{c_1}{2} = \frac{2a^2 + |a^2-1|}{4a^2}c.$$

Note that

$$c_1 = \frac{2a^2 + |a^2-1|}{2a^2}c = \begin{cases} \frac{3a^2-1}{2a^2}c & \text{if } |a| > 1 \\ \frac{a^2+1}{2a^2}c & \text{if } |a| < 1; a \neq 0. \end{cases}$$

Therefore

$$\big\|f(ax) - a^4 f(x)\big\| \leq c_2 = \frac{c_1}{2} = \frac{3a^2-1}{4a^2}c, \quad \text{if } |a| > 1,$$

and

$$\big\|f(ax) - a^4 f(x)\big\| \leq c_3 = \frac{c_1}{2} = \frac{a^2+1}{4a^2}c, \quad \text{if } |a| < 1; a \neq 0.$$

Therefore we get

$$\big\|f(x) - a^{-4} f(ax)\big\| \leq a^{-4}c_2 = \frac{3a^2-1}{4a^6}c, \quad \text{if } |a| > 1,$$

and

$$\big\|f(x) - a^4 f(a^{-1}x)\big\| \leq c_3 = \frac{a^2+1}{4a^2}c, \quad \text{if } |a| < 1; a \neq 0.$$

Thus we can easily obtain the following general inequality

$$\left\|f(x)-a^{-4n}f(a^nx)\right\| \leq \frac{3a^2-1}{4a^2(a^4-1)}(1-a^{-4n})c, \quad \text{if } \ |a|>1. \tag{1.11}$$

In fact,

$$\begin{aligned}\left\|f(x)-a^{-4n}f(a^nx)\right\| &\leq \left\|f(x)-a^{-4}f(a^1x)\right\|+a^{-4}\left\|f(ax)-a^{-4}f(a^2x)\right\|+\dots \\ &\quad \dots+a^{-4(n-2)}\left\|f(a^{n-2}x)-a^{-4}f(a^{n-1}x)\right\| \\ &\quad +a^{-4(n-1)}\left\|f(a^{n-1}x)-a^{-4}f(a^nx)\right\| \qquad (1.12)\\ &\leq \left(1+a^{-4}+a^{-8}+\dots+a^{-4(n-2)}+a^{-4(n-1)}\right)\frac{3a^2-1}{4a^6}c \\ &\leq \frac{3a^2-1}{4a^2(a^4-1)}(1-a^{-4n})c, \quad \text{if } \ |a|>1.\end{aligned}$$

Also,

$$\begin{aligned}\left\|f(x)-\alpha^{4n}f(\alpha^{-n}x)\right\| &\leq \left\|f(x)-\alpha^4f(\alpha^{-1}x)\right\|+\alpha^4\left\|f(\alpha^{-1}x)-\alpha^4f(\alpha^{-2}x)\right\| \\ &\quad +\dots+\alpha^{4(n-1)}\left\|f(\alpha^{-(n-1)}x)-\alpha^4f(\alpha^{-n}x)\right\| \\ &\leq \left(1+\alpha^4+\dots+\alpha^{4(n-1)}\right)\frac{\alpha^2+1}{4\alpha^2}c \\ &\leq \frac{\alpha^2+1}{4\alpha^2}\frac{1}{1-\alpha^4}(1-\alpha^{4n})c, \quad \text{if } \ |\alpha|<1.\end{aligned}$$

The "*altemative*" general inequality for $|\alpha|<1$; $\alpha\neq 0$ is similarly established.

Note 1. (i) Assume $|\alpha|>1$, and denote

$$Q_n : Q_n(x)=\alpha^{-4n}f(\alpha^nx).$$

Claim that sequence $\{Q_n\}$, $|\alpha|>1$ is a Cauchy sequence.

In fact, if $m>n>0$, then

$$\begin{aligned}0\leq \left\|Q_n(x)-Q_m(x)\right\| &= \left\|\alpha^{-4n}f(\alpha^nx)-\alpha^{-4m}f(\alpha^mx)\right\| \\ &= |\alpha|^{-4n}\left\|f(\alpha^nx)-\alpha^{-4(m-n)}f(\alpha^{m-n}.\alpha^nx)\right\| \\ &\leq |\alpha|^{-4n}\frac{3\alpha^2-1}{4\alpha^2(\alpha^4-1)}\left(1-\alpha^{-4(m-n)}\right)c \\ &= \frac{3\alpha^2-1}{4\alpha^2(\alpha^4-1)}\left(|\alpha|^{-4n}-|\alpha|^{-4n}\alpha^{-4(m-n)}\right)c \\ &\to 0, \quad \text{as} \quad n\to\infty \text{ (and } m\to\infty).\end{aligned}$$

Thus $\{Q_n\}$, $|\alpha|>1$, is Cauchy sequence.

Similarly, if $|\alpha|<1$, $\alpha\neq 0$, one proves that

$$\{Q_n\}, |\alpha|<1, \alpha\neq 0$$

is Cauchy sequence, as well.

(ii) Claim the quarticness of

$$Q : Q(x)=\lim_{n\to\infty}Q_n(x)=\lim_{n\to\infty}\alpha^{-4n}f(\alpha^nx), \ \alpha>1.$$

In fact, replacing $x\to\alpha^nx$, $y\to\alpha^ny$ in the α-quartic functional inequality (1.9) and then multiplying by $|\alpha|^{-4n}$, and taking limit $n\to\infty$, we obtain

$$\begin{aligned}0 &\leq \big\|2[Q(\alpha x+y)+Q(x+\alpha y)]+\alpha(\alpha-1)^2Q(x-y) \\ &\quad -2(\alpha^2-1)^2[Q(x)+Q(y)]-\alpha(\alpha+1)^2Q(x+y)\big\| \\ &\leq |\alpha|^{-4n}c\to 0, \quad n\to\infty.\end{aligned}$$

Thus

$$2[Q(\alpha x+y)+Q(x+\alpha y)]+\alpha(\alpha-1)^2Q(x-y)=2(\alpha^2-1)^2[Q(x)+Q(y)]+\alpha(\alpha+1)^2Q(x+y)$$

leading to (1.1) and thus the quarticness of $Q:|\alpha|>1$.

Similarly, we prove that Q is quartic for $|\alpha|<1$, $\alpha\neq 0$. Thus the existence of Q is complete.

If

$$Q(x)=\lim_{n\to\infty}Q_n(x)=\lim_{n\to\infty}\begin{cases} a^{-4n}f(a^nx) & \text{if } |a|>1 \\ a^{4n}f(a^{-n}x) & \text{if } |a|<1; a\neq 0 \end{cases}, \tag{1.13}$$

then

$$\|f(x)-Q(x)\|\leq\frac{2a^2+|a^2-1|}{4a^2}c\cdot\begin{cases} \frac{1}{a^4-1} & \text{if } |a|>1 \\ \frac{1}{1-a^4} & \text{if } |a|<1; a\neq 0. \end{cases} \tag{1.14}$$

Note 2. Claim the uniqueness of

$$Q:Q(x)=\lim_{n\to\infty}Q_n(x)=\lim_{n\to\infty}\alpha^{-4n}f(\alpha^nx),\ \alpha>1.$$

In fact, if there is another quartic mapping Q' satisfying (1.14), then

$$\begin{aligned} 0 &\leq \|Q(x)-Q'(x)\| \\ &\leq \|Q(x)-f(x)\|+\|f(x)-Q'(x)\|, \end{aligned}$$

or

$$\begin{aligned} 0 &\leq \|Q(x)-Q'(x)\| \\ &= |\alpha|^{-4n}\|Q(\alpha^nx)-Q'(\alpha^nx)\| \\ &= \|\alpha^{-4n}Q(\alpha^nx)-\alpha^{-4n}Q'(\alpha^nx)\| \\ &\leq \|\alpha^{-4n}Q(\alpha^nx)-\alpha^{-4n}f(\alpha^nx)\|+\|\alpha^{-4n}f(\alpha^nx)-\alpha^{-4n}Q'(\alpha^nx)\| \\ &= |\alpha|^{-4n}\Big\{\|Q(\alpha^nx)-f(\alpha^nx)\|+\|f(\alpha^nx)-Q'(\alpha^nx)\|\Big\} \\ &\leq \frac{2\alpha^2+|\alpha^2-1|}{2\alpha^2}|\alpha|^{-4n}c\to 0,\quad n\to\infty \end{aligned}$$

or

$$Q(x)=Q'(x),$$

proving uniqueness of $Q:|\alpha|>1$. Similarly, one proves uniqueness of $Q:|\alpha|<1$, $\alpha\neq 0$.

Theorem 1.1. Let X be a normed space and Y be a Banach space. If $f:X\to Y$ is a mapping satisfying (1.9), then there exists a unique quartic mapping $Q:X\to Y$, satisfying inequality (1.14).

If $f(0)=0$, then $\|f(x)-Q(x)\|\leq c\big/2|1-a^4|$, for $\forall a\neq 0;\pm 1$.

2. Stability of General (a,b)-Quartic Functional Equation

$$2\big[f(ax+by)+f(bx+ay)\big]+ab(a-b)^2f(x-y)=2(a^2-b^2)^2\big[f(x)+f(y)\big]+ab(a+b)^2f(x+y), \tag{2.1}$$

where $a\neq\pm b$, $a,b\neq 0,\pm 1$, $k=a+b\neq 1$, $l=a-b\neq 1$, and $a^4+b^4-a^2b^2-1\neq 0$.

Replacing $x=y=0$ in this equation, one gets

$$4(a^4+b^4-a^2b^2-1)f(0)=0,\ \text{or}$$

$$f(0)=0 \tag{2.2}$$

Similarly substituting $x = x, y = 0$ in (2.1), we obtain

$$f(ax) + f(bx) = (a^4 + b^4)f(x) + (a^2 - b^2)^2 f(0) = (a^4 + b^4)f(x). \tag{2.3}$$

From (2.3), we observe that

$$\text{“}f(ax) = a^4 f(x) \quad \text{if and only if} \quad f(bx) = b^4 f(x).\text{”} \tag{2.4}$$

Let us introduce the following “(a, b)-quartic functional mean”

$$\overline{f_{(a,b)}}(x) = \frac{f(ax) + f(bx)}{a^4 + b^4}. \tag{2.5}$$

From (2.3) and (2.5), we find the quartic functional mean equation

$$\overline{f_{a,b)}}(x) = f(x) \tag{2.6}$$

From (2.4)-(2.5)-(2.6), one establishes

$$\text{“}\overline{f_{(a,0)}}(x) = \frac{f(ax)}{a^4} = f(x) \ \text{ iff } \ \overline{f_{(0,b)}}(x) = \frac{f(bx)}{b^4} = f(x).\text{”}$$

Also assuming $f(2x) = 16f(x)$, replacing $x = x, y = x$ in (2.1) and setting $k = a + b \neq 0, \pm 1$, one obtains

$$4f(k) + ab(a - b)^2 f(0) = 4(a^2 - b^2)^2 f(x) + ab(a + b)^2 f(2x), \tag{2.8}$$

or

$$f(k) + \frac{1}{4}ab(a - b)^2 f(0) = (a^2 - b^2)^2 f(x) + 4ab(a + b)^2 f(x) = k^4 f(x),$$

or

$$f(k) = k^4 f(x) - \frac{1}{4}ab(a - b)^2 f(0) = k^4 f(x). \tag{2.9}$$

Without assuming $f(2x) = 16f(x)$, replacing $x = x, y = -x$ in (2.1) and setting $l = a - b \neq 0, \pm 1$ with $a \neq \pm b$, one obtains

$$2\big[f(lx) + f(-lx)\big] + ab(a - b)^2 f(2x) = 2(a^2 - b^2)^2 \big[f(x) + f(-x)\big] + ab(a + b)^2 f(0) \tag{2.10}$$

Placing $-x$ on x in (2.10), and then subtracting the new equation from (2.10), we get $ab(a-b)^2\big[f(2x) - f(-2x)\big] = 0$. Letting $x/2$ on x, we find that f is an “*even function*”, such that $f(-x) = f(x)$. Thus from (2.10), we obtain

$$4f(lx) + ab(a - b)^2 f(2x) = 4(a^2 - b^2)^2 f(x) + ab(a + b)^2 f(0). \tag{2.11}$$

Assuming $f(2x) = 16f(x)$, we get

$$f(lx) = l^4 f(x) + \frac{1}{4}ab(a + b)^2 f(0) = l^4 f(x).$$

Without assuming $f(2x) = 16f(x)$, subtracting (2.11) from (2.8), we obtain from (2.2) that

$$f(kx) - f(lx) = \frac{1}{2}ab(a^2 + b^2)f(2x) = \left[\left(\frac{k}{2}\right)^4 - \left(\frac{l}{2}\right)^4\right] f(2x).$$

Replacing $x \to x/2$, we obtain that

$$f\left(\frac{k}{2}x\right) - \left(\frac{k}{2}\right)^4 f(x) = f\left(\frac{l}{2}x\right) - \left(\frac{l}{2}\right)^4 f(x).$$

Therefore, we observe that

$$\text{“}f\left(\frac{k}{2}x\right) = \left(\frac{k}{2}\right)^4 f(x) \quad \text{if and only if} \quad f\left(\frac{l}{2}x\right) = \left(\frac{l}{2}\right)^4 f(x)\text{”}. \tag{2.12}$$

Employing the "quartic mean", we have equivalently that

$$\text{“}\overline{f}_{\frac{k}{2}}(x)\frac{f\left(\frac{k}{2}x\right)}{\left(\frac{k}{2}\right)^4} = f(x) \text{ iff } \overline{f}_{\frac{l}{2}}(x) = \frac{f\left(\frac{l}{2}x\right)}{\left(\frac{l}{2}\right)^4} = f(x)\text{”}.$$

Let X be a real normed linear space and let Y be a real complete normed linear space. Assume $f : X \to Y$, satisfying the following general (a, b)-quartic functional inequality

$$\left\|2[f(ax+by)+f(bx+ay)]+ab(a-b)^2 f(x-y)-2(a^2-b^2)^2[f(x)+f(y)]-ab(a+b)^2 f(x+y)\right\| \leq c, \tag{2.13}$$

where $a, b \neq 0; a, b \neq \pm 1$. Replacing $x = y = 0$ in (2.13), one gets

$$\|f(0)\| \leq c/4|a^4 + b^4 - a^2b^2 - 1|. \tag{2.14}$$

Substituting $x = x, y = 0$ in (2.13), and employing the triangle inequality and (2.14), we obtain

$$\|f(ax) + f(bx) - (a^4 + b^4)f(x)\| \leq \frac{c}{2} + (a^2 - b^2)^2|f(0)|,$$

or

$$\|f(ax) + f(bx) - (a^4 + b^4)f(x)\| \leq c\frac{(a^2 - b^2)^2 + 2|a^4 + b^4 - a^2b^2 - 1|}{4|a^4 + b^4 - a^2b^2 - 1|}. \tag{2.15}$$

Substituting $x = x, y = x$ in (2.13), and employing the triangle inequality and (2.14), as well as the following hypothesis

$$\|f(2x) - 16f(x)\| \leq c_1 \ (\geq 0), \tag{2.16}$$

and denoting $k = a + b \neq 0, \pm 1$, we obtain:

$$\|f(kx) - k^4 f(x)\| \leq \frac{1}{4}\left(c + ab(a-b)^2|f(0)| + ab(a+b)^2 c_1\right),$$

or

$$\|f(kx) - k^4 f(x)\| \leq c_2 = \frac{1}{4}\left(\frac{ab(a-b)^2 + 4|a^4 + b^4 - a^2b^2 - 1|}{4|a^4 + b^4 - a^2b^2 - 1|}c + ab(a+b)^2 c_1\right), \tag{2.17}$$

or

$$\|f(x) - k^4 f(kx)\| \leq k^{-4} c_2 \tag{2.18}$$

if $|k| > 1$.

Thus we easily obtain, the following general inequality:

$$\|f(x) - k^{-4n} f(k^n x)\| \leq \frac{1}{k^4 - 1}(1 - k^{-4n})c_2, \quad \text{if } |k| > 1. \tag{2.19}$$

In fact,

$$\begin{aligned}
\left\|f(x) - k^{-4n} f(k^n x)\right\| &\leq \left\|f(x) - k^{-4} f(k^1 x)\right\| + k^{-4}\left\|f(k^1 x) - k^{-4} f(k^2 x)\right\| + \cdots \\
&\quad \cdots + k^{-4(n-2)}\left\|f(k^{n-2}x) - k^{-4} f(k^{n-1}x)\right\| \\
&\quad + k^{-4(n-1)}\left\|f(k^{n-1}x) - k^4 f(k^n x)\right\| \qquad (2.20) \\
&\leq k^{-4}\left(1 + k^{-4} + k^{-8} + \cdots + k^{-4(n-2)} + k^{-4(n-1)}\right) c_2 \\
&= \frac{k^{-4}}{1 - k^{-4}}(1 - k^{-4n})c_2 = \frac{1}{k^4 - 1}(1 - k^{-4n})c_2, \quad \text{if } |k| > 1.
\end{aligned}$$

The"*altemative*" general inequality is similarly established, as follows

$$\|f(x) - k^{4n} f(k^{-n} x)\| \leq \frac{1}{1 - k^4}(1 - k^{4n})c_2, \quad \text{if } |k| < 1; k \neq 0. \tag{2.21}$$

In fact,

$$\begin{aligned}\left\|f(x)-k^{4n}f(k^{-n}x)\right\| &\leq \left\|f(x)-k^4f(k^{-1}x)\right\|+k^4\left\|f(k^{-1}x)-k^4f(k^{-2}x)\right\|+\dots\\ &\quad \dots+k^{4(n-2)}\left\|f(k^{-(n-2)}x)-k^4f(k^{-(n-1)}x)\right\|\\ &\quad +k^{4(n-1)}\left\|f(k^{-(n-1)}x)-k^4f(k^{-n}x)\right\| \qquad (2.22)\\ &\leq (1+k^4+k^8+\dots+k^{4(n-2)}+k^{4(n-1)})c_2\\ &\leq \frac{1}{1-k^4}(1-k^{4n})c_2, \quad \text{if } \ |k|<1; k\neq 0.\end{aligned}$$

If we denote

$$Q(x)=\lim_{n\to\infty}Q_n(x)=\lim_{n\to\infty}\begin{cases}k^{-4n}f(k^nx) & \text{if } \ |k|>1\\ k^{4n}f(k^{-n}x) & \text{if } \ |k|<1; k\neq 0.\end{cases}$$

It follows

$$|f(x)-Q(x)|\leq c_2\cdot\begin{cases}\frac{1}{k^4-1} & \text{if } \ |k|>1\\ \frac{1}{1-k^4} & \text{if } \ |k|<1; k\neq 0.\end{cases} \qquad (2.23)$$

Note 3. Following Notes 1-2, we establish the existence and uniqueness of the quartic mapping Q.

If $f(0)=0$, and $f(2x)=16f(x)$, then

$$\|f(x)-Q(x)\|\leq\frac{c}{2|1-k^4|}, \quad \text{for} \quad \forall k\neq 0;\pm 1. \qquad (2.24)$$

Theorem 2.1. Let X be normed space and Y a Banach space. If $f:X\to Y$ is a mapping satisfying (2.13) then there exists a unique quartic mapping $Q:X\to Y$, satisfying inequality (2.23).
If $f(0)=0$, and $f(2x)=16f(x)$, then

$$\|f(x)-Q(x)\|\leq\frac{c}{2|1-k^4|}, \quad \text{for } \forall k=a+b\neq 0;\pm 1.$$

3. General Alternative a-Quartic Functional Equation

$$\begin{aligned}&2f(ax+y)+f(x+ay)+f(ax-y)\\ &\quad =\frac{a}{2}\left[(a^2+4a+1)f(x+y)-(a^2-4a+1)f(x-y)\right]\\ &\quad +(3a^4-4a^2+1)f(x)+(a^4-4a^2+3)f(y), \qquad (3.1)\end{aligned}$$

where $a\neq 0, a\neq\pm 1$.
Replacing $x=y=0$ in (3.1), one gets

$$4a^2(a^2-1)f(0)=0,$$

or

$$f(0)=0. \qquad (3.2)$$

Substituting $x=x$, $y=0$ in (3.1), we obtain

$$f(ax)=a^4f(x)+\frac{1}{3}(a^4-4a^2+3)f(0)=a^4f(x). \qquad (3.3)$$

Let X be a real normed linear space and let Y be a real complete normed linear space. Assume $f:X\to Y$, satisfying the following general alternative a-quartic functional inequality

$$\begin{aligned}&\Big\|2f(ax+y)+f(x+ay)+f(ax-y)\\ &\quad -\frac{a}{2}\left[(a^2+4a+1)f(x+y)-(a^2-4a+1)f(x-y)\right]\end{aligned}$$

$$-(3a^4-4a^2+1)f(x)-(a^4-4a^2+3)f(y)\Big\| \le c, \tag{3.4}$$

where $a \neq 0, a \neq \pm 1$. Replacing $x = y = 0$ in (3.4), one gets

$$4a^2|a^2-1|\big\|f(0)\big\| \le c,$$

or

$$\big\|f(0)\big\| \le \frac{c}{a^2|a^2-1|}. \tag{3.5}$$

Substituting $x = x$, $y = 0$ in (3.4) and employing (3.5), we obtain

$$\big\|f(x)-a^{-4}f(ax)\big\| \le \frac{3a^2+|a^2-3|}{3a^2}a^{-4}c = a^{-4}c_1. \tag{3.6}$$

Thus

$$\big\|f(x)-a^{-4n}f(a^nx)\big\| \le \frac{1}{a^4-1}\Big(1-a^{-4(n+1)}\Big)c_1. \tag{3.7}$$

Similarly, we obtain

$$\big\|f(x)-a^4f(a^{-1}x)\big\| \le \frac{3a^2+|a^2-3|}{3a^2}c = c_1. \tag{3.8}$$

Thus

$$\big\|f(x)-a^{4n}f(a^{-n}x)\big\| \le \frac{1}{1-a^4}\Big(1-a^{4(n+1)}\Big)c_1. \tag{3.9}$$

If

$$Q(x) = \lim_{n\to\infty} Q_n(x) = \lim_{n\to\infty}\begin{cases} a^{-4n}f(a^nx) & \text{if } |a|>1 \\ a^{4n}f(a^{-n}x) & \text{if } |a|<1; a\neq 0 \end{cases}, \text{ then}$$

$$\big\|f(x)-Q(x)\big\| \le \frac{3a^2+|a^2-3|}{3a^2}c\cdot\begin{cases} \frac{1}{a^4-1} & \text{if } |a|>1 \\ \frac{1}{1-a^4} & \text{if } |a|<1; a\neq 0. \end{cases} \tag{3.10}$$

Note 4. Following Notes 1-2, we establish existence and uniqueness of Q.

If $f(0) = 0$, then $\big\|f(x)-Q(x)\big\| \le \frac{c}{2|1-a^4|}$, for $\forall a \neq 0; \pm 1$.

Theorem 3.1. Let X be a normed space and Y a Banach space. If $f : X \to Y$ is a mapping satisfying (3.1), then there exists a unique quartic mapping $Q : X \to Y$, satisfying inequality (3.4).

If $f(0) = 0$, then $\big\|f(x)-Q(x)\big\| \le c\Big/2\big|1-a^4\big|$, for $\forall a \neq 0; \pm 1$.

OPEN RESEARCH PROBLEMS

OPEN PROBLEM A.

Employing both the "*Hyers' direct method*" and the "*fixed point method*", it is still "*open*", the investigation of "*generalized Ulam stabilities*" and "*generalized Ulam superstabilities*" of these quartic functional equations in various normed spaces, domains and groups such as in

1. Banach spaces;
2. Banach algebras; $C*$-algebras;
3. N-multi-Banach spaces; multi-Banach spaces;
4. Multi-normed spaces;
5. Quasi-Banach spaces;
6. Quasi-β[beta]-normed spaces;
7. Non-Archimedean normed spaces;
8. Fuzzy normed spaces;

9. Quasi fuzzy normed spaces;
10. Non- Archimedean fuzzy normed spaces;
11. Intuitionistic normed spaces;
12. Random normed spaces; and probabilistic normed spaces;
13. Non-Archimedean RN[Random Normed]-spaces;
14. Intuitionistic random normed spaces;
15. Intuitionistic fuzzy normed spaces;
16. intuitionistic fuzzy Banach algebras;
17. Intuitionistic Non-Archimedean fuzzy normed spaces;
18. Menger normed spaces;
19. Menger probabilistic normed spaces;
20. Non-Archimedean Menger normed spaces;
21. Intuitionistic Menger normed spaces;
22. L-non-Archimedean- fuzzy Euclidean normed spaces;
23. F-spaces; Fréchet spaces;
24. Banach modules;
25. Distributions and Hyperfunctions;
    as well as, on:
26. Restricted domains;
27. Heisenberg groups.

OPEN PROBLEM B.

28. Exploiting the elementary "*M. Hosszu's method*", due to M. Hosszu (see: "On the Fréchet's functional equation"", Bull. Inst. Politech. Iasi 10, (1964), 1-2, 27-28), determine the general solution and the Ulam stability of each one of these quartic functional equations. The advantage of this method is that we do not assume any regularity conditions on the unknown function f. See also the paper of the authors Xu et al.: [31].

29. Employing two "*L. Szekelyhidi's fundamental results*", due to L. Szekelyhidi (see: "Convolution type functional equation on topological abelian groups", World Scientific, Singapore, 1991), determine the general solution and the Ulam stability of each one of these quartic functional equations in certain types of groups, such as, "commutative groups". See: [31].

OPEN PROBLEM C.

30. Exploiting the elementary "*M. Hosszu's method*", due to M. Hosszu (see: "On the Fréchet's functional equation", Bull. Inst. Politech. Iasi 10, (1964), 1-2, 27-28), determine the general solution and the Ulam stability of each one of these pertinent "*Pexider quartic*" functional equations "with or without *involution*" [32]. We do not assume any regularity conditions on the unknown functions. For "*Pexider quartic*" equations [31]:

$$f(ax+y)+f(x+ay)=g(x+y)+\bar{g}(x-y)+h(x)+\bar{h}(y), \tag{1}$$

$$f(ax+by)+f(bx+ay)=g(x+y)+\bar{g}(x-y)+h(x)+\bar{h}(y), \tag{2}$$

with fixed integers $a, b \neq 0, \pm 1$ and unkown functions $f, \bar{f}, g, \bar{g}, h, \bar{h}$. See also the papers of the author, et al.: ([31, 33]).

31. Employing two "*L. Szekelyhidi's fundamental results*", due to L. Szekelyhidi (see: "Convolution type functional equation on topological abelian groups", World Scientific, Singapore, 1991), determine the general solution and the Ulam stability of each one of these pertinent "*Pexider quartic*" functional equations with or without "*involution*" in certain types of groups, such as, "commutative groups". See: [31, 33].

32. Investigate Ulam-Hyers stabilities of pertinent "*quartic derivations*" from a Banach algebra into its Banach modules. See: ([34, 35]).

33. Establish the solution and stabilities of "*conditionally*" quartic functional equations, for instance: $\|x\| = \|y\|$.

34. Prove stabilities of "*orthogonally*" quartic functional equations, "*in the sense of J. Rätz*" [36]: "$x \perp y \iff \langle x, y\rangle = 0$".

35. Work on stabilities of "*quartic-like*" functional equations, such as:

    1. $$2\big[f_1(ax+y) + f_2\big(x+ay+\sigma(y)\big)\big] + a(a-1)^2 g_1(x-y)$$
    $$= 2(a^2-1)^2\big[h_1(x)+h_2(y)\big] + a(a+1)^2 g_2(x+y),$$
    $\sigma = \sigma(y)$ is "*involution*": $\sigma(x+y) = \sigma(x) + \sigma(y)$; $\sigma(\sigma(x)) = x$.

    2. $$2f(ax+y+\rho) + f(x+ay+\tau) + f(ax-y)$$
    $$= \frac{a}{2}\big[(a^2+4a+1)f(x+y) - (a^2-4a+1)f(x-y)\big]$$
    $$+(3a^4-4a^2+1)f(x) + (a^4-4a^2+3)f(y).$$

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Existence of positive solutions for summation boundary value problem for a fourth-order difference equations

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Abstract

In this paper, we study the existence of positive solutions to the difference-summation boundary value problem

$$\Delta^4 u(t-2) + a(t)f(u) = 0, \qquad t \in \{2,3,...,T\},$$

$$u(0) = \Delta u(0) = \Delta^2 u(0) = 0, \qquad u(T+2) = \alpha \sum_{s=4}^{\eta} u(s),$$

where f is continuous, $T \geq 5$ is a fixed positive integer, $\eta \in \{4,5,...,T-1\}$, $0 < \alpha < \frac{4T(T+1)(T+2)}{(\eta-3)(\eta+2)(\eta^2-\eta+4)}$. We show the existence of at least one positive solution if f is either superlinear or sublinear by applying Guo–Krasnoselskii fixed point theorem in cones.

Keywords : Positive solution; Boundary value problem; Fixed point theorem; Cone
2010 Mathematics Subject Classification: 39A15, 34B15

1 Introduction

The existence of solutions for boundary value problems of difference equations has received much attention. For example, see [4-15] and the references therein.

1

Liang *et al.* in [4] considered the fourth-order boundary value problem of the form

$$\begin{cases} \Delta^4 x(t-2) + a(t)f(x) = 0, & t \in \{2,3,...,T\}, \\ x(0) = x(T+2) = 0, \quad \Delta^2 x(0) = \Delta^2 x(T) = 0, \end{cases}$$

where $T > 2$. Existence and uniqueness of solutions are obtained by a fixed point theorem.

Ma *et al.* in [5] considered the fourth-order boundary value problem of the form

$$\begin{cases} \Delta^4 u(t-2) - \lambda f(t,u(t)) = 0, & t \in \{2,3,...,T\}, \\ u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0, \end{cases}$$

where λ is a parameter, $T > 5$. Existence and uniqueness of solutions are obtained by the theory of fixed-point index in cones.

In this paper, we consider the existence of positive solutions to the equation

$$\Delta^4 u(t-2) + a(t)f(u) = 0, \qquad t \in \{2,3,...,T\}, \tag{1.1}$$

with summation boundary condition

$$u(0) = \Delta u(0) = \Delta^2 u(0) = 0, \qquad u(T+2) = \alpha \sum_{s=4}^{\eta} u(s), \tag{1.2}$$

where f is continuous.

The aim of this paper is to give some results for existence of positive solutions to (1.1)-(1.2).

Let $\mathbb{N}$ be the nonnegative integer, we let $\mathbb{N}_{i,j} = \{k \in \mathbb{N} | \quad i \le k \le j\}$ and $\mathbb{N}_p = \mathbb{N}_{0,p}$. By the positive solution of (1.1)-(1.2) we mean that a function $u(t) : \mathbb{N}_{T+2} \to [0,\infty)$ and satisfies the problem (1.1)-(1.2).

Throughout this paper, we suppose the following conditions hold:
$(H1)$ $T \ge 5$ is a fixed positive integer, $\eta \in \{4,5,...,T-1\}$, constant $\alpha > 0$ such that $0 < \alpha < \frac{4T(T+1)(T+2)}{(\eta-3)(\eta+2)(\eta^2-\eta+4)}$.
$(H2)$ $f \in C([0,\infty),[0,\infty))$, f is either superlinear or sublinear. Set

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, \qquad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}.$$

Then $f_0 = 0$ and $f_\infty = \infty$ correspond to the superlinear case, and $f_0 = \infty$ and $f_\infty = 0$ correspond to the sublinear case.
$(H3)$ $a \in C(\mathbb{N}_{2,T},[0,\infty))$, a is not identical zero.

The proof of the main theorem is based upon an application of the following Guo-Krasnoselskii's fixed point theorem in a cone.

Theorem 1.1. *Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow K$$

be a completely continuous operator such that
(i) $\|Au\| \leqslant \|u\|$, $\quad u \in K \cap \partial\Omega_1$, and $\|Au\| \geqslant \|u\|$, $\quad u \in K \cap \partial\Omega_2$; or
(ii) $\|Au\| \geqslant \|u\|$, $\quad u \in K \cap \partial\Omega_1$, and $\|Au\| \leqslant \|u\|$, $\quad u \in K \cap \partial\Omega_2$.
Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2 Preliminaries

We now state and prove several lemmas before stating our main results.

Lemma 2.1. *Suppose that $y(t) \in C(\mathbb{N}_{2,T})$ and $y(t) > 0$. Then the linear boundary value problem*

$$\Delta^4 u(t-2) + y(t) = 0, \qquad t \in \mathbb{N}_{2,T}, \tag{2.1}$$

$$u(0) = \Delta u(0) = \Delta^2 u(0) = 0, \qquad u(T+2) = \alpha \sum_{s=4}^{\eta} u(s), \tag{2.2}$$

has a unique solution

$$\begin{aligned} u(t) =& \frac{t^{\underline{3}}}{6\Lambda}\Big[\sum_{s=2}^{T}(T-s+3)^{\underline{3}} y(s) - \frac{\alpha}{4}\sum_{s=2}^{\eta-2}(\eta-s+2)^{\underline{4}} y(s)\Big] \\ & - \frac{1}{6}\sum_{s=2}^{t-2}(t-s+1)^{\underline{3}} y(s), \end{aligned} \tag{2.3}$$

where

$$\Lambda := T(T+1)(T+2) - \frac{\alpha}{4}(\eta-3)(\eta+2)(\eta^2-\eta+4). \tag{2.4}$$

Proof. In fact, if $u(t)$ is a solution of problem (2.1), by the discrete Taylor expansion formula, we have

$$u(t) = C_1 t^{\underline{3}} + C_2 t^{\underline{2}} + C_3 t^{\underline{1}} + C_4 - \frac{1}{6}\sum_{s=0}^{t-4}(t-s-1)^{\underline{3}} y(s+2), \quad t \in \mathbb{N}_{T+2}.$$

Applying the first boundary condition $u(0) = \Delta u(0) = \Delta^2 u(0) = 0$ in (2.2), we obtain

$$C_2 = C_3 = C_4 = 0.$$

So,

$$u(t) = C_1 t^{\underline{3}} - \frac{1}{6}\sum_{s=0}^{t-4}(t-s-1)^{\underline{3}} y(s+2), \tag{2.5}$$

From (2.5) and the second boundary condition in (2.2) implies

$$\begin{aligned}\alpha\sum_{s=4}^{\eta} u(s) &= \alpha C_1 \sum_{s=4}^{\eta}(s)^{\underline{3}} - \frac{\alpha}{6}\sum_{s=4}^{\eta}\sum_{\xi=0}^{s-4} (s-\xi-1)^{\underline{3}} y(\xi+2)\\ &= \alpha C_1 \sum_{s=0}^{\eta-4}(s+4)^{\underline{3}} - \frac{\alpha}{6}\sum_{s=0}^{\eta-4}\sum_{\xi=0}^{\eta-s-4} (\xi+3)^{\underline{3}} y(s+2)\\ &= C_1(T+2)^{\underline{3}} - \frac{1}{6}\sum_{s=0}^{T-2}(T-s+1)^{\underline{3}} y(s+2).\end{aligned}$$

Solving the above equation for a constant C_1, we get

$$C_1 = \frac{1}{6\Lambda}\sum_{s=0}^{T-2}(T-s+1)^{\underline{3}} y(s+2) - \frac{\alpha}{6\Lambda}\sum_{s=0}^{\eta-4}\sum_{\xi=0}^{\eta-s-4} (\xi+3)^{\underline{3}} y(s+2)$$

where Λ is defined by (2.4)

Therefore, (2.1)-(2.2) has a unique solution

$$\begin{aligned}u(t) =& \frac{t^{\underline{3}}}{6\Lambda}\left[\sum_{s=2}^{T}(T-s+3)^{\underline{3}} y(s) - \frac{\alpha}{4}\sum_{s=2}^{\eta-2}(\eta-s+2)^{\underline{4}} y(s)\right]\\ &- \frac{1}{6}\sum_{s=2}^{t-2}(t-s+1)^{\underline{3}} y(s).\end{aligned}$$

□

Lemma 2.2. *The function*

$$G(t,s) = \frac{1}{6\Lambda}\begin{cases} -\Lambda(t-s+1)^{\underline{3}} + t^{\underline{3}}(T-s+3)^{\underline{3}} - \frac{\alpha t^{\underline{3}}}{4}(\eta-s+2)^{\underline{4}}, & s\in\mathbb{N}_{2,t-2}\cap\mathbb{N}_{2,\eta-2}\\ -\Lambda(t-s+1)^{\underline{3}} + t^{\underline{3}}(T-s+3)^{\underline{3}}, & s\in\mathbb{N}_{\eta-1,t-2}\\ t^{\underline{3}}(T-s+3)^{\underline{3}} - \frac{\alpha t^{\underline{3}}}{4}(\eta-s+2)^{\underline{4}}, & s\in\mathbb{N}_{t-1,\eta-2}\\ t^{\underline{3}}(T-s+3)^{\underline{3}}, & s\in\mathbb{N}_{t-1,T}\cap\mathbb{N}_{\eta-1,T}\end{cases} \tag{2.6}$$

where Λ *is defined by* (2.4), *is the Green's function of the problem*

$$
\begin{aligned}
&-\Delta^4 u(t-2)=0, \qquad t\in\mathbb{N}_{2,T},\\
&u(0)=\Delta u(0)=\Delta^2 u(0)=0, \qquad u(T+2)=\alpha\sum_{s=4}^{\eta}u(s).
\end{aligned}
\tag{2.7}
$$

Proof. Suppose $t<\eta$. The unique solution of problem (2.1)-(2.2) can be written

$$
\begin{aligned}
u(t)=&-\frac{1}{6}\sum_{s=2}^{t-2}(t-s+1)^{\underline{3}}y(s)\\
&+\frac{t^{\underline{3}}}{6\Lambda}\Bigg[\sum_{s=2}^{t-2}(T-s+3)^{\underline{3}}y(s)+\sum_{s=t-1}^{\eta-2}(T-s+3)^{\underline{3}}y(s)+\sum_{s=\eta-1}^{T}(T-s+3)^{\underline{3}}y(s)\Bigg]\\
&-\frac{\alpha t^{\underline{3}}}{24\Lambda}\Bigg[\sum_{s=2}^{t-2}(\eta-s+2)^{\underline{4}}y(s)+\sum_{s=t-1}^{\eta-2}(\eta-s+2)^{\underline{4}}y(s)\Bigg]\\
=&\frac{1}{6\Lambda}\sum_{s=2}^{t-2}\Bigg[-\Lambda(t-s+1)^{\underline{3}}+t^{\underline{3}}(T-s+3)^{\underline{3}}-\frac{\alpha t^{\underline{3}}}{4}(\eta-s+2)^{\underline{4}}\Bigg]y(s)\\
&+\frac{1}{6\Lambda}\sum_{s=t-1}^{\eta-2}\Bigg[t^{\underline{3}}(T-s+3)^{\underline{3}}-\frac{\alpha t^{\underline{3}}}{4}(\eta-s+2)^{\underline{4}}\Bigg]y(s)\\
&+\frac{1}{6\Lambda}\sum_{s=\eta-1}^{T}t^{\underline{3}}(T-s+3)^{\underline{3}}y(s)\\
=&\sum_{s=2}^{T}G(t,s)y(s).
\end{aligned}
$$

Suppose $t\geq\eta$. The unique solution of problem (2.1)-(2.2) can be written

$$
\begin{aligned}
u(t)=&-\frac{1}{6}\Bigg[\sum_{s=2}^{\eta-2}(t-s+1)^{\underline{3}}y(s)+\sum_{s=\eta-1}^{t-2}(t-s+1)^{\underline{3}}y(s)\Bigg]\\
&+\frac{t^{\underline{3}}}{6\Lambda}\Bigg[\sum_{s=2}^{\eta-2}(T-s+3)^{\underline{3}}y(s)+\sum_{s=\eta-1}^{t-2}(T-s+3)^{\underline{3}}y(s)+\sum_{s=t-1}^{T}(T-s+3)^{\underline{3}}y(s)\Bigg]\\
&-\frac{\alpha t^{\underline{3}}}{24\Lambda}\sum_{s=2}^{\eta-2}(\eta-s+2)^{\underline{4}}y(s)
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{6\Lambda}\sum_{s=2}^{\eta-2}\left[-\Lambda(t-s+1)^{\underline{3}}+t^{\underline{3}}(T-s+3)^{\underline{3}}-\frac{\alpha t^{\underline{3}}}{4}(\eta-s+2)^{\underline{4}}\right]y(s)\\
&+\frac{1}{6\Lambda}\sum_{s=\eta-1}^{t-2}\left[-\Lambda(t-s+1)^{\underline{3}}+t^{\underline{3}}(T-s+3)^{\underline{3}}\right]y(s)\\
&+\frac{1}{6\Lambda}\sum_{s=t-1}^{T}t^{\underline{3}}(T-s+3)^{\underline{3}}y(s)\\
=&\sum_{s=2}^{T}G(t,s)y(s).
\end{aligned}
$$

Then the unique solution of problem (2.1)-(2.2) can be written as $u(t)=\sum_{s=2}^{T}G(t,s)y(s)$. The proof is complete. □

We observe that the condition $0<\alpha<\frac{4T(T+1)(T+2)}{(\eta-3)(\eta+2)(\eta^2-\eta+4)}$. implies $G(t,s)$ is positive on $\mathbb{N}_{2,T}\times\mathbb{N}_{2,T}$.

Let

$$M_1=\min\left\{\frac{G(t,s)}{G(t,t)}:t\in\mathbb{N}_{2,T},s\in\mathbb{N}_{2,T}\right\} \tag{2.8}$$

$$M_2=\max\left\{\frac{G(t,s)}{G(t,t)}:t\in\mathbb{N}_{T+2},s\in\mathbb{N}_{2,T}\right\} \tag{2.9}$$

Lemma 2.3. *Let $(t,s)\in\mathbb{N}_{2,T}\times\mathbb{N}_{2,T}$. Then we have*

$$G(t,s)\geq M_1G(t,t.) \tag{2.10}$$

where $0<M_1<1$ is a constant given by

$$
\begin{aligned}
M_1=\min\Bigg\{&\frac{4(T-\eta+7)^{\underline{3}}-4\Lambda\left(\frac{\eta-5}{\eta-2}\right)-\alpha\eta^{\underline{4}}}{4(T+1)^{\underline{3}}-24\alpha},\frac{4(T-\eta+5)^{\underline{3}}-\alpha(\eta+1)^{\underline{4}}}{4(T+1)^{\underline{3}}-24\alpha},\\
&\frac{6}{4(T+1)^{\underline{3}}-24\alpha},\frac{4(T-\eta-5)^{\underline{3}}-4\Lambda\left(\frac{T-3}{T}\right)-\alpha\eta^{\underline{4}}}{4(T-\eta+4)^{\underline{3}}},\\
&\frac{60(\eta-1)^{\underline{3}}-\Lambda(T-\eta+2)^{\underline{3}}}{T^{\underline{3}}(T-\eta+4)^{\underline{3}}},\frac{6}{(T-\eta+4)^{\underline{3}}}\Bigg\}
\end{aligned}
\tag{2.11}
$$

Proof. In order that 2.10 holds, it is sufficient that M_1 satisfies

$$M_1\leq\min_{(t,s)\in\mathbb{N}_{2,T}\times\mathbb{N}_{2,T}}\frac{G(t,s)}{G(t,t)}. \tag{2.12}$$

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Then we may choose

$$M_1 \le \min\left\{ \min_{(t,s)\in\mathbb{N}_{2,\eta-2}\times\mathbb{N}_{2,T}} \frac{G(t,s)}{G(t,t)}, \min_{(t,s)\in\mathbb{N}_{\eta-1,T}\times\mathbb{N}_{2,T}} \frac{G(t,s)}{G(t,t)} \right\}. \tag{2.13}$$

since

$$\begin{aligned}
&\min_{(t,s)\in\mathbb{N}_{2,\eta-2}\times\mathbb{N}_{2,T}} \frac{G(t,s)}{G(t,t)} \\
&= \min_{t\in\mathbb{N}_{2,\eta-2}} \left\{ \min_{s\in\mathbb{N}_{2,t-2}} \frac{-\Lambda(t-s+1)^{\underline{3}} + t^{\underline{3}}(T-s+3)^{\underline{3}} - \frac{\alpha t^{\underline{3}}}{4}(\eta-s+2)^{\underline{4}}}{t^{\underline{3}}(T-t+3)^{\underline{3}} - \frac{\alpha t^{\underline{3}}}{4}(\eta-t+2)^{\underline{4}}}, \right. \\
&\qquad \min_{s\in\mathbb{N}_{t-1,\eta-2}} \frac{t^{\underline{3}}(T-s+3)^{\underline{3}} - \frac{\alpha t^{\underline{3}}}{4}(\eta-s+2)^{\underline{4}}}{t^{\underline{3}}(T-t+3)^{\underline{3}} - \frac{\alpha t^{\underline{3}}}{4}(\eta-t+2)^{\underline{4}}}, \\
&\qquad \left. \min_{s\in\mathbb{N}_{\eta-1,T}} \frac{t^{\underline{3}}(T-s+3)^{\underline{3}}}{t^{\underline{3}}(T-t+3)^{\underline{3}} - \frac{\alpha t^{\underline{3}}}{4}(\eta-t+2)^{\underline{4}}} \right\} \\
&\ge \min_{t\in\mathbb{N}_{2,\eta-2}} \left\{ \frac{-\Lambda(t-1)^{\underline{3}} + t^{\underline{3}}(T-t+5)^{\underline{3}} - \frac{\alpha t^{\underline{3}}}{4}\eta^{\underline{4}}}{t^{\underline{3}}(T-t+3)^{\underline{3}} - \frac{\alpha t^{\underline{3}}}{4}(\eta-t+2)^{\underline{4}}}, \frac{(T-\eta+5)^{\underline{3}} - \frac{\alpha}{4}(\eta-t+3)^{\underline{4}}}{(T-t+3)^{\underline{3}} - \frac{\alpha}{4}(\eta-t+2)^{\underline{4}}}, \right. \\
&\qquad \left. \frac{3^{\underline{3}}}{(T-t+3)^{\underline{3}} - \frac{\alpha}{4}(\eta-t+2)^{\underline{4}}} \right\} \\
&\ge \min\left\{ \frac{-\Lambda\left(1-\frac{3}{\eta-2}\right) + (T-\eta+7)^{\underline{3}} - \frac{\alpha}{4}(\eta)^{\underline{4}}}{(T+1)^{\underline{3}} - \frac{\alpha 4^{\underline{4}}}{4}}, \frac{(T-\eta+5)^{\underline{3}} - \frac{\alpha}{4}(\eta+1)^{\underline{4}}}{(T+1)^{\underline{3}} - \frac{\alpha 4^{\underline{4}}}{4}}, \frac{6}{(T+1)^{\underline{3}} - \frac{\alpha 4^{\underline{4}}}{4}} \right\} \\
&= \min\left\{ \frac{4(T-\eta+7)^{\underline{3}} - 4\Lambda\left(\frac{\eta-5}{\eta-2}\right) - \alpha\eta^{\underline{4}}}{4(T+1)^{\underline{3}} - 24\alpha}, \frac{4(T-\eta+5)^{\underline{3}} - \alpha(\eta+1)^{\underline{4}}}{4(T+1)^{\underline{3}} - 24\alpha}, \frac{6}{4(T+1)^{\underline{3}} - 24\alpha} \right\}
\end{aligned} \tag{2.14}$$

Similarly, we get

$$\begin{aligned}
&\min_{(t,s)\in\mathbb{N}_{\eta-1,T}\times\mathbb{N}_{2,T}} \frac{G(t,s)}{G(t,t)} \\
&\ge \min\left\{ \frac{4(T-\eta-5)^{\underline{3}} - 4\Lambda\left(\frac{T-3}{T}\right) - \alpha\eta^{\underline{4}}}{4(T-\eta+4)^{\underline{3}}}, \frac{60(\eta-1)^{\underline{3}} - \Lambda(T-\eta+2)^{\underline{3}}}{T^{\underline{3}}(T-\eta+4)^{\underline{3}}}, \frac{6}{(T-\eta+4)^{\underline{3}}} \right\}
\end{aligned} \tag{2.15}$$

The (2.11) is immediate from (2.14)-(2.15) □

Lemma 2.4. *Let* $(t,s) \in \mathbb{N}_{T+2} \times \mathbb{N}_{2,T}$. *Then we have*

$$G(t,s) \le M_2 G(t,t.) \tag{2.16}$$

where $M_2 \geq 1$ *is a constant given by*

$$M_2 = \max\left\{\frac{24\Lambda + (\eta-2)^{\underline{3}}(T+1)^{\underline{3}}}{\alpha(\eta-2)^{\underline{3}}\eta^{\underline{4}}}, \frac{4(T-\eta+6)^{\underline{3}} - 24\alpha}{4(T-\eta+6)^{\underline{3}} - \alpha\eta^{\underline{4}}}, \frac{4(T-\eta+4)^{\underline{3}}}{4(T-\eta+5)^{\underline{3}} - \alpha\eta^{\underline{4}}}, \frac{4T^{\underline{3}} - 24\alpha(\eta-1)^{\underline{3}}}{6(\eta-1)^{\underline{3}}}, \frac{T^{\underline{3}}(T-\eta+4)^{\underline{3}} - 6\Lambda}{6(\eta-1)^{\underline{3}}}, \frac{(T-\eta+5)^{\underline{3}}}{6}\right\} \tag{2.17}$$

Proof. For $t = 0, 1$, from (2.6) we get

$$G(0,s) = G(0,0) = 0; \quad G(1,s) = G(1,1) = 0.$$

Then we may choose $M_2 = 1$. For $t \in \mathbb{N}_{2,T}$, if 2.16 holds, it is sufficient that M_2 satisfies

$$M_2 \geq \max_{(t,s)\in\mathbb{N}_{2,T}\times\mathbb{N}_{2,T}} \frac{G(t,s)}{G(t,t)}. \tag{2.18}$$

Then we may choose

$$M_2 \geq \max\left\{\max_{(t,s)\in\mathbb{N}_{2,\eta-2}\times\mathbb{N}_{2,T}} \frac{G(t,s)}{G(t,t)}, \max_{(t,s)\in\mathbb{N}_{\eta-1,T}\times\mathbb{N}_{2,T}} \frac{G(t,s)}{G(t,t)}\right\}. \tag{2.19}$$

since

$$\begin{aligned}
&\max_{(t,s)\in\mathbb{N}_{2,\eta-2}\times\mathbb{N}_{2,T}} \frac{G(t,s)}{G(t,t)} \\
&= \max_{t\in\mathbb{N}_{2,\eta-2}}\left\{\max_{s\in\mathbb{N}_{2,t-2}} \frac{-\Lambda(t-s+1)^{\underline{3}} + t^{\underline{3}}(T-s+3)^{\underline{3}} - \frac{\alpha t^{\underline{3}}}{4}(\eta-s+2)^{\underline{4}}}{t^{\underline{3}}(T-t+3)^{\underline{3}} - \frac{\alpha t^{\underline{3}}}{4}(\eta-t+2)^{\underline{4}}},\right. \\
&\qquad \max_{s\in\mathbb{N}_{t-1,\eta-2}} \frac{t^{\underline{3}}(T-s+3)^{\underline{3}} - \frac{\alpha t^{\underline{3}}}{4}(\eta-s+2)^{\underline{4}}}{t^{\underline{3}}(T-t+3)^{\underline{3}} - \frac{\alpha t^{\underline{3}}}{4}(\eta-t+2)^{\underline{4}}}, \\
&\qquad \left.\max_{s\in\mathbb{N}_{\eta-1,T}} \frac{t^{\underline{3}}(T-s+3)^{\underline{3}}}{t^{\underline{3}}(T-t+3)^{\underline{3}} - \frac{\alpha t^{\underline{3}}}{4}(\eta-t+2)^{\underline{4}}}\right\} \\
&\leq \max_{t\in\mathbb{N}_{2,\eta-2}}\left\{\frac{-\Lambda 3^{\underline{3}} + t^{\underline{3}}(T+1)^{\underline{3}} - \frac{\alpha t^{\underline{3}}}{4}(\eta-t+4)^{\underline{4}}}{t^{\underline{3}}(T-t+3)^{\underline{3}} - \frac{\alpha t^{\underline{3}}}{4}(\eta-t+2)^{\underline{4}}}, \frac{(T-t+4)^{\underline{3}} - \frac{\alpha 4^{\underline{4}}}{4}}{(T-t+3)^{\underline{3}} - \frac{\alpha}{4}(\eta-t+2)^{\underline{4}}},\right. \\
&\qquad \left.\frac{(T-\eta+4)^{\underline{3}}}{(T-t+3)^{\underline{3}} - \frac{\alpha}{4}\eta^{\underline{4}}}\right\} \\
&\leq \max\left\{\frac{24\Lambda + (\eta-2)^{\underline{3}}(T+1)^{\underline{3}}}{\alpha(\eta-2)^{\underline{3}}\eta^{\underline{4}}}, \frac{4(T-\eta+6)^{\underline{3}} - 24\alpha}{4(T-\eta+6)^{\underline{3}} - \alpha\eta^{\underline{4}}}, \frac{4(T-\eta+4)^{\underline{3}}}{4(T-\eta+5)^{\underline{3}} - \alpha\eta^{\underline{4}}}\right\}
\end{aligned} \tag{2.20}$$

Similarly, we get

$$\begin{aligned}
&\max_{(t,s)\in\mathbb{N}_{\eta-1,T}\times\mathbb{N}_{2,T}} \frac{G(t,s)}{G(t,t)} \\
&\leq \max\left\{\frac{4T^{\underline{3}} - 24\alpha(\eta-1)^{\underline{3}}}{6(\eta-1)^{\underline{3}}}, \frac{T^{\underline{3}}(T-\eta+4)^{\underline{3}} - 6\Lambda}{6(\eta-1)^{\underline{3}}}, \frac{(T-\eta+5)^{\underline{3}}}{6}\right\}
\end{aligned} \tag{2.21}$$

For $t = T+1, T+2$ from (2.6) we get

$$\begin{aligned}
G(T+1,s) <& \frac{1}{6\Lambda}\left[-\Lambda(T-s+2)^{\underline{3}} + (T+1)^{\underline{3}}\left((T-s+3)^{\underline{3}} - \frac{\alpha}{4}(\eta-s+2)^{\underline{4}}\right)\right] \\
=& -\frac{\alpha}{24\Lambda}(T+1)^{\underline{3}}(\eta-s+2)^{\underline{4}} \\
<& -\frac{\alpha}{24\Lambda}(T+1)^{\underline{3}}(\eta-T+1)^{\underline{4}} \\
=& G(T+1,T+1),
\end{aligned}$$

$$\begin{aligned}
G(T+2,s) <& \frac{1}{6\Lambda}\left[-\Lambda(T-s+3)^{\underline{3}} + (T+2)^{\underline{3}}\left((T-s+3)^{\underline{3}} - \frac{\alpha}{4}(\eta-s+2)^{\underline{4}}\right)\right] \\
=& \frac{1}{6\Lambda}\left[(T-s+2)^{\underline{3}}\left(\frac{T-s+3}{T-s}\right)\left((T+2)^{\underline{3}} - \Lambda\right) - \frac{\alpha}{4}(\eta-s+2)^{\underline{4}}\right] \\
=& -\frac{\alpha}{24\Lambda}(T+2)^{\underline{3}}(\eta-s+2)^{\underline{4}} \\
<& -\frac{\alpha}{24\Lambda}(T+2)^{\underline{3}}(\eta-T)^{\underline{4}} \\
=& G(T+2,T+2).
\end{aligned}$$

Then we choose $M_2 = 1$. So (2.17) is immediate from (2.20)-(2.21). $\square$

3 Main Results

Now we are in the position to establish the main result.

Theorem 3.1. *Assume* $(H1)$ - $(H3)$ *hold. Then the problem* (1.1)-(1.2) *has at least one positive solution.*

Proof. In the following, we denote

$$m = \min_{t\in\mathbb{N}_{\eta-1,T}} G(t,t), \quad M = \max_{t\in\mathbb{N}_{T+2}} G(t,t).$$

Then $0 < m < M$.

Let E be the Banach's space defined by $E = \{u : \mathbb{N}_{T+2} \to R\}$. Define

$$K = \{u \in E : u \geqslant 0, t \in \mathbb{N}_{T+2} \text{ and } \min_{t\in\mathbb{N}_{2,T}} u(t) \geq \sigma \parallel u \parallel\}.$$

where $\sigma = \frac{M_1 m}{M_2 M} \in (0,1)$, $\parallel u \parallel = \max_{t\in\mathbb{N}_{T+2}} \mid u(t) \mid$. It is obvious that K is a cone in E.

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We define the operator $A : K \to E$ by

$$(Au)(t) = \sum_{s=2}^{T} G(t,s)a(s)f(u(s)), t \in \mathbb{N}_{T+2}.$$

It is clear that problem (1.1)-(1.2) has a solution u if and only if $u \in K$ is a fixed point of operator A. We shall now show that the operator A maps K to itself. For this, let $u \in K$, from $(H_2) - (H_3)$, we get

$$(Au)(t) = \sum_{s=1}^{T} G(t,s)a(s)f(u(s)) \geq 0, t \in \mathbb{N}_{T+2}. \tag{3.1}$$

from (2.9), we obtain

$$\begin{aligned}(Au)(t) =& \sum_{s=2}^{T} G(t,s)a(s)f(u(s)) \leq M_2 \sum_{s=2}^{T} G(t,t)a(s)f(u(s)) \\ \leq& M_2 M \sum_{s=2}^{T} a(s)f(u(s)), \quad t \in \mathbb{N}_{T+2}.\end{aligned}$$

Therefore

$$\| Au \| \leq M_2 M \sum_{s=2}^{T} a(s)f(u(s)). \tag{3.2}$$

Now from $(H_2), (H_3)$, (2.8) and (3.2), for $t \in \mathbb{N}_{\eta,T}$, we have

$$\begin{aligned}(Au)(t) \geq& M_1 \sum_{s=2}^{T} G(t,t)a(s)f(u(s)) \geq M_1 m \sum_{s=2}^{T} a(s)f(u(s)) \\ \geq& \frac{M_1 m}{M_2 M} \| Au \| = \sigma \| u \| .\end{aligned}$$

Then

$$\min_{t \in \mathbb{N}_{\eta,T}} (Au)(t) \geq \sigma \| u \| . \tag{3.3}$$

From (3.1)-(3.2), we obtain $Au \in K$, Hence $A(K) \subseteq K$. So $F : k \to K$ is completely continuous.

Superlinear case. $f_0 = 0$ and $f_\infty = \infty$. Since $f_0 = 0$, we may choose $H_1 > 0$ so that $f(u) \leqslant \epsilon_1 u$, for $0 < u \leqslant H_1$, where $\epsilon_1 > 0$ satisfies

$$\epsilon_1 M_2 M \sum_{s=2}^{T} a(s) \leq 1. \tag{3.4}$$

Thus, if we let

$$\Omega_1 = \{u \in E : \|u\| < H_1\},$$

then for $u \in K \cap \partial\Omega_1$, we get

$$\begin{aligned}(Au)(t) \leq & M_2 \sum_{s=2}^{T} G(t,t)a(s)f(u(s)) \leq \epsilon_1 M_2 M \sum_{s=2}^{T} a(s)u(s) \\ \leq & \epsilon_1 M_2 M \sum_{s=2}^{T} a(s)\|u\| \leq \|u\|.\end{aligned}$$

Thus $\|Au\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$.

Further, since $f_\infty = \infty$, there exists $\widehat{H}_2 > 0$ such that $f(u) \geq \epsilon_2 u$, for $u \geq \widehat{H}_2$, where $\epsilon_2 > 0$ satisfies

$$\epsilon_2 M_1 \sigma \sum_{s=\eta-1}^{T} G(\eta-1,\eta-1)a(s) \geq 1. \tag{3.5}$$

Let $H_2 = \max\{2H_1, \frac{\widehat{H}_2}{\sigma}\}$ and $\Omega_2 = \{u \in E : \|u\| < H_2\}$. Then $u \in K \cap \partial\Omega_2$ implies

$$\min_{t\in\mathbb{N}_{\eta-1,T}} u(t) \geq \sigma\|u\| \geq \widehat{H}_2.$$

Applying (2.8) and (3.5), we get

$$\begin{aligned}(Au)(\eta-1) = & M_1 \sum_{s=2}^{T} G(\eta-1,s)a(s)f(u(s)) \geq M_1 \sum_{s=\eta-1}^{T} G(\eta-1,\eta-1)a(s)f(u(s)) \\ \geq & \varepsilon_2 M_1 \sum_{s=\eta-1}^{T} G(\eta-1,\eta-1)a(s)y(s) \geq \varepsilon_2 M_1 \sigma \sum_{s=\eta-1}^{T} G(\eta-1,\eta-1)a(s)\|u\| \\ \geq & \|u\|.\end{aligned}$$

Hence, $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$. By the first part of Theorem 1.1, A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ such that $H_1 \leqslant \|u\| \leqslant H_2$.

Sublinear case. $f_0 = \infty$ and $f_\infty = 0$. Since $f_0 = \infty$, choose $H_3 > 0$ such that $f(u) \geqslant \epsilon_3 u$ for $0 < u \leqslant H_3$, where $\varepsilon_3 > 0$ satisfies

$$\epsilon_3 M_1 \sigma \sum_{s=\eta-1}^{T} G(\eta-1,\eta-1)a(s) \geqslant 1. \tag{3.6}$$

Let

$$\Omega_3 = \{u \in E : \|u\| < H_3\},$$

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then for $u \in K \cap \partial\Omega_3$, we get

$$\begin{aligned}(Au)(\eta-1) \geq & M_1 \sum_{s=\eta-1}^{T} G(\eta-1,\eta-1)a(s)f(u(s)) \geq \epsilon_3 M_1 \sum_{s=\eta-1}^{T} G(\eta-1,\eta-1)a(s)y(s) \\ \geq & \varepsilon_3 M_1 \sigma \sum_{s=\eta-1}^{T} G(\eta-1,\eta-1)a(s)\|u\| \geq \|u\|.\end{aligned}$$

Thus, $\|Au\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_3$.

Now, since $f_\infty = 0$, there exists $\widehat{H}_4 > 0$ so that $f(u) \leqslant \epsilon_4 u$ for $u \geqslant \widehat{H}_4$, where $\epsilon_4 > 0$ satisfies

$$\epsilon_4 M_2 M \sum_{s=\eta-1}^{T} a(s) \geqslant 1. \tag{3.7}$$

Subcase 1. Suppose f is bounded, $f(u) \leq L$ for all $u \in [0,\infty)$ for some $L > 0$. Let $H_4 = \max\{2H_3, LM_2M \sum_{s=1}^{T} a(s)\}$.

Then for $u \in K$ and $\|u\| = H_4$, we get

$$\begin{aligned}(Au)(\eta) \leq & M_2 \sum_{s=2}^{T} G(t,t)a(s)f(u(s)) \leq LM_2M \sum_{s=2}^{T} a(s) \\ \leq & H_4 = \|u\|\end{aligned}$$

Thus $(Au)(t) \leq \|u\|$.

Subcase 2. Suppose f is unbounded, there exist $H_4 > \max\{2H_3, \frac{\widehat{H}_4}{\sigma}\}$ such that $f(u) \leq f(H_4)$ for all $0 < u \leq H_4$. Then for $u \in K$ with $\|u\| = H_4$ from (2.9) and (3.7), we have

$$\begin{aligned}(Au)(t) \leq & M_2 \sum_{s=2}^{T} G(t,t)a(s)f(u(s)) \leq M_2M \sum_{s=2}^{T} a(s)f(H_4) \\ \leq & \epsilon_4 M_2 M \sum_{s=2}^{T} a(s)H_4 \leq H_4 = \|u\|.\end{aligned}$$

Thus in both cases, we may put $\Omega_4 = \{u \in E : \|u\| < H_4\}$. Then

$$\|Au\| \leqslant \|u\|, u \in K \cap \partial\Omega_4.$$

By the second part of Theorem 1.1, A has a fixed point u in $K \cap (\overline{\Omega}_4 \setminus \Omega_3)$, such that $H_3 \leqslant \|u\| \leqslant H_4$. This completes the sublinear part of the theorem. Therefore, the problem (1.1)-(1.2) has at least one positive solution. □

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4 Some examples

In this section, in order to illustrate our result, we consider some examples.

Example 4.1 Consider the BVP

$$\Delta^4 u(t-2) + t^2 u^k = 0, \qquad t \in N_{2,6}, \tag{4.1}$$

$$u(0) = \Delta u(0) = \Delta^2 u(0) = 0, \qquad u(8) = \frac{2}{3}\sum_{s=4}^{5} u(s). \tag{4.2}$$

Set $\alpha = \frac{2}{3},\ \eta = 5,\ T = 6,\ a(t) = t^2,\ f(u) = u^k$.
We can show that

$$T(T+1)(T+2) - \frac{\alpha}{4}(\eta-3)(\eta+2)(\eta^2-\eta+4) = 280 > 0.$$

Case I : $k \in (1,\infty)$. In this case, $f_0 = 0,\ f_\infty = \infty$ and (i) of theorem 3.1 holds. Then BVP (4.1)-(4.2) has at least one positive solution.

Case II : $k \in (0,1)$. In this case, $f_0 = \infty,\ f_\infty = 0$ and (ii) of theorem 3.1 holds. Then BVP (4.1)-(4.2) has at least one positive solution.

Example 4.2 Consider the BVP

$$\Delta^4 u(t-2) + e^t t^e \left(\frac{\pi \sin u + 2\cos u}{u^2}\right) = 0, \qquad t \in N_{2,8}, \tag{4.3}$$

$$u(0) = \Delta u(0) = \Delta^2 u(0) = 0, \qquad u(10) = \frac{1}{3}\sum_{s=4}^{6} u(s), \tag{4.4}$$

Set $\alpha = \frac{1}{3},\ \eta = 6,\ T = 8,\ a(t) = e^t t^e,\ f(u) = \frac{\pi \sin u + 2\cos u}{u^2}$.
We can show that

$$T(T+1)(T+2) - \frac{\alpha}{4}(\eta-3)(\eta+2)(\eta^2-\eta+4) = 596 > 0.$$

Through a simple calculation we can get $f_0 = \infty,\ f_\infty = 0$. Thus, by (ii) of theorem 3.1, we can get BVP (4.3)-(4.4) has at least one positive solution.

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Similarity measure between generalized intuitionistic fuzzy sets and its application to pattern recognition

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Abstract

This paper presents new methods for measuring similarity between generalized intuitionistic fuzzy sets (GIFSs) and its application to pattern recognition. Firstly, the geometrical interpretation of GIFSs is carefully reviewed and then the results of the interpretation is utilized to generate new methods for measuring similarity in order to calculate the degree of similarity between GIFSs. Numerical example is given to illustrate the application of the proposed similarity measures. Finally, we also use the proposed similarity measures to characterize the similarity between linguistic variables.

1 Introduction

As a generalization of fuzzy sets, intuitionistic fuzzy sets (IFSs) were presented by Atanassov [1, 2, 3]. Since IFSs can present the degrees of membership and non-membership with a degree of hesitancy, the knowledge and semantic representation become more meaningful and applicable. These IFSs have been widely studied and applied in various areas, such as logic programming [4], decision making [7, 22], pattern recognition [12, 14, 15, 16, 19, 26] and medical diagnosis

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[9, 25], and seem to have more popular than fuzzy sets technology. Mondal and Samanta [18] introduced generalized intuitionistic fuzzy sets (GIFSs) as a generalization of IFSs and studied their basic properties. Park et al. [20] proposed a method to calculate the correlation coefficient of GIFSs. There is a little investigation on GIFSs. Similarity assessment plays a fundamental and important role in inference and approximate reasoning in all applications of intuitionistic fuzzy logic [4]. For different purposes different similarity measures should be used. Based on the importance of the problem, the effectiveness and properties of the different similarity measures for IFSs have been compared and examined by many researchers (e.g. Hung and Yang [12], Li and Cheng [14], Li et al. [15], Liang and Shi [16], Mitchell [19], Szmidt and Baldwin [23]). The analysis of similarity is also a fundamental issue while employing GIFSs. Recently, Park et al. [21] proposed and applied similarity measure to compare generalized intuitionistic fuzzy preferences given by individuals (experts) and evaluated an extent of a group agreement. In this paper, we propose new similarity measures based on the geometrical representation for GIFS. The proposed similarity measures depend on the triplet, membership degree, nonmembership degree, and hesitation margin. This paper proves that the proposed similarity measures satisfy the properties of axiomatic definition for similarity measures. Numerical example is given to illustrate the application of the developed similarity measures. Furthermore, we use the proposed similarity measures to characterize the similarity between linguistic variables.

2 Brief introduction of GIFSs

In the following, we firstly recall basic notions and definitions of GIFSs which can be found in [18].

Let X be the universe of discourse. A generalized intuitionistic fuzzy set (GIFS) A in X is an object having the form

$$A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\} \tag{1}$$

where $\mu_A, \nu_A : X \to [0, 1]$ denote membership function and non-membership function, respectively, of A and satisfy $\min\{\mu_A(x), \nu_A(x)\} \leq 0.5$ for all $x \in X$. Let GIFS(X) denote the set of all GIFSs in X.

For an IFS $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$, it is observed that $\mu_A(x) + \nu_A(x) \leq 1$ implies $\min\{\mu_A(x), \nu_A(x)\} \leq 0.5$ for each $x \in X$. Thus, every IFS is GIFS.

For each GIFS A in X, we call

$$\phi_A(x) = 1 - \mu_A(x) - \nu_A(x) \tag{2}$$

a generalized intuitionistic fuzzy index (or a hesitation margin) of x in A and it expresses a lack/excess of knowledge of whether x belongs to A or not. (see, [20]). It is obvious that $-0.5 \leq \phi_A(x) \leq 1$ for each $x \in X$.

Having in mind that for each element x belonging to a GIFS A, the values of membership, non-membership and generalized intuitionistic fuzzy index add up to one, i.e.

$$\mu_A(x) + \nu_A(x) + \phi_A(x) = 1 \tag{3}$$

and that each of the membership and non-membership are from $[0, 1]$ and the generalized intuitionistic fuzzy index is from $[-0.5, 1]$, we can imagine a cuboid (Figure 1) inside which there is a polygon $ADBEGF$ where the above equation is fulfilled. In other words, the polygon $ADBEGF$ represents a surface where coordinates of any element belonging to a GIFS can be represented. Each point belonging to the polygon $ADBEGF$ is described via three coordinates: (μ, ν, ϕ). Points A and B represent crisp elements. Point $A(1, 0, 0)$ represents elements fully belonging to a GIFS as $\mu = 1$. Point $B(0, 1, 0)$ represents elements fully not belonging to a GIFS as $\nu = 1$. Point $D(0, 0, 1)$ represents element about which we are not able to say if they belong or not belong to a GIFS (generalized intuitionistic fuzzy index $\phi = 1$). Point $E(1, 0.5, -0.5)$ represents element about which we can say to belong to a GIFS ($\phi = -0.5$). Point $F(0.5, 1, -0.5)$ represents element about which we can say to not belong to a GIFS ($\phi = -0.5$). Such an interpretation is intuitively appealing and provides means for the representation of many aspects of imperfect information. Segment AB (where $\phi = 0$) represents elements belonging to classical fuzzy sets ($\mu + \nu = 1$). Triangle ADB (where $0 \leq \phi \leq 1$) represents elements belonging to IFSs ($0 \leq \mu + \nu \leq 1$). Any other combination of the values characterizing a GIFS can be represented inside the triangles AGF and BEG. In other words, each element belonging to a GIFS can be represented as a point (μ, ν, ϕ) belonging to the polygon $ADBEGF$ (cf. Figure 1).

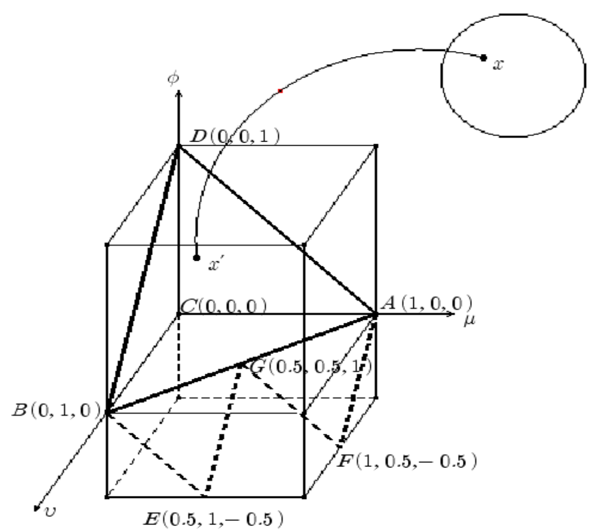

Figure 1: A geometrical interpretation of a GIFS

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It is worth mentioning that the geometrical interpretation is directly related to the definition of a GIFS, and it does not need any additional assumptions. By employing the above geometrical representation, a GIFS A can be expressed as

$$A = \{(\mu_A(x), \nu_A(x), \phi_A(x)) \mid x \in X\}. \tag{4}$$

Therefore, this representation of a GIFS will be a point of departure for considering the our method in calculating the degree of similarity between GIFSs.

For $A, B \in \text{GIFS}(X)$, Mondal and Samanta [18] defined the notion of containment as follows:

$$A \subseteq B \iff \mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) \geq \nu_B(x)\ \forall x \in X. \tag{5}$$

As above-mentioned, we can not omit the third parameter (hesitancy degree) in the representation of GIFSs and then redefine the notion of containment as follows:

$$A \subseteq B \iff \mu_A(x) \leq \mu_B(x),\ \nu_A(x) \geq \nu_B(x) \text{ and } \phi_A(x) \geq \phi_B(x)\ \forall x \in X. \tag{6}$$

Definition 2.1 Let $S : \text{GIFS}(X) \times \text{GIFS}(X) \to [0, 1]$ be a mapping. $S(A, B)$ is said to be the degree of similarity between $A \in \text{GIFS}(X)$ and $B \in \text{GIFS}(X)$ if $S(A, B)$ satisfies the properties (SP1)-(SP4):
(SP1) $0 \leq S(A, B) \leq 1$;
(SP2) $S(A, B) = 1$ if and only if $A = B$;
(SP3) $S(A, B) = S(B, A)$;
(SP4) $S(A, C) \leq S(A, B)$ and $S(A, C) \leq S(B, C)$ if $A \subseteq B \subseteq C$, $A, B, C \in \text{GIFS}(X)$.

Definition 2.2 Let $D : \text{GIFS}(X) \times \text{GIFS}(X) \to [0, 1]$ be a mapping. $D(A, B)$ is called a distance $A \in \text{GIFS}(X)$ and $B \in \text{GIFS}(X)$ if $D(A, B)$ satisfies the properties (DP1)-(DP4):
(DP1) $0 \leq D(A, B) \leq 1$;
(DP2) $D(A, B) = 0$ if and only if $A = B$;
(DP3) $D(A, B) = D(B, A)$;
(DP4) $D(A, B) \leq D(A, C)$ and $D(B, C) \leq D(A, C)$ if $A \subseteq B \subseteq C$, $A, B, C \in \text{GIFS}(X)$.

Because distance and similarity measures are complementary concepts, similarity measures can be used to define distance measures and vice verses.

3 New similarity measures between GIFSs

In this section, we take into account three parameters describing GIFSs to propose a new similarity measures between GIFSs based on the geometrical representation of GIFSs.

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Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ and $B = \{(x, \mu_B(x), \nu_B(x)) \mid x \in X\}$ be two GIFSs in $X = \{x_1, x_2, \ldots, x_n\}$. We propose a new similarity measure:

$$\begin{aligned} S_g(A,B) &\\ = 1 - \frac{1}{n}\sum_{i=1}^{n}\Bigg(&\frac{|\mu_A(x_i)-\mu_B(x_i)|+|\nu_A(x_i)-\nu_B(x_i)|+|\phi_A(x_i)-\phi_B(x_i)|}{4} \\ &+\frac{\max(|\mu_A(x_i)-\mu_B(x_i)|,|\nu_A(x_i)-\nu_B(x_i)|,|\phi_A(x_i)-\phi_B(x_i)|)}{2}\Bigg), \end{aligned} \tag{7}$$

where $\phi_A(x_i)$ and $\phi_B(x_i)$ are, respectively, the hesitancy degree of the element $x_i \in X$ to the sets A and B.

Theorem 3.1 *$S_g(A,B)$ is the similarity measure between two GIFSs A and B.*

Proof For the sake of simplicity, IFSs A and B are denoted by $A = \{(\mu_A(x_i), \nu_A(x_i), \phi(x_i)) \mid x_i \in X\}$ and $B = \{(\mu_B(x_i), \nu_B(x_i), \phi_B(x_i)) \mid x_i \in X\}$, respectively. Obviously, $S_g(A,B)$ satisfies (SP1) and (SP3) of Definition 1. We only need to prove that $S_g(A,B)$ satisfies (SP2) and (SP4).

(SP2): From (6), we have

$$\begin{aligned} &S_g(A,B) = 1 \\ &\quad \Leftrightarrow \mu_A(x_i) = \mu_B(x_i), \nu_A(x_i) = \nu_B(x_i), \phi_A(x_i) = \phi_B(x_i), \forall x_i \in X \\ &\quad \Leftrightarrow A = B. \end{aligned}$$

(SP4): For any IFS $C = \{(\mu_C(x_i), \nu_C(x_i), \phi_C(x_i)) \mid x_i \in X\}$, if $A \subseteq B \subseteq C$, then we have

$$\begin{aligned} S_g(A,C) = 1 - \frac{1}{n}\sum_{i=1}^{n}\Bigg(&\frac{|\mu_A(x_i)-\mu_C(x_i)|+|\nu_A(x_i)-\nu_C(x_i)|+|\phi_A(x_i)-\phi_C(x_i)|}{4} \\ &+\frac{\max(|\mu_A(x_i)-\mu_C(x_i)|,|\nu_A(x_i)-\nu_C(x_i)|,|\phi_A(x_i)-\phi_C(x_i)|)}{2}\Bigg). \end{aligned}$$

It is easy to see that

$$\begin{aligned} &|\mu_A(x_i)-\mu_C(x_i)| \geq |\mu_A(x_i)-\mu_B(x_i)|, |\nu_A(x_i)-\nu_C(x_i)| \geq |\nu_A(x_i)-\nu_B(x_i)|, \\ &|\phi_A(x_i)-\phi_C(x_i)| \geq |\phi_A(x_i)-\phi_B(x_i)|. \end{aligned}$$

So we have

$$\begin{aligned} &\frac{|\mu_A(x_i)-\mu_C(x_i)|+|\nu_A(x_i)-\nu_C(x_i)|+|\phi_A(x_i)-\phi_C(x_i)|}{4} \\ &\qquad +\frac{\max(|\mu_A(x_i)-\mu_C(x_i)|,|\nu_A(x_i)-\nu_C(x_i)|,|\phi_A(x_i)-\phi_C(x_i)|)}{2} \\ &\geq \frac{|\mu_A(x_i)-\mu_B(x_i)|+|\nu_A(x_i)-\nu_B(x_i)|+|\phi_A(x_i)-\phi_B(x_i)|}{4} \\ &\qquad +\frac{\max(|\mu_A(x_i)-\mu_B(x_i)|,|\nu_A(x_i)-\nu_B(x_i)|,|\phi_A(x_i)-\phi_B(x_i)|)}{2} \end{aligned}$$

and thus we get $S_g(A,C) \le S_g(A,B)$. By the same reason, we can get $S_g(A,C) \le S_g(B,C)$.

However, the elements in the universe may have different importance in pattern recognition. We should consider the weight of the elements so that we can obtain more reasonable results in pattern recognition.

Assume that the weight of x_i in X is w_i, where $w_i \in [0,1]$ $(i = 1,2,\ldots,n)$ and $\sum_{i=1}^{n} w_i = 1$. The similarity measure between GIFSs A and B can be obtained by the following form:

$$\begin{aligned} &S_{gw}(A,B) \\ &\quad = 1 - \sum_{i=1}^{n} w_i \Bigg(\frac{|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\phi_A(x_i) - \phi_B(x_i)|}{4} \\ &\qquad + \frac{\max(|\mu_A(x_i) - \mu_B(x_i)|, |\nu_A(x_i) - \nu_B(x_i)|, |\phi_A(x_i) - \phi_B(x_i)|)}{2} \Bigg). \quad (8) \end{aligned}$$

Likewise, for $S_{gw}(A,B)$, the following theorem holds.

Theorem 3.2 *$S_{gw}(A,B)$ is the similarity measure between two GIFSs A and B.*

Proof The proof is similar to that of Theorem 3.1.

Remark 3.3 Obviously, if $w_i = 1/n$ $(i = 1,2,\ldots,n)$, (8) becomes (7). So, (7) is a special case of (8).

Now, we propose another new similarity measure between GIFSs $A = \{(x, \mu_A(x), \nu_A(x)) | x \in X\}$ and $B = \{(x, \mu_B(x), \nu_B(x)) | x \in X\}$ in $X = \{x_1, x_2, \ldots, x_n\}$ as follows:

Let $\varphi_{\mu_{AB}}(i) = |\mu_A(x_i) - \mu_B(x_i)|/2$, $\varphi_{\nu_{AB}}(i) = |\nu_A(x_i) - \nu_B(x_i)|/2$, $\varphi_{\phi_{AB}}(i) = |\phi_A(x_i) - \phi_B(x_i)|/2$ and $x_i \in X$. Then

$$S_d^p(A,B) = 1 - \frac{1}{\sqrt[p]{n}} \sqrt[p]{\sum_{i=1}^{n} (\varphi_{\mu_{AB}}(i) + \varphi_{\nu_{AB}}(i) + \varphi_{\pi_{AB}}(i))^p}, \quad (9)$$

where $1 \le p < \infty$.

Theorem 3.4 *$S_d^p(A,B)$ is the similarity measure between two GIFSs A and B.*

Proof Obviously, $S_d^p(A,B)$ satisfies (SP1) and (SP3). As to (SP2) and (SP4), we give the following proof.

(SP2): From (8), we have

$$\begin{aligned} &S_d^p(A,B) = 1 \\ &\quad \Leftrightarrow \varphi_{\mu_{AB}}(i) = 0, \varphi_{\nu_{AB}}(i) = 0, \varphi_{\phi_{AB}}(i) = 0, \ \forall i = 1,\ldots,n \\ &\quad \Leftrightarrow \mu_A(x_i) = \mu_B(x_i), \nu_A(x_i) = \nu_B(x_i), \phi_A(x_i) = \phi_B(x_i), \ \forall x_i \in X \\ &\quad \Leftrightarrow A = B. \end{aligned}$$

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(SP4): Since $A \subseteq B \subseteq C$, we have $\mu_A(x_i) \leq \mu_B(x_i) \leq \mu_C(x_i)$, $\nu_A(x_i) \geq \nu_B(x_i) \geq \nu_C(x_i)$ and $\phi_A(x_i) \geq \phi_B(x_i) \geq \phi_C(x_i)$ for any $x_i \in X$. Then we have

$$\begin{aligned}
&\varphi_{\mu_{AB}}(i) + \varphi_{\nu_{AB}}(i) + \varphi_{\phi_{AB}}(i) \\
&\quad = |\mu_A(x_i) - \mu_B(x_i)|/2 + |\nu_A(x_i) - \nu_B(x_i)|/2 + |\phi_A(x_i) - \phi_B(x_i)|/2 \\
&\quad \leq |\mu_A(x_i) - \mu_C(x_i)|/2 + |\nu_A(x_i) - \nu_C(x_i)|/2 + |\phi_A(x_i) - \phi_C(x_i)|/2 \\
&\quad = \varphi_{\mu_{AC}}(i) + \varphi_{\nu_{AC}}(i) + \varphi_{\phi_{AC}}(i).
\end{aligned}$$

So we have

$$\sqrt[p]{\sum_{i=1}^{n} (\varphi_{\mu_{AC}}(i) + \varphi_{\nu_{AC}}(i) + \varphi_{\phi_{AC}}(i))^p} \geq \sqrt[p]{\sum_{i=1}^{n} (\varphi_{\mu_{AB}}(i) + \varphi_{\nu_{AB}}(i) + \varphi_{\phi_{AB}}(i))^p}.$$

Therefore, $S_d^p(A, C) \leq S_d^p(A, B)$. In the similar way, it is easy to prove $S_d^p(A, C) \leq S_d^p(B, C)$.

Similar to (8), considering the weight w_i of $x_i \in X$, the similarity measure of GIFSs A and B can be obtained as following form.

$$S_{dw}^p(A, B) = 1 - \sqrt[p]{\sum_{i=1}^{n} w_i(\varphi_{\mu_{AB}}(i) + \varphi_{\nu_{AB}}(i) + \varphi_{\phi_{AB}}(i))^p}, \tag{10}$$

where $1 \leq p < \infty$.

Likewise, for $S_{dw}^p(A, B)$, the following theorem holds.

Theorem 3.5 $S_{dw}^p(A, B)$ *is the similarity measure between two GIFSs A and B.*

Proof The proof is similar to that of Theorem 3.4.

4 An application to pattern recognition problem

Assume that a question related to pattern recognition is given using GIFSs.

Assume that there exist m patterns which are represented by GIFSs $A_i = \{(x_i,\ \mu_A(x_i), \nu_A(x_i)) \mid x_i \in X\}$ $(i = 1, 2, \ldots, m)$, where $X = \{x_1, x_2, \ldots, x_n\}$. Suppose that there be a sample to be recognized which is represented by GIFS $B = \{(x_i, \mu_B(x_i), \nu_B(x_i)) \mid x_i \in X\}$. Set

$$S(A_{i_0}, B) = \max_{1 \leq i \leq n} \{S(A_i, B)\}, \tag{11}$$

where $S(A_i, B)$ is the similarity measure between A_i and B $(i = 1, 2, \ldots, n)$ given by (8) or (10).

According to the principle of the maximum degree of similarity between GIFSs, it can be decided that the sample B belongs to some pattern A_{i_0}.

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Example 4.1 Assume that there are three patterns denoted with GIFSs in $X = \{x_1, x_2, x_3\}$. Three patterns A_1, A_2 and A_3 are denoted as follows:

$$\begin{aligned} A_1 &= \{(x_1, 0.6, 0.3), (x_2, 0.8, 0.3), (x_3, 0.7, 0.4)\}; \\ A_2 &= \{(x_1, 0.5, 0.6), (x_2, 0.5, 0.4), (x_3, 0.7, 0.5)\}; \\ A_3 &= \{(x_1, 0.6, 0.5), (x_2, 0.7, 0.4), (x_3, 0.8, 0.4)\}. \end{aligned}$$

Assume that a sample $B = \{(x_1, 0.6, 0.3), (x_2, 0.7, 0.5), (x_3, 0.7, 0.5)\}$ is given. Given three kinds of mineral fields, each is featured by the content of three minerals and contains one kind of typical hybrid minerals. The three kinds of typical hybrid minerals are represented by GIFSs A_1, A_2 and A_3 in X, respectively. Given another kind of hybrid mineral B, to which field does this kind of mineral B most probably belong to ?

For convenience, assume that the weight w_i of x_i in X are equal and $p = 2$. By (8) and (10), we have

$$\begin{aligned} &S_g(A_1, B) = 0.900,\ S_g(A_2, B) = 0.800,\ S_g(A_3, B) = 0.866; \\ &S_d^2(A_1, B) = 0.971,\ S_d^2(A_2, B) = 0.755,\ S_d^2(A_3, B) = 0.859. \end{aligned}$$

From this data, the proposed similarity measures S_g and S_d^p show the same classification according to the principle of the maximum degree of similarity between GIFSs. That is, the sample B belongs to the pattern A_1.

The results of above example indicates the proposed similarity measure to be good in pattern recognition problems. In the following example, the proposed similarity measure is used to characterize the similarity between linguistic variables.

Example 4.2 Let $F = \{(x, \mu_F(x), \nu_F(x)) : x \in X\}$ be a GIFS in X. For any positive real number n, We define the GIFS F^n as follows:

$$F^n = \{(x, (\mu_F(x))^n, 1 - (1 - \nu_F(x))^n) \mid x \in X\}.$$

Using the above operation, we also define the concentration and dilation of F as follows:

- concentration: $\mathrm{CON}(F) = F^2$;
- dilation: $\mathrm{DIL}(F) = F^{1/2}$.

Like the fuzzy sets, $\mathrm{CON}(F)$ and $\mathrm{DIL}(F)$ may be treated as "very (F)" and "more or less (F)", respectively.

In the next, we consider a GIFS F in $X = \{6, 7, 8, 9, 10\}$ defined by

$$F = \{\{(6, 0.2, 0.9), (7, 0.4, 0.7), (8, 0.7, 0.4), (9, 0.9, 0.1), (10, 1, 0)\}.$$

With taking into account the characterization of linguistic variables, we regard F as "LARGE" in X. Using the operations of concentration and dilation

- $F^{1/2}$ may be treated as "More or less LARGE",
- F^2 may be treated as "Very LARGE",
- F^4 may be treated as "Very very LARGE".

The proposed similarity measure is utilized to calculate the degree of similarity between these GIFSs. The results are summarized in Table 1. In Table 1, L., V.L., V.V.L. and M.L.L. denote LARGE, Very LARGE, Very very LARGE and More or less LARGE, respectively.

Table 1: The values calculated by the proposed similarity measure S_d^1

S_d^1	M.L.L.	L.	V.L.	V.V.L.
M.L.L.	1	0.8562	0.7134	0.6022
L.	0.8562	1	0.8540	0.7428
V.L.	0.7134	0.8540	1	0.8848
V.V.L.	0.6022	0.7428	0.8848	1

From the viewpoint of mathematical operations, the similarities between the above GIFSs require the following conditions:

$$S(\text{M.L.L.}, \text{L.}) > S(\text{M.L.L.}, \text{V.L.}) > S(\text{M.L.L.}, \text{V.V.L.}), \tag{12}$$
$$S(\text{L.}, \text{M.L.L.}) > S(\text{L.}, \text{V.L.}) > S(\text{L.}, \text{V.V.L.}), \tag{13}$$
$$S(\text{V.L.}, \text{V.V.L.}) > S(\text{V.L.}, \text{L.}) > S(\text{V.L.}, \text{M.L.L.}), \tag{14}$$
$$S(\text{V.V.L.}, \text{V.L.}) > S(\text{V.V.L.}, \text{L.}) > S(\text{V.V.L.}, \text{M.L.L.}). \tag{15}$$

From Table 1, it can be seen that the proposed similarity measure S_d^1 satisfies the requirements (12)-(15). Therefore, the proposed similarity measure S_d^1 is reliable in applications with compound linguistic variables.

5 Conclusions

We apply the principle of maximum degree of similarity measures between GIFSs to solve the pattern recognition problem. Based on the geometrical interpretation of GIFSs, we take into account three parameters (membership, non-membership, hesitation margin) to propose a similarity measure for calculating the degree of similarity between GIFSs. Numerical example is given to illustrate the application of the developed similarity measures. Furthermore, we use the proposed similarity measures to characterize the similarity between linguistic variables.

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ON THE STABILITY OF THE GENERALIZED QUADRATIC SET-VALUED FUNCTIONAL EQUATION

HAHNG-YUN CHU† AND SEUNG KI YOO*

ABSTRACT. In this article, we focus on the n-dimensional quadratic set-valued functional equation $(4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j) = 4\sum_{i=1}^{n} f(x_i)$, where $n \geq 2$ is an integer. We prove the Hyers-Ulam stability for the set-valued functional equation.

1. INTRODUCTION

The stability problem of functional equation concerning group homomorphisms had been first raised by S. M. Ulam [18] in 1940.

Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exists a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x*y), h(x) \diamond h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

The first partial solution to Ulam's question was provided by D. H. Hyers [8] for Banach spaces. Hyers' theorem was generalized by T. Aoki [1] for additive mapping. Th. M. Rassias [15] generalized the result of Hyers as follows:

Let $f : X \to Y$ be a mapping between Banach spaces and let $0 \leq p < 1$ be fixed. If f satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{1.1}$$

for some $\theta \geq 0$ and for all $x, y \in X$, then there exists a unique additive mapping $A : X \to Y$ such that $\|A(x) - f(x)\| \leq \frac{2\theta}{2-2^p}\|x\|^p$ for all $x \in X$. If $f(tx)$ is continuous in t for each fixed $x \in X$, then A is linear.

Thereafter, P. Găvruta [7] provided a generalization of Th. M. Rassias' theorem, more precisely speaking, in which he replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ in (1.1) by control functions $\phi(x, y)$ with more general types for the existence of a unique linear mapping. The functional equation $f(x+y)+f(x-y) =$

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* *Corresponding author*

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Key words and Phrases. Hyers-Ulam stability, generalized quadratic set-valued functional equation.

$2f(x)+2f(y)$ is called the *quadratic functional equation* and every solution of the quadratic functional equation is called a *quadratic function*.

The Hyers-Ulam stsbility of quadratic functional equation was proved by F. Skof [17] for a function $f : E_1 \to E_2$ where E_1 is a normed space and E_2 is a Banach space. P. W. Cholewa [3] considered Skof's theorem to a version of abelian groups. Skof's result was generalized by S. Czerwik [6] who proved the generalized Hyers-Ulam stability of quadratic functional equation in the spirit of Rassias approach. Kang and Chu [10] extended the quadratic functional equation to the generalized form $(4-n)f(\sum_{i=1}^n x_i)+\sum_{i=1}^n f(\sum_{j=1}^n \theta(i,j)x_j) = 4\sum_{i=1}^n f(x_i)$ where $n \geq 2$ is an integer and the function θ is defined by

$$\theta(i,j) = \begin{cases} 1 \text{ if } i \neq j \\ -1 \text{ if } i = j \end{cases}$$

and also investigated the Hyers-Ulam stability for the generalized quadratic functional equation. In [12], Lu and Park defined the additive set-valued functional equations $f(\alpha x + \beta y) = rf(x) + sf(y)$ and $f(x + y + z) = 2f(\frac{x+y}{2}) + f(z)$ and proved the Hyers-Ulam stability of the set-valued functional equations. In [14], Park et al. investigated stability problems of the Jensen additive, quadratic, cubic and quartic set-valued functional equation. Kenary et al. [11] proved the stability for various types of the set-valued functional equation using the fixed point alternative. In recent years, Chu and Yoo [5] studied the Hyers-Ulam stability of the n-dimensional additive set-valued functional equation. In [4], they also investigated the Hyers-Ulam stability of the n-dimensional cubic set-valued functional equation.

Let $CB(Y)$ be the set of all closed bounded subsets of Y and $CC(Y)$ the set of all closed convex subsets of Y. Let $CBC(Y)$ be the set of all closed bounded convex subsets of Y. For any elements A, B of $CC(Y)$, we denote $A \oplus B = \overline{A + B}$. If A is convex, then we obtain that $(\alpha + \beta)A = \alpha A + \beta A$ for all $\alpha, \beta \in \mathbb{R}^+$. Let $f : X \to CBC(Y)$ be a mapping. The *quadratic set-valued functional equation* is defined by $f(x + y) \oplus f(x - y) = 2f(x) \oplus 2f(y)$ for all $x, y \in X$. Every solution of the quadratic set-valued functional equation is said to be a *quadratic set-valued mapping*.

In this paper, we introduce the generalized n-dimensional quadratic set-valued functional equation

$$(4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j) = 4\sum_{i=1}^{n} f(x_i) \tag{1.2}$$

where $n \geq 2$ is an integer and the function θ is defined by

$$\theta(i,j) = \begin{cases} 1 \text{ if } i \neq j \\ -1 \text{ if } i = j \end{cases}$$

and investigate the Hyers-Ulam stability of the functional equation.

In the set-valued dynamics, every solution of the generalized n-dimensional quadratic set-valued functional equation is called a *n-dimensional quadratic set-valued mapping*.

For a subset $A \subset Y$, the distance function $d(\cdot, A)$ is defined by $d(x, A) := inf\{\| x - y \|: y \in A\}$ for $x \in Y$. For $A, B \in CB(Y)$, the *Hausdorff distance* $h(A, B)$ is defined by

$$h(A, B) := inf\{\alpha \geq 0 | \ A \subseteq B + \alpha B_Y, B \subseteq A + \alpha B_Y\},$$

where B_Y is the closed unit ball in Y. In [2], it was proved that $(CBC(Y), \oplus, h)$ is a complete metric semigroup. Rådström [16] proved that $(CBC(Y), \oplus, h)$ is isometrically embedded in a Banach space. The following remark is easily proved by using the notion of the Hausdorff distance.

Remark 1.1. Let $A, A', B, B', C \in CBC(Y)$ and $\alpha > 0$. Then we have that

(1) $h(A \oplus A', B \oplus B') \leq h(A, B) + h(A', B')$;

(2) $h(\alpha A, \alpha B) = \alpha h(A, B)$;

(3) $h(A, B) = h(A \oplus C, B \oplus C)$.

This paper is organized as follows. In section 2, we prove that the generalized n-dimensional set-valued mapping is actually general type of the quadratic set-valued mapping. We also investigate Hyers-Ulam stability for the generalized n-dimensional set-valued funational equation.

As applications of the stability, we take to change the control function and obtain the different approaches to unique generalized n-dimensional functional equation. In section 3, we also get the Hyers-Ulam staility for the generalized n-dimensional set-valued functional equation by using the fixed point method which is developed by Margolis and Diaz.

2. Stability of the set-valued functional equation

In this section, we mainly deal with the Hyers-Ulam stability for the generalized n-dimensional quadratic set-valued functional equation. We first study for properties of the n-dimensional quadratic set-valued mapping. Next we prove the Hyers-Ulam stabilities for the generalized n-dimensional quadratic set-valued equation. Especially when n is an even numbers, we find the precise control function depending upon the original function and n-dimensional quadratic set-valued mapping. Similarly we also obtain the precise control function in the odd case for the generalized n-dimensional quadratic set-valued functional equation.

Proposition 2.1. *Suppose that a mapping $f : X \to CBC(Y)$ defined by*

$$(4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j) = 4\sum_{i=1}^{n} f(x_i) \tag{2.1}$$

for all $x_1, \ldots, x_n \in X$. Then f has the following properties:

(1) $f(0) = \{0\}$

(2) $f(x) = f(-x)$ *for all $x \in X$*

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(3) *f is a quadratic set-valued mapping.*

Proof. (1) Putting $x_i = 0$ $(i = 1, \dots, n)$ in (2.1), we have $f(0) = \{0\}$.
(2) Putting $x_1 = x$ and $x_i = 0$ $(i = 2, \dots, n)$ in (2.1), we get $(4-n)f(x) \oplus f(-x) \oplus (n-1)f(x) = 4f(x)$. Thus $f(x) = f(-x)$ for all $x \in X$.
(3) Replacing $x_1 = x$, $x_2 = y$ and $x_i = 0$ $(i = 3, \dots, n)$, we have $(4-n)f(x+y) \oplus f(-x+y) \oplus f(x-y) \oplus (n-2)f(x+y) = 4f(x) \oplus 4f(y) \oplus (n-2)f(0)$. So we conclude that $f(x+y) \oplus f(x-y) = 2f(x) \oplus 2f(y)$. This completes the proof. □

Next, we prove the stability of the generalized n-dimensional quadratic set-valued functional equation. To extended precisely to the stability theory for the set-valued functional equation, we state the stability according to dimensions of the equation.

Theorem 2.2. *Let $n \geq 2$ be an integer and let $\phi : X^n \to [0, \infty)$ be a function such that*

$$\tilde{\phi}(x_1, \dots, x_n) := \sum_{i=0}^{\infty} \frac{1}{4^i} \phi(2^i x_1, \dots, 2^i x_n) < \infty \tag{2.2}$$

for all $x_1, \dots, x_n \in X$. Suppose that $f : X \longrightarrow (CBC(Y), h)$ is a mapping with $f(0) = \{0\}$ and

$$h\Big((4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j),\ 4\sum_{i=1}^{n} f(x_i)\Big) \leq \phi(x_1, \dots, x_n) \tag{2.3}$$

for all $x_1, \dots, x_n \in X$. Then there exists a unique n-dimensional quadratic set-valued mapping $T : X \to (CBC(Y), h)$ such that

$$h(f(x), T(x)) \leq \frac{1}{8}\tilde{\phi}(x, x, 0, \dots, 0) \tag{2.4}$$

for all $x \in X$.

Proof. Putting $x_1 = x_2 = x$ and $x_3 = \cdots = x_n = 0$ in (2.3), we have

$$h(\frac{f(2x)}{4}, f(x)) \leq \frac{1}{8}\phi(x, x, 0, \dots, 0) \tag{2.5}$$

for all $x \in X$. Replacing x by $2x$ and dividing by 4 in (2.5)

$$h(\frac{f(4x)}{4^2}, f(2x)) \leq \frac{1}{32}\phi(2x, 2x, 0, \dots, 0) \tag{2.6}$$

for all $x \in X$. By (2.5) and (2.6), we get

$$h(\frac{f(4x)}{4^2}, f(x)) \leq \frac{1}{8}\phi(x, x, 0, \dots, 0) + \frac{1}{4 \cdot 8}\phi(2x, 2x, 0, \dots, 0) \tag{2.7}$$

for all $x \in X$. Using the induction on i, we have that

$$h(\frac{f(2^r x)}{4^r}, f(x)) \leq \frac{1}{8}\sum_{i=0}^{r-1} \frac{1}{4^i}\phi(2^i x, 2^i x, 0, \dots, 0) \tag{2.8}$$

for any positive integer r and for all $x \in X$.

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Now, we show that the sequence $\{\frac{f(2^r x)}{4^r}\}$ converges for all $x \in X$. For any positive integer r and s, we divide inequality (2.8) by 4^s and replace x by $2^s x$. Then we obtain that the following inequality

$$h(\frac{f(2^{r+s}x)}{4^{r+s}}, \frac{f(2^s x)}{4^s}) \leq \frac{1}{4^s}\frac{1}{8}\sum_{i=0}^{r-1}\frac{1}{4^i} \leq \phi(2^{s+i}x, 2^{s+i}x, 0, \ldots, 0) \tag{2.9}$$

for all $x \in X$. Since the right-hand side of the inequality (2.9) tends to zero as s tends to infinity, the sequence $\{\frac{f(2^r x)}{4^r}\}$ is a Cauchy sequence in (CBC(Y), h). Therefore, we can define a mapping $T : X \to (CBC(Y), h)$ as $T(x) := \lim_{r\to\infty} \frac{f(2^r x)}{4^r}$ for all $x \in X$. It follows from the definition of T and (2.2) that

$$h\Big((4-n)T(\sum_{i=1}^n x_i) \oplus \sum_{i=1}^n T(\sum_{j=1}^n \theta(i,j)x_j),\ 4\sum_{i=1}^n T(x_i)\Big) \leq \lim_{r\to\infty}\frac{1}{4^r}\phi(2^r x_1, \ldots, 2^r x_n) = 0 \tag{2.10}$$

for all $x_1, \ldots, x_n \in X$. Hence, we claim that T is an n-dimensional quadratic set-valued mapping. By letting $r \to \infty$ in (2.8), we have the desired inequality (2.4). Now we prove the uniqueness of T. Let $T' : X \to (CBC(Y), h)$ be another n-dimensional quadratic set-valued mapping satisfying (2.4). Therefore, we get the following inequality

$$h(T(x), T'(x)) = \frac{1}{4^r}h(T(2^r x), T'(2^r x)) \leq \frac{1}{4^r}\frac{1}{8}\tilde{\phi}(2^r x, 2^r x, 0, \ldots, 0)$$

for all $x \in X$. Hence, letting $r \to \infty$, the right-hand side of above inequality goes to zero, and it follows that $T(x) = T'(x)$ for all $x \in X$. □

Corollary 2.3. *Let $n \geq 2$ be an integer, $0 < p < 2$ and $\theta \geq 0$ be real numbers and let X be a real normed space. Suppose that $f : X \to (CBC(Y), h)$ is a mapping satisfying*

$$h\Big((4-n)f(\sum_{i=1}^n x_i) \oplus \sum_{i=1}^n f(\sum_{j=1}^n \theta(i,j)x_j),\ 4\sum_{i=1}^n f(x_i)\Big) \leq \theta\sum_{i=1}^n \|x_i\|^p$$

for all $x_1, \ldots, x_n \in X$. Then there exists a unique n-dimensional quadratic set-valued mapping $T : X \to (CBC(Y), h)$ that satisfies

$$h(f(x), T(x)) \leq \frac{\theta}{2^2 - 2^p}\|x\|^p$$

for all $x \in X$.

Proof. The result follows Theorem 2.2 by setting $\phi(x_1, \ldots, x_n) = \theta\sum_{i=1}^n \|x_i\|^p$ for all $x_1, \ldots, x_n \in X$. □

Theorem 2.4. *Let $n \geq 2$ be an integer and let $\phi : X^n \to [0, \infty)$ be a function such that*

$$\tilde{\phi}(x_1, \ldots, x_n) := \sum_{i=1}^{\infty} 4^i \phi(\frac{x_1}{2^i}, \ldots, \frac{x_n}{2^i}) < \infty \tag{2.11}$$

for all $x_1, \dots, x_n \in X$. Suppose that $f : X \longrightarrow (CBC(Y), h)$ is a mapping and

$$h\Big((4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j),\ 4\sum_{i=1}^{n} f(x_i)\Big) \leq \phi(x_1, \dots, x_n) \tag{2.12}$$

for all $x_1, \dots, x_n \in X$. Then there exists a unique n-dimensional quadratic set-valued mapping $T : X \to (CBC(Y), h)$ such that

$$h(f(x), T(x)) \leq \frac{1}{8}\tilde{\phi}(x, x, 0, \dots, 0) \tag{2.13}$$

for all $x \in X$.

Proof. By (2.11) and (2.12), we get $f(0) = \{0\}$. Replacing x by $\frac{x}{2}$ and multiplying by 4 in (2.5), we have the following inequality

$$h(f(x), 4f(\frac{x}{2})) \leq \frac{1}{2}\phi(\frac{x}{2}, \frac{x}{2}, 0, \dots, 0)$$

for all $x \in X$. The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 2.5. *Let $n \geq 2$ be an integer, $p > 2$ and $\theta \geq 0$ be real numbers and let X be a real normed space. Suppose that $f : X \to (CBC(Y), h)$ is a mapping satisfying*

$$h\Big((4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j),\ 4\sum_{i=1}^{n} f(x_i)\Big) \leq \theta\sum_{i=1}^{n} \|x_i\|^p$$

for all $x_1, \dots, x_n \in X$. Then there exists a unique n-dimensional quadratic set-valued mapping $T : X \to (CBC(Y), h)$ that satisfies

$$h(f(x), T(x)) \leq \frac{\theta}{2^p - 2^2}\|x\|^p$$

for all $x \in X$.

Proof. The result follows Theorem 2.4 by setting $\phi(x_1, \dots, x_n) = \theta\sum_{i=1}^{n} \|x_i\|^p$ for all $x_1, \dots, x_n \in X$. □

Let n be an even positive integer. In this case, we can obtain the control function for the Hausdorff distence between the original mapping and n-dimensional quadratic set-valued mapping.

Theorem 2.6. *Let $n \geq 2$ be even and let $\phi : X^n \to [0, \infty)$ be a function such that*

$$\tilde{\phi}(x_1, \dots, x_n) := \sum_{i=0}^{\infty} \frac{1}{4^i}\phi(2^i x_1, \dots, 2^i x_n) < \infty \tag{2.14}$$

for all $x_1, \dots, x_n \in X$. Suppose that $f : X \longrightarrow (CBC(Y), h)$ is an even mapping with $f(0) = \{0\}$ and

$$h\Big((4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j),\ 4\sum_{i=1}^{n} f(x_i)\Big) \leq \phi(x_1, \dots, x_n) \tag{2.15}$$

for all $x_1, \ldots, x_n \in X$. *Then there exists a unique* n*-dimensional quadratic set-valued mapping* $T : X \to (CBC(Y), h)$ *such that*

$$h(f(x), T(x)) \leq \frac{1}{4n}\tilde{\phi}(x, -x, x, -x, \ldots, x, -x) \tag{2.16}$$

for all $x \in X$.

Proof. Put $x_k = (-1)^{k-1}x$ $(k = 1, \ldots, n)$ in (2.15). Since f is even and the range of f is convex, we have that

$$h(\frac{f(2x)}{4}, f(x)) \leq \frac{1}{4n}\phi(x, -x, x, -x, \ldots, x, -x) \tag{2.17}$$

for all $x \in X$. The rest of the proof is similar to proof of Theorem 2.2. □

Corollary 2.7. *Let* $n \geq 2$ *be even,* $0 < p < 2$ *and* $\theta \geq 0$ *be real numbers and let* X *be a real normed space. Suppose that* $f : X \to (CBC(Y), h)$ *is an even mapping satisfying*

$$h\Big((4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j),\ 4\sum_{i=1}^{n} f(x_i)\Big) \leq \theta \sum_{i=1}^{n} \|x_i\|^p$$

for all $x_1, \ldots, x_n \in X$. *Then there exists a unique* n*-dimensional quadratic set-valued mapping* $T : X \to (CBC(Y), h)$ *that satisfies*

$$h(f(x), T(x)) \leq \frac{\theta}{2^2 - 2^p}\|x\|^p$$

for all $x \in X$.

Proof. The result follows Theorem 2.6 by setting $\phi(x_1, \ldots, x_n) = \theta \sum_{i=1}^{n} \|x_i\|^p$ for all $x_1, \ldots, x_n \in X$. □

Theorem 2.8. *Let* $n \geq 2$ *be even and let* $\phi : X^n \to [0, \infty)$ *be a function such that*

$$\tilde{\phi}(x_1, \ldots, x_n) := \sum_{i=0}^{\infty} 4^i \phi(\frac{x_1}{2^{i+1}}, \ldots, \frac{x_n}{2^{i+1}}) < \infty \tag{2.18}$$

for all $x_1, \ldots, x_n \in X$. *Suppose that* $f : X \longrightarrow (CBC(Y), h)$ *is an even mapping with* $f(0) = \{0\}$ *and*

$$h\Big((4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j),\ 4\sum_{i=1}^{n} f(x_i)\Big) \leq \phi(x_1, \ldots, x_n) \tag{2.19}$$

for all $x_1, \ldots, x_n \in X$. *Then there exists a unique* n*-dimensional quadratic set-valued mapping* $T : X \to (CBC(Y), h)$ *such that*

$$h(f(x), T(x)) \leq \frac{1}{n}\tilde{\phi}(x, -x, x, -x, \ldots, x, -x) \tag{2.20}$$

for all $x \in X$.

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Proof. Replacing x by $\frac{x}{2}$ and multiplying by 4 in (2.17), we have the following inequality

$$h(f(x), 4f(\frac{x}{2})) \leq \frac{1}{n}\phi(\frac{x}{2}, -\frac{x}{2}, \frac{x}{2}, -\frac{x}{2}, \dots, \frac{x}{2}, -\frac{x}{2})$$

for all $x \in X$. The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 2.9. *Let $n \geq 2$ be even, $p > 2$ and $\theta \geq 0$ be real numbers and let X be a real normed space. Suppose that $f : X \to (CBC(Y), h)$ is an even mapping satisfying*

$$h\Big((4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j),\ 4\sum_{i=1}^{n} f(x_i)\Big) \leq \theta \sum_{i=1}^{n} \|x_i\|^p$$

for all $x_1, \dots, x_n \in X$. Then there exists a unique n-dimensional quadratic set-valued mapping $T : X \to (CBC(Y), h)$ that satisfies

$$h(f(x), T(x)) \leq \frac{\theta}{2^p - 2^2}\|x\|^p$$

for all $x \in X$.

Proof. The result follows Theorem 2.8 by setting $\phi(x_1, \dots, x_n) = \theta \sum_{i=1}^{n} \|x_i\|^p$ for all $x_1, \dots, x_n \in X$. □

As applications for the theorem, we get the Hyers-Ulam stability for the generalized n-dimensional set-valued functional equation and especially we deal with the odd case for n.

Theorem 2.10. *Let $n \geq 2$ be odd and let $\phi : X^n \to [0, \infty)$ be a function such that*

$$\tilde{\phi}(x_1, \dots, x_n) := \sum_{i=0}^{\infty} \frac{1}{9^i}\phi(3^i x_1, \dots, 3^i x_n) < \infty \tag{2.21}$$

for all $x_1, \dots, x_n \in X$. Suppose that $f : X \longrightarrow (CBC(Y), h)$ is an even mapping with $f(0) = \{0\}$ and

$$h\Big((4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j),\ 4\sum_{i=1}^{n} f(x_i)\Big) \leq \phi(x_1, \dots, x_n) \tag{2.22}$$

for all $x_1, \dots, x_n \in X$. Then there exists a unique n-dimensional quadratic set-valued mapping $T : X \to (CBC(Y), h)$ such that

$$h(f(x), T(x)) \leq \frac{2}{9(n-1)}\tilde{\phi}(x, -x, x, -x, \dots, -x, x) \tag{2.23}$$

for all $x \in X$.

Proof. Put $x_k = (-1)^{k-1}x$ $(k = 1, \dots, n)$ in (2.22). Since f is even and the range of f is convex, we have that

$$h(\frac{f(3x)}{9}, f(x)) \leq \frac{2}{9(n-1)}\phi(x, -x, x, -x, \dots, -x, x)$$

for all $x \in X$. The rest of the proof is similar to proof of Theorem 2.2. □

Corollary 2.11. *Let $n > 2$ be odd, $0 < p < 2$ and $\theta \geq 0$ be real numbers and let X be a real normed space. Suppose that $f : X \to (CBC(Y), h)$ is an even mapping satisfying*

$$h\Big((4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j),\ 4\sum_{i=1}^{n} f(x_i)\Big) \leq \theta \sum_{i=1}^{n} \|x_i\|^p$$

for all $x_1, \ldots, x_n \in X$. Then there exists a unique n-dimensional quadratic set-valued mapping $T : X \to (CBC(Y), h)$ that satisfies

$$h(f(x), T(x)) \leq \frac{2n\theta}{(n-1)(3^2 - 3^p)} \|x\|^p$$

for all $x \in X$.

Proof. The result follows Theorem 2.10 by setting $\phi(x_1, \ldots, x_n) = \theta \sum_{i=1}^{n} \|x_i\|^p$ for all $x_1, \ldots, x_n \in X$. □

Theorem 2.12. *Let $n > 2$ be odd and let $\phi : X^n \to [0, \infty)$ be a function such that*

$$\tilde{\phi}(x_1, \ldots, x_n) := \sum_{i=0}^{\infty} 9^i \phi(\frac{x_1}{3^{i+1}}, \ldots, \frac{x_n}{3^{i+1}}) < \infty \tag{2.24}$$

for all $x_1, \ldots, x_n \in X$. Suppose that $f : X \longrightarrow (CBC(Y), h)$ is an even mapping with $f(0) = \{0\}$ and

$$h\Big((4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j),\ 4\sum_{i=1}^{n} f(x_i)\Big) \leq \phi(x_1, \ldots, x_n) \tag{2.25}$$

for all $x_1, \ldots, x_n \in X$. Then there exists a unique n-dimensional quadratic set-valued mapping $T : X \to (CBC(Y), h)$ such that

$$h(f(x), T(x)) \leq \frac{2}{n-1} \tilde{\phi}(x, -x, x, -x, \ldots, -x, x) \tag{2.26}$$

for all $x \in X$.

Proof. Put $x_k = (-1)^{k-1}x$ $(k = 1, \ldots, n)$ in (2.25). Since f is even and the range of f is convex, we have that

$$h(9f(\frac{x}{3}), f(x)) \leq \frac{2}{n-1} \phi(\frac{x}{3}, -\frac{x}{3}, \frac{x}{3}, -\frac{x}{3}, \ldots, -\frac{x}{3}, \frac{x}{3})$$

for all $x \in X$. The rest of the proof is similar to proof of Theorem 2.2. □

Corollary 2.13. *Let $n > 2$ be odd, $p > 2$ and $\theta \geq 0$ be real numbers and let X be a real normed space. Suppose that $f : X \to (CBC(Y), h)$ is an even mapping satisfying*

$$h\Big((4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j),\ 4\sum_{i=1}^{n} f(x_i)\Big) \leq \theta \sum_{i=1}^{n} \|x_i\|^p$$

for all $x_1, \dots, x_n \in X$. Then there exists a unique n-dimensional quadratic set-valued mapping $T : X \to (CBC(Y), h)$ that satisfies

$$h(f(x), T(x)) \leq \frac{2n\theta}{(n-1)(3^p - 3^2)} \|x\|^p$$

for all $x \in X$.

Proof. The result follows Theorem 2.12 by setting $\phi(x_1, \dots, x_n) = \theta \sum_{i=1}^{n} \|x_i\|^p$ for all $x_1, \dots, x_n \in X$. □

3. Stability of the set-valued functional equation by fixed point method

As using the fixed point method, we get plenty of the results related to the generalized n-dimensional quadratic set-valued functional equation. We first introduce the generalized metric on the given phase space and recall fundamental results for the fixed point theory. Let X be a set. A function $d : X \times X \to [0, \infty)$ is the *generalized metric* on X if d satisfies the following properties:

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The following theorem is very useful for proving Hyers-Ulam stability which is due to Margolis and Diaz [13].

Theorem 3.1. *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) *$d(J^n x, J^{n+1} x) < \infty, \ \forall n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X | d(J^{n_0} x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

Using the alternative fixed point theorem, we investigate the stability of the even dimensional quadratic set-valued functional equation.

Theorem 3.2. *Let $n \geq 2$ be even. Suppose that an even mapping $f : X \longrightarrow (CBC(Y), h)$ with $f(0) = \{0\}$ satisfies the inequality*

$$h\Big((4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j),\ 4\sum_{i=1}^{n} f(x_i)\Big) \leq \phi(x_1, \dots, x_n) \tag{3.1}$$

for all $x_1, \dots, x_n \in X$, and there exists a constant L with $0 < L < 1$ for which the function $\phi : X^n \to [0, \infty)$ satisfies

$$\phi(2x, -2x, 2x, -2x, \dots, 2x, -2x) \leq 4L\phi(x, -x, x, -x, \dots, x, -x) \tag{3.2}$$

for all $x \in X$. Then there exists a n-dimensional quadratic set-valued mapping $T : X \to (CBC(Y), h)$ given by $T(x) = \lim_{k\to\infty} \frac{f(2^k x)}{4^k}$ such that

$$h(f(x), T(x)) \leq \frac{1}{4n(1-L)}\phi(x, -x, x, -x, \dots, x, -x) \tag{3.3}$$

for all $x \in X$.

Proof. Put $x_k = (-1)^{k-1}x$ $(k = 1, \dots, n)$ in (3.1). Since f is even and the range of f is convex, we have that

$$h(\frac{f(2x)}{4}, f(x)) \leq \frac{1}{4n}\phi(x, -x, x, -x, \dots, x, -x) \tag{3.4}$$

for all $x \in X$.

Let $S := \{g \mid g : X \to CBC(Y),\ g(0) = \{0\}\}$. We define a generalized metric on S defined by

$$d(g_1, g_2) := \inf\{\mu \in (0, \infty) \mid h(g_1(x), g_2(x)) \leq \mu\phi(x, -x, x, -x, \dots, x, -x), x \in X\}.$$

It is easy to show that (S, d) is complete (see [9]). Now, we define the mapping $J : S \to S$ given by $Jg(x) = \frac{1}{4}g(2x)$ for all $x \in X$. For $g_1, g_2 \in S$, let $d(g_1, g_2) = \mu$. Then

$$h(\frac{1}{4}g_1(2x), \frac{1}{4}g_2(2x)) \leq \frac{1}{4}\mu\phi(2x, -2x, 2x, -2x \dots, 2x, -2x)$$

for all $x \in X$. Then by (3.2), we have $h(Jg_1(x), Jg_2(x)) \leq \mu L\phi(x, -x, x, -x, \dots, x, -x)$ for all $x \in X$. Therefore, we get $d(Jg_1, Jg_2) \leq Ld(g_1, g_2)$ for any $g_1, g_2 \in S$. Hence J is a strictly contractive mapping with Lipschitz constant L. It follows from (3.4) that $d(Jf, f) \leq \frac{1}{4n}$. By Theorem 3.1, the sequence $\{J^k f\}$ converges to a fixed point $T : X \to (CBC(Y), h)$ of J in the set $\{g \in S \mid d(f, g) < \infty\}$ such that $\{J^k f\} \to 0$ as $k \to \infty$. This implies $T(x) = \lim_{k\to\infty} \frac{f(2^k x)}{4^k}$ for all $x \in X$. And we also have $d(f, T) \leq \frac{1}{1-L}d(Jf, f) \leq \frac{1}{4n(1-L)}$. This means that the inequality (3.3) holds. By (3.1),

$$h\Big((4-n)T(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} T(\sum_{j=1}^{n} \theta(i,j)x_j),\ 4\sum_{i=1}^{n} T(x_i)\Big) \leq \lim_{k\to\infty} \frac{1}{4^k}\phi(x, -x, x, -x, \dots, x, -x) = 0$$

Therefore, T is a unique n-dimensional quadratic set-valued mapping as desired. □

Corollary 3.3. *Let $n \geq 2$ be even, $0 < p < 2$ and $\theta \geq 0$ be real numbers and let $n \geq 2$ be even. Suppose that $f : X \longrightarrow (CBC(Y), h)$ is an even mapping satisfying*

$$h\Big((4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j),\ 4\sum_{i=1}^{n} f(x_i)\Big) \leq \theta \sum_{i=1}^{n} \|x_i\|^p$$

for all $x_1, \dots, x_n \in X$. Then there exists a unique n-dimensional quadratic set-valued mapping $T : X \to (CBC(Y), h)$ that satisfies

$$h(f(x), T(x)) \leq \frac{\theta}{2^2 - 2^p}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.2 by setting $\phi(x_2, \dots, x_n) = \theta \sum_{i=1}^{n} \|x_i\|^p$ for every $x_1, \dots, x_n \in X$. Then we can choose $L = 2^{p-2}$ and we get the desired result. □

Theorem 3.4. *Let $n \geq 2$ be even. Suppose that an even mapping $f : X \longrightarrow (CBC(Y), h)$ with $f(0) = \{0\}$ satisfies the inequality*

$$h\Big((4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j),\ 4\sum_{i=1}^{n} f(x_i)\Big) \leq \phi(x_1, \dots, x_n) \tag{3.5}$$

for all $x_1, \dots, x_n \in X$, and there exists a constant L with $0 < L < 1$ for which the function $\phi : X^n \to [0, \infty)$ satisfies

$$\phi(\frac{x}{2}, -\frac{x}{2}, \frac{x}{2}, -\frac{x}{2}, \dots, \frac{x}{2}, -\frac{x}{2}) \leq \frac{L}{4}\phi(x, -x, x, -x, \dots, x, -x) \tag{3.6}$$

for all $x \in X$. Then there exists a n-dimensional quadratic set-valued mapping $T : X \to (CBC(Y), h)$ given by $T(x) = \lim_{k\to\infty} 4^k f(\frac{x}{2^k})$ such that

$$h(f(x), T(x)) \leq \frac{L}{4n(1-L)}\phi(x, -x, x, -x, \dots, x, -x) \tag{3.7}$$

for all $x \in X$.

Proof. Replacing x by $\frac{x}{2}$ and multiplying 4 in (3.4), we have

$$h(f(x), 4f(\frac{x}{2})) \leq \frac{1}{n}\phi(\frac{x}{2}, -\frac{x}{2}, \frac{x}{2}, -\frac{x}{2}, \dots, \frac{x}{2}, -\frac{x}{2}) \leq \frac{L}{4n}\phi(x, -x, x, -x, \dots, x, -x)$$

for all $x \in X$. The rest of the proof is similar to proof of Theorem 2.2. □

Corollary 3.5. *Let $n \geq 2$ be even, $p > 2$ and $\theta \geq 0$ be real numbers and let $n \geq 2$ be even. Suppose that $f : X \longrightarrow (CBC(Y), h)$ is an even mapping satisfying*

$$h\Big((4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j),\ 4\sum_{i=1}^{n} f(x_i)\Big) \leq \theta \sum_{i=1}^{n} \|x_i\|^p$$

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for all $x_1, \ldots, x_n \in X$. Then there exists a unique n-dimensional quadratic set-valued mapping $T : X \to (CBC(Y), h)$ that satisfies

$$h(f(x), T(x)) \leq \frac{\theta}{n(2^p - 2^2)}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.9 by setting $\phi(x_2, \ldots, x_n) = \theta \sum_{i=1}^{n} \|x_i\|^p$ for every $x_1, \ldots, x_n \in X$. Then we can choose $L = 2^{2-p}$ and we get the desired result. □

Finally, we deal with the Hyers-Ulam stability for the odd dimensional quadratic set-valued functional equation.

Theorem 3.6. *Let $n > 2$ be odd. Suppose that an even mapping $f : X \longrightarrow (CBC(Y), h)$ with $f(0) = \{0\}$ satisfies the inequality*

$$h\Big((4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j),\ 4\sum_{i=1}^{n} f(x_i)\Big) \leq \phi(x_1, \ldots, x_n) \tag{3.8}$$

for all $x_1, \ldots, x_n \in X$, and there exists a constant L with $0 < L < 1$ for which the function $\phi : X^n \to [0, \infty)$ satisfies

$$\phi(3x, -3x, 3x, -3x, \ldots, -3x, 3x) \leq 9L\phi(x, -x, x, -x, \ldots, -x, x) \tag{3.9}$$

for all $x \in X$. Then there exists a n-dimensional quadratic set-valued mapping $T : X \to (CBC(Y), h)$ given by $T(x) = \lim_{k\to\infty} \frac{f(3^k x)}{9^k}$ such that

$$h(f(x), T(x)) \leq \frac{2}{9(n-1)(1-L)}\phi(x, -x, x, -x, \ldots, -x, x) \tag{3.10}$$

for all $x \in X$.

Proof. Put $x_k = (-1)^{k-1}x$ $(k = 1, \ldots, n)$ in (3.8). Since f is even and the range of f is convex, we have that

$$h(9f(\frac{x}{3}), f(x)) \leq \frac{2}{n-1}\phi(\frac{x}{3}, -\frac{x}{3}, \frac{x}{3}, -\frac{x}{3}, \ldots, -\frac{x}{3}, \frac{x}{3})$$

for all $x \in X$. The rest of the proof is similar to proof of Theorem 2.2.

□

Corollary 3.7. *Let $n > 2$ be odd, $0 < p < 2$ and $\theta \geq 0$ be real numbers and let $n \geq 2$ be odd. Suppose that $f : X \longrightarrow (CBC(Y), h)$ is an even mapping satisfying*

$$h\Big((4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j),\ 4\sum_{i=1}^{n} f(x_i)\Big) \leq \theta\sum_{i=1}^{n} \|x_i\|^p$$

for all $x_1, \ldots, x_n \in X$*. Then there exists a unique* n*-dimensional quadratic set-valued mapping* $T : X \to (CBC(Y), h)$ *that satisfies*

$$h(f(x), T(x)) \le \frac{2\theta}{(n-1)(3^2 - 3^p)} \|x\|^p$$

for all $x \in X$*.*

Proof. The proof follows from Theorem 3.4 by setting $\phi(x_2, \ldots, x_n) = \theta \sum_{i=1}^{n} \|x_i\|^p$ for every $x_1, \ldots, x_n \in X$. Then we can choose $L = 3^{p-2}$ and we get the desired result. □

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COMMON BEST PROXIMITY POINTS FOR PROXIMALLY COMMUTING MAPPINGS IN NON-ARCHIMEDEAN PM-SPACES

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Abstract. In this paper, we prove new common best proximity point theorems for proximally commuting mappings in complete non-Archimedean PM-spaces. Our results generalized the recent results of S. Basha [Common best proximity points: global minimization of multi-objective functions, J. Global Optim. 49(2011), 15–21] and C. Mongkolkeha, P. Kumam [Some common best proximity points for proximity commuting mappings, Optim. Lett. 7 (2013), 1825–1836].

1. Introduction

Best proximity point theorems provide sufficient conditions that ensure the existence of approximate solutions which are optimal as well. In fact, if there is no solution to the fixed point equation $Tx = x$ for a non-self mapping $T : A \to B$, then it is desirable to determine an approximate solution x such that the error $F_{x,Tx}(t)$ is maximum.

A classical best approximation theorem was introduced by Fan [13], that is, if A is a nonempty compact convex subset of a Hausdorff locally convex topological vector space B and $T : A \to B$ is a continuous mapping, then there exists an element $x \in A$ such that $d(x, Tx) = d(Tx, A)$. Afterward, several authors, including Prolla [22], Reich [23], Sehgal and Singh [32, 33] and others, have derived some extensions of Fan's theorem in many directions. Other works of the existence of a best proximity point for contractions can be seen in [2, 5, 12, 15].

In 2005, Anthony Eldred, Kirk and Veeramani [6] have obtained best proximity point theorems for relatively nonexpansive mappings. Since then, best proximity point theorems for several types of contractions have been established in [3, 4, 8, 12, 16, 17, 19, 20, 26, 27, 28, 29, 30, 36, 37, 38, 39, 40].

2. Preliminaries

Throughout this paper, the space of all probability distribution functions (briefly, d.f.'s) is denoted by $\Delta^+ = \{F : \mathbb{R} \cup \{-\infty, +\infty\} \longrightarrow [0, 1] : F$ is left-continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1\}$ and the subset $D^+ \subseteq \Delta^+$ is the set $D^+ = \{F \in \Delta^+ : l^- F(+\infty) = 1\}$. Here $l^- f(x)$ denotes the left limit of the function f at the point x and $l^- f(x) = \lim_{t\to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leq G$ if and only if $F(t) \leq G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ^+ in this order is the d.f. given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 2.1. ([31]) A mapping $* : [0, 1] \times [0, 1] \to [0, 1]$ is a *continuous t–norm* if $*$ satisfies the following conditions:

(a) $*$ is commutative and associative;
(b) $*$ is continuous;
(c) $a * 1 = a$ for all $a \in [0, 1]$;

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(d) $a * b \leq c * d$ whenever $a \leq c$ and $c \leq d$, and $a, b, c, d \in [0, 1]$.

Two typical examples of continuous t–norm are $a * b = ab$ and $a * b = \min(a, b)$.

A t–norm $*$ is said to be *positive* ([31]) if $a * b > 0$ whenever $a, b \in (0, 1]$. The notation $* < *'$ means that $a * b < a *' b$ for all $a, b \in (0, 1)$.

Definition 2.2. (1) A *Probabilistic Metric space* (briefly, PM-space) is a triple $(X, F, *)$, where X is a nonempty set, T is a continuous t–norm and F is a mapping from $X \times X$ into D^+ such that, if $F_{x,y}$ denotes the value of F at the pair (x, y), the following conditions hold:

(PM1) $F_{x,y}(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = y$;
(PM2) $F_{x,y}(t) = F_{y,x}(t)$;
(PM3) $F_{x,z}(t + s) \geq F_{x,y}(t) * F_{y,z}(s)$ for all $x, y, z \in X$ and $t, s \geq 0$.

(2) If, in the above definition, the triangular inequality (PM3) is replaced by

(PM4) $F_{x,z}(\max\{t, s\}) \geq F_{x,y}(t) * F_{y,z}(s)$ for all $x, y, z \in X$ and $t, s \geq 0$,

then the triple $(X, F, *)$ is called a *non-Archimedean PM-space* (briefly, NA-PM-space).

It is easy to check that the triangular inequality (PM4) implies (PM3), that is, every NA-PM-space is itself a PM-space. It is easy to show that (PM4) is equivalent to the following condition:

(PM5) $F_{x,z}(t) \geq F_{x,y}(t) * F_{y,z}(t)$ for all $x, y, z \in X$ and $t \geq 0$.

Example 2.3. Let (X, d) be an ordinary metric space and let θ be a nondecreasing and continuous function from $(0, \infty)$ into $(0, 1)$ such that $\lim_{t\to\infty} \theta(t) = 1$. Some examples of these functions are as follows:
$$\theta(t) = \frac{t}{t+1}, \quad \theta(t) = 1 - e^{-t}, \quad \theta(t) = e^{-1/t}.$$
Let $a * b \leq ab$ for each $a, b \in [0, 1]$. For each $t \in (0, \infty)$, define
$$F_{x,y}(t) = [\theta(t)]^{d(x,y)}$$
for all $x, y \in X$. Then $(X, F, *)$ is a NA-PM-space ([1]).

For more details and examples of these spaces see also [7], [9], [10], [11], [14], [18], [21], [24], [25], [34], [35], [41] and [42].

Definition 2.4. Let (X, F, T) be a NA-PM-space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to a point $x \in X$ if, for any $\epsilon > 0$ and $\lambda > 0$, there exists positive integer N such that
$$F_{x_n,x}(\epsilon) > 1 - \lambda$$
whenever $n \geq N$;

(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for any $\epsilon > 0$ and $\lambda > 0$, there exists positive integer N such that
$$F_{x_{n+p},x_n}(\epsilon) > 1 - \lambda$$
whenever $n \geq N$ and $p \in \mathcal{N}$;

(3) A PM-space $(X, F, *)$ is said to be *complete* if every Cauchy sequence in X is convergent to a point in X.

Definition 2.5. Let $(X, F, *)$ be a PM-space. For each p in X and $\lambda > 0$, the strong λ-neighborhood of p is the set
$$N_p(\lambda) = \{q \in X : F_{p,q}(\lambda) > 1 - \lambda\}$$
and the strong neighborhood system for X is the union $\bigcup_{p\in V} \mathcal{N}_p$, where
$$\mathcal{N}_p = \{N_p(\lambda) : \lambda > 0\}.$$

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The strong neighborhood system for X determines a Hausdorff topology for X.

Theorem 2.6. ([31]) *If $(X, F, *)$ is a PM-space and $\{p_n\}$ and $\{q_n\}$ are sequences such that $p_n \to p$ and $q_n \to q$ as $n \to \infty$, then $\lim_{n\to\infty} F_{p_n,q_n}(t) = F_{p,q}(t)$ for all continuity point t of $F_{p,q}$.*

Let A and B be two nonempty subsets of a PM-space and $t > 0$. The following notions and notations are used in the sequel.

$$\begin{aligned} F_{A,B}(t) &:= \sup\{F_{x,y}(t) : x \in A, y \in B\}, \\ A_0 &:= \{x \in A : F_{x,y}(t) = F_{A,B}(t) \text{ for some } y \in B\}, \\ B_0 &:= \{y \in B : F_{x,y}(t) = F_{A,B}(t) \text{ for some } x \in A\}. \end{aligned}$$

Definition 2.7. A mapping $T : X \to X$ is said to be a *contraction* if there exists a constant $k \in [0, 1)$ such that

$$F_{Tx,Ty}(kt) \geq F_{x,y}(t) \tag{2.1}$$

for all $x, y \in X$ and $t > 0$.

Definition 2.8. A mapping $T : X \to X$ is said to be a *weak contraction* if

$$F_{Tx,Ty}(\phi(t)) * F_{x,y}(t) \geq F_{x,y}(\phi(t)) \tag{2.2}$$

for all $x, y \in X$ and $t > 0$, where $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that, for any $t > 0$, $0 < \varphi(t) < t$.

Definition 2.9. A point $x \in A$ is said to be a *best proximity point* of a mapping $S : A \to B$ if it satisfies the following condition:

$$F_{x,Sx}(t)) = F_{A,B}(t)$$

for all $x, y \in X$ and $t > 0$.

It can be observed that a best proximity reduces to a fixed point if the underlying mapping is a self-mapping.

Definition 2.10. Let $S : A \to B$ and $T : A \to B$ be two mappings. An element $x^* \in A$ is said to be a *common best proximity point* if it satisfies the following condition:

$$F_{x^*,Sx^*}(t) = F_{x^*,Tx^*}(t) = F_{A,B}(t)$$

for each $t > 0$.

Observe that a common best proximity point is an element at which the multi-objective functions $x \to F_{x,Sx}(t)$ and $x \to F_{x,Tx}(t)$ attain a common global maximum since $F_{x,Sx}(t) \leq F_{A,B}(t)$ and $F_{x,Tx}(t) \leq F_{A,B}(t)$ for all x and $t > 0$.

Definition 2.11. A mapping $S : A \to B$ and $T : A \to B$ is said to be a *proximally commuting* if they satisfy the following condition:

$$[F_{u,Sx}(t) = F_{v,Tx}(t) = F_{A,B}(t)] \implies Sv = Tu$$

for all $u, v, x, \in A$ and $t > 0$.

It is easy to see that the proximal commutativity of self-mappings become commutativity of the mappings.

Definition 2.12. Two mappings $S : A \to B$ and $T : A \to B$ are said to be a *proximally swapped* if they satisfy the following condition:

$$[F_{y,u}(t) = F_{y,v}(t) = F_{A,B}(t), \quad Su = Tv] \implies Sv = Tu$$

for all $u, v, \in A$, $y \in B$ and $t > 0$.

Definition 2.13. A set A is said to be *approximatively compact* with respect to a set B if every sequence $\{x_n\}$ in A satisfies the condition that $F_{y,x_n}(t) \to F_{y,A}(t)$ for some $y \in B$ and for each $t > 0$ has a convergent subsequence.

Observe that every set is approximatively compact with respect to itself. Also, every compact set is approximatively compact with respect to any set. Moreover, A_0 and B_0 are nonempty set if A is compact and B is approximatively compact with respect to A.

3. Main result

Now, we give our main results in this paper.

Theorem 3.1. *Let A and B be nonempty closed subsets of a complete NA-PM-space $(X, F, *)$ in which the t–norm $*$ is positive and $* <$ min such that A is approximatively compact with respect to B. Also, assume that A_0 and B_0 are nonempty. Let $S : A \to B$, $T : A \to B$ be the non-self mappings satisfying the following conditions:*

(a) *for each x and y are elements in A and $t > 0$,*

$$F_{Sx,Sy}(\phi(t)) * F_{Tx,Ty}(t) \geq F_{Tx,Ty}(\phi(t)),$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that, for any $t > 0$, $0 < \varphi(t) < t$;

(b) *T is continuous;*

(c) *S and T commute proximally;*

(d) *S and T can be swapped proximally;*

(e) *$S(A_0) \subseteq B_0$ and $S(A_0) \subseteq T(A_0)$.*

Then there exists an element $x \in A$ such that $F_{x,Tx}(t) = F_{A,B}(t)$ and $F_{x,Sx}(t) = F_{A,B}(t)$.

Moreover, if x^* is another common best proximity point of the mappings S and T, then it is necessary that $F_{x,x^*}(t) \geq F_{A,B}(t) * F_{A,B}(t)$ for all $t > 0$.

Proof. Let x_0 be a fixed element in A_0. In view of the fact that $S(A_0) \subseteq T(A_0)$, there exists an element $x_1 \in A_0$ such that $Sx_0 = Tx_1$. Again, since $S(A_0) \subseteq T(A_0)$, there exists an element $x_2 \in A_0$ such that $Sx_1 = Tx_2$. By the similar fashion, we can find a sequence $\{x_n\}$ in A_0 such that

$$Sx_{n-1} = Tx_n \tag{3.1}$$

for all $n \in \mathbb{N}$. It follows that

$$F_{Sx_n,Sx_{n+1}}(\phi(t)) * F_{Tx_n,Tx_{n+1}}(t) \geq F_{Tx_n,Tx_{n+1}}(\phi(t)) \tag{3.2}$$

and

$$F_{Sx_n,Sx_{n+1}}(\phi(t)) * F_{Sx_{n-1},Sx_n}(t) \geq F_{Sx_{n-1},Sx_n}(\phi(t)) \tag{3.3}$$

for all $t > 0$. Thus we have

$$F_{Sx_n,Sx_{n+1}}(t) \geq F_{Sx_{n-1},Sx_n}(t) \tag{3.4}$$

for all $t > 0$, which means that the sequence $\{F_{Sx_{n-1},Sx_n}(t)\}$ is non-decreasing and bounded above. Hence there exists $r \leq 1$ such that, for any $t > 0$,

$$\lim_{n\to\infty} F_{Sx_{n-1},Sx_n}(t) = r. \tag{3.5}$$

If $r < 1$, then we have

$$F_{Sx_n,Sx_{n+1}}(\phi(t)) * F_{Sx_{n-1},Sx_n}(t) \geq F_{Sx_{n-1},Sx_n}(\phi(t)) \tag{3.6}$$

for all $t > 0$. Taking $n \to \infty$ in the inequality (3.6), by the continuity of φ, we get $a * r \geq a$, where $a = \lim_{n\to\infty} F_{Sx_{n-1},Sx_n}(\phi(t))$, which is a contradiction unless $r = 1$. Therefore, it follows that

$$\lim_{n\to\infty} F_{Sx_{n-1},Sx_n}(t) = 1. \tag{3.7}$$

By the property of F, we conclude that $F_{Sx_{n-1},Sx_n}(t)$ tend to 1 for all $t > 0$.

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Next, we prove that $\{Sx_n\}$ is a Cauchy sequence. We consider two cases.

Case I. Suppose that there exits $n \in \mathbb{N}$ such that $Sx_n = Sx_{n+1}$. Then we observe that

$$F_{Sx_{n+1},Sx_{n+2}}(\phi(t)) * F_{Tx_{n+1},Tx_{n+2}}(t) \geq F_{Tx_{n+1},Tx_{n+2}}(\phi(t))$$

and

$$F_{Sx_{n+1},Sx_{n+2}}(\phi(t)) * F_{Sx_n,Sx_{n+1}}(t) \geq F_{Sx_n,Sx_{n+1}}(\phi(t))$$

for all $t > 0$. Then we have

$$F_{Sx_{n+1},Sx_{n+2}}(t) = 1$$

for all $t > 0$, which implies that $Sx_{n+1} = Sx_{n+2}$ and so, for each $m > n$, we conclude that $Sx_m = Sx_n$. Hence $\{Sx_n\}$ is a Cauchy sequence in B.

Case II. The successive terms of $\{Sx_n\}$ are different. Suppose that $\{Sx_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$, $t > 0$ and the subsequences $\{Sx_{m_k}\}$, $\{Sx_{n_k}\}$ of $\{Sx_n\}$ with $n_k > m_k \geq k$ such that

$$F_{Sx_{m_k},Sx_{n_k}}(t) \leq 1-\varepsilon, \quad F_{Sx_{m_k},Sx_{n_k-1}}(t) > 1-\varepsilon. \tag{3.8}$$

By using (3.8) and the triangular inequality, we have

$$\begin{aligned} 1-\varepsilon &\geq F_{Sx_{m_k},Sx_{n_k}}(t) \\ &\geq F_{Sx_{m_k},Sx_{n_k-1}}(t) * F_{Sx_{n_k-1},Sx_{n_k}}(t) \\ &\geq (1-\varepsilon) * F_{Sx_{n_k-1},Sx_{n_k}}(t). \end{aligned} \tag{3.9}$$

Thus, using (3.9) and (3.7), we have

$$F_{Sx_{m_k},Sx_{n_k}}(t) \to 1-\varepsilon \tag{3.10}$$

as $k \to \infty$. Again, by the triangular inequality, we have

$$F_{Sx_{m_k},Sx_{n_k}}(t) \geq F_{Sx_{m_k},Sx_{m_k+1}}(t) * F_{Sx_{m_k+1},Sx_{n_k+1}}(t) * F_{Sx_{n_k+1},Sx_{n_k}}(t) \tag{3.11}$$

and

$$F_{Sx_{m_k+1},Sx_{n_k+1}}(t) \geq F_{Sx_{m_k+1},Sx_{m_k}}(t) * F_{Sx_{m_k},Sx_{n_k}}(t) * F_{Sx_{n_k},Sx_{n_k+1}}(t). \tag{3.12}$$

From (3.7), (3.10), (3.11) and (3.12), it follows that

$$F_{Sx_{m_k+1},Sx_{n_k+1}}(t) \to 1-\varepsilon \tag{3.13}$$

as $k \to \infty$. In view of the fact that

$$F_{Sx_{m_k+1},Sx_{n_k+1}}(\phi(t)) * F_{Tx_{m_k+1},Tx_{n_k+1}}(t) \geq F_{Tx_{m_k+1},Tx_{n_k+1}}(\phi(t)), \tag{3.14}$$

we have

$$F_{Sx_{m_k+1},Sx_{n_k+1}}(\phi(t)) * F_{Sx_{m_k},Sx_{n_k}}(t) \geq F_{Sx_{m_k},Sx_{n_k}}(\phi(t)). \tag{3.15}$$

Letting $k \to \infty$ in the inequality (3.15), we obtain

$$a * (1-\varepsilon) \geq a,$$

where $a = F_{Sx_{m_k+1},Sx_{n_k+1}}(\phi(t))$, which is a contradiction by the property of φ. Then we deduce that $\{Sx_n\}$ is a Cauchy sequence in B. Since B is a closed subset a complete NA-PM-space X, there exists $y \in B$ such that $Sx_n \to y$ as $n \to \infty$. Consequently, it follows that the sequence $\{Tx_n\}$ also converges to y. From $S(A_0) \subseteq B_0$, there exists an element $u_n \in A$ such that

$$F_{Sx_n,u_n}(t) = F_{A,B}(t) \tag{3.16}$$

for all $n \in \mathbb{N}$ and $t > 0$. Thus it follows from (3.1) and (3.16) that

$$F_{Tx_n,u_{n-1}}(t) = F_{Sx_{n-1},u_{n-1}}(t) = F_{A,B}(t) \tag{3.17}$$

for all $n \in \mathbb{N}$ and $t > 0$. By (3.16), (3.17) and the fact that the mappings S and T are proximally commuting, we obtain

$$Tu_n = Su_{n-1} \tag{3.18}$$

for all $n \in \mathbb{N}$. Moreover, we have

$$\begin{aligned} F_{y,A}(t) &\geq F_{y,u_n}(t) \\ &\geq F_{y,Sx_n}(t) * F_{Sx_n,u_n}(t) \\ &= F_{y,Sx_n}(t) * F_{A,B}(t) \\ &\geq F_{y,Sx_n}(t) * F_{y,A}(t), \end{aligned} \tag{3.19}$$

for all $t > 0$. Therefore, it follows that, for all $t > 0$, $F_{y,u_n}(t) \to F_{y,A}(t)$ as $n \to \infty$. Since A is approximatively compact with respect to B, there exists a subsequence $\{u_{n_k}\}$ of the sequence $\{u_n\}$ such that $\{u_{n_k}\}$ converges to some element $u \in A$. Further, since $F_{y,u_{n_k-1}}(t) \to F_{y,A}(t)$ for all $t > 0$ and A is approximatively compact with respect to B, there exists a subsequence $\{u_{n_{k_j}-1}\}$ of the sequence $\{u_{n_k-1}\}$ such that $\{u_{n_{k_j}-1}\}$ converges to some element $v \in A$. By the continuity of the mappings S and T, we have

$$Tu = \lim_{j\to\infty} Tu_{n_{k_j}} = \lim_{k\to\infty} Su_{n_{k_j}-1} = Sv \tag{3.20}$$

and

$$\begin{aligned} F_{y,u}(t) &= \lim_{k\to\infty} F_{Sx_{n_k},u_{n_k}}(t) = F_{A,B}(t), \\ F_{y,v}(t) &= \lim_{j\to\infty} F_{Tx_{n_{k_j}},u_{n_{k_j}-1}}(t) = F_{A,B}(t). \end{aligned} \tag{3.21}$$

Since S and T can be swapped proximally, we have

$$Tv = Su. \tag{3.22}$$

Next, we prove that $Su = Sv$. Suppose that $Su \neq Sv$. Then, by (3.20), (3.21), (3.22) and the property of φ, we have

$$F_{Su,Sv}(\phi(t)) * F_{Tu,Tv}(t) \geq F_{Tu,Tv}(\phi(t))$$

and so

$$F_{Su,Sv}(\phi(t)) * F_{Su,Sv}(t) \geq F_{Su,Sv}(\phi(t))$$

for all $t > 0$, which is a contradiction. Thus $Su = Sv$ and also $Tu = Su$. Since $S(A_0)$ is contained in B_0, there exists an element $x \in A$ such that $F_{x,Tu}(t) = F_{A,B}(t)$ and $F_{x,Su}(t) = F_{A,B}(t)$. Since S and T are proximally commuting, we have $Sx = Tx$. Consequently, we have

$$F_{Su,Sx}(\phi(t)) * F_{Tu,Tx}(t) \geq F_{Tu,Tx}(\phi(t)) \tag{3.23}$$

and so

$$F_{Su,Sx}(\phi(t)) * F_{Su,Sx}(t) \geq F_{Su,Sx}(\phi(t)) \tag{3.24}$$

for all $t > 0$, which is impossible if $Su \neq Sx$. Thus we have $Su = Sx$ and hence $Tu = Tx$. It follows that

$$F_{x,Tx}(t) = F_{x,Tu}(t) = F_{A,B}(t)$$

and

$$F_{x,Sx}(t) = F_{x,Su}(t) = F_{A,B}(t)$$

for all $t > 0$. Therefore, x is a common best proximity point of S and T.

To prove the uniqueness of the point x, suppose that x^* is another common best proximity point of the mappings S and T. Then we have

$$F_{x^*,Tx^*}(t) = F_{A,B}(t), \quad F_{x^*,Sx^*}(t) = F_{A,B}(t)$$

for all $t > 0$. Since S and T are proximally commuting, we get $Sx = Tx$ and $Sx^* = Tx^*$. Consequently, we have

$$F_{Sx^*,Sx}(\phi(t)) * F_{Tx^*,Tx}(t) \geq F_{Tx^*,Tx}(\phi(t)) \tag{3.25}$$

and so

$$F_{Sx^*,Sx}(\phi(t)) * F_{Sx^*,Sx}(t) \geq F_{Sx^*,Sx}(\phi(t)) \tag{3.26}$$

for all $t > 0$, which is impossible if $Sx^* \neq Sx$. Thus we have $Sx = Sx^*$. Moreover, it can be concluded that

$$\begin{aligned} F_{x,x^*}(t) &\geq F_{x,Sx}(t) * F_{Sx,Sx^*}(t) * F_{Sx^*,x^*}(t) \\ &\geq F_{A,B}(t) * F_{A,B}(t) \end{aligned}$$

for all $t > 0$. This completes the proof. □

Corollary 3.2. *Let A be a nonempty closed subset of a complete NA-PM-space $(X, F, *)$ in which the t–norm $*$ is positive and $* <$ min such that A is compact. Let $S : A \to A$, $T : A \to A$ be the self mappings satisfying the following conditions:*

(a) *for each x and y are elements in A and $t > 0$,*

$$F_{Sx,Sy}(\phi(t)) * F_{Tx,Ty}(t) \geq F_{Tx,Ty}(\phi(t)),$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that, for any $t > 0$, $0 < \varphi(t) < t$;

(b) *T is continuous;*

(c) *S and T commutative;*

(e) *$S(A) \subseteq A$ and $S(A) \subseteq T(A)$.*

Then S and T have common fixed point.

4. An example

Now, we give an example to illustrate Theorem 3.1.

Example 4.1. Consider the complete metric space $\mathbb{R}^2$ with Euclidean metric. Define

$$F_{(x_1,x_2),(y_1,y_2)}(t) = \frac{t}{t + |x_1 - y_1| + |x_2 - y_2|}$$

for all $t > 0$ and

$$F_{(x_1,x_2),(y_1,y_2)}(t) = 0$$

for all $t \leq 0$. It is easy to show that $(X, F, \cdot)$ is a NA-PM-space. Let

$$A = \{(x, 1) : 0 \leq x \leq 1\}, \quad B = \{(x, -1) : 0 \leq x \leq 1\}.$$

Define two mappings $S : A \to B$, $T : A \to B$ as follows:

$$S(x, 1) = \big(0, -1\big), \quad T(x, 1) = \big(x, -1\big),$$

respectively. It is easy to see that $F_{A,B}(t) = \frac{t}{t+2}$, $A_0 = A$ and $B_0 = B$. Further, S and T are continuous and A is approximatively compact with respect to B.

First, we show that S and T are satisfy the condition (a) of Theorem 3.1 with $\varphi : [0, \infty) \to [0, \infty)$ defined by $\varphi(t) = \dfrac{t}{2}$ for all $t \in [0, \infty)$. Let $(x, 1), (y, 1) \in A$. Without loss of generality, we can take $x > y$. Then we have

$$\begin{aligned} F_{S(x,1),S(y,1)}(\phi(t)) * F_{T(x,1),T(y,1)}(t) &= 1 \cdot \tfrac{t}{t+|x-y|} \\ &\geq \tfrac{\frac{t}{2}}{\frac{t}{2}+|x-y|} \\ &= F_{T(x,1),T(y,1)}(\phi(t)) \end{aligned}$$

for all $t > 0$.

Next, we show that S and T are proximally commuting. Let $(u, 1), (v, 1), (x, 1) \in A$ be such that

$$F_{(u,1),S(x,1)}(t) = F_{A,B}(t) = \frac{t}{t+2}, \quad F_{(v,1),T(x,1)}(t) = F_{A,B}(t) = \frac{t}{t+2}$$

for all $t > 0$. It follows that $u = 0$ and $v = x$ and hence

$$S(v, 1) = \big(0, -1\big) = (u, -1) = T(u, 1).$$

Finally, we show that S and T are proximally swapped. If it is true that

$$F_{(u,1),(y,-1)}(t) = F_{(v,1),(y,-1)}(t) = F_{A,B}(t) = \frac{t}{t+2}, \quad S(u,1) = T(v,1)$$

for some $(u,1),(v,1) \in A$ and $(y,-1) \in B$, then we get $u = v = 0$ and so

$$S(v,1) = T(u,1).$$

Therefore, all the hypothesis of Theorem 3.1 are satisfied. Furthermore, $(0,1) \in A$ is a common best proximity point of the mappings S and T since

$$F_{(0,1),S(0,1)}(t) = F_{(0,1),(0,-1)}(t) = F_{(0,1),T((0,1)}(t) = F_{A,B}(t)$$

for all $t > 0$.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors carried out the proof. All authors conceived of the study, and participated in its design and coordination. All authors read and approved the final manuscript.

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The Value Distribution of Some Difference Polynomials of Meromorphic Functions *

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Abstract

The purpose of this paper is to investigate the value distribution of some difference polynomials $G_1(z)=\prod_{j=1}^{m} f(z+c_j)-af(z)^n$, $G_2(z)=f(z)^n\prod_{j=1}^{m} f(z+c_j)$ and $G_3(z)=f(z)^n\prod_{j=1}^{m}(f(z+c_j)-f(z))$, where $f(z)$ is a meromorphic function and $a\in\mathbb{C}\setminus\{0\}$ and $c_j, j=1,2,\ldots,m$ are complex constants.

Key words: meromorphic function; difference polynomial; zeros.

Mathematical Subject Classification (2010): 30D35, 39A10.

1 Introduction and main results

This purpose of this paper is to study some properties of value distribution of some complex difference polynomials of meromorphic functions. The fundamental theorems and the standard notations of the Nevanlinna value distribution theory of meromorphic functions will be used(see [7, 15]). In addition, for meromorphic function f, we will use $S(r,f)$ to denote any quantity satisfying $S(r,f)=o(T(r,f))$ for all r outside a possible exceptional set E of finite logarithmic measure $\lim_{r\to\infty}\int_{[1,r)\cap E}\frac{dt}{t}<\infty$. We use $\rho(f),\lambda(f)$ and $\lambda(\frac{1}{f})$ to denote the order, the exponent of convergence of zeros and the exponent of convergence of poles of $f(z)$ respectively.

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Many people were interested in the value distribution of different expressions of meromorphic function and obtained lots of valuable theorems. In 1959, Hayman [8] studied value distribution of meromorphic function and its derivatives, and obtained the following famous theorems.

Theorem 1.1 *[8]. Let $f(z)$ be a transcendental entire function. Then*

(i) for $n \geq 3$ and $a \neq 0, \Psi(z) = f'(z) - af(z)^n$ assumes all finite values infinitely often.

(ii) for $n \geq 2$, $\Phi(z) = f'(z)f(z)^n$ assumes all finite values except possibly zero infinitely often.

However, Mues [12] proved that the conclusion of Theorem 1.1 is not true for $n = 3$ by providing a counter example and proved that $f'(z) - af(z)^4$ has infinitely many zeros.

Recently, the topic of difference product in the complex plane $\mathbb{C}$ has attracted many researchers, a number of papers have focused on value distribution of differences and differences operator analogues of Nevanlinna theory (including [2, 4, 5, 6, 11]).

In 2007, Laine and Yang [9] proved the following result, which is regarded as a difference counterpart of Theorem 1.1.

Theorem 1.2 *[9]. Let $f(z)$ be a transcendental entire function of finite order, and c be a non-zero complex constant. Then for $n \geq 2$, $\Phi_1(z) = f(z+c)f(z)^n$ assumes every non-zero value $a \in \mathbb{C}$ infinitely often.*

It is well known that $\Delta f(z) = f(z+c) - f(z)$, where $c \in \mathbb{C} \setminus \{0\}$ is a constant satisfying $f(z+c) - f(z) \not\equiv 0$, which can be considered as the difference counterpart of $f'(z)$. Similarly, $\Delta f(z) - af(z)^n$ can be considered as the difference counterpart $f'(z) - af(z)^n$, where $a \in \mathbb{C} \setminus \{0\}$.

In 2011, Chen [1] considered the difference counterpart of Theorem 1.1 and obtained the following theorems.

Theorem 1.3 *[1]. Let $f(z)$ be a transcendental entire function of finite order, and let $a, c \in \mathbb{C} \setminus \{0\}$ be constants, with c such that $f(z+c) \not\equiv f(z)$. Set $\Psi_n(z) = \Delta f(z) - af(z)^n$ and $n \geq 3$ is an integer. Then $\Psi_n(z)$ assumes all finite values infinitely often, and for every $b \in \mathbb{C}$ one has $\lambda(\Psi_n(z) - b) = \rho(f)$.*

Theorem 1.4 *[1]. Let $f(z)$ be a transcendental entire function of finite order with a Borel exceptional value 0, and let $a, c \in \mathbb{C} \setminus \{0\}$ be constants, with c such that $f(z+c) \not\equiv f(z)$. Then $\Psi_2(z)$ assumes all finite values infinitely often, and for every $b \in \mathbb{C}$ one has $\lambda(\Psi_2(z) - b) = \rho(f)$.*

Theorem 1.5 *[1]. Let $f(z)$ be a transcendental entire function of finite order with a finite nonzero Borel exceptional value d, and let $a, c \in \mathbb{C} \setminus \{0\}$ be constants, with c such that $f(z+c) \not\equiv f(z)$. Then for every $b \in \mathbb{C}$ with $b \neq -ad^2$, $\Psi_2(z)$ assumes the value b infinitely often, and $\lambda(\Psi_2(z) - b) = \rho(f)$.*

In 2013, Zheng and Chen [16] further investigated the value distribution of some difference polynomial of entire function and obtained the following theorem.

Theorem 1.6 *[16]. Let $f(z)$ be a transcendental entire function of finite order with a finite nonzero Borel exceptional value d, and let $a \in \mathbb{C} \setminus \{0\}$, $c_1, c_2, \ldots, c_m$ be complex constants satisfying that at least one of them is non-zero. Then for $1 \leq m < n$ and every $b(\neq d^m - ad^n) \in \mathbb{C}$, $G_1(z) = \prod_{j=1}^{m} f(z+c_j) - af(z)^n$ assumes the value b infinitely often and $\lambda(G_1(z) - b) = \rho(f)$.*

Thus, it is natural to ask: *On What condition can Theorem 1.6 still hold when $f(z)$ is a transcendental meromorphic function?*

The main purpose of this article is to study the above questions and obtain the following theorem.

Theorem 1.7 *Let $f(z)$ be a transcendental meromorphic function of finite order with two Borel exceptional values d, ∞, and let $a \in \mathbb{C} \setminus \{0\}$, $c_1, c_2, \ldots, c_m$ be complex constants satisfying that at least one of them is non-zero. Then for $1 \leq m < n$ and every $b(\neq d^m - ad^n) \in \mathbb{C}$, $G_1(z) = \prod_{j=1}^{m} f(z+c_j) - af(z)^n$ assumes the value b infinitely often and $\lambda(G_1(z) - b) = \rho(f)$.*

In addition, we further study the value distribution of some difference polynomials of meromorphic function of more general form

$$G_2(z) = f(z)^n \prod_{j=1}^{m} f(z+c_j), \quad G_3(z) = f(z)^n \prod_{j=1}^{m} [f(z+c_j) - f(z)]$$

and obtain the following results:

Theorem 1.8 *Let $f(z)$ be a transcendental meromorphic function of finite order with two Borel exceptional values d, ∞, and let $c_1, c_2, \ldots, c_m$ be nonzero complex constants. Then for $n \geq 1$, $G_2(z)$ assumes every value $b(\neq d^{n+m}) \in \mathbb{C}$ infinitely often and $\lambda(G_2(z) - b) = \rho(f)$.*

Corollary 1.1 *Let $f(z)$ be a transcendental meromorphic function of finite order with two Borel exceptional values $0, \infty$, and let $c_1, c_2, \ldots, c_m$ be nonzero complex constants. Then for $n \geq 1$, $G_2(z)$ assumes every nonzero value $b \in \mathbb{C}$ infinitely often and $\lambda(G_2(z) - b) = \rho(f)$.*

Example 1.1 *Let $f(z) = \frac{e^z - 2}{e^z + 2}$, it is easy to see that $0, \infty$ are not Borel exceptional values. Let $n = 1, m = 2, c_1 = \pi i, c_2 = -\pi i$ and $b = 1$, then we have $G_2(z) = f(z)f(z+c_1)f(z+c_2) - 1 = \frac{4}{e^z - 2}$ has no zeros. Let $n = 2, m = 2, c_1 = \pi i, c_2 = -\pi i$ and $b = 1$, then we have $G_2(z) = f(z)^3 f(z+c_1)f(z+c_2) - 1 = \frac{-4}{e^z + 1}$ has no zeros. Hence, this shows that the condition in Corollary 1.1 is sharp in a sense.*

Theorem 1.9 *Let $f(z)$ be a transcendental meromorphic function of finite order with two Borel exceptional values d, ∞, and let $c_1, c_2, \ldots, c_m$ be nonzero complex constants and $f(z + c_j) \not\equiv f(z)$ for $j = 1, 2, \ldots, m$. Then for $n, m \geq 1$ are two integers, $G_3(z)$ assumes every value $b \in \mathbb{C} \setminus \{0\}$ infinitely often and $\lambda(G_3(z) - b) = \rho(f)$.*

Remark 1.1 *When $b = 0$, the conclusion may not hold. For example, let $f(z) = e^z$, $G_3(z) = f(z)^n[f(z + \pi i) - f(z)]$. Then $G_3(z) = -2e^{(n+1)z}$ has no zeros.*

Corollary 1.2 *Let $f(z)$ be a transcendental entire function of finite order with a Borel exceptional values d, and let $c_1, c_2, \ldots, c_m$ be nonzero complex constants and $f(z+c_j) \not\equiv f(z)$ for $j = 1, 2, \ldots, m$. Then for $n, m \geq 1$ are two integers, $G_3(z)$ assumes every value $b \in \mathbb{C}$ infinitely often and $\lambda(G_3(z) - b) = \rho(f)$.*

Remark 1.2 *It is easily to see that Theorem 1.9 is an improvement of the result in [10, Theorem 1.4], where they consider the case of $m = 1$ and the value b can replaced by a small function $\alpha(z)$. In fact, our results also can allow the value b to be a polynomial, even be a meromorphic function $\alpha(z) \not\equiv 0$ satisfying $\rho(\alpha) < \rho(f)$.*

Example 1.2 *Let $f(z) = e^z + 2z, c = 2\pi i, \alpha(z) = 4cz, n = 1$ and $m = 1$. Then we know that $f(z)$ has no Borel exceptional value, and we have $G_3(z) = f(z)\Delta f(z) - 4cz = 2ce^z$, which has no zeros. Hence, the condition on $f(z)$ having a Borel exceptional value is necessary in Corollary 1.2.*

The following result of this paper is the value distribution of differential and difference polynomial of entire function.

Theorem 1.10 *Let $f(z)$ be a transcendental entire function of finite order, and $a, c_1, \ldots, c_m$ be nonzero complex constants. Then for any positive integers $n \geq 2m+3$, $\Psi(z) = f^{(k)}(z)\prod_{j=1}^{m} f(z+c_j) - af(z)^n$ assumes all finite values $b \in \mathbb{C}$ infinitely often.*

Regarding Theorem 1.2, we pose the following question.

Question 1.1 *What can be said if the condition $n \geq 2m+3$ in Theorem 1.10 is replaced with $1 \leq n \leq 2m+2$?*

2 Some Lemmas

The following lemma is important in the fields of factorization and uniqueness theory of meromorphic functions which is given by Gross [3]. In 2010, Xu and Yi [13] made a small changed form as follows.

Lemma 2.1 *[13]. Suppose that $f_j(z)(j = 1, 2, \ldots, n+1)$ are meromorphic functions and $g_j(j = 1, 2 \ldots, n)$ are entire functions satisfying the following conditions.*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv f_{n+1}$.

(ii) If $1 \leq j \leq n+1, 1 \leq k \leq n$, the order of f_j is less than the order of $e^{g_k(z)}$. If $n \geq 2$, $1 \leq j \leq n+1, 1 \leq h < k \leq n$, and the order of $f_j(z)$ is less than the order of $e^{g_h - g_k}$.

Then $f_j(z) \equiv 0(j = 1, 2, \ldots, n+1)$.

Lemma 2.2 *[15]. Let f be a nonconstant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \cdots + a_n f^n$, where $a_0, a_1, a_2, \cdots, a_n$ are constants and $a_n \neq 0$. Then*

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

Lemma 2.3 *[2, Theorem 2.1]. Let $f(z)$ be a meromorphic function of finite order ρ and let c be a fixed nonzero complex number, then for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = O(r^{\rho-1+\varepsilon}) = S(r, f).$$

Lemma 2.4 *[2, Corollary 2.5]. Let $f(z)$ be a meromorphic function with order $\rho = \rho(f), \rho < +\infty$, and let η be a fixed nonzero complex number, then for each $\varepsilon > 0$, we have*

$$T(r, f(z+\eta)) = T(r, f) + O(r^{\rho-1+\varepsilon}) + O(\log r).$$

Lemma 2.5 *[2, Theorem 2.2]. Let f be a meromorphic function with exponent of convergence of poles $\lambda(\frac{1}{f}) = \lambda < +\infty$, $c \neq 0$ be fixed, then for each $\varepsilon > 0$,*

$$N(r, f(z+\eta)) = N(r, f) + O(r^{\lambda-1+\varepsilon}) + O(\log r).$$

Lemma 2.6 *[14]. If $f(z)$ is a transcendental meromorphic function with exponent of convergence of poles $\lambda(\frac{1}{f}) = \lambda < +\infty$, $c \neq 0$. Then, for each $\varepsilon > 0$, one has*

$$\lambda\left(\frac{1}{f(z+c)}\right) = \lambda\left(\frac{1}{f(z)}\right) = \lambda, \quad \lambda\left(\frac{1}{\Delta f}\right) \leq \lambda.$$

Lemma 2.7 *Let $f(z)$ be a transcendental meromorphic function with exponent of convergence of poles $\lambda(\frac{1}{f}) = \lambda < \rho(f) = \rho < +\infty$, and let $c_1, c_2, \ldots, c_m$ be nonzero complex constants, and $n, m \geq 1$ be integers. Then $\rho(G_2) = \rho(f)$.*

Proof: We firstly prove that $\rho(G_2) \leq \rho(f)$. We can rewrite $G_2(z)$ as the form

$$G_2(z) = f(z)^{n+m} \prod_{j=1}^{m} \frac{f(z+c_j)}{f(z)}. \tag{1}$$

For each $\varepsilon (0 < \varepsilon < \rho - \lambda)$, it follows by Lemma 2.3 and (1) that

$$\begin{aligned} m(r, G_2) &\leq (n+m)m(r, f) + \sum_{j=1}^{m} m(r, \frac{f(z+c_j)}{f(z)}) + O(1) \\ &= (n+m)m(r, f) + O(r^{\rho-1+\varepsilon}). \end{aligned} \tag{2}$$

By Lemma 2.5, we have

$$\begin{aligned} N(r, G_2) &\leq nN(r, f) + \sum_{j=1}^{m} N(r, f(z+c_j)) \\ &\leq (n+m)N(r, f) + O(r^{\lambda-1+\varepsilon}) + O(\log r). \end{aligned} \tag{3}$$

Since $\lambda < \rho$, it follows from (2) and (3) that

$$T(r, G_2) \leq (n+m)T(r, f) + O(r^{\rho-1+\varepsilon}) + O(\log r). \tag{4}$$

So, we can get that $\rho(G_2) \leq \rho(f)$ easily.

Next, we prove that $\rho(G_2) \geq \rho(f)$. From Lemma 2.3 and (1), we have

$$(n+m)m(r,f) = m(r,f^{n+m}) \leq m(r,G_2) + \sum_{j=1}^{m} m(r,\frac{f(z)}{f(z+c_j)}) + O(1) \qquad (5)$$
$$= m(r,G_2) + O(r^{\rho-1+\varepsilon}).$$

Since $\lambda(1/f) = \lambda < \rho$, for any given $\varepsilon > 0$ there exists $r_0 > 0$ such that for all $r > r_0$ we have

$$N(r,f) \leq r^{\lambda+\varepsilon}. \qquad (6)$$

Thus, it follows from (5) and (6) that

$$T(r,f) \leq \frac{1}{n+m} m(r,G_2) + O(r^{\rho-1+\varepsilon}) + O(r^{\lambda+\varepsilon}), \quad r > r_0. \qquad (7)$$

Since $\lambda < \rho$ and $0 < \varepsilon < \rho - \lambda$, it follows from (7) that $\rho(G_2) \geq \rho(f)$.

Hence, the proof of Lemma 2.7 is proved. □

By using the same argument as in Lemma 2.7, we can prove the following lemma easily.

Lemma 2.8 *Let $f(z)$ be a transcendental meromorphic function with exponent of convergence of poles $\max\{\lambda(f), \lambda(\frac{1}{f})\} = \lambda < \rho(f) = \rho < +\infty$, and let $c_1, c_2, \ldots, c_m$ be nonzero complex constants such that $f(z+c_j) \not\equiv f(z) (j = 1, 2, \ldots, m)$, and $n, m \geq 1$ be integers. Then $\rho(G_3) = \rho(f)$.*

Lemma 2.9 *[15, page 37]. Let $f(z)$ be a nonconstant meromorphic function in the complex plane and l be a positive integer. Then*

$$T(r, f^{(l)}(z)) \leq (l+1)T(r,f) + S(r,f), \quad N(r, f^{(l)}(z)) = N(r,f) + l\overline{N}(r,f).$$

3 Proofs of Theorems 1.7, 1.8 and 1.9

3.1 The Proof of Theorem 1.7

We first prove $\rho(G_1) = \rho(f)$. By Lemma 2.2 and Lemma 2.4, we have $\rho(G_1) \leq \rho(f)$. On the other hand, it follows from Lemma 2.4 that

$$nT(r,f) = T(r, af^n) + O(1) = T\left(r, \prod_{j=1}^{m} f(z+c_j) - G_1(z)\right) + O(1)$$
$$\leq \sum_{j=1}^{m} T(r, f(z+c_j)) + T(r, G_1(z)) + O(1)$$
$$= mT(r,f) + T(r, G_1(z)) + O(r^{\rho-1+\varepsilon}) + O(\log r),$$

that is,

$$(n-m)T(r,f) \leq T(r, G_1(z)) + O(r^{\rho-1+\varepsilon}) + O(\log r). \qquad (8)$$

Since $1 \leq m < n$, it follows from (8) that $\rho(f) \leq \rho(G_1)$. Hence, we can prove that $\rho(G_1) = \rho(f)$.

Since $f(z)$ has two Borel exceptional values d, ∞, then $f(z), f(z+c_j)$ can be written as the form

$$f(z) = d + \frac{g(z)}{p(z)} \exp\{\alpha z^k\}, \quad f(z+c_j) = d + \frac{g(z+c_j)}{p(z+c_j)} h_j(z) \exp\{\alpha z^k\}, \tag{9}$$

where $\alpha \neq 0$ is a constant, $k(\geq 1)$ is an integer satisfying $\rho(f) = k$, and $g(z), h_j(z) = e^{\alpha k c_j z^{k-1} + \cdots + \alpha c_j^k}$ are entire functions such that $g(z)h_j(z) \not\equiv 0, \rho(g) < k, \rho(h_j) \leq k-1$, $j = 1, 2 \ldots, m$, and $p(z)$ is the canonical product formed with the poles of $f(z)$ satisfying $\rho(p) = \lambda(p) = \lambda(\frac{1}{f}) < \rho(f)$. Set $H(z) = \frac{g(z)}{p(z)} \not\equiv 0$, then we can see that $\rho(H) < \rho(f)$.

Now we prove that $\lambda(G_1 - b) = \rho(f)$. Suppose that $\lambda(G_1 - b) < \rho(f)$. Since $\rho(G_1) = \rho(f) = \rho(G_1 - b)$, then $\lambda(G_1 - b) < \rho(G_1 - b) = \rho(f) = k$ and $G_1(z) - b$ can be rewritten as the form

$$G_1(z) - b = \frac{g_1^*(z)}{p_1^*(z)} \exp\{\beta z^k\} = H_1^*(z) \exp\{\beta z^k\}, \tag{10}$$

where $\beta(\neq 0)$ is a constant, $g_1^*(z)(\not\equiv 0)$ is an entire function satisfying $\rho(g_1^*) < k$. Thus, by Lemma 2.6, we have $\rho(p_1^*) = \lambda(p_1^*) \leq \max\{\lambda(\frac{1}{f(z)}), \lambda(\frac{1}{f(z+c_j)}), j = 1, 2, \ldots, m\} = \lambda(\frac{1}{f}) < \rho(f) = k$. So, we have $\rho(H_1^*) < \rho(f) = k$.

Thus, from (9), (10) and the definition of $G_1(z)$, we have

$$\begin{aligned}
&\prod_{j=1}^{m} H(z+c_j) h_j(z) e^{m\alpha z^k} + \cdots + d^{m-2} \left(\sum_{1 \leq i < j \leq m} H(z+c_i) H(z+c_j) h_i(z) h_j(z) \right) e^{2\alpha z^k} \\
&\quad + d^{m-1} \left(\sum_{j=1}^{m} H(z+c_j) h_j(z) \right) e^{\alpha z^k} + d^m \\
&\quad - a \left(d^n + n d^{n-1} H(z) e^{\alpha z^k} + \cdots + H(z)^n e^{n\alpha z^k} \right) = b + H_1^*(z) e^{\beta z^k}.
\end{aligned}$$

Since $1 \leq m < n, aH(z)H_1^*(z) \not\equiv 0$, by comparing growths of both sides of the above equality, we have $\beta = n\alpha$. Thus, we can rewrite the above equality as the form

$$f_n(z) e^{n\alpha z^k} + f_{n-1}(z) e^{(n-1)\alpha z^k} + \cdots + f_1(z) e^{\alpha z^k} = f_{n+1}(z), \tag{11}$$

where $f_{n+1}(z) = b - d^m + ad^n$, and $f_1, \ldots, f_n$ are algebraic expressions in the terms $a, d, n, m, H(z), H_1^*(z), H(z+c_j), h_j(z), j = 1, 2, \ldots, m$, such as addition, subtraction and multiplication. Since $\rho(H) < \rho(f) = k, \rho(h_j) \leq k-1$ and $\rho(H_1^*) < \rho(f) = k$, then we have $\rho(f_t) < k = \rho(e^{t\alpha z^k})$ for $t = 1, 2, \ldots, n$. Thus, by Lemma 2.1 and (11), we have $f_t(z) \equiv 0$ for $t = 1, 2, \ldots, n+1$, that is, $b - d^m + ad^n \equiv 0$, which is a contradiction with the assumption $b \neq d^m - ad^n$. Hence, we have that $\lambda(G_1 - b) = \rho(f)$.

This completes the proof of Theorem 1.7.

3.2 The proof of Theorem 1.8

Similar to the proof of Theorem 1.7, we can obtain (9).

Now we prove that $\lambda(G_2 - b) = \rho(f)$. Suppose that $\lambda(G_2 - b) < \rho(f)$.

By Lemma 2.7, we have $\rho(G_2) = \rho(f) = \rho(G_2 - b)$, then $\lambda(G_2 - b) < \rho(G_2 - b) = \rho(f) = k$ and $G_2(z) - b$ can be rewritten as the form

$$G_2(z) - b = \frac{g_2^*(z)}{p_2^*(z)} \exp\{\beta z^k\} = H_2^*(z) \exp\{\beta z^k\}, \tag{12}$$

where $\beta(\neq 0)$ is a constant, $g_2^*(z)(\not\equiv 0)$ is an entire function satisfying $\rho(g_2^*) < k$. Thus, by Lemma 2.6, we have $\rho(p_2^*) = \lambda(p_2^*) \leq \max\{\lambda(\frac{1}{f(z)}), \lambda(\frac{1}{f(z+c_j)}), j = 1, 2, \ldots, m\} = \lambda(\frac{1}{f}) < \rho(f) = k$. So, we have $\rho(H_2^*) < \rho(f) = k$.

From (9), (12) and the definition of $G_2(z)$, we have

$$(d + H(z)e^{\alpha z^k})^n \prod_{j=1}^{m} \left[d + H(z + c_j)h_j(z)e^{\alpha z^k}\right] = b + H_2^*(z) \exp\{\beta z^k\}.$$

By simple calculation, we can rewrite the above equation as the form

$$f_{n+m}(z)e^{(n+m)\alpha z^k} + \cdots + f_1(z)e^{\alpha z^k} + d^{n+m} - b = H_2^*(z) \exp\{\beta z^k\}, \tag{13}$$

where $f_{n+m}(z) = H(z)^n \prod_{j=1}^{m} H(z + c_j)h_j(z) \not\equiv 0$ and $f_1, f_2, \ldots, f_{n+m-1}$ are algebraic expressions in the terms $d, n, m, H(z), H(z + c_j), h_j(z), j = 1, 2, \ldots, m$, such as addition, subtraction and multiplication. Since $\rho(H) < \rho(f) = k, \rho(h_j) \leq k - 1$ and $\rho(H_2^*) < \rho(f) = k$, then we have $\rho(f_t) < k = \rho(e^{t\alpha z^k})$ for $t = 1, 2, \ldots, n + m$. By comparing growths of both sides of the above equality, we have $\beta = (n+m)\alpha$. Thus, it follows from (13) that

$$[f_{n+m}(z) - H_2^*(z)]e^{(n+m)\alpha z^k} + \cdots + f_1(z)e^{\alpha z^k} = b - d^{n+m}, \tag{14}$$

By Lemma 2.1, we have $b = d^{n+m}$, a contradiction. Hence, we have $\lambda(G_2 - b) = \rho(f)$.

This completes the proof of Theorem 1.8.

3.3 The proof of Theorem 1.9

Similar to the proof of Theorem 1.7, we can obtain (9).

Now we prove that $\lambda(G_3 - b) = \rho(f)$. Suppose that $\lambda(G_3 - b) < \rho(f)$.

By Lemma 2.8, we have $\rho(G_3) = \rho(f) = \rho(G_3 - b)$, then $\lambda(G_3 - b) < \rho(G_3 - b) = \rho(f) = k$ and $G_3(z) - b$ can be rewritten as the form

$$G_3(z) - b = \frac{g_3^*(z)}{p_3^*(z)} \exp\{\beta z^k\} = H_3^*(z) \exp\{\beta z^k\}, \tag{15}$$

where $\beta(\neq 0)$ is a constant, $g_3^*(z)(\not\equiv 0)$ is an entire function satisfying $\rho(g_3^*) < k$. Thus, by Lemma 2.6, we have $\rho(p_3^*) = \lambda(p_3^*) \leq \max\{\lambda(\frac{1}{f(z)}), \lambda(\frac{1}{\Delta f(z)})\} = \lambda(\frac{1}{f}) < \rho(f) = k$. So, we have $\rho(H_3^*) < \rho(f) = k$.

From (9), (13) and the definition of $G_3(z)$, we have

$$\left(d+H(z)e^{\alpha z^k}\right)^n \prod_{j=1}^{m}\left\{[H(z+c_j)h_j(z)-H(z)]e^{\alpha z^k}\right\} = b + H_3^*(z)\exp\{\beta z^k\}.$$

By simple calculation, we can rewrite the above equation as the form

$$f_{n+1}(z)e^{(n+m)\alpha z^k} + \cdots + f_1(z)e^{\alpha z^k} - b = H_3^*(z)\exp\{\beta z^k\}, \tag{16}$$

where $f_{n+1}(z) = H(z)^n \prod_{j=1}^{m}[H(z+c_j)h_j(z)-H(z)] \not\equiv 0$ and

$$f_{i+1}(z) = C_n^i d^{n-i} H(z)^i \prod_{j=1}^{m}[H(z+c_j)h_j(z)-H(z)], \quad i = 0,1,\ldots,n.$$

Since $\rho(H) < \rho(f) = k, \rho(h_j) \le k-1$ and $\rho(H_3^*) < \rho(f) = k$, then we have $\rho(f_i) < k = \rho(e^{t\alpha z^k})$ for $i = 1,2,\ldots,n+1$. By comparing growths of both sides of (16), we have $\beta = (n+m)\alpha$. Thus, it follows from (13) that

$$[f_{n+1}(z) - H_3^*(z)]e^{(n+m)\alpha z^k} + \cdots + f_1(z)e^{\alpha z^k} = b, \tag{17}$$

By Lemma 2.1, we have $f_1(z) \equiv 0$, that is,

$$d^n \prod_{j=1}^{m}[H(z+c_j)h_j(z)-H(z)] \equiv 0,$$

which is a contradiction with the assumptions $f(z+c_j) \not\equiv f(z)$ for $j = 1,2,\ldots,m$. Hence, we have $\lambda(G_3 - b) = \rho(f)$.

This completes the proof of Theorem 1.9.

4 The proof of Theorem 1.10

We will take two case as follows into consideration by using the idea of Theorem 1 in [16].

Case 1. Suppose that $0 < \sigma(f) < \infty$. We assume that there exists $b \in \mathbb{C}$ such that $\Psi(z) - b$ has finitely many zeros only. Set

$$F(z) = \frac{f^{(k)}(z)\prod\limits_{j=1}^{m} f(z+c_j) - b}{af(z)^n}. \tag{18}$$

It follows from (18) that $T(r,F) \le (n+m+1)T(r,f) + S(r,f)$ and $F(z)$ has only finite 1-points, *i.e.*,

$$\overline{N}(r, \frac{1}{F-1}) = O(\log r). \tag{19}$$

Since $f(z)$ is entire, from (18), we have that the poles of $F(z)$ occur only at zeros of $f(z)$, and those poles which are not zeros of $f^{(k)}(z)\prod\limits_{j=1}^{m} f(z+c_j) - b$ having multiplicities $\ge n$

at the same time. Moreover, the zeros of $F(z)$ can only occur at zeros of $f^{(k)}(z)\prod_{j=1}^{m} f(z+c_j)-b$ which are not poles of $F(z)$. Thus, it follows by Lemma 2.3 and Lemma 2.9 that

$$\begin{aligned}\overline{N}(r,F)+\overline{N}(r,\frac{1}{F}) &\le \frac{1}{n}N(r,F)+\overline{N}(r,\frac{1}{f^{(k)}(z)\prod_{j=1}^{m} f(z+c_j)-b}) \\ &\le \frac{1}{n}T(r,F)+T(r,f^{(k)}(z)\prod_{j=1}^{m} f(z+c_j))+O(1) \\ &\le \frac{1}{n}T(r,F)+(m+1)T(r,f)+S(r,f).\end{aligned} \tag{20}$$

By the second fundamental theorem, it follows from (19) and (20) that

$$\begin{aligned}T(r,F) &\le \overline{N}(r,F)+\overline{N}(r,\frac{1}{F})+\overline{N}(r,\frac{1}{F-1})+S(r,F) \\ &\le \frac{1}{n}T(r,F)+(m+1)T(r,f)+S(r,f),\end{aligned}$$

i.e.,

$$(1-\frac{1}{n})T(r,F) \le (m+1)T(r,f)+S(r,f). \tag{21}$$

On the other hand, we have

$$\begin{aligned}nT(r,f)=T(r,f^n) &= T(r,\frac{f^{(k)}(z)\prod_{j=1}^{m} f(z+c_j)-b}{aF(z)}) \\ &\le T(r,f^{(k)}(z)\prod_{j=1}^{m} f(z+c_j))+T(r,F)+O(1) \\ &\le (m+1)T(r,f)+T(r,F)+S(r,f),\end{aligned}$$

i.e.,

$$(n-m-1)T(r,f) \le T(r,F)+S(r,f). \tag{22}$$

Thus, it follows from (21) and (22) that

$$(1-\frac{1}{n}-\frac{m+1}{n-m-1})T(r,F) \le S(r,f). \tag{23}$$

Since $f(z)$ is a transcendental entire function and $n \ge 2m+3$, from (23) we can deduce a contradiction. Hence, for any $b \in \mathbb{C}$, $\Psi(z)-b$ has infinitely many zeros.

Case 2. Suppose that $\sigma(f)=0$, then $\Psi(z)$ is also of zero order. We assume that $\Psi(z)$ is a polynomial, then

$$T(r,\Psi(z))=O(\log r). \tag{24}$$

Thus, it follows from (24) and Lemma 2.4 that

$$T(r,f^{(k)}(z)\prod_{j=0}^{m} f(z+c_j)-b)=T(r,\Psi(z)+af(z)^n)=nT(r,f)+S(r,f). \tag{25}$$

On the other hand, we have

$$\begin{aligned} T(r, f^{(k)}(z)\prod_{j=1}^{m} f(z+c_j)) &= m(r, f^{(k)}(z)\prod_{j=1}^{m} f(z+c_j)) \qquad (26) \\ &= m\left(r, f^{m+1}(z)\frac{f^{(k)}(z)\prod_{j=1}^{m} f(z+c_j)}{f^{m+1}(z)}\right) \\ &\leq (m+1)m(r,f) + m\left(r, \frac{f^{(k)}(z)}{f(z)}\right) + \sum_{j=1}^{m} m\left(r, \frac{f(z+c_j)}{f(z)}\right) \\ &\quad + O(1) + S(r,f) \\ &= (m+1)T(r,f) + S(r,f). \end{aligned}$$

From (25), (26) and $n \geq m+2$, we can deduce a contradiction with the assumption that f is transcendental.

Thus, it is easy to see that for any $b \in \mathbb{C}$, $\Psi(z) - b$ is a transcendental entire function with zero order and has infinitely many zeros.

Therefore, this completes the proof of Theorem 1.10.

Competing interests

The authors declare that they have no competing interests.

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On properties of decomposable measures and pseudo-integrals

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Abstract

In this paper, we mainly discuss two classes of σ-$\oplus$-decomposable measures and the corresponding pseudo-integrals: one is based on the generated pseudo-addition (g-case) and the other is based on the idempotent pseudo-operation (sup and inf). In particular, we obtained the correlation between the measure zero sets with respect to a σ-$\oplus$-decomposable measure and the corresponding pseudo-integrals on them. As an application of the main results, we generalized the classical Radon-Nikodym theorem to the decomposable measure theory based on pseudo-integrals.

Keywords: Pseudo-addition; Pseudo-multiplication; Pseudo-integral; Radon-Nikodym theorem

1 Introduction

Pseudo-analysis is a generalization of the classical analysis, where instead of the field of real numbers a semiring is taken on a real interval $[a, b] \subset [-\infty, +\infty]$ endowed with pseudo-addition $\oplus$ and with pseudo-multiplication $\odot$ (see [6, 22, 24, 25, 36]). Based on this structure there were developed the concepts of $\oplus$-measure (pseudo-additive measure), pseudo-integral, pseudo-convolution, pseudo-Laplace transform, etc. The advantage of the pseudo-analysis is that there are covered with one theory, and so with unified methods, problems (usually nonlinear and under uncertainty) from many different fields (system theory, optimization, decision making, control theory, differential equations, difference equations, etc.). Pseudo-analysis uses many mathematical tools from different field as functional equations, variational calculus, measure theory, functional analysis, optimization theory, semiring theory, etc.

Similar ideas were developed independently by Maslov and his collaborators in the framework of idempotent analysis and idempotent mathematics, with important applications [14, 15]. In particular, idempotent analysis is fundamental for the theory of weak solutions to Hamilton-Jacobi equations with non-smooth Hamiltonian, see [14, 15] and also [26, 27] (in the framework of pseudo-analysis). In some cases, this theory enables one to obtain exact solutions in the similar form as for the linear equations. Some further developments relate more general pseudo-operations with applications to nonlinear partial differential equations, see [29]. Recently, these applications have become important also in the field of image processing [27].

The classical measure theory is one of the most important theories in mathematics and based on countable additive measures [11, 40]. Although the additive measures are widely used, they do not allow modeling many phenomena involving interaction between criteria. For this reason, the fuzzy measure proposed by Sugeno as an extension of classical measure in which the additivity is replaced by a weaker condition, i.e., monotonicity [39]. So far, there have been many different fuzzy measures, such as the decomposable measure, the λ-additive measure, the belief measure,the possibility measure and the plausibility measure, etc. Among the fuzzy measure mentioned before, the decomposable measure was independently introduced by Dubois and Prade [8] and Weber [42], because of the close relation with the classical measure theory. Further developments of decomposable measures and related integrals have been extensive studied [6, 23, 31, 32, 33, 35]. Decomposable measures include several well-known fuzzy measures such as the λ-additive measure and probability and possibility measures, and they are a natural setting for relaxing probabilistic assumptions regarding the modeling of uncertainty [9, 34]. Decomposable measures and the corresponding integrals are very useful in decision theory and the theory of nonlinear differential and integral equations [26, 28, 30, 38].

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Based on the above these, the notions of σ-$\oplus$-decomposable measure (pseudo-additive measure) and corresponding integral (pseudo-integral) based on this measure were introduced [22, 24, 25, 27, 36]. Since integrals bases on non-additive measure have wide application, there were obtained generalizations of the classical integral inequalities for integrals with respect to non-additive measures, such as the ineqalities for Choqut and Sugeno integral were given in [2, 13, 18, 19, 41, 44] and the inequalities for the pseudo-integrals with respect to σ-$\oplus$-decomposable measure were considered in [1, 3, 4, 17, 20, 21]. In [37], Sugeno generalize the classical Radon-Nikodym derivatives for functions with respect to fuzzy measures.

In this paper, we will discuss two classes of σ-$\oplus$-decomposable measures and the corresponding pseudo-integrals: one is based on the generated pseudo-addition (g-case, see [16, 22]) and the other is based on the idempotent pseudo-operation (sup and inf, see [23, 39]). In Section 2, we recall the notions of pseudo-addition $\oplus$ and pseudo-multiplication $\odot$ forming a real semiring on the interval $[a, b] \subset [-\infty, +\infty]$. Then we will give the notion of σ-$\oplus$-decomposable measure and corresponding pseudo-integral based on this measure. In Section 3, we will discuss several important properties as monotonicity, continuous from above and continuous from below for σ-$\oplus$-decomposable measures, and will show the relationship between the measure zero sets with respect to a σ-$\oplus$-decomposable measure and the corresponding pseudo-integrals on them. Finally, we will generalize the classical Radon-Nikodym theorem to decomposable measure theory based on pseudo-integrals.

2 Preliminaries

Let $[a, b]$ be a closed subinterval of $[-\infty, \infty]$ (in some cases we will also take semiclosed subintervals). The total order on $[a, b]$ will be denoted by $\preceq$. This can be the usual order of the real line, but it can also be another order.

Definition 2.1 *[17] A binary operation $\oplus : [a, b] \times [a, b] \to [a, b]$ is called a pseudo-addition, if it satisfies the following conditions, for all $x, y, z, w \in [a, b]$:*

(1) $x \oplus y = y \oplus x$; (commutativity)
(2) $x \oplus z \preceq y \oplus w$ whenever $x \preceq y$ and $z \preceq w$; (monotonicity)
(3) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$; (associativity)
(4) $\mathbf{0} \oplus x = x$, where $\mathbf{0}$ is a zero element (usually $\mathbf{0}$ is either a or b). (boundary condition)

Let $[a, b]_+ = \{x \mid x \in [a, b], \mathbf{0} \preceq x\}$.

Definition 2.2 *[17] A binary operation $\odot : [a, b] \times [a, b] \to [a, b]$ is called a pseudo-multiplication, if it satisfies the following conditions, for all $x, y, z \in [a, b]$:*

(1) $x \odot y = y \odot x$; (commutativity)
(2) $x \odot z \preceq y \odot z$ whenever $x \preceq y$ and $z \in [a, b]_+$; (positively monotonicity)
(3) $(x \odot y) \odot z = x \odot (y \odot z)$; (associativity)
(4) $\mathbf{1} \odot x = x$, where $\mathbf{1} \in [a, b]$ is an unit element. (boundary condition)

We assume also $\mathbf{0} \odot x = \mathbf{0}$ and that $\odot$ is a distributive pseudo-multiplication with respect to $\oplus$, i.e.,

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z).$$

The structure $([a, b], \oplus, \odot)$ is called a real semiring [1, 20]. In this paper we will consider semirings with the following continuous operations:

Case 1: The pseudo-addition is an idempotent operation and the pseudo-multiplication is not.

(a) $x \oplus y = \sup(x, y)$, $\odot$ is an arbitrary non-idempotent pseudo-multiplication on the interval $[a, b]$.

We have $\mathbf{0} = a$ and the idempotent operation sup induces a total order in the following way: $x \preceq y$ if and only if $\sup(x, y) = y$. In order to keep the semiring structure, $\odot$ has to be pseudo-multiplication of the first type, i.e., $a \odot b = a$ and then $a \neq \mathbf{1}$. Special important case is when this pseudo-multiplication can be represented by a strictly increasing and continuous generator surjective function $g : [a, b] \to [0, \infty]$, i.e., $\odot$ is given with

$$x \odot y = g^{-1}(g(x) \cdot g(y)),$$

such that $g(\mathbf{0}) = g(a) = 0$, with the usual convention $0 \cdot \infty = 0$.

(b) $x \oplus y = \inf(x, y)$, $\odot$ is an arbitrary non-idempotent pseudo-multiplication on the interval $[a, b]$.

We have $\mathbf{0} = b$ and the idempotent operation inf induces a total order in the following way: $x \preceq y$ if and only if $\inf(x, y) = y$. In order to keep the semiring structure, $\odot$ has to be pseudo-multiplication of the second type, i.e., $a \odot b = b$ and then $b \neq \mathbf{1}$. Special important case is when this pseudo-multiplication can be represented by a strictly decreasing and continuous generator surjective function $g : [a, b] \to [0, \infty]$, i.e., $\odot$ is given with

$$x \odot y = g^{-1}(g(x) \cdot g(y)),$$

such that $g(\mathbf{0}) = g(b) = 0$.

Case 2: The pseudo-operations are defined by a strictly monotone and continuous generator surjective function $g : [a, b] \to [0, \infty]$, i.e., pseudo-operations are given with

$$x \oplus y = g^{-1}(g(x) + g(y)) \text{ and } x \odot y = g^{-1}(g(x) \cdot g(y)),$$

such that $g(\mathbf{0}) = 0$.

If the zero element for the pseudo-addition is a, we will consider increasing generators. Then $g(a) = 0$ and $g(b) = \infty$. If the zero element for the pseudo-addition is b, we will consider decreasing generators. Then $g(b) = 0$ and $g(a) = \infty$. If the generator g is increasing (decreasing), then the operation $\oplus$ induces the usual order (opposite to the usual order) on the interval $[a, b]$ in the following way: $x \preceq y$ if and only if $g(x) \leq g(y)$.

Case 3: Both operations are idempotent. We have

(a) $x \oplus y = \sup(x, y)$, $x \odot y = \inf(x, y)$, on the interval $[a, b]$. We have $\mathbf{0} = a$ and $\mathbf{1} = b$. The idempotent operation sup induces the usual order ($x \preceq y$ if and only if $\sup(x, y) = y$).

(b) $x \oplus y = \inf(x, y)$, $x \odot y = \sup(x, y)$, on the interval $[a, b]$. We have $\mathbf{0} = b$ and $\mathbf{1} = a$. The idempotent operation inf induces the usual order ($x \preceq y$ if and only if $\inf(x, y) = y$).

Let X be a non-empty set, we shall denote by $\mathscr{A}$ and $\mathcal{B}$ are σ-algebra and Borel σ-algebra of subsets of a set X, respectively.

Definition 2.3 *[3] A set function $m : \mathscr{A} \to [a, b]_+$ (or semiclosed interval) is called a σ-$\oplus$-decomposable measure if it satisfies the following conditions:*

(1) $m(\emptyset) = \mathbf{0}$;

(2) $m(\bigcup\limits_{i=1}^{\infty} E_i) = \bigoplus\limits_{i=1}^{\infty} m(E_i)$ for any sequence $\{E_i\}_{i \in \mathbb{N}}$ of pairwise disjoint sets from $\mathscr{A}$, where $\bigoplus\limits_{i=1}^{\infty} x_i = \lim\limits_{n \to \infty} \bigoplus\limits_{i=1}^{n} x_i$ for all $\{x_i\} \subset [a, b]$.

A σ-$\oplus$-decomposable measure m also is called σ-sup-decomposable measure if $x \oplus y = \sup(x, y)$ on the interval $[a, b]$. A set E is called σ-$\oplus$-decomposable measure zero set if $m(E) = \mathbf{0}$. It is obvious that $\emptyset$ is σ-$\oplus$-decomposable measure zero set.

We notice that if m is a σ-$\oplus$-decomposable measure, where $\oplus$ has a generator g, then $\mu = g \circ m$ is a σ-additive measure, and if a set E is σ-$\oplus$-decomposable measure zero set, then E is a measure zero set with respect to μ. In fact, we have that

(1) $\mu(\emptyset) = g(m(\emptyset)) = g(\mathbf{0}) = 0$;

(2) $\mu(\bigcup\limits_{i=1}^{\infty} E_i) = g\left(m(\bigcup\limits_{i=1}^{\infty} E_i)\right) = g(\bigoplus\limits_{i=1}^{\infty} m(E_i)) = \sum\limits_{i=1}^{\infty} g(m(E_i)) = \sum\limits_{i=1}^{\infty} \mu(E_i)$ for any sequence $\{E_i\}_{i \in \mathbb{N}}$ of pairwise disjoint sets from $\mathscr{A}$;

(3) If E is a σ-$\oplus$-decomposable measure zero set, then $\mu(E) = g(m(E)) = g(\mathbf{0}) = 0$.

We call that m is g-finite, σ-g-finite, totally g-finite, totally σ-g-finite and g-complete, if $\mu = g \circ m$ is finite, σ-finite, totally finite, totally σ-finite and complete (see [11]), respectively.

Definition 2.4 *If $(X, \mathscr{A})$ is a measurable space and m, ν are two σ-$\oplus$-decomposable measure on $\mathscr{A}$. ν is called absolutely $\oplus$-continuous with respect to m, if $\nu(E) = \mathbf{0}$ for every measurable set E for which $m(E) = \mathbf{0}$.*

It should be noted that if ν is absolutely $\oplus$-continuous with respect to m, where $\oplus$ has a generator g, then $g \circ \nu$ is absolutely continuous with respect to $g \circ m$ (see [11]).

Let f and h be two functions defined on X and with values in $[a,b]$. Then, for any $x \in X$ and $\lambda \in [a,b]$ we define

$$(f \oplus h)(x) = f(x) \oplus h(x),$$
$$(f \odot h)(x) = f(x) \odot h(x),$$
$$(\lambda \odot f)(x) = \lambda \odot f(x).$$

Definition 2.5 *[4] The pseudo-characteristic function of a set E is defined with:*

$$\chi_E(x) = \begin{cases} \mathbf{0}, & x \notin E, \\ \mathbf{1}, & x \in E, \end{cases}$$

where $\mathbf{0}$ is zero element for $\oplus$ and $\mathbf{1}$ is unit element for $\odot$.

Definition 2.6 *A set function $m : \mathscr{A} \to [a,b]$ (or semiclosed interval) is monotone if*

$$m(E) \preceq m(F)$$

whenever $E, F \in \mathscr{A}$ and $E \subset F$.

Denote by μ the usual Lebesgue measure on $\mathbb{R}$. We have

$$m(E) = ess \sup_{\mu}\{x|x \in E\} = \sup\{a|\mu(\{x|x \in E, x > a\}) > 0\}.$$

Further, m is a σ-sup-decomposable measure [17]. More, Mesiar and Pap(see [17]) have showed that any σ-sup-decomposable measure generated as essential supremum of a continuous density can be obtained as a limit of pseudo-additive measures with respect to generted pseudo-addition.

In this paper we will consider the semiring $([a,b], \oplus, \odot)$ for three (with completely different behavior) cases, namely Case 1(a), 2 and 3(a). Observe that the Case 1(b) and 3(b) are linked to Case 1(a) and 3(a) by duality [21].

First, if the pseudo-operations are defined by a monotone and continuous surjective function $g : [a,b] \to [0,\infty]$ (i.e., Case 2), then the pseudo-integral for a measurable function $f : X \to [a,b]$ is given by

$$\int_X^{\oplus} f \odot dm = g^{-1}\left(\int_X (g \circ f)\, d\,(g \circ m)\right),$$

where the integral applied on the right side is the standard Lebesgue integral. In a special case, when $X = [c,d], \mathscr{A} = \mathcal{B}(X)$ and $m = g^{-1} \circ \mu$, then the pseudo-integral reduces on the g-integral

$$\int_{[c,d]}^{\oplus} f dx = g^{-1}\left(\int_c^d g(f(x))dx\right).$$

Second, if the semiring is of the form $([a,b], \sup, \odot)$ (i.e., Case 1(a) and Case 3(a)), then we shall consider complete sup-measure (shortly sup-measure) m only and $\mathscr{A} = 2^X$, i.e., for any family $\{E_i\}_{i \in I}$ of measurable sets,

$$m(\bigcup_{i \in I} E_i) = \sup_{i \in I} m(E_i).$$

If X is countable (especially, if X is finite) then any σ-sup-decomposable measure m is complete and, moreover, $m(E) = \sup_{x \in E} \psi(x)$, where $\psi : X \to [a,b]$ is a density function given by $\psi(x) = m(\{x\})$. Then the pseudo-integral for a function $f : X \to [a,b]$ is given by

$$\int_X^{\oplus} f \odot dm = \sup_{x \in X}(f(x) \odot \psi(x)),$$

where function ψ defines σ-sup-decomposable measure m.

3 Main results

Theorem 3.1 *A σ-$\oplus$-decomposable measure $m : \mathscr{A} \to [a,b]$ is monotone, if $\oplus$ satisfies one of the following conditions:*
(1) $x \oplus y = \sup(x,y)$ on the interval $[a,b]$;
(2) $\oplus$ has a strictly monotone and continuous surjective generator g.

Proof. If $E, F \in \mathscr{A}$ and $E \subset F$, then $F - E \in \mathscr{A}$. Since $F = E \cup (F - E)$, we get

$$m(F) = m(F - E) \oplus m(E).$$

If $\oplus$ satisfies Condition (1), then we have

$$m(F) = \sup\{m(F - E), m(E)\} \geq m(E),$$

i.e., $\sup\{m(E), m(F)\} = m(F)$. Hence, by $x \preceq y$ if and only if $\sup(x,y) = y$ for all $x, y \in [a,b]$, we have

$$m(E) \preceq m(F).$$

If $\oplus$ satisfies Condition (2), then we have

$$g(m(F)) = g(m(F - E)) + g(m(E)).$$

Since $g(x) \geq 0$ for all $x \in [a,b]$, we have $g(m(F)) \geq g(m(E))$. Hence, by $x \preceq y$ if and only if $g(x) \leq g(y)$ for all $x, y \in [a,b]$, we have

$$m(E) \preceq m(F).$$

□

It is easy to see that if F is a σ-$\oplus$-decomposable measure zero set with respect to m, where m is a σ-$\oplus$-decomposable measure, then E is a σ-$\oplus$-decomposable measure zero set with respect to m, for all $E \subset F$.

Theorem 3.2 *Let $(X, \mathscr{A})$ be a measurable space. If $m : \mathscr{A} \to [a,b]$ is a σ-$\oplus$-decomposable measure and $\{E_n\} \subset \mathscr{A}(X)$ is an increasing sequence for which $\lim\limits_{n\to\infty} E_n \in \mathscr{A}(X)$, then*

$$m\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} m(E_n).$$

Proof. We might write $E_0 = \emptyset$. Since $\{E_n\}$ is an increasing sequence, $\{E_i - E_{i-1}\}$ is a sequence of pairwise disjoint sets from $\mathscr{A}$, then

$$E_n = \bigcup_{i=1}^{n} (E_i - E_{i-1}) \text{ and } \lim_{n\to\infty} E_n = \bigcup_{i=1}^{\infty} (E_i - E_{i-1}).$$

Hence, we have

$$m(E_n) = m\left(\bigcup_{i=1}^{n} (E_i - E_{i-1})\right) = \bigoplus_{i=1}^{n} m(E_i - E_{i-1})$$

and

$$m\left(\lim_{n\to\infty} E_n\right) = m\left(\bigcup_{i=1}^{\infty} (E_i - E_{i-1})\right) = \bigoplus_{i=1}^{\infty} m(E_i - E_{i-1}).$$

By $\bigoplus\limits_{i=1}^{\infty} x_i = \lim\limits_{n\to\infty} \bigoplus\limits_{i=1}^{n} x_i$ for all $\{x_i\} \subset [a,b]$, we have

$$\bigoplus_{i=1}^{\infty} m(E_i - E_{i-1}) = \lim_{n\to\infty} \bigoplus_{i=1}^{n} m(E_i - E_{i-1}).$$

Consequently, we get $m\left(\lim\limits_{n\to\infty} E_n\right) = \lim\limits_{n\to\infty} m(E_n)$. □

Theorem 3.3 *Let $(X, \mathscr{A})$ be a measurable space and $m : \mathscr{A} \to [a,b]$ be a σ-$\oplus$-decomposable measure, where $\oplus$ has a strictly increasing (or decreasing) and continuous surjective generator g. If $\{E_n\} \subset \mathscr{A}(X)$ is a decreasing sequence, and there exists at least one $l \in \mathbb{N}$ such that $m(E_l) \prec b$ (or $m(E_l) \prec a$). Then*

$$m\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} m\left(E_n\right).$$

Proof. Suppose $m(E_l) \prec b$ for some $l \in \mathbb{N}$. For the case $m(E_l) \prec a$, we can prove it by a similar proof. By Theorem 3.1 and $\{E_n\}$ is a decreasing sequence, we have

$$m(E_n) \preceq m(E_l) \prec b \text{ for all } n \geq l,$$

and therefore

$$m\left(\lim_{n\to\infty} E_n\right) \prec b.$$

By the monotonicity of g, we get

$$g(m(E_l)) < +\infty \text{ and } g\left(m\left(\lim_{n\to\infty} E_n\right)\right) < +\infty.$$

Since $\{E_n\}$ ia a decreasing sequence, $\{E_l - E_n\}$ is an increasing sequence. By Theorem 3.2, we obtain

$$m\left(E_l - \lim_{n\to\infty} E_n\right) = m\left(\lim_{n\to\infty}\left(E_l - E_n\right)\right) = \lim_{n\to\infty} m\left(E_l - E_n\right),$$

i.e.,

$$g\left(m\left(E_l - \lim_{n\to\infty} E_n\right)\right) = g\left(\lim_{n\to\infty} m\left(E_l - E_n\right)\right).$$

By the continuity of g and $g \circ m$ is a σ-additive measure, we have

$$\begin{aligned}
g\left(m\left(E_l\right)\right) - g\left(m\left(\lim_{n\to\infty} E_n\right)\right) &= g\left(m\left(E_l - \lim_{n\to\infty} E_n\right)\right) \\
&= g\left(\lim_{n\to\infty} m\left(E_l - E_n\right)\right) \\
&= \lim_{n\to\infty} g\left(m\left(E_l - E_n\right)\right) \\
&= g\left(m\left(E_l\right)\right) - \lim_{n\to\infty} g\left(m\left(E_n\right)\right) \\
&= g\left(m\left(E_l\right)\right) - g\left(\lim_{n\to\infty} m\left(E_n\right)\right).
\end{aligned}$$

Since $g(m(E_l)) < +\infty$, we have

$$g\left(m\left(\lim_{n\to\infty} E_n\right)\right) = g\left(\lim_{n\to\infty} m\left(E_n\right)\right).$$

By the strictly monotonicity of g, we get $m\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} m\left(E_n\right)$. $\square$

Lemma 3.1 *[17] Let m be a σ-*sup*-decomposable measure which is defined by*

$$m(E) = ess\sup_{\mu}\{\psi(x) | x \in E\},$$

on $([0,\infty], \mathcal{B})$, where $\psi : [0,\infty] \to [0,\infty]$ is a continuous density. Then for any generator g there exist a family $\{m_\lambda\}$ of σ-$\oplus_\lambda$-decomposable measures on $([0,\infty], \mathcal{B})$, where $\oplus_\lambda$ is generated by g^λ (the function g on the power λ), $\lambda \in (0,\infty)$, such that $\lim_{\lambda\to\infty} m_\lambda = m$.

Theorem 3.4 *Let $([0,\infty], \sup, \odot)$ be a semiring with $\odot$ generated by a strictly increasing and continuous surjective generator g. Let m be the same as in Lemma 3.1. If $\{E_n\} \subset \mathcal{B}([0,\infty])$ is a decreasing sequence, $m(E_n) \prec b$ for at least one $n \in \mathbb{N}$, then*

$$m\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} m\left(E_n\right).$$

Proof. Since m is the same one in Lemma 3.1, for the generator g there exist a family $\{m_\lambda\}$ of σ-$\oplus_\lambda$-decomposable measures on $([0,\infty], \mathcal{B})$, where $\oplus_\lambda$ is generated by g^λ, $\lambda \in (0,\infty)$, such that

$$\lim_{\lambda\to\infty} m_\lambda = m.$$

Let $l \in \mathbb{N}$ such that $m(E_l) \prec b$. Thus we have

$$m_\lambda(E_l) \prec b,\ \lambda \in (0,\infty).$$

Since $f(x) = x^\mu\ (\mu \neq 0)$ is strictly increasing, whenever $\mu > 0$, g^λ and g are comonotone functions. Hence, by Theorem 3.3, we have

$$m_\lambda\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} m_\lambda(E_n),\ \lambda \in (0,\infty).$$

Consequently, we obtain that

$$\lim_{\lambda\to\infty} m_\lambda\left(\lim_{n\to\infty} E_n\right) = \lim_{\lambda\to\infty}\lim_{n\to\infty} m_\lambda(E_n) = \lim_{n\to\infty}\lim_{\lambda\to\infty} m_\lambda(E_n),$$

i.e., $m\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} m(E_n)$. □

Theorem 3.5 *Let $(X,\oplus,\odot)$ be a semiring with generated pseudo-operations by a strictly monotone and continuous surjective generator g, and let $m : \mathscr{A}(X) \to [a,b]$ be a σ-$\oplus$-decomposable measure. If $f : X \to [a,b]$ is a measurable function with respect to m and E is a σ-$\oplus$-decomposable measure zero set on $\mathscr{A}(X)$, then*

$$\int_E^{\oplus} f \odot dm = \boldsymbol{0}.$$

Proof. If the pseudo-operations are generated by a strictly monotone and continuous surjective generator g, i.e., $x\oplus y = g^{-1}(g(x)+g(y))$ and $x\odot y = g^{-1}(g(x)\cdot g(y))$ for every $x,y \in [a,b]$. Then the pseudo-integral for a measurable function $f : X \to [a,b]$ is given by

$$\int_E^{\oplus} f \odot dm = g^{-1}\left(\int_E g\circ f d\mu\right),$$

for every measurable set E, where $\mu = g\circ m$ is a σ-additive measure. Since E is a set of σ-$\oplus$-decomposable measure zero on $\mathscr{A}(X)$ and $g(\mathbf{0}) = 0$, we have that E is a measure zero set with respect to μ. Hence, by Theorem **C** of § 25 of [11], we have $\int_E g\circ f d\mu = 0$. Consequently, by the strictly monotonicity of g, we obtain that $\int_E^{\oplus} f\odot dm = g^{-1}(0) = \mathbf{0}$. □

Example 3.1 *Let $X = \mathbb{R}$ and let the pseudo-addition be represented by a strictly increasing and continuous generator surjective function $g : [0,+\infty] \to [0,+\infty]$, which is defined by $g(x) = x$ for all $x \in [0,+\infty]$. It is easy to see that $x\oplus y = x+y$ and $x\odot y = x\cdot y$ for all $x,y \in [0,+\infty]$. Hence, we have $\boldsymbol{0} = 0, \boldsymbol{1} = 1$. We define a set function m on $\mathcal{B}(X)$ by*

$$m(E) = \mu(E)$$

for all $E \in \mathcal{B}(X)$, where μ is a Lebesgue measure. It is obvious that m satisfies (1) and (2) of Definition 2.3. Consequently, the set function m is a σ-$\oplus$-decomposable measure. Let E be the set of rational numbers. Since $\mu(E) = 0$, thus $m(E) = \boldsymbol{0}$, i.e., E is a set of σ-$\oplus$-decomposable measure zero on $\mathcal{B}(X)$. Hence, for any measurable function f we have

$$\int_E^{\oplus} f \odot dm = \boldsymbol{0}.$$

Theorem 3.6 *Let $(X,\sup,\odot)$ be a semiring with $\odot$ generated by a strictly increasing and continuous surjective generator g, and let $m : \mathscr{A}(X) \to [a,b]$ be a σ-$\sup$-decomposable measure which is defined by $m(E) = \sup_{x\in E}\psi(x)$, where $\psi : X \to [a,b]$ is a density function given by $\psi(x) = m(\{x\})$. If $f : X \to [a,b]$ is a measurable function with respect to m and E is a set of σ-$\sup$-decomposable measure zero on $\mathscr{A}(X)$, then*

$$\int_E^{\sup} f \odot dm = \boldsymbol{0}.$$

Proof. Let $m : \mathscr{A} \to [a,b]$ be a σ-sup-decomposable measure which is defined by $m(E) = \sup_{x\in E} \psi(x)$, where $\psi : X \to [a,b]$ is a density function given by $\psi(x) = m(\{x\})$. Then the pseudo-integral for a measurable function $f : X \to [a,b]$ is given by

$$\int_E^{\sup} f \odot dm = \sup_{x\in E}(f(x) \odot \psi(x)),$$

for every measurable set E, where function ψ defines σ-sup-decomposable measure m. Since $f(x) \preceq \sup_{x\in E} f(x)$ and $\psi(x) \preceq \sup_{x\in E} \psi(x) = m(E)$ for all $x \in E$, by the monotonicity of g, we have

$$g\left(f\left(x\right)\right) \leq g\left(\sup_{x\in E} f\left(x\right)\right) \text{ and } g\left(\psi\left(x\right)\right) \leq g\left(m\left(E\right)\right),$$

for all $x \in E$. Hence, by $g(x) \geq 0$ for all $x \in [a,b]$, we have

$$g\left(f\left(x\right)\right) \cdot g\left(\psi\left(x\right)\right) \leq g\left(\sup_{x\in E} f\left(x\right)\right) \cdot g\left(m\left(E\right)\right),$$

which implies that

$$g\left(g^{-1}\left(g\left(f\left(x\right)\right) \cdot g\left(\psi\left(x\right)\right)\right)\right) \leq g\left(g^{-1}\left(g\left(\sup_{x\in E} f\left(x\right)\right) \cdot g\left(m\left(E\right)\right)\right)\right),$$

for all $x \in E$. Since $\odot$ is generated by a generator g, i.e., $y \odot z = g^{-1}(g(y)g(z))$ for all $y, z \in [a,b]$, we get that

$$g\left(f(x) \odot \psi(x)\right) \leq g\left(\sup_{x\in E} f(x) \odot m(E)\right),$$

for all $x \in E$. Hence, we obtain that

$$f(x) \odot \psi(x) \preceq \sup_{x\in E} f(x) \odot m(E)$$

for all $x \in E$, which implies that

$$\sup_{x\in E}(f(x) \odot \psi(x)) \preceq \sup_{x\in E} f(x) \odot m(E).$$

Since E is a set of σ-sup-decomposable measure zero on $\mathscr{A}(X)$, we have $\sup_{x\in E} f(x) \odot m(E) = \mathbf{0}$. Consequently, we have $\sup_{x\in E}(f(x) \odot \psi(x)) \preceq \mathbf{0}$. By the monotonicity of g and $g(y) \geq 0$ for $y \in [a,b]$, we have $0 \leq g\left(\sup_{x\in E}(f(x) \odot \psi(x))\right) \leq g(\mathbf{0}) = 0$, i.e., $g\left(\sup_{x\in E}(f(x) \odot \psi(x))\right) = 0$, which implies that $\int_E^{\sup} f \odot dm = \mathbf{0}$. □

Theorem 3.7 *Let $(X, \sup, \inf)$ be a semiring, and let m be the same as in Theorem 3.6. If $f : X \to [a,b]$ is a measurable function with respect to m and E is a set of σ-sup-decomposable measure zero on $\mathscr{A}(X)$, then*

$$\int_E^{\sup} f \odot dm = \mathbf{0}.$$

Proof. Let m be the same as in Theorem 3.6. Then the pseudo-integral for a measurable function $f : X \to [a,b]$ is given by

$$\int_E^{\sup} f \odot dm = \sup_{x\in E}(f(x) \odot \psi(x)),$$

for every measurable set E, where function ψ defines σ-sup-decomposable measure m. Since $f(x) \preceq \sup_{x\in E} f(x)$ and $\psi(x) \preceq \sup_{x\in E} \psi(x) = m(E)$ for all $x \in E$, and $y \odot z = \inf(y,z)$ for all $y, z \in [a,b]$, we have

$$f(x) \odot \psi(x) = \inf\{f(x), \psi(x)\} \preceq \inf\left\{\sup_{x\in E} f(x), m(E)\right\} = \sup_{x\in E} f(x) \odot m(E),$$

for all $x \in E$, which implies that

$$\sup_{x\in E}(f(x) \odot \psi(x)) \preceq \sup_{x\in E} f(x) \odot m(E).$$

Since E is a set of σ-sup-decomposable measure zero on $\mathscr{A}(X)$, we have $\sup_{x\in E} f(x) \odot m(E) = \mathbf{0}$. Hence, we have $\sup_{x\in E}(f(x) \odot \psi(x)) \preceq \mathbf{0}$. By $y \preceq z$ if and only if $\sup(y, z) = z$ and $y \oplus z = \sup(y, z)$ for all $y, z \in [a, b]$, we obtain that

$$\sup_{x\in E}(f(x) \odot \psi(x)) \oplus \mathbf{0} = \mathbf{0}.$$

By (4) of Definition 2.1, we have $\sup_{x\in E}(f(x) \odot \psi(x)) = \mathbf{0}$, which implies that $\int_E^{\sup} f \odot dm = \mathbf{0}$. □

Theorem 3.8 *Let $(X, \oplus, \odot)$ be a semiring with generated pseudo-operations by a strictly monotone and continuous surjective generator g, and let $m : \mathscr{A}(X) \to [a, b]$ be a σ-$\oplus$-decomposable measure. If $f : X \to [a, b]$ is a measurable function and $\mathbf{0} \prec f$ a.e. on a measurable set E with respect to m, and if*

$$\int_E^{\oplus} f \odot dm = \mathbf{0},$$

then $m(E) = \mathbf{0}$.

Proof. If the pseudo-operations are generated by a strictly monotone and continuous surjective generator g, i.e., $y \oplus z = g^{-1}(g(y) + g(z))$ and $y \odot z = g^{-1}(g(y) \cdot g(z))$ for every $y, z \in [a, b]$. Then the pseudo-integral for a measurable function $f : X \to [a, b]$ is given by

$$\int_E^{\oplus} f \odot dm = g^{-1}\left(\int_E g \circ f d\mu\right),$$

for every measurable set E, where $\mu = g \circ m$ is a σ-additive measure.

Let $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \emptyset$, where $E_2 = \{x \in E \,|\, \mathbf{0} \prec f(x)\}$. By $y \preceq z$ if and only if $g(y) \le g(z)$ for all $y, z \in [a, b]$, the strictly monotonicity of g and $g(\mathbf{0}) = 0$, we obtain that $g(f(x)) > 0$ for all $x \in E_2$. Since $\mathbf{0} \prec f$ a.e. on a measurable set E with respect to m, we have $m(E_1) = \mathbf{0}$. Hence, by Theorem 3.5, we have $\int_{E_1}^{\oplus} f \odot dm = \mathbf{0}$, which implies that

$$\int_E^{\oplus} f \odot dm = \int_{E_1}^{\oplus} f \odot dm \oplus \int_{E_2}^{\oplus} f \odot dm = \int_{E_2}^{\oplus} f \odot dm.$$

If $\int_E^{\oplus} f \odot dm = \mathbf{0}$, then we get that

$$\int_{E_2}^{\oplus} f \odot dm = g^{-1}\left(\int_{E_2} g \circ f d\mu\right) = \mathbf{0},$$

i.e., $\int_{E_2} g \circ f d\mu = g(\mathbf{0}) = 0$. By Theorem **D** of § 25 of [11], we have $\mu(E_2) = 0$. Hence, we obtain that $m(E_2) = g^{-1}(\mu(E)) = \mathbf{0}$. Consequently, $m(E) = m(E_1) \oplus m(E_2) = \mathbf{0}$. □

Theorem 3.9 *Let $(X, \sup, \odot)$ be a semiring with $\odot$ generated by a strictly increasing and continuous surjective generator g, and let m be the same as in Theorem 3.6. If $f : X \to [a, b]$ is a measurable function and $\mathbf{0} \prec f$ a.e. on a measurable set E with respect to m, and if*

$$\int_E^{\sup} f \odot dm = \mathbf{0},$$

then $m(E) = \mathbf{0}$.

Proof. Let m be the same as in Theorem 3.6. Then the pseudo-integral for a measurable function $f : X \to [a, b]$ is given by

$$\int_E^{\sup} f \odot dm = \sup_{x \in E}(f(x) \odot \psi(x)),$$

for every measurable set E, where function ψ defines σ-sup-decomposable measure m. Let $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \emptyset$, where $E_2 = \{x \in E \,|\, \mathbf{0} \prec f(x)\}$. If $\mathbf{0} \prec f$ a.e. on a measurable set E with respect to m, then $m(E_1) = \mathbf{0}$, which implies that

$$m(E) = \sup\{m(E_1), m(E_2)\} = m(E_2).$$

If $\int_E^{\sup} f \odot dm = \mathbf{0}$, then by $g(\mathbf{0}) = 0$, we have $g\left(\int_E^{\sup} f \odot dm\right) = g\left(\sup_{x \in E}(f(x) \odot \psi(x))\right) = 0$. By the strictly monotonicity of g, we have

$$g(f(x) \odot \psi(x)) \leq g\left(\sup_{x \in E}(f(x) \odot \psi(x))\right) = 0,$$

for all $x \in E$. Since $g(y) \geq 0$ for all $y \in [a, b]$, we have $g(f(x) \odot \psi(x)) = 0$, for all $x \in E$. If $\odot$ is generated by a generator g, i.e., $y \odot z = g^{-1}(g(y)g(z))$ for all $y, z \in [a, b]$, then we get that

$$g(f(x)) \cdot g(\psi(x)) = g(f(x) \odot \psi(x)) = 0.$$

Since $\mathbf{0} \prec f(x)$ for all $x \in E_2$, we get $g(f(x)) > 0$ for all $x \in E_2$. Thus, we have $g(\psi(x)) = 0$, i.e., $\psi(x) = \mathbf{0}$ for all $x \in E_2 \subset E$. Consequently, we obtain taht $m(E_2) = \sup_{x \in E_2} \psi(x) = \mathbf{0}$, which implies that $m(E) = \mathbf{0}$. $\square$

Theorem 3.10 *Let $(X, \sup, \inf)$ be a semiring, and let m be the same as in Theorem 3.6. If $f : X \to [a, b]$ is a measurable function and $\mathbf{0} \prec f$ a.e. on a measurable set E with respect to m, and if*

$$\int_E^{\sup} f \odot dm = \mathbf{0},$$

then $m(E) = \mathbf{0}$.

Proof. Let m be the same as in Theorem 3.6. Then the pseudo-integral for a measurable function $f : X \to [a, b]$ is given by

$$\int_E^{\sup} f \odot dm = \sup_{x \in E}(f(x) \odot \psi(x)),$$

for every measurable set E, where function ψ defines σ-sup-decomposable measure m.

Let $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \emptyset$, where $E_2 = \{x \in E \,|\, \mathbf{0} \prec f(x)\}$. If $\mathbf{0} \prec f$ a.e. on a measurable set E with respect to m, then $m(E_1) = \mathbf{0}$. Hence, by Theorem 3.7, we have $\int_{E_1}^{\sup} f \odot dm = \mathbf{0}$, which implies that

$$\int_E^{\sup} f \odot dm = \int_{E_1}^{\sup} f \odot dm \oplus \int_{E_2}^{\sup} f \odot dm = \int_{E_2}^{\sup} f \odot dm.$$

If $\int_E^{\sup} f \odot dm = \mathbf{0}$, then we get that $\int_{E_2}^{\sup} f \odot dm = \sup_{x \in E_2}(f(x) \odot \psi(x)) = \mathbf{0}$. Hence, we obtain that

$$f(x) \odot \psi(x) \preceq \sup_{x \in E_2}(f(x) \odot \psi(x)) = \mathbf{0} = f(x) \odot \mathbf{0},$$

for all $x \in E_2$. By $\mathbf{0} \prec f(x)$ for all $x \in E_2$ and (2) of Definition 2.2, we have $\psi(x) \preceq \mathbf{0}$ for all $x \in E_2$, which implies that $m(E_2) = \sup_{x \in E_2} \psi(x) \preceq \mathbf{0}$. Since $f(x) \preceq \sup_{x \in E_2} f(x)$ and $\psi(x) \preceq \sup_{x \in E_2} \psi(x) = m(E_2)$ for all $x \in E_2$, and $y \odot z = \inf(y, z)$ for all $y, z \in [a, b]$, we have

$$f(x) \odot \psi(x) = \inf\{f(x), \psi(x)\} \preceq \inf\left\{\sup_{x \in E_2} f(x), m(E_2)\right\} = \sup_{x \in E_2} f(x) \odot m(E_2),$$

for all $x \in E_2$, which implies that

$$\sup_{x\in E_2} f(x)\odot \mathbf{0} = \mathbf{0} = \sup_{x\in E_2}(f(x)\odot\psi(x)) \preceq \sup_{x\in E_2} f(x)\odot m(E_2).$$

By $\mathbf{0}\prec f(x)$ for all $x\in E_2$ and (2) of Definition 2.2, we have $\mathbf{0}\preceq m(E_2)$. Consequently, we obtain that $m(E_2)=\mathbf{0}$, which implies that $m(E)=m(E_1)\oplus m(E_2)=\mathbf{0}$. □

Theorem 3.11 *Let $(X,\oplus,\odot)$ be a semiring with generated pseudo-operations by a strictly monotone and continuous surjective generator g, and let m be a σ-$\oplus$-decomposable measure on $\mathscr{A}(X)$ and f be a measurable function with respect to m on X. Then $\int_E^{\oplus} f\odot dm = \boldsymbol{0}$ if and only if $f=\boldsymbol{0}$ a.e. for every measurable set E.*

Proof. If the pseudo-operations are generated by a strictly monotone and continuous surjective generator g, i.e., we have

$$y\oplus z = g^{-1}(g(y)+g(z)) \text{ and } x\odot y = g^{-1}(g(y)\cdot g(z))$$

for every $y,z\in[a,b]$. Then the pseudo-integral for a measurable function $f: X\to[a,b]$ is given by

$$\int_E^{\oplus} f\odot dm = g^{-1}\left(\int_E g\circ f d\mu\right)$$

for every measurable set E, where $\mu=g\circ m$ is a σ-additive measure.

Suppose $\int_E^{\oplus} f\odot dm=\mathbf{0}$ for every measurable set E. Since $g(\mathbf{0})=0$, we have $\int_E g\circ f d\mu = g(\mathbf{0})=0$ for every measurable set E. By Theorem **E** of § 25 of [11], we have $g\circ f=0$ a.e. for every measurable set E, which implies that $f=\mathbf{0}$ a.e. for every measurable set E.

Suppose $f=\mathbf{0}$ a.e. for every measurable set E. Let $E_1\cup E_2=E$ and $E_1\cap E_2=\emptyset$, where $E_1=\{x\in E|f(x)=\mathbf{0}\}$. Then, we have $m(E_2)=\mathbf{0}$. By Theorem 3.5, we have $\int_{E_2}^{\oplus} f\odot dm=\mathbf{0}$. Hence, we get that

$$\int_E^{\oplus} f\odot dm = \int_{E_1}^{\oplus} f\odot dm \oplus \int_{E_2}^{\oplus} f\odot dm = \int_{E_1}^{\oplus} f\odot dm.$$

Since $f(x)=\mathbf{0}$ for all $x\in E_1$, we have $g(f(x))=0$ for all $x\in E_1$. Hence, we obtain that $\int_{E_1} g\circ f d\mu=0$, i.e., $\int_{E_1}^{\oplus} f\odot dm = g^{-1}(0)=\mathbf{0}$. Consequently, $\int_E^{\oplus} f\odot dm = g^{-1}(0)=\mathbf{0}$. □

Theorem 3.12 *Let $(X,\sup,\odot)$ be a semiring with $\odot$ generated by a strictly increasing and continuous surjective generator g, and let m be the same as in Theorem 3.6 and f be a measurable function with respect to m on X. Then $\int_E^{\sup} f\odot dm=\boldsymbol{0}$ if and only if $f=\boldsymbol{0}$ a.e. for every measurable set E.*

Proof. Let m be the same as in Theorem 3.6. Then the pseudo-integral for a measurable function $f: X\to[a,b]$ is given by

$$\int_E^{\sup} f\odot dm = \sup_{x\in E}(f(x)\odot\psi(x)),$$

for every measurable set E, where function ψ defines σ-sup-decomposable measure m.

Suppose $\int_E^{\sup} f\odot dm=\mathbf{0}$ for every measurable set E. By $g(\mathbf{0})=0$, we have

$$g\left(\int_E^{\sup} f\odot dm\right) = g\left(\sup_{x\in E}(f(x)\odot\psi(x))\right)=0.$$

By the strictly monotonicity of g, we have

$$g(f(x) \odot \psi(x)) \leq g\left(\sup_{x\in E}(f(x) \odot \psi(x))\right) = 0,$$

for all $x \in E$. Since $g(y) \geq 0$ for all $y \in [a,b]$, we have $g(f(x) \odot \psi(x)) = 0$, for all $x \in E$. Since $\odot$ is generated by a generator g, i.e., $y \odot z = g^{-1}(g(y)g(z))$ for all $y, z \in [a,b]$, we get that

$$g(f(x)) \cdot g(\psi(x)) = g(f(x) \odot \psi(x)) = 0,$$

for all $x \in E$. Let $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \emptyset$, where $E_1 = \{x \in E | f(x) = \mathbf{0}\}$. Then, we have $f(x) \neq \mathbf{0}$ for all $x \in E_2$, which implies that $g(f(x)) \neq 0$ for all $x \in E_2$. Hence, we have $g(\psi(x)) = 0$, i.e., $\psi(x) = \mathbf{0}$ for all $x \in E_2 \subset E$, which implies that $m(E_2) = \sup_{x\in E_2} \psi(x) = \mathbf{0}$. Hence, $f = \mathbf{0}$ a.e. for every measurable set E.

Suppose $f = \mathbf{0}$ a.e. for every measurable set E. Let $E_1 \cup E_2 = E$ and $E_1 \cap E_2 = \emptyset$, where $E_1 = \{x \in E | f(x) = \mathbf{0}\}$, then $m(E_2) = \mathbf{0}$. By Theorem 3.6, we have $\int_{E_2}^{\sup} f \odot dm = \mathbf{0}$. Hence, we get that

$$\int_{E}^{\sup} f \odot dm = \int_{E_1}^{\sup} f \odot dm \oplus \int_{E_2}^{\sup} f \odot dm = \int_{E_1}^{\sup} f \odot dm.$$

Since $f(x) = \mathbf{0}$ for all $x \in E_1$, we have $f(x) \odot \psi(x) = \mathbf{0}$ for all $x \in E_1$. Hence, we obtain that $\int_{E_1}^{\sup} f \odot dm = \sup_{x\in E_1} (f(x) \odot \psi(x)) = \mathbf{0}$, which implies that $\int_{E}^{\sup} f \odot dm = \mathbf{0}$. □

Theorem 3.13 *Let $(X, \sup, \inf)$ be a semiring, and let m be the same as in Theorem 3.6 and f be a measurable function with respect to m and $\mathbf{0} \preceq f$. Then $\int_{E}^{\sup} f \odot dm = \mathbf{0}$ if and only if $f = \mathbf{0}$ a.e. for every measurable set E.*

Proof. Let m be the same as in Theorem 3.6. Then the pseudo-integral for a measurable function $f : X \to [a,b]$ is given by

$$\int_{E}^{\sup} f \odot dm = \sup_{x\in E}(f(x) \odot \psi(x)),$$

for every measurable set E, where function ψ defines σ-sup-decomposable measure m. For every measurable set E, let $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \emptyset$, where $E_1 = \{x \in E | f(x) = \mathbf{0}\}$.

Suppose $\int_{E}^{\sup} f \odot dm = \mathbf{0}$ for every measurable set E. By the proof of Theorem 3.10, we have $m(E_2) = \mathbf{0}$. Hence, $f = \mathbf{0}$ a.e. for every measurable set E.

Suppose $f = \mathbf{0}$ a.e. for every measurable set E, then $m(E_2) = \mathbf{0}$. By Theorem 3.7, we have $\int_{E_2}^{\sup} f \odot dm = \mathbf{0}$. Hence, we get that

$$\int_{E}^{\sup} f \odot dm = \int_{E_1}^{\sup} f \odot dm \oplus \int_{E_2}^{\sup} f \odot dm = \int_{E_1}^{\sup} f \odot dm.$$

Since $f(x) = \mathbf{0}$ for all $x \in E_1$, we have $f(x) \odot \psi(x) = \mathbf{0}$ for all $x \in E_1$. Hence, we obtain that $\int_{E_1}^{\sup} f \odot dm = \sup_{x\in E_1} (f(x) \odot \psi(x)) = \mathbf{0}$, which implies that $\int_{E}^{\sup} f \odot dm = \mathbf{0}$. □

Theorem 3.14 *Let $(X, \oplus, \odot)$ be a semiring with generated pseudo-operations by a strictly monotone and continuous surjective generator $g : [a,b] \to [0, +\infty]$, and let $m : \mathscr{A}(X) \to [a,b]$ be a σ-$\oplus$-decomposable measure. For a measurable function $f : X \to [a,b]$ with respect to m, define the set function $\nu : X \to [a,b]$ by*

$$\nu(E) = \int_{E}^{\oplus} f \odot dm,$$

for any measurable set E. Then ν is a σ-$\oplus$-decomposable measure and absolutely $\oplus$-continuous with respect to m.

Proof. If the pseudo-operations are generated by a strictly monotone and continuous surjective generator $g : [a, b] \to [0, +\infty]$, i.e., $y \oplus z = g^{-1}(g(y) + g(z))$ and $y \odot z = g^{-1}(g(y) \cdot g(z))$ for every $y, z \in [a, b]$. Then the pseudo-integral for a measurable function $f : X \to [a, b]$ is given by

$$\nu(E) = \int_E^{\oplus} f \odot dm = g^{-1}\left(\int_E g \circ f d\mu\right),$$

for every measurable set E, where $\mu = g \circ m$ is a σ-additive measure. Hence, ν is a set function of $\mathscr{A}$ to [a,b].

(1) By $\emptyset$ is σ-$\oplus$-decomposable measure zero set and Theorem 3.5, we have

$$\nu(\emptyset) = \int_{\emptyset}^{\oplus} f \odot dm = \mathbf{0};$$

(2) For any sequence $\{E_i\}_{i\in\mathbb{N}}$ of pairwise disjoint sets from $\mathscr{A}$, we have

$$\begin{aligned}
\nu\left(\bigcup_{i=1}^{\infty} E_i\right) &= \int_{\bigcup_{i=1}^{\infty} E_i}^{\oplus} f \odot dm = g^{-1}\left(\int_{\bigcup_{i=1}^{\infty} E_i} g \circ f d\mu\right) \\
&= g^{-1}\left(\sum_{i=1}^{\infty} \int_{E_i} g \circ f d\mu\right) = g^{-1}\left(\sum_{i=1}^{\infty} g\left(\int_{E_i}^{\oplus} f \odot dm\right)\right) \\
&= \bigoplus_{i=1}^{\infty} \int_{E_i}^{\oplus} f \odot dm = \bigoplus_{i=1}^{\infty} \nu(E_i).
\end{aligned}$$

Consequently, ν is a σ-$\oplus$-decomposable measure. By Theorem 3.5, we obtain that ν is absolutely $\oplus$-continuous with respect to m. □

Theorem 3.15 *Let $(X, \oplus, \odot)$ be a semiring with generated pseudo-operations by a strictly monotone and continuous surjective generator g, and let $m : \mathscr{A}(X) \to [a, b]$ be a totally σ-g-finite-decomposable measure. If the σ-g-finite-decomposable measure ν is absolutely $\oplus$-continuous with respect to m, then there exists a measurable function f on X and $g(f(x)) < +\infty$ for all $x \in X$, such that*

$$\nu(E) = \int_E^{\oplus} f \odot dm,$$

for every measurable set E. The function f is unique in the sense that if also $\nu(E) = \int_E^{\oplus} h \odot dm$, then $f = h$ a.e. with respect to m.

Proof. If the pseudo-operations are generated by a strictly monotone and continuous surjective generator g, i.e.,

$$y \oplus z = g^{-1}(g(y) + g(z)) \text{ and } x \odot y = g^{-1}(g(y) \cdot g(z))$$

for every $y, z \in [a, b]$. Then the pseudo-integral for a measurable function $f : X \to [a, b]$ is given by

$$\int_E^{\oplus} f \odot dm = g^{-1}\left(\int_E g \circ f d\mu\right)$$

for every measurable set E, where $\mu = g \circ m$ is a σ-additive measure.

Since m is a totally σ-g-finite-decomposable measure, we have μ is a totally σ-finite measure. If the σ-g-finite-decomposable measure ν is absolutely continuous with respect to m, then $g \circ \nu$ is σ-finite measure and absolutely $\oplus$-continuous with respect to μ. Hence, by Theorem **B** of § 31 of [11], there exists a finite valued measurable function p with respect to μ on X such that

$$g(\nu(E)) = \int_E p d\mu,$$

for every measurable set E. By the strictly monotonicity of g, we have $p(x) = (g(g^{-1}(p(x))))$ for all $x \in X$. Let $f = g^{-1} \circ p$, then

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$$g(\nu(E)) = \int_E p d\mu = \int_E g \circ f d\mu,$$

for all measurable set E, which implies that

$$\nu(E) = g^{-1}\left(\int_E g \circ f d\mu\right) = \int_E^{\oplus} f \odot dm,$$

for every measurable set E. If $\nu(E) = \int_E^{\oplus} h \odot dm$, then $g(\nu(E)) = \int_E g \circ h d\mu$. Hence $g \circ f = g \circ h$ a.e. with respect to μ, i.e., $f = h$ a.e. with respect to m. □

4 Conclusions

In this paper, we mainly discussed two classes of σ-$\oplus$-decomposable measures and the corresponding pseudo-integrals: one is based on the generated pseudo-addition (g-case, see [16, 22]) and the other is based on the idempotent pseudo-operation (sup and inf, see [23, 39]). We got several properties as monotonicity, continuous from above and continuous from below for σ-$\oplus$-decomposable measures. In particular, we obtained the correlation between the measure zero sets with respect to a σ-$\oplus$-decomposable measure and the corresponding pseudo-integrals on them. As an application of the main results, we generalized the classical Radon-Nikodym theorem, which has been extensively studied and discussed [5, 7, 10, 12, 43], to the decomposable measure theory based on pseudo-integrals. We also hope that our results in this paper may lead to significant, new and innovative results in other related fields.

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COMPOSITION OPERATOR ON ZYGMUND-ORLICZ SPACE

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Abstract. In this paper, we use Young's function to define Zygmund-Orlicz space. We study boundedness and compactness of composition operator on Zygmund-Orlicz space.

1. Introduction

Let $\mathbb{D}$ be the unit disk of the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the space of all holomorphic function on $\mathbb{D}$. Let μ be a bounded, continuous and positive function denfined on $\mathbb{D}$. A function $f \in H(\mathbb{D})$ belongs to $\mu-$Zygmund space , denoted as $f \in Z^\mu$, if

$$\|f\|_\mu := \sup_{z\in\mathbb{D}} \mu(z)|f''(z)| < \infty.$$

Clearly, if $\mu(z) = 1 - |z|^2$, the space Z^μ is just the Zygmund space, which is denoted by Z, while when $\mu(z) = (1 - |z|^2)^\alpha$ with $\alpha > 0$, the space Z^μ becomes the α-Zygmund space which is denoted by Z^α. It is readily seen that Z^μ is a Banach space with the norm

$$\|f\|_{Z^\mu} = |f(0)| + |f'(0)| + \|f\|_\mu.$$

For some more information of μ-Zygmund space on the unit disk see [3], while for composition and integral-type operators between them on the unit disk see for example [4, 6, 7, 8, 9].

Let A_1, A_2 be two linear subspaces of $H(\mathbb{D})$. If ϕ is a holomorphic self-map of $\mathbb{D}$, such that $f \circ \phi$ belongs to A_2 for all $f \in A_1$, then ϕ induces a linear operator $C_\phi : A_1 \to A_2$ defined as

$$C_\phi f = f \circ \phi,$$

called the composition operator with symbol ϕ. This type of operator appears in studies on isometries of various function spaces. Composition operator has been studied by numerous authors on many subspaces of $H(\mathbb{D})$ and in paticular on Zygmund spaces and μ-Zygmund spaces. In [5], Julio C. Ramos Fernández characterized boundedness and compactness of composition operators on Bloch-Orlicz spaces denoted by $\mathcal{B}^\varphi$, where φ is Young's function. More precisely, let $\varphi : [0, \infty) \to [0, \infty)$ be a strictly increasing convex function such that $\varphi(0) = 0$ and $\lim\limits_{t\to\infty} \frac{t}{\varphi(t)} = \lim\limits_{t\to 0+} \frac{\varphi(t)}{t} = 0$,

$$\mathcal{B}^\varphi = \{f \in H(D) : \sup_{z\in D}(1 - |z|^2)\varphi(\lambda|f'(z)|) < \infty\}$$

for some $\lambda > 0$ depending on f.

It is easy to see that $\mathcal{B}^\varphi$ is a Banach space with the norm

$$\|f\|_{\mathcal{B}^\varphi} = |f(0)| + \|f\|_\varphi,$$

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where $\|f\|_\varphi = \inf\{k > 0 : S_\varphi(\frac{f'}{k}) \le 1\}$ is a Minkowski's functional and $S_\varphi(f) = \sup_{z\in\mathbb{D}}(1-|z|^2)\varphi(|f(z)|)$.

This paper is organized as follows: In section 2, we use Young's function to define the Zygmund-Orlicz space, as a generalization of Zygmund space. The spaces is defined in a similar way as Korenblem-Orlicz space and Bloch-Orlicz space in [1, 2, 5]. We study some of its properties and show that the Zygmund-Orlicz space is isometrically equal to certain μ-Zygmund space for a very special weight μ. In section 3, we characterize boundedness and compactness of composition operator on Zygmund-Orlicz space.

Throughout the rest of this paper, C will denote a finite positive constant, and it may differ from one occurrence to the other.

2. Zygmund-Orlicz spaces

In this section, we define the Zygmund-Orlicz space Z^φ using Young's function. More precisely, Z^φ is the class of all analytic functions f in $\mathbb{D}$ such that

$$\sup_{z\in D}(1-|z|^2)\varphi(\lambda|f''(z)|) < \infty$$

for some $\lambda > 0$ depending on f. The set Z^φ is an F-space which we call Zygmund-Orlicz space associated to the function φ. We can observe that when $\varphi(t) = t$ with $t \ge 0$, we get back the Zygmund spaces Z.

It is not hard to see that

$$\|f\|_\varphi = \inf\{k > 0 : S_\varphi(\frac{f''}{k}) \le 1\}$$

define a seminorm for Z^φ, where $S_\varphi(f) = \sup_{z\in\mathbb{D}}(1-|z|^2)\varphi(|f(z)|)$.

In fact, it can be show that Z^φ is a Banach space with the norm

$$\|f\|_{Z^\varphi} = |f(0)| + |f'(0)| + \|f\|_\varphi. \tag{2.1}$$

Also, we can observe that for any $f \in Z^\varphi \setminus \{0\}$, the following relation

$$S_\varphi(\frac{f''}{\|f\|_{Z^\varphi}}) \le S_\varphi(\frac{f''}{\|f\|_\varphi}) \le 1 \tag{2.2}$$

holds. The inequality above allow us to obtain that

$$|f''(z)| \le \varphi^{-1}(\frac{1}{1-|z|^2})\|f\|_\varphi$$

for all $f \in Z^\varphi$ and for all $z \in \mathbb{D}$. Furthermore, we have

$$|f'(z)| \le |f'(0)| + |\int_0^{|z|} |f''(\zeta)||d\zeta| \le (1 + \int_0^1 \varphi^{-1}(\frac{1}{1-|z|^2t^2})dt)\|f\|_{Z^\varphi}. \tag{2.3}$$

Lemma 1. *The Zygmund-Orlicz space is isometrically equal to μ-Zygmund space, where*

$$\mu(z) = \frac{1}{\varphi^{-1}(\frac{1}{1-|z|^2})}$$

with $z \in \mathbb{D}$. Thus, for any $f \in Z^\varphi$, we have $\|f\|_\varphi = \|f\|_\mu = \sup_{z\in\mathbb{D}} \mu(z)|f''(z)|$.

Proof. From (2.2), for any $f \in Z^\varphi \setminus \{0\}$ and any $z \in \mathbb{D}$, we have

$$(1-|z|^2)\varphi(\frac{|f''(z)|}{\|f\|_\varphi}) \le 1$$

which implies that $\mu(z)|f''(z)| \le \|f\|_\varphi$ for all $z \in \mathbb{D}$. Thus $Z^\varphi \subset Z^\mu$ and $\|f\|_\mu \le \|f\|_\varphi$.

Conversely, if $f \in Z^\mu$, then $\mu(z)|f''(z)| \le \|f\|_\mu$, for all $z \in \mathbb{D}$. From here, we have $S_\varphi(\frac{f''}{\|f\|_\varphi}) \le 1$. Thus, $f \in Z^\varphi$ and $\|f\|_\varphi \le \|f\|_\mu$. ☐

The following result will be very useful in the next section and it is a version of Lemma 6 in [5]. for completeness, we include an outline of its proof.

Lemma 2. *Let $a \in \mathbb{D}$ fixed. There exists a holomorphic function $f_a \in H(\mathbb{D})$, such that*

$$\varphi(|f_a'(z)|) = \frac{1-|a|^2}{|1-\bar{a}z|^2}$$

for all $z \in \mathbb{D}$.

Proof. For $z \in \mathbb{D}$, we set $u(z) = \varphi^{-1}(\frac{1-|a|^2}{|1-\bar{a}z|^2})$, then u is a real and continuously differentiable function. Therefore, its partial derivatives exist and are continuous throughout $\mathbb{D}$. It is clear that function u satisfies $u(z) \geq \varphi^{-1}(\frac{1}{|1-|a|^2}) > 0$ for all $z \in \mathbb{D}$. Now we set $f_a'(z) = u(z)e^{iv(z)}$ where v is a real function defined on $\mathbb{D}$. Then, in order for f_a' to be an analytic function on $\mathbb{D}$, its real parts $U(z) = u(z)\cos v(z)$ and its imaginary parts $V(z) = u(z)\sin v(z)$ must satisfy the Cauchy-Riemann equations. We get the relation $uv_x = -u_y$ and $uv_y = u_x$. We can choose a C^1 real function v defined on $\mathbb{D}$ such that f_a' is an analytic function on $\mathbb{D}$ satisfying $\varphi(|f_a'(z)|) = \frac{1-|a|^2}{|1-\bar{a}z|^2}$. Of course, $f_a(z) = \int_0^z f_a'(\zeta)d\zeta + f_a(0)$ is an analytic function on $\mathbb{D}$, too. □

Remark. It is clear that for any $a \in \mathbb{D}$, the function $g_a(z) = \int_0^z f_a(s)ds$ with $z \in \mathbb{D}$ and f_a is the function found in Lemma 2.2, belongs to the space Z^φ.

$$S(g_a'') = \sup_{z\in\mathbb{D}}(1-|z|^2)\frac{1-|a|^2}{|1-\bar{a}z|^2} = \sup_{z\in\mathbb{D}}(1-|\sigma_a(z)|^2) = 1 \tag{2.4}$$

where $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ denote the automorphism of the disk $\mathbb{D}$. From (2.4), we get $\|g_a\|_\varphi = 1$ for all $a \in \mathbb{D}$.

Lemma 3. *The composition opreator C_ϕ is compact on Z^φ if and only if given a bounded sequence $\{f_n\}$ in Z^φ such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, then $\|C_\phi f_n\|_\varphi \to 0$ as $n \to \infty$.*

3. Main results

Theorem 1. *Let ϕ be a holomorphic self-map of $\mathbb{D}$. Then $C_\phi : Z^\varphi \to Z^\varphi$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)}{\mu(\phi(z))}|\phi'(z)|^2 < \infty \quad \textit{and}$$

$$\sup_{z\in\mathbb{D}} \mu(z)|\phi''(z)| \left(1 + \int_0^1 \varphi^{-1}(\frac{1}{1-|\phi(z)t|^2})dt\right) < \infty.$$

Proof. Suppose that

$$L_1 = \sup_{z\in\mathbb{D}} \frac{\mu(z)}{\mu(\phi(z))}|\phi'(z)|^2 < \infty,$$

$$L_2 = \sup_{z\in\mathbb{D}} \mu(z)|\phi''(z)| \left(1 + \int_0^1 \varphi^{-1}(\frac{1}{1-|\phi(z)t|^2})\right) dt < \infty. \tag{3.5}$$

Then for all $f \in Z^\varphi \setminus \{0\}$, We have the following estimate

$$\begin{aligned}
S_\varphi\Big(\frac{(f\circ\phi)''(z)}{(L_1+L_2)\|f\|_{Z^\varphi}}\Big) &= \sup_{z\in\mathbb{D}}(1-|z|^2)\varphi\Big(\frac{|f''(\phi(z))\phi'^2(z) + f'(\phi(z))\phi''(z)|}{(L_1+L_2)\|f\|_{Z^\varphi}}\Big) \\
&\leq \sup_{z\in\mathbb{D}}(1-|z|^2)\varphi\Big(\frac{|f''(\phi(z))||\phi'(z)|^2 + |f'(\phi(z))||\phi''(z)|}{(L_1+L_2)\|f\|_{Z^\varphi}}\Big) \\
&\leq \sup_{z\in\mathbb{D}}(1-|z|^2)\varphi\Big(\frac{\frac{|\phi'(z)|^2}{\mu(\phi(z))}\|f\|_\varphi + (1+\int_0^1\varphi^{-1}(\frac{1}{1-|\phi(z)t|^2})dt)|\phi''(z)|\|f\|_{Z^\varphi}}{(L_1+L_2)\|f\|_{Z^\varphi}}\Big) \\
&\leq \sup_{z\in\mathbb{D}}(1-|z|^2)\varphi\Big(\frac{1}{\mu(z)}\frac{(L_1+L_2)\|f\|_{Z^\varphi}}{(L_1+L_2)\|f\|_{Z^\varphi}}\Big) \\
&= \sup_{z\in\mathbb{D}}(1-|z|^2)\varphi\Big(\frac{1}{\mu(z)}\Big) = 1
\end{aligned}$$

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where we have used the relations (2.1) (2.2) (2.3) and Lemma 2.1. From here, we can conclude that $\|C_\phi f\|_\varphi \leq (L_1+L_2)\|f\|_{Z^\varphi}$. Moreover from (3.5), take $z=0$, we have

$$\varphi^{-1}(1)|\phi''(0)|\Big(1+\int_0^1 \varphi^{-1}(\frac{1}{1-|\phi(z)t|^2})dt\Big)<\infty,$$

and therefore

$$1+\int_0^1 \varphi^{-1}(\frac{1}{1-|\phi(0)t|^2})dt<\infty. \tag{3.6}$$

Hence with (2.1) (2.3)

$$\begin{aligned} |f(\phi(0))|+|f'(\phi(0))\phi'(0)| &\leq |f(0)|+\int_0^{|\phi(0)|}|f'(\xi)||d\xi|+|f'(0)|+\int_0^{|\phi(0)|}|f''(\zeta)||d\zeta| \\ &\leq \|f\|_{Z^\varphi}+2(1+\int_0^{|\phi(0)|}|f''(\zeta)||d\zeta|)\|f\|_{Z^\varphi} \\ &= \|f\|_{Z^\varphi}+2(1+\int_0^1 \varphi^{-1}(\frac{1}{1-|\phi(0)t|^2})dt)\|f\|_{Z^\varphi}. \end{aligned} \tag{3.7}$$

Combined with (3.6) (3.7), we have $\|C_\phi f\|_{Z^\varphi}\leq C\|f\|_{Z^\varphi}$ for all $f\in Z^\varphi$, and $C_\phi: Z^\varphi\to Z^\varphi$ is bounded.

Conversely, suppose that there exists a constant $C>0$ such that

$$\|f\circ\phi\|_\varphi \leq C\|f\|_\varphi$$

for all $f\in Z^\varphi$. By taking the function $f(z)=z\in Z^\varphi$, we obtain

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\varphi(\frac{|(C_\phi f)''(z)|}{k})=\sup_{z\in\mathbb{D}}(1-|z|^2)\varphi(\frac{|\phi''(z)|}{k})<\infty.$$

That is

$$\sup_{z\in\mathbb{D}}\mu(z)|\phi''(z)|<\infty. \tag{3.8}$$

For $a\in\mathbb{D}$, set $g_a(z)=\int_0^z f_a(s)ds$ and from the remark , we see that $g_a\in Z^\varphi$ and $\|g_a\|_\varphi=1$. With (2.2), we have that

$$1\geq S_\varphi(\frac{(g_a\circ\phi)''(z)}{C\|g_a\|_\varphi})=\sup_{z\in\mathbb{D}}(1-|z|^2)\varphi(\frac{|f_a'(\phi(z))\phi'^2(z)+f_a(\phi(z))\phi''(z)|}{C}).$$

Hence

$$\begin{gathered}\mu(z)|f_a'(\phi(z))||\phi'(z)|^2-\mu(z)|f_a(\phi(z))||\phi''(z)| \\ \leq \mu(z)|f_a'(\phi(z))\phi'^2(z)+f_a(\phi(z))\phi''(z)|\leq C,\end{gathered}$$

or

$$\mu(z)|f_a'(\phi(z))||\phi'(z)|^2\leq C+\mu(z)|f_a(\phi(z))||\phi''(z)|.$$

For $g_a\in Z^\varphi$, we have $g_a'\in\mathcal{B}^\varphi$. That is to say $f_a\in\mathcal{B}^\varphi$ and from Lemma 2.1 we get $f_a\in\mathcal{B}_\mu$, where $\mu(z)=\dfrac{1}{\varphi^{-1}(\frac{1}{1-|z|^2})}$. Thus, from [11], we have

$$|f_a(z)|\leq C(1+\int_0^{|z|}\frac{1}{\mu(t)}dt)\|f_a\|_{\mathcal{B}_\mu}. \tag{3.9}$$

Hence

$$\sup_{z\in\mathbb{D}}\mu(z)|f_a'(\phi(z))||\phi'(z)|^2\leq C+C\mu(z)(1+\int_0^{|\phi(z)|}\frac{1}{\mu(t)}dt)|\phi''(z)|\|f_a\|_{\mathcal{B}_\mu}. \tag{3.10}$$

It is obvious that $\sup_{z\in\mathbb{D}}\mu(z)|f_a'(\phi(z))||\phi'(z)|^2<\infty$ when $|\phi(z)|\leq\frac{1}{\sqrt{2}}$.

Now for $\frac{1}{\sqrt{2}}<|\phi(z)|<1$, fix $a\in\mathbb{D}$, we set

$$h_a(z)=(1-|a|^2)\int_0^z(\int_0^{\bar a\zeta}h(t)dt-\frac{1}{2}\frac{(\int_0^{\bar a\zeta}h(t)dt)^2}{\int_0^{|a|^2}h(t)dt})d\zeta,$$

where $h(t) = \frac{1}{\mu(t)}$. Then

$$h_a'(z) = (1-|a|^2)\Big(\int_0^{\bar{a}z} h(t)dt - \frac{1}{2}\frac{(\int_0^{\bar{a}z} h(t)dt)^2}{\int_0^{|a|^2} h(t)dt}\Big),$$
$$h_a''(z) = (1-|a|^2)\Big(\bar{a}h(\bar{a}z) - \frac{\bar{a}h(\bar{a}z)\int_0^{\bar{a}z} h(t)dt}{\int_0^{|a|^2} h(t)dt}\Big).$$

It is easy to see that $h_a'(a) = \frac{1-|a|^2}{2}\int_0^{|a|^2} h(t)dt$ and $h_a''(a) = 0$. From above, we can see that $h_a \in Z^\varphi$, and $\|h_a\|_\varphi \le C$. Therefore, $\|C_\phi h_a\|_\varphi \le C$. Hence, we have

$$\begin{aligned} &\sup_{z\in\mathbb{D}}(1-|z|^2)\varphi\Big(\frac{|(C_\phi h_a)''(z)|}{C}\Big) \\ = \;& \sup_{z\in\mathbb{D}}(1-|z|^2)\varphi\Big(\frac{|h_a''(\phi(z))||\phi'(z)|^2 + |h_a'(\phi(z))||\phi''(z)|}{C}\Big) < \infty. \end{aligned}$$

From above, put $a = \phi(z)$, we obtain

$$\sup_{z\in\mathbb{D}} \mu(z)|\phi''(z)|\int_0^{|\phi(z)|^2}\frac{1}{\mu(t)}dt \le C.$$

Combined with the boundedness of ϕ, (3.8) and the following inequality of [12]

$$\int_0^{|\phi(z)|^2}\frac{1}{\mu(t)}dt \le \int_0^{|\phi(z)|}\frac{1}{\mu(t)}dt \le C + C\int_0^{|\phi(z)|^2}\frac{1}{\mu(t)}dt, \tag{3.11}$$

we have

$$\begin{aligned} &\sup_{z\in\mathbb{D}} \mu(z)|\phi''(z)||\phi(z)|\int_0^1 \varphi^{-1}\Big(\frac{1}{1-|\phi(z)t|^2}\Big)dt \\ \le\;& \sup_{z\in\mathbb{D}} \mu(z)|\phi''(z)|\int_0^{|\phi(z)|}\frac{1}{\mu(t)}dt \\ \le\;& \sup_{z\in\mathbb{D}} \mu(z)|\phi''(z)|\Big(C + C\int_0^{|\phi(z)|^2}\frac{1}{\mu(t)}dt\Big) < \infty \end{aligned} \tag{3.12}$$

Hence with (3.10), we have

$$\sup_{z\in\mathbb{D}} \mu(z)|f_a'(\phi(z))||\phi'(z)|^2 = \sup_{z\in D}\frac{\mu(z)}{\mu(\phi(z))}|\phi'(z)|^2 \le C,$$

$$\sup_{z\in\mathbb{D}} \mu(z)|\phi''(z)|\Big(1 + \int_0^1 \varphi^{-1}\Big(\frac{1}{1-|\phi(z)t|^2}\Big)dt\Big) \le C.$$

This completes the proof. □

Theorem 2. *Let ϕ be a holomorphic self-map of $\mathbb{D}$. Then $C_\phi : Z^\varphi \to Z^\varphi$ is compact if and only if C_ϕ is bounded and*

$$\lim_{|\phi(z)|\to 1}\frac{\mu(z)}{\mu(\phi(z))}|\phi'(z)|^2 = 0, \tag{3.13}$$

$$\lim_{|\phi(z)|\to 1}\mu(z)|\phi''(z)|\Big(1 + \int_0^1 \varphi^{-1}\Big(\frac{1}{1-|\phi(z)t|^2}\Big)dt\Big) = 0. \tag{3.14}$$

Proof. Suppose first that C_ϕ is bounded and (3.5) holds. Let $\{f_n\}$ be a bounded sequence in Z^φ converging to 0 uniformly on compact subsets of $\mathbb{D}$. Then, by Lemma 2.3, it is sufficient to show that$\|C_\phi f_n\|_\varphi \to 0$ as $n \to \infty$. From (3.13) and (3.14), for $\varepsilon_1, \varepsilon_2 > 0$, we can find an $r \in (0,1)$ such that

$$\frac{\mu(z)}{\mu(\phi(z))}|\phi'(z)|^2 < \varepsilon_1 \quad \textit{and} \quad \mu(z)|\phi''(z)|\Big(1 + \int_0^1 \varphi^{-1}\Big(\frac{1}{1-|\phi(z)t|^2}\Big) < \varepsilon_2,$$

whenever $r < |\phi(z)| < 1$. From here, we have that

$$\begin{aligned}
\mu(z)|(C_\phi f_n)''(z)| &\leq \mu(z)|f_n''(\phi(z)||\phi'(z)|^2 + |f_n'(\phi(z))||\phi''(z)|) \\
&\leq \frac{\mu(z)}{\mu(\phi(z))}|\phi'(z)|^2\|f_n\|_\varphi \\
&+ \mu(z)|\phi''(z)|(1+\int_0^1 \varphi^{-1}(\frac{1}{1-|\phi(z)t|^2})dt\|f_n\|_{Z^\varphi} \\
&< \|f_n\|_\varphi\varepsilon_1 + \|f_n\|_{Z^\varphi}\varepsilon_2. \qquad (3.15)
\end{aligned}$$

On the other hand, since $\{f_n\}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$, $\sup\limits_{|\phi(r)|\leq r} |f_n''(\phi(z))| \to 0$, and $\sup\limits_{|\phi(r)|\leq r} |f_n'(\phi(z))| \to 0$ as $n \to \infty$. From the boundedness of C_ϕ, set $f(z) = z \in Z^\varphi$ and $f(z) = z^2 \in Z^\varphi$, we have

$$M_1 = \sup_{z\in D}\mu(z)|\phi'(z)|^2 < \infty, \qquad M_2 = \sup_{z\in D}\mu(z)|\phi''(z)| < \infty. \qquad (3.16)$$

Hence, we have

$$\begin{aligned}
&\sup_{|\phi(r)|\leq r} \mu(z)|(C_\phi f_n)''(z)| \\
\leq& \sup_{|\phi(r)|\leq r} \mu(z)|\phi'(z)|^2|f_n''(\phi(z))| + \sup_{|\phi(r)|\leq r} \mu(z)|\phi''(z)||f_n'(\phi(z))| \\
\leq& M_1 \sup_{|\phi(r)|\leq r} |f_n''(\phi(z))| + M_2 \sup_{|\phi(r)|\leq r} |f_n'(\phi(z))| \to 0
\end{aligned}$$

With (3.15), we obtain that C_ϕ is a compact operator on Z^φ.

Suppose that $C_\phi : Z^\varphi \to Z^\varphi$ is compact. It is clear that $C_\phi : Z^\varphi \to Z^\varphi$ is bounded. Let $\{z_n\}$ be a sequence in $\mathbb{D}$ such that $|w_n| = |\phi(z_n)| \to 1$ as $n \to \infty$. Set

$$g_n(z) = \int_0^z (f_{w_n}(s) - f_{w_n}(0))ds$$

then $g_n \in Z^\varphi$ and $\|g_n\|_\varphi = 1$. Furthermore, $g_n \to 0$ uniformly on compact subsets of D as $n \to \infty$. Hence

$$\begin{aligned}
0 &\leftarrow \|C_\phi g_n\|_\mu \geq \mu(z_n)|g_n''(\phi(z_n))\phi'^2(z_n) + g_n'(\phi(z_n))\phi''(z_n)| \\
&\geq \mu(z_n)|f'_{\phi(z_n)}(\phi(z_n))||\phi'(z_n)|^2 \\
&- \mu(z_n)|f_{\phi(z_n)}(\phi(z_n)) - f_{\phi(z_n)}(0)||\phi''(z_n)|. \qquad (3.17)
\end{aligned}$$

Since C_ϕ is compact operator on Z^φ, set

$$s_{w_n}(z) = \frac{1}{\ln(1-|w_n|^2)}\int_0^z \Big(\frac{(\ln(1-\bar{w_n}\zeta))^2}{2\ln(1-|w_n|^2)} - \ln(1-\bar{w_n}\zeta)\Big)d\zeta.$$

It is easy to see that $s'_{w_n}(w_n) = -\frac{1}{2}$ and $s''_{w_n}(w_n) = 0$. So $s_{w_n} \in Z^\varphi$ and $s_{w_n} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$. For $\varepsilon_3 > 0$, we have

$$\varepsilon_3 > \|C_\phi s_{w_n}\|_\mu = \frac{1}{2}\sup_{z\in D}\mu(z_n)|\phi''(z_n)|,$$

which means

$$\lim_{n\to\infty}\mu(z_n)|\phi''(z_n)| = 0. \qquad (3.18)$$

Set

$$h_n(z) = (1-|w_n|^2)\int_0^z (\int_0^{\bar{w_n}\zeta} h(t)dt - \frac{1}{2}\frac{(\int_0^{\bar{w_n}\zeta}h(t)dt)^2}{\int_0^{|w_n|^2}h(t)dt})d\zeta,$$

then $\{h_n\} \subset Z^{\varphi}$ and $\{h_n\}$ is a sequence converging to 0 uniformly on compact subsets of $\mathbb{D}$. Furthermore, for $\varepsilon_4 > 0$, we have

$$\begin{aligned}
& \sup_{z_n \in \mathbb{D}} (1-|z_n|^2)\varphi\Big(\frac{|(C_\phi h_n)''(z_n)|}{k}\Big) \\
= & \sup_{z_n \in \mathbb{D}} (1-|z_n|^2)\varphi\Big(\frac{|h_n''(\phi(z_n))\phi'^2(z_n) + h_n'(\phi(z_n))\phi''(z_n)|}{k}\Big) \\
= & \sup_{z_n \in \mathbb{D}} (1-|z_n|^2)\varphi\Big(\frac{\frac{1-|\phi(z_n)|^2}{2}|\phi''(z_n)|\int_0^{|\phi(z_n)|^2}\frac{1}{\mu(t)}dt}{k}\Big) < \varepsilon_4.
\end{aligned}$$

That is to say, for $\varepsilon_4 > 0, \exists N$, such that $\frac{1}{\sqrt{2}} < |\phi(z_n)| < 1$ whenever $n > N$, we have

$$\frac{1-|\phi(z_n)|^2}{2}\mu(z_n)|\phi''(z_n)|\int_0^{|\phi(z_n)|^2}\frac{1}{\mu(t)}dt < \varepsilon_4. \tag{3.19}$$

Hence, with (3.18) (3.19), we obtain that

$$\begin{aligned}
\mu(z_n)|\phi''(z_n)|\int_0^{|\phi(z_n)|}\frac{1}{\mu(t)}dt &\le \mu(z_n)|\phi''(z_n)|(C_1 + C_2\int_0^{|\phi(z_n)|^2}\frac{1}{\mu(t)}dt) \\
&\le C\varepsilon_3 + C\frac{1-|\phi(z_n)|^2}{2}\mu(z_n)|\phi''(z_n)|\int_0^{|\phi(z_n)|^2}\frac{1}{\mu(t)}dt \\
&\le C\varepsilon_3 + C\varepsilon_4. \qquad (3.20)
\end{aligned}$$

Therefore, with the boundedness of ϕ and (3.18) (3.20), we have

$$\lim_{|\phi(z)|\to 1}\mu(z)|\phi''(z)|(1+\int_0^1\varphi^{-1}(\frac{1}{1-|\phi(z)t|^2})dt) = 0$$

and (3.14) hold. Moreover, with (3.17) we have that

$$\begin{aligned}
\mu(z_n)|f_{w_n}(w_n) - f_{w_n}(0)||\phi''(z_n)| &\le \mu(z_n)|\phi''(z_n)|(1+\int_0^{|w_n|}\frac{1}{\mu(t)}dt)\|f_{w_n} - f_{w_n}(0)\|_{\mathcal{B}_\mu} \\
&\le (C\varepsilon_3 + C\varepsilon_4)\|f_{w_n} - f_{w_n}(0)\|_{\mathcal{B}_\mu}. \qquad (3.21)
\end{aligned}$$

From (3.17) (3.21), we obtain that

$$\begin{aligned}
\frac{\mu(z_n)}{\mu(\phi(z_n))}|\phi'(z_n)|^2 &= \mu(z_n)|f_{w_n}'(w_n)||\phi'(z_n)|^2 \\
&\le \|C_\phi g_n\|_\mu + \mu(z_n)|f_{w_n}(w_n) - f_{w_n}(0)||\phi''(z_n)| \\
&\le \|C_\phi g_n\|_\mu + (C\varepsilon_3 + C\varepsilon_4)\|f_{w_n} - f_{w_n}(0)\|_{\mathcal{B}_\mu}.
\end{aligned}$$

This implies that

$$\lim_{|\phi(z)|\to 1}\frac{\mu(z)}{\mu(\phi(z))}|\phi'(z)|^2 = 0,$$

and (3.13) holds. This completes the proof. □

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Multiple positive solutions for m-point boundary value problems with one-dimensional p-Laplacian systems and sign changing nonlinearity *

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Abstract: In this paper, we consider the multipoint boundary value problem for the one-dimensional p-Laplacian system

$$\begin{cases} (\phi_{p_1}(u'))' + q_1(t)f(t,u,v) = 0, \ t \in (0,1), \\ (\phi_{p_2}(v'))' + q_2(t)g(t,u,v) = 0, \ t \in (0,1), \end{cases}$$

$$\begin{cases} u(0) = \sum\limits_{i=1}^{m-2} a_i u(\xi_i), \ u'(1) = \beta u'(0), \\ v(0) = \sum\limits_{i=1}^{m-2} a_i v(\xi_i), \ v'(1) = \beta v'(0), \end{cases}$$

where $\phi_{p_i}(s) = |s|^{p_i-2}s$, $p_i > 1, \ i = 1,2, \ \xi_i \in (0,1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ and $a_i \in [0,1), \ \sum\limits_{i=1}^{m-2} a_i < 1, \ \beta \in (0,1)$. By using the fixed point index theorem on cones, we study the existence of positive solutions for the m-point boundary value problem with sign changing nonlinear term. Some sufficient conditions for the existence of multiple positive solutions are obtained. Finally, an example is also included to illustrate the importance of the main result obtained.

Keywords: Multipoint boundary value problem, Fixed point theorem, Cone, Positive solution, One-dimensional p-Laplacian.

2010 MR Subject Classification: 34B10, 34B15, 34B18

1 Introduction

In this paper, we study the existence of multiple positive solutions to the boundary value problem (BVP for short) for the one-dimensional p-Laplacian system

$$\begin{cases} (\phi_{p_1}(u'))' + q_1(t)f(t,u,v) = 0, \ t \in (0,1), \\ (\phi_{p_2}(v'))' + q_2(t)g(t,u,v) = 0, \ t \in (0,1), \end{cases} \tag{1.1}$$

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$$\begin{cases} u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i),\ u'(1) = \beta u'(0), \\ v(0) = \sum_{i=1}^{m-2} a_i v(\xi_i),\ v'(1) = \beta v'(0), \end{cases} \tag{1.2}$$

where $\phi_{p_i}(s) = |s|^{p_i-2}s,\ p_i > 1,\ i = 1, 2,\ \xi_i \in (0, 1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$.

Multipoint boundary value problems of ordinary differential equations arise in a variety of areas of applied mathematics and physics. For example, the vibrations of a guy wire of a uniform cross-section and composed of N parts of different densities can be set up as a multipoint boundary value problem (see [10]). The study of multipoint boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [2]. Since then there has been much current attention focused on the study of nonlinear multipoint boundary value problems, see ([1, 3, 4, 5, 6, 8, 9, 11, 12, 13, 14, 15]).

Karakostas [4] proved the existence of positive solutions for the two-point boundary value problem

$$x''(t) - \text{sign}(1-\alpha)q(t)f(x, x')x' = 0,\ t \in (0, 1),$$

with one of the following sets of boundary conditions:

$$x(0) = 0,\ x'(1) = \alpha x'(0),$$

or

$$x(1) = 0,\ x'(1) = \alpha x'(0),$$

where $\alpha > 0,\ \alpha \neq 1$. By using indices of convergence of the nonlinearities at 0 and at $+\infty$, they provided a priori upper and lower bounds for the slope of the solutions.

Ma [11] proved the existence of positive solutions for the multipoint boundary value problem

$$x''(t) - q(t)f(x, x')x' = 0,\ t \in (0, 1),$$

$$x(0) = \sum_{i=1}^{n-2} b_i x(\xi_i),\ x'(1) = \alpha x'(0),$$

where $\xi_i \in (0, 1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1,\ b_i \in [0, 1),\ \alpha > 1$. They provided sufficient conditions for the existence of multiple positive solutions to the above BVP by applying the fixed point theorem in cones.

Recently, Ji [3] investigated the following m-point boundary value problem

$$(\phi_p(u'))' + q(t)f(t, u) = 0,\ t \in (0, 1),$$

$$u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i),\ u(1) = \sum_{i=1}^{m-2} \beta_i u(\xi_i).$$

They obtained sufficient conditions that guarantee the existence of positive solutions by using fixed point theorems on cones.

Motivated by these results, our purpose of this paper is to show the existence of multiple positive solutions to multipoint BVP (1.1), (1.2). To date no paper has appeared in the literature which discusses the multipoint boundary value problem for one-dimensional p-Laplacian systems when nonlinearity in the differential equation may change sign. This paper attempts to fill this gap in the literature. The interesting point of this paper is the nonlinear terms f and g may change sign.

For convenience, we list the following assumptions:

(H_1) $a_i \in [0,1)$ satisfies $\sum\limits_{i=1}^{m-2} a_i < 1,\ \beta \in (0,1)$;
(H_2) $f,\ g \in C([0,1] \times [0,+\infty) \times [0,+\infty),\ (-\infty,+\infty))$;
(H_3) $q_1,\ q_2 \in L^1[0,1]$ are nonnegative on $(0,1)$ and $q_1,\ q_2$ are not identically zero on any subinterval of $(0,1)$. Furthermore, $q_1,\ q_2$ satisfy $0 < \int_0^1 q_1(t)dt < +\infty,\ 0 < \int_0^1 q_2(t)dt < +\infty$.

2 Preliminaries

For the convenience of readers, we provide some background material from the theory of cones in Banach spaces. We also state in this section the fixed point index theorem on cones.
Definition 2.1. Let E be a real Banach space over R. A nonempty closed set $K \subset E$ is said to be a cone provide that
(i) $au + bv \in K$ for all $u, v \in K$ and all $a \geq 0, b \geq 0$, and
(ii) $u, -u \in K$ implies $u = 0$.
Every cone $K \subset E$ induces an ordering in E given by $x \leq y$ if and only if $y - x \in K$.

Definition 2.2. The map α is said to be a nonnegative continuous concave functional on a cone K of a real Banach space E provided that $\alpha : K \to [0,\infty)$ is continuous and

$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$

for all $x,\ y \in K$ and $0 \leq t \leq 1$. Similarly, we say the map γ is a nonnegative continuous convex functional on a cone K of a real Banach space E provided that $\gamma : P \to [0,\infty)$ is continuous and

$\gamma(tx + (1-t)y) \leq t\gamma(x) + (1-t)\gamma(y)$

for all $x,\ y \in K$ and $0 \leq t \leq 1$.

To prove our results, the following fixed point theorem in cones is fundamental.

Theorem 2.1. ([7]) Let K be a cone in a real Banach space E. Let D be an open bounded subset of E with $D_k = D \cap K \neq \emptyset$. Assume that $A : \overline{D}_k \to K$ is completely continuous such that $A \neq Ax$ for $x \in D_k K$. Then the following results hold:
(1) If $\|Ax\| \leq \|x\|,\ x \in \partial D_k$, then $i_k(A, D_k) = 1$.
(2) If there exists $e \in K\backslash\{0\}$ such that $x \neq Ax + \lambda e$ for all $x \in \partial D_k$ and all $\lambda > 0$, then $i_k(A, D_k) = 0$.
(3) Let U be open in X such that $\overline{U} \subset D_k$. If $i_k(A, D_k) = 1$ and $i_k(A, U_k) = 0$, then A has a fixed point in $D_k \backslash \overline{U}_k$. The same result holds if $i_k(A, D_k) = 0$ and $i_k(A, U_k) = 1$.

3 Related lemmas

In this paper, we denote $C^+[0,1] = \{x \in C[0,1] : x(t) \geq 0,\ t \in [0,1]\}$. $\phi_{p_1}^{-1},\ \phi_{p_2}^{-1}$ are, respectively, the inverse function to $\phi_{p_1},\ \phi_{p_2}$.

Let $E = C[0,1] \times C[0,1]$, define norm $\|(u,v)\| = \|u\| + \|v\|$, where $\|u\| = \max\limits_{t\in[0,1]} |u(t)|$, $\|v\| = \max\limits_{t\in[0,1]} |v(t)|$, then E is a Banach space.

Define the cone $K \subset E$ by

$K = \{(u,v) \in E \mid u(t),\ v(t) \geq 0,\ u \text{ and } v \text{ are concave and nondecreasing on } [0,1]\}$.

Lemma 3.1. Assume that $(H_1) - (H_3)$ hold. Then, For any $x,\ y \in C^+[0,1]$, the problem

$$\begin{cases} (\phi_{p_1}(u'))' + q_1(t)f(t,x,y) = 0,\ t \in (0,1), \\ (\phi_{p_2}(v'))' + q_2(t)g(t,x,y) = 0,\ t \in (0,1), \end{cases} \tag{3.1}$$

$$\begin{cases} u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \ u'(1) = \beta u'(0), \\ v(0) = \sum_{i=1}^{m-2} a_i v(\xi_i), \ v'(1) = \beta v'(0), \end{cases} \tag{3.2}$$

has the unique solution (u, v) as

$$\begin{aligned} u(t) = &\int_0^t \phi_{p_1}^{-1} \left(\int_s^1 q_1(\tau) f(\tau, x(\tau), y(\tau)) d\tau + \frac{\phi_{p_1}(\beta)}{1-\phi_{p_1}(\beta)} \int_0^1 q_1(\tau) f(\tau, x(\tau), y(\tau)) d\tau \right) ds \\ &+ \frac{1}{1-\sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_{p_1}^{-1} \left(\int_s^1 q_1(\tau) f(\tau, x(\tau), y(\tau)) d\tau \right. \\ &\left. + \frac{\phi_{p_1}(\beta)}{1-\phi_{p_1}(\beta)} \int_0^1 q_1(\tau) f(\tau, x(\tau), y(\tau)) d\tau \right) ds, \end{aligned} \tag{3.3}$$

$$\begin{aligned} v(t) = &\int_0^t \phi_{p_2}^{-1} \left(\int_s^1 q_2(\tau) g(\tau, x(\tau), y(\tau)) d\tau + \frac{\phi_{p_2}(\beta)}{1-\phi_{p_2}(\beta)} \int_0^1 q_2(\tau) g(\tau, x(\tau), y(\tau)) d\tau \right) ds \\ &+ \frac{1}{1-\sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_{p_2}^{-1} \left(\int_s^1 q_2(\tau) g(\tau, x(\tau), y(\tau)) d\tau \right. \\ &\left. + \frac{\phi_{p_2}(\beta)}{1-\phi_{p_2}(\beta)} \int_0^1 q_2(\tau) g(\tau, x(\tau), y(\tau)) d\tau \right) ds. \end{aligned} \tag{3.4}$$

Proof. For any $x, y \in C^+[0,1]$, suppose (u, v) is a solution of BVP (3.1), (3.2). By integration of (3.1), it follows that

$$\begin{aligned} u'(t) &= \phi_{p_1}^{-1} \left(\phi_{p_1}(u'(0)) - \int_0^t q_1(\tau) f(\tau, x(\tau), y(\tau)) d\tau \right), \\ u(t) &= u(0) + \int_0^t \phi_{p_1}^{-1} \left(\phi_{p_1}(u'(0)) - \int_0^s q_1(\tau) f(\tau, x(\tau), y(\tau)) d\tau \right) ds. \end{aligned}$$

Using the boundary condition (3.2), we can easily have

$$\begin{aligned} u(t) = &\int_0^t \phi_{p_1}^{-1} \left(\int_s^1 q_1(\tau) f(\tau, x(\tau), y(\tau)) d\tau + \frac{\phi_{p_1}(\beta)}{1-\phi_{p_1}(\beta)} \int_0^1 q_1(\tau) f(\tau, x(\tau), y(\tau)) d\tau \right) ds \\ &+ \frac{1}{1-\sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_{p_1}^{-1} \left(\int_s^1 q_1(\tau) f(\tau, x(\tau), y(\tau)) d\tau \right. \\ &\left. + \frac{\phi_{p_1}(\beta)}{1-\phi_{p_1}(\beta)} \int_0^1 q_1(\tau) f(\tau, x(\tau), y(\tau)) d\tau \right) ds. \end{aligned}$$

In a similar way, we can prove

$$\begin{aligned} v(t) = &\int_0^t \phi_{p_2}^{-1} \left(\int_s^1 q_2(\tau) g(\tau, x(\tau), y(\tau)) d\tau + \frac{\phi_{p_2}(\beta)}{1-\phi_{p_2}(\beta)} \int_0^1 q_2(\tau) g(\tau, x(\tau), y(\tau)) d\tau \right) ds \\ &+ \frac{1}{1-\sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_{p_2}^{-1} \left(\int_s^1 q_2(\tau) g(\tau, x(\tau), y(\tau)) d\tau \right. \\ &\left. + \frac{\phi_{p_2}(\beta)}{1-\phi_{p_2}(\beta)} \int_0^1 q_2(\tau) g(\tau, x(\tau), y(\tau)) d\tau \right) ds. \end{aligned}$$

Lemma 3.2. Assume that $(H_1)-(H_3)$ hold. If $f(t,x,y),\ g(t,x,y)>0$, for $x,\ y\in C^+[0,1]$, $t\in[0,1]$, for the unique solution (u,v) of BVP (3.1), (3.2), then $u(t)$ and $v(t)$ are concave, and $u(t),\ v(t)\geq 0,\ u'(t),\ v'(t)\geq 0,\ t\in[0,1]$.

Proof. From the fact that $(\phi_p(u'))'(t)=-q(t)f(t,x(t),y(t))\leq 0$, we have $\phi_p(u'(t))$ is nonincreasing. It follows that $u'(t)$ is also nonincreasing. Thus, we know that the graph of $u(t)$ is concave down on $(0,1)$. Then the concavity of u together with boundary $u'(1)=\beta u'(0)$ implies that $u'(t)\geq 0$ for $t\in[0,1]$. Similarly, we can prove the graph of $v(t)$ is concave down on (0,1) and $v'(t)\geq 0$ for $t\in[0,1]$.

From $u'(t)\geq 0$, we know that

$$u(\xi_i)\geq u(0),\ \text{for } i=1,2,\ldots,m-2.$$

This implies

$$u(0)=\sum_{i=1}^{m-2}a_iu(\xi_i)\geq\sum_{i=1}^{m-2}a_iu(0).$$

By $1-\sum\limits_{i=1}^{m-2}a_i>0$, it is obvious that $u(0)\geq 0$. Hence $u(1)\geq u(0)\geq 0$. So from the concavity of u, we know that $u(t)\geq 0$, $t\in[0,1]$. In a similar way, we can know $v(t)\geq 0$, $t\in[0,1]$.

Lemma 3.3. If $(u,v)\in K,\ \eta\in(0,1)$, then $u(t)\geq\eta\|u\|,\ v(t)\geq\eta\|v\|,\ t\in[\eta,1]$.

Proof. For $u\in K$, we know $u(t)$ and $v(t)$ are nonnegative, nondecreasing and concave on $[0,1]$, then

$$u(t)\geq tu(1)\geq\eta u(1)=\eta\|u\|,\ v(t)\geq tv(1)\geq\eta v(1)=\eta\|v\|,\ t\in[\eta,1].$$

We define

$$\varphi(t)=\theta t,\ \theta\in(0,1),$$

$$L=1+\frac{\sum\limits_{i=1}^{m-2}a_i\xi_i}{1-\sum\limits_{i=1}^{m-2}a_i},$$

$$\gamma_1=\min\left\{\frac{\eta\int_\eta^1\phi_{p_1}^{-1}\left(\frac{1}{1-\phi_{p_1}(\beta)}\int_s^1q_1(\tau)d\tau\right)ds}{L\phi_{p_1}^{-1}\left(\frac{1}{1-\phi_{p_1}(\beta)}\int_0^1q_1(\tau)d\tau\right)},\ \frac{\eta\int_\eta^1\phi_{p_2}^{-1}\left(\frac{1}{1-\phi_{p_2}(\beta)}\int_s^1q_2(\tau)d\tau\right)ds}{L\phi_{p_2}^{-1}\left(\frac{1}{1-\phi_{p_2}(\beta)}\int_0^1q_2(\tau)d\tau\right)}\right\},$$

$$\gamma=\eta\gamma_1,$$

$$K_\rho=\{(u,v)\in K:\|(u,v)\|<\rho\},$$

$$K_\rho^*=\{(u,v)\in K:\rho\varphi(t)<u(t)+v(t)<\rho\},$$

$$\begin{aligned}\Omega_\rho&=\{(u,v)\in K:\min_{\eta\leq t\leq 1}(u(t)+v(t))<\gamma\rho\}\\&=\{(u,v)\in K:\gamma\|(u,v)\|\leq\min_{\eta\leq t\leq 1}(u(t)+v(t))<\gamma\rho\}.\end{aligned}$$

Lemma 3.4. ([7]) Ω_ρ has the following properties:

(a) Ω_ρ is open relative to K.

(b) $K_{\gamma\rho}\subset\Omega_\rho\subset K_\rho$.

(c) $u\in\partial\Omega_\rho$ if and only if $\min\limits_{\eta\leq t\leq 1}(u(t)+v(t))=\gamma\rho$.

(d) If $u\in\partial\Omega_\rho$, then $\gamma\rho\leq u(t)\leq\rho$, for $t\in[\eta,1]$.

Now for convenience we introduce the following notations. Let

$$f_{\gamma\rho}^{\rho}=\left\{\min_{t\in[\eta,1]}\frac{f(t,u,v)}{\phi_{p_1}(\rho)}:u+v\in[\gamma\rho,\rho]\right\},\ g_{\gamma\rho}^{\rho}=\left\{\min_{t\in[\eta,1]}\frac{g(t,u,v)}{\phi_{p_2}(\rho)}:u+v\in[\gamma\rho,\rho]\right\},$$

$$f_{\rho\varphi(t)}^{\rho}=\left\{\max_{t\in[0,1]}\frac{f(t,u,v)}{\phi_{p_1}(\rho)}:u+v\in[\rho\varphi(t),\rho]\right\},\ g_{\rho\varphi(t)}^{\rho}=\left\{\max_{t\in[0,1]}\frac{g(t,u,v)}{\phi_{p_2}(\rho)}:u+v\in[\rho\varphi(t),\rho]\right\},$$

$$f_\infty=\lim_{(u,v)\to\infty} inf\min_{t\in[\eta,1]}\frac{f(t,u,v)}{\phi_{p_1}(u+v)},\ g_\infty=\lim_{(u,v)\to\infty} inf\min_{t\in[\eta,1]}\frac{g(t,u,v)}{\phi_{p_2}(u+v)},$$

$$f^\infty=\lim_{(u,v)\to\infty} sup\max_{t\in[0,1]}\frac{f(t,u,v)}{\phi_{p_1}(u+v)},\ g^\infty=\lim_{(u,v)\to\infty} sup\max_{t\in[0,1]}\frac{g(t,u,v)}{\phi_{p_2}(u+v)},$$

$$\frac{1}{m_1}=2L\phi_{p_1}^{-1}\left(\frac{1}{1-\phi_{p_1}(\beta)}\int_0^1 q_1(\tau)d\tau\right),\ \frac{1}{m_2}=2L\phi_{p_2}^{-1}\left(\frac{1}{1-\phi_{p_2}(\beta)}\int_0^1 q_2(\tau)d\tau\right),$$

$$\frac{1}{M_1}=2\eta\int_\eta^1\phi_{p_1}^{-1}\left(\frac{1}{1-\phi_{p_1}(\beta)}\int_s^1 q_1(\tau)d\tau\right)ds,\ \frac{1}{M_2}=2\eta\int_\eta^1\phi_{p_2}^{-1}\left(\frac{1}{1-\phi_{p_2}(\beta)}\int_s^1 q_2(\tau)d\tau\right)ds,$$

where $(u,v)\to\infty\Leftrightarrow\|u\|+\|v\|\to\infty$.

Remark 2.1. By (H_3), it is easy to see that $0<m_1,m_2,M_1,M_2<\infty$, and $M_1\gamma=M_1\eta\gamma_1\le\eta m_1<m_1,\ M_2\gamma=M_2\eta\gamma_1\le\eta m_2<m_2$.

4 Existence of positive solutions

We now give our results on the existence of multiple positive solutions of BVP (1.1), (1.2).

Theorem 4.1. Assume $(H_1)-(H_3)$ hold. In addition, the following condition (H_4) holds:

(H_4) There exist $\rho_1,\rho_2,\rho_3\in(0,\infty)$, with $\rho_1<\gamma\rho_2<\rho_2<\rho_3$ such that

(1) $f(t,u,v),\ g(t,u,v)>0,\ t\in[0,1],\ u+v\in[\rho_1\varphi(t),\infty)$;

(2) $f_{\rho_1\varphi(t)}^{\rho_1}<\phi_{p_1}(m_1),\ g_{\rho_1\varphi(t)}^{\rho_1}<\phi_{p_2}(m_2),\ f_{\gamma\rho_2}^{\rho_2}\ge\phi_{p_1}(M_1\gamma),\ g_{\gamma\rho_2}^{\rho_2}\ge\phi_{p_2}(M_2\gamma)$, $f_{\rho_3\varphi(t)}^{\rho_3}\le\phi_{p_1}(m_1),\ g_{\rho_3\varphi(t)}^{\rho_3}\le\phi_{p_2}(m_2)$.

Then BVP (1.1), (1.2) has at least three positive solutions in K.

Proof. We assume that (H_4) holds. Denote

$$f^*(t,u,v)=\begin{cases} f(t,u,v), & u+v\ge\rho_1\varphi(t),\\ f(t,u,\rho_1\varphi(t)-u), & 0\le u+v<\rho_1\varphi(t).\end{cases}$$

$$g^*(t,u,v)=\begin{cases} g(t,u,v), & u+v\ge\rho_1\varphi(t),\\ g(t,u,\rho_1\varphi(t)-u), & 0\le u+v<\rho_1\varphi(t).\end{cases}$$

We can see that $f^*(t,u,v),\ g^*(t,u,v)\in C([0,1]\times[0,+\infty)\times[0,+\infty),\ (0,+\infty))$.

Define the following integral equation systems:

$$\begin{aligned}A(u,v)(t)=&\int_0^t\phi_{p_1}^{-1}\left(\int_s^1 q_1(\tau)f^*(\tau,u(\tau),v(\tau))d\tau\right.\\ &\left.+\frac{\phi_{p_1}(\beta)}{1-\phi_{p_1}(\beta)}\int_0^1 q_1(\tau)f^*(\tau,u(\tau),v(\tau))d\tau\right)ds\\ &+\frac{1}{1-\sum_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\phi_{p_1}^{-1}\left(\int_s^1 q_1(\tau)f^*(\tau,u(\tau),v(\tau))\,d\tau\right.\\ &\left.+\frac{\phi_{p_1}(\beta)}{1-\phi_{p_1}(\beta)}\int_0^1 q_1(\tau)f^*(\tau,u(\tau),v(\tau))d\tau\right)ds,\end{aligned}\tag{4.1}$$

$$
\begin{aligned}
B(u,v)(t) = & \int_0^t \phi_{p_2}^{-1}\left(\int_s^1 q_2(\tau)g^*(\tau,u(\tau),v(\tau))d\tau\right. \\
& \left. + \frac{\phi_{p_2}(\beta)}{1-\phi_{p_2}(\beta)}\int_0^1 q_2(\tau)g^*(\tau,u(\tau),v(\tau))d\tau\right) ds \\
& + \frac{1}{1-\sum\limits_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_{p_2}^{-1}\left(\int_s^1 q_2(\tau)g^*(\tau,u(\tau),v(\tau))\, d\tau\right. \\
& \left. + \frac{\phi_{p_2}(\beta)}{1-\phi_{p_2}(\beta)}\int_0^1 q_2(\tau)g^*(\tau,u(\tau),v(\tau))d\tau\right) ds. \qquad (4.2)
\end{aligned}
$$

Define operator

$F(u,v)(t) = (A(u,v)(t), B(u,v)(t))$.

According to the definition of F and Lemma 3.2, it is easy to show that $F(K) \subset K$. By similar arguments in [5, 12], $F : K \to K$ is completely continuous.

Now we consider the following modified problem of (1.1) and (1.2):

$$
\begin{cases} (\phi_{p_1}(u'))' + q_1(t)f^*(t,u,v) = 0,\ t \in (0,1), \\ (\phi_{p_2}(v'))' + q_2(t)g^*(t,u,v) = 0,\ t \in (0,1), \end{cases} \qquad (4.3)
$$

$$
\begin{cases} u(0) = \sum\limits_{i=1}^{m-2} a_i u(\xi_i),\ u'(1) = \beta u'(0), \\ v(0) = \sum\limits_{i=1}^{m-2} a_i v(\xi_i),\ v'(1) = \beta v'(0), \end{cases} \qquad (4.4)
$$

From the condition (H_4), we have

$f^{*\rho_1}_{\rho_1\varphi(t)} < \phi_{p_1}(m_1),\ g^{*\rho_1}_{\rho_1\varphi(t)} < \phi_{p_2}(m_2),\ f^{*\rho_2}_{\gamma\rho_2} \geq \phi_{p_1}(M_1\gamma),\ g^{*\rho_2}_{\gamma\rho_2} \geq \phi_{p_2}(M_2\gamma)$,
$f^{*\rho_3}_{\rho_3\varphi(t)} \leq \phi_{p_1}(m_1),\ g^{*\rho_3}_{\rho_3\varphi(t)} \leq \phi_{p_2}(m_2)$.

Firstly, we show that $i_k(F, K^*_{\rho_1}) = 1$.

In fact, by (4.1), (4.2), $f^{*\rho_1}_{\rho_1\varphi(t)} < \phi_{p_1}(m_1)$ and $g^{*\rho_1}_{\rho_1\varphi(t)} < \phi_{p_2}(m_2)$, for $(u,v) \in \partial K^*_{\rho_1}$, we have

$$
\begin{aligned}
\|A(u,v)(t)\| &= \max_{0\leq t\leq 1} |A(u,v)(t)| = A(u,v)(1) \\
&= \int_0^1 \phi_{p_1}^{-1}\left(\int_s^1 q_1(\tau)f^*(\tau,u(\tau),v(\tau))d\tau\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\phi_{p_1}(\beta)}{1-\phi_{p_1}(\beta)}\int_0^1 q_1(\tau)f^*(\tau,u(\tau),v(\tau))d\tau\Bigg)ds\\
&+\frac{1}{1-\sum\limits_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\phi_{p_1}^{-1}\Bigg(\int_s^1 q_1(\tau)f^*(\tau,u(\tau),v(\tau))d\tau\\
&+\frac{\phi_{p_1}(\beta)}{1-\phi_{p_1}(\beta)}\int_0^1 q_1(\tau)f^*(\tau,u(\tau),v(\tau))d\tau\Bigg)ds\\
\le&\int_0^1\phi_{p_1}^{-1}\Bigg(\int_0^1 q_1(\tau)f^*(\tau,u(\tau),v(\tau))d\tau\\
&+\frac{\phi_{p_1}(\beta)}{1-\phi_{p_1}(\beta)}\int_0^1 q_1(\tau)f^*(\tau,u(\tau),v(\tau))d\tau\Bigg)ds\\
&+\frac{1}{1-\sum\limits_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\phi_{p_1}^{-1}\Bigg(\int_0^1 q_1(\tau)f^*(\tau,u(\tau),v(\tau))d\tau\\
&+\frac{\phi_{p_1}(\beta)}{1-\phi_{p_1}(\beta)}\int_0^1 q_1(\tau)f^*(\tau,u(\tau),v(\tau))d\tau\Bigg)ds\\
=&L\phi_{p_1}^{-1}\left(\frac{1}{1-\phi_{p_1}(\beta)}\int_0^1 q_1(\tau)f^*(\tau,u(\tau),v(\tau))d\tau\right)\\
<&L\phi_{p_1}^{-1}\left(\frac{\phi_{p_1}(\rho_1)\phi_{p_1}(m_1)}{1-\phi_{p_1}(\beta)}\int_0^1 q_1(\tau)d\tau\right)=\frac{\rho_1}{2m_1}m_1=\frac{\rho_1}{2}=\frac{\|(u,v)\|}{2},
\end{aligned}
$$

$$
\begin{aligned}
\|B(u,v)(t)\|=&\max_{0\le t\le 1}|B(u,v)(t)|=B(u,v)(1)\\
=&\int_0^1\phi_{p_2}^{-1}\Bigg(\int_s^1 q_2(\tau)g^*(\tau,u(\tau),v(\tau))d\tau\\
&+\frac{\phi_{p_2}(\beta)}{1-\phi_{p_2}(\beta)}\int_0^1 q_2(\tau)g^*(\tau,u(\tau),v(\tau))d\tau\Bigg)ds\\
&+\frac{1}{1-\sum\limits_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\phi_{p_2}^{-1}\Bigg(\int_s^1 q_2(\tau)g^*(\tau,u(\tau),v(\tau))d\tau\\
&+\frac{\phi_{p_2}(\beta)}{1-\phi_{p_2}(\beta)}\int_0^1 q_2(\tau)g^*(\tau,u(\tau),v(\tau))d\tau\Bigg)ds\\
\le&\int_0^1\phi_{p_2}^{-1}\Bigg(\int_0^1 q_2(\tau)g^*(\tau,u(\tau),v(\tau))d\tau\\
&+\frac{\phi_{p_2}(\beta)}{1-\phi_{p_2}(\beta)}\int_0^1 q_2(\tau)g^*(\tau,u(\tau),v(\tau))d\tau\Bigg)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{1-\sum\limits_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\phi_{p_2}^{-1}\left(\int_0^1 q_2(\tau)g^*(\tau,u(\tau),v(\tau))d\tau\right.\\
&\left.+\frac{\phi_{p_2}(\beta)}{1-\phi_{p_2}(\beta)}\int_0^1 q_2(\tau)g^*(\tau,u(\tau),v(\tau))d\tau\right)ds\\
=&L\phi_{p_2}^{-1}\left(\frac{1}{1-\phi_{p_2}(\beta)}\int_0^1 q_2(\tau)g^*(\tau,u(\tau),v(\tau))d\tau\right)\\
<&L\phi_{p_2}^{-1}\left(\frac{\phi_{p_2}(\rho_1)\phi_{p_2}(m_1)}{1-\phi_{p_2}(\beta)}\int_0^1 q_2(\tau)d\tau\right)=\frac{\rho_1}{2m_2}m_2=\frac{\rho_1}{2}=\frac{\|(u,v)\|}{2}.
\end{aligned}
$$

Therefore, $\|F(u,v)(t)\|=\|(A(u,v)(t),B(u,v)(t))\|=\|A(u,v)(t)\|+\|B(u,v)(t)\|<\|(u,v)\|$ for $(u,v)\in\partial K^*_{\rho_1}$. By Theorem 2.1, we have $i_k(F,K^*_{\rho_1})=1$.

Secondly, we show that $i_k(F,\Omega_{\rho_2})=0$.

Let $(e_1(t),e_2(t))\equiv(\frac{1}{2},\frac{1}{2})$ for $t\in[0,1]$, then $(e_1(t),e_2(t))\in\partial K_1$. We claim that

$(u(t),v(t))\neq F(u,v)(t)+\lambda(e_1(t),e_2(t)),\ (u,v)\in\partial\Omega_{\rho_2},\ \lambda\geq 0.$

In fact, if not, there exist $(u_0,v_0)\in\partial\Omega_{\rho_2}$ and $\lambda_0\geq 0$ such that

$(u_0(t),v_0(t))=F(u_0,v_0)(t)+\lambda_0(e_1(t),e_2(t)).$

Hence, from Lemma 3.3 and $f^{*\rho_2}_{\gamma\rho_2}\geq\phi_{p_1}(M_1\gamma)$, we have that for $t\in[\eta,1]$,

$$
\begin{aligned}
u_0(t)=&A(u_0,v_0)(t)+\lambda_0e_1(t)\geq\eta\|A(u_0,v_0)(t)\|+\frac{\lambda_0}{2}=\eta A(u_0,v_0)(1)+\frac{\lambda_0}{2}\\
=&\eta\int_0^1\phi_{p_1}^{-1}\left(\int_s^1 q_1(\tau)f^*(\tau,u_0(\tau),v_0(\tau))d\tau\right.\\
&\left.+\frac{\phi_{p_1}(\beta)}{1-\phi_{p_1}(\beta)}\int_0^1 q_1(\tau)f^*(\tau,u_0(\tau),v_0(\tau))d\tau\right)ds\\
&+\frac{\eta}{1-\sum\limits_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\phi_{p_1}^{-1}\left(\int_s^1 q_1(\tau)f^*(\tau,u_0(\tau),v_0(\tau))d\tau\right.\\
&\left.+\frac{\phi_{p_1}(\beta)}{1-\phi_{p_1}(\beta)}\int_0^1 q_1(\tau)f^*(\tau,u_0(\tau),v_0(\tau))d\tau\right)ds+\frac{\lambda_0}{2}\\
\geq&\eta\int_0^1\phi_{p_1}^{-1}\left(\int_s^1 q_1(\tau)f^*(\tau,u_0(\tau),v_0(\tau))d\tau\right.\\
&\left.+\frac{\phi_{p_1}(\beta)}{1-\phi_{p_1}(\beta)}\int_0^1 q_1(\tau)f^*(\tau,u_0(\tau),v_0(\tau))d\tau\right)ds+\frac{\lambda_0}{2}\\
>&\eta\int_\eta^1\phi_{p_1}^{-1}\left(\int_s^1 q_1(\tau)f^*(\tau,u_0(\tau),v_0(\tau))d\tau\right.\\
&\left.+\frac{\phi_{p_1}(\beta)}{1-\phi_{p_1}(\beta)}\int_s^1 q_1(\tau)f^*(\tau,u_0(\tau),v_0(\tau))d\tau\right)ds+\frac{\lambda_0}{2}\\
=&\eta\int_\eta^1\phi_{p_1}^{-1}\left(\frac{1}{1-\phi_{p_1}(\beta)}\int_s^1 q_1(\tau)f^*(\tau,u_0(\tau),v_0(\tau))d\tau\right)ds+\frac{\lambda_0}{2}\\
\geq&\eta\int_\eta^1\phi_{p_1}^{-1}\left(\frac{\phi_{p_1}(\rho_2)\phi_{p_1}(M_1\gamma)}{1-\phi_{p_1}(\beta)}\int_s^1 q_1(\tau)d\tau\right)ds+\frac{\lambda_0}{2}\\
=&M_1\gamma\rho_2\frac{1}{2M_1}+\frac{\lambda_0}{2}=\frac{\gamma\rho_2}{2}+\frac{\lambda_0}{2}.
\end{aligned}
$$

Similarly, from Lemma 3.3 and $g^{*\rho_2}_{\gamma\rho_2} \geq \phi_{p_2}(M_2\gamma)$, we have that for $t \in [\eta, 1]$, We can prove $v_0(t) > \dfrac{\gamma\rho_2}{2} + \dfrac{\lambda_0}{2}$. This implies that $\min\limits_{\eta\leq t\leq 1}(u(t) + v(t)) = \gamma\rho_2 > \gamma\rho_2 + \lambda_0$, which is a contradiction. Hence, by Theorem 2.1, it follows that $i_k(F, \Omega_{\rho_2}) = 0$.

Finally, similar to the proof of $i_k(F, K^*_{\rho_1}) = 1$, we can show that $i_k(F, K^*_{\rho_3}) = 1$. We can get the BVP (4.3), (4.4) has at least three positive solutions (u_1, v_1), (u_2, v_2) and (u_3, v_3) such that

$(u_1, v_1) \in K^*_{\rho_1}$, $(u_1, v_1) \in \Omega_{\rho_2}\backslash\overline{K^*_{\rho_1}}$, $(u_3, v_3) \in K^*_{\rho_3}\backslash\overline{\Omega_{\rho_2}}$.

As a result, the BVP (4.3), (4.4) has at least three positive solutions (u_1, v_1), (u_2, v_2) and (u_3, v_3) such that $u_1 + v_1, u_2 + v_2, u_3 + v_3 \in [\rho_1\varphi(t), \infty)$, and

$f^*(t, u, v) = f(t, u, v)$, $g^*(t, u, v) = g(t, u, v)$, $u + v \geq \rho_1\varphi(t)$,

which mean (u_1, v_1), (u_2, v_2) and (u_3, v_3) are also solutions of BVP (1.1), (1.2).

Similarly, we can obtain the following conclusions.

Theorem 4.2. Assume $(H_1) - (H_3)$ hold. In addition, the following condition (H_5) holds:
(H_5) There exist $\rho_1, \rho_2, \rho_3 \in (0, \infty)$, with $\rho_1 < \rho_2 < \gamma\rho_3$ such that

(1) $f(t, u, v)$, $g(t, u, v) > 0$, $t \in [0, 1]$, $u + v \in [\min\{\gamma\rho_1, \rho_2\varphi(t)\}, \infty)$;

(2) $f^{\rho_1}_{\gamma\rho_1} \geq \phi_{p_1}(M_1\gamma)$, $g^{\rho_1}_{\gamma\rho_1} \geq \phi_{p_2}(M_2\gamma)$, $f^{\rho_2}_{\rho_2\varphi(t)} < \phi_{p_1}(m_1)$, $g^{\rho_2}_{\rho_2\varphi(t)} < \phi_{p_2}(m_2)$,
$f^{\rho_3}_{\gamma\rho_3} \geq \phi_{p_1}(M_1\gamma)$, $g^{\rho_3}_{\gamma\rho_3} \geq \phi_{p_2}(M_2\gamma)$.

Then BVP (1.1) and (1.2) has at least two positive solutions in K.

Theorem 4.3. Assume $(H_1) - (H_3)$ hold. In addition, the following condition (H_6) holds:
(H_6) There exist $\rho_1, \rho_2 \in (0, \infty)$, with $\rho_1 < \gamma\rho_2$ such that

(1) $f(t, u, v)$, $g(t, u, v) > 0$, $t \in [0, 1]$, $u + v \in [\rho_1\varphi(t), \infty)$;

(2) $f^{\rho_1}_{\rho_1\varphi(t)} < \phi_{p_1}(m_1)$, $g^{\rho_1}_{\rho_1\varphi(t)} < \phi_{p_2}(m_2)$, $f^{\rho_2}_{\gamma\rho_2} \geq \phi_{p_1}(M_1\gamma)$, $g^{\rho_2}_{\gamma\rho_2} \geq \phi_{p_2}(M_2\gamma)$,
$0 \leq f^\infty < \phi_{p_1}(m_1)$, $0 \leq g^\infty < \phi_{p_2}(m_2)$.

Then BVP (1.1) and (1.2) has at least three positive solutions in K.

Proof. We show that (H_6) implies (H_4). Let $k \in (f^\infty, \phi_{p_1}(m_1))$. Then there exists $r > \rho_2$, such that $\max\limits_{t\in[0,1]} f(t, u, v) \leq k\phi_{p_1}(u + v)$ for $u + v \in [r, \infty)$ since $0 \leq f^\infty < \phi_{p_1}(m_1)$. Let

$$\alpha = \left\{\max_{t\in[0,1]} f(t, u, v) : \rho_1\varphi(t) \leq u + v \leq r\right\} \text{ and } \rho_3 > \max\left\{\phi^{-1}_{p_1}\left(\frac{\alpha}{\phi_{p_1(m)} - k}\right), \rho_2\right\}.$$

Then we have

$$\max_{t\in[0,1]} f(t, u, v) \leq k\phi_{p_1}(u+v)+\alpha \leq k\phi_{p_1}(\rho_3)+\alpha < \phi_{p_1}(m_1)\phi_{p_1}(\rho_3), \text{ for } u+v \in [\rho_1\varphi(t), \rho_3).$$

This implies that $f^{\rho_3}_{\rho_3\varphi(t)} < \phi_{p_1}(m_1)$. Similarly, $0 \leq g^\infty < \phi_{p_2}(m_2)$ implies that $g^{\rho_3}_{\rho_3\varphi(t)} < \phi_{p_2}(m_2)$. Hence, (H_4) holds, by Theorem 4.1, BVP (1.1), (1.2) has at least three positive solutions in K.

Theorem 4.4. Assume $(H_1) - (H_3)$ hold. In addition, the following condition (H_7) holds:
(H_7) There exist $\rho_1, \rho_2 \in (0, \infty)$, with $\rho_1 < \rho_2$ such that

(1) $f(t, u, v)$, $g(t, u, v) > 0$, $t \in [0, 1]$, $u + v \in [\min\{\gamma\rho_1, \rho_2\varphi(t)\}, \infty)$;

(2) $f^{\rho_1}_{\gamma\rho_1} \geq \phi_{p_1}(M_1\gamma)$, $g^{\rho_1}_{\gamma\rho_1} \geq \phi_{p_2}(M_2\gamma)$, $f^{\rho_2}_{\rho_2\varphi(t)} < \phi_{p_1}(m_1)$, $g^{\rho_2}_{\rho_2\varphi(t)} < \phi_{p_2}(m_2)$,
$\phi_{p_1}(M_1) < f_\infty \leq \infty$, $\phi_{p_2}(M_2) < g_\infty \leq \infty$.

Then BVP (1.1) and (1.2) has at least two positive solutions in K.

Proof. We show that (H_7) implies (H_5). Since $\phi_{p_1}(M_1) < f_\infty \leq \infty$, then there exists $\rho_3 > \dfrac{\rho_2}{\gamma}$, such that

$$\min_{t\in[\eta,1]} f(t, u, v) \geq \phi_{p_1}(u+v)\phi_{p_1}(M_1) \geq \phi_{p_1}(\gamma\rho_3)\phi_{p_1}(M_1) = \phi_{p_1}(\rho_3)\phi_{p_1}(M_1\gamma), u+v \in [\gamma\rho_3, \rho_3).$$

This implies that $f^{\rho_3}_{\gamma\rho_3} \geq \phi_{p_1}(M_1\gamma)$. Similarly, $\phi_{p_2}(M_2) < g_\infty \leq \infty$ implies that $g^{\rho_3}_{\gamma\rho_3} \geq \phi_{p_2}(M_2\gamma)$. Hence, (H_5) holds, by Theorem 4.2, BVP (1.1), (1.2) has at least two positive solutions in K.

By the arguments similar to that of Theorem 3.1 and Theorem 3.2, we obtain the following

results.
Theorem 4.5. Assume $(H_1) - (H_3)$ hold. In addition, the following condition (H_8) holds:
(H_8) There exist $\rho_1, \rho_2 \in (0, \infty)$, with $\rho_1 < \gamma\rho_2$ such that
(1) $f(t, u, v),\ g(t, u, v) > 0,\ t \in [0, 1],\ u + v \in [\rho_1\varphi(t), \infty)$;
(2) $f^{\rho_1}_{\rho_1\varphi(t)} \leq \phi_{p_1}(m_1),\ g^{\rho_1}_{\rho_1\varphi(t)} \leq \phi_{p_2}(m_2),\ f^{\rho_2}_{\gamma\rho_2} \geq \phi_{p_1}(M_1\gamma),\ g^{\rho_2}_{\gamma\rho_2} \geq \phi_{p_2}(M_2\gamma)$.
Then BVP (1.1), (1.2) has at least one positive solutions in K.
Theorem 4.6. Assume $(H_1) - (H_3)$ hold. In addition, the following condition (H_9) holds:
(H_9) There exist $\rho_1, \rho_2 \in (0, \infty)$, with $\rho_1 < \rho_2$ such that
(1) $f(t, u, v),\ g(t, u, v) > 0,\ t \in [0, 1],\ u + v \in [\min\{\gamma\rho_1, \rho_2\varphi(t)\}, \infty)$;
(2) $f^{\rho_1}_{\gamma\rho_1} \geq \phi_{p_1}(M_1\gamma),\ g^{\rho_1}_{\gamma\rho_1} \geq \phi_{p_2}(M_2\gamma),\ f^{\rho_2}_{\rho_2\varphi(t)} \leq \phi_{p_1}(m_1),\ g^{\rho_2}_{\rho_2\varphi(t)} \leq \phi_{p_2}(m_2)$.
Then BVP (1.1) and (1.2) has at least one positive solutions in K.

5 Example

Now we present an example to illustrate the main result.
Example 5.1. Consider the following BVP

$$\begin{cases} (|u'(t)|u'(t))' + q_1(t)f(t, u, v) = 0,\ t \in (0, 1), \\ (|v'(t)|v'(t))' + q_2(t)g(t, u, v) = 0,\ t \in (0, 1), \end{cases} \tag{5.1}$$

$$\begin{cases} u(0) = \frac{1}{2}u(\frac{1}{3}) + \frac{1}{4}u(\frac{2}{3}),\ u'(1) = \frac{1}{2}u'(0), \\ v(0) = \frac{1}{2}v(\frac{1}{3}) + \frac{1}{4}v(\frac{2}{3}),\ v'(1) = \frac{1}{2}v'(0), \end{cases} \tag{5.2}$$

where

$$f(t, u, v) = \begin{cases} \frac{1}{80}(1 + t)^{\frac{1}{2}}(u + v - \frac{t}{4})^{21} + \frac{1}{10^{35}}, & 0 \leq u + v \leq 2, \\ \frac{1}{80}(1 + t)^{\frac{1}{2}}(2 - \frac{t}{4})^{21} + \frac{1}{10^{35}}, & u + v > 2, \end{cases}$$

$$g(t, u, v) = \begin{cases} \frac{1}{40}(1 + t)^{\frac{1}{4}}(u + v - \frac{t}{4})^{19} + \frac{1}{10^{40}}, & 0 \leq u + v \leq 2, \\ \frac{1}{40}(1 + t)^{\frac{1}{4}}(2 - \frac{t}{4})^{19} + \frac{1}{10^{40}}, & u + v > 2, \end{cases}$$

$q_1(t) = q_2(t) = 1$.
Obviously, $p_1 = p_2 = 3,\ \beta = 1,\ \xi_1 = \frac{1}{3},\ \xi_2 = \frac{2}{3},\ a_1 = \frac{1}{2},\ a_2 = \frac{1}{4}$. Choose $\rho_1 = 1,\ \rho_2 = \frac{448\sqrt{3}}{9},\ \rho_3 = 1200,\ \eta = \frac{1}{4},\ \theta = \frac{1}{2}$, then $\varphi(t) = \frac{t}{2}$. We note $\gamma = \frac{3\sqrt{3}}{224},\ m_1 = m_2 = \frac{3\sqrt{3}}{28},\ M_1 = M_2 = 4$. Consequently, $f(t, u, v)$ satisfies
(1) $f(t, u, v),\ g(t, u, v) > 0,\ t \in [0, 1],\ u + v \in [\frac{t}{2}, \infty)$;
(2) $f^{\rho_1}_{\rho_1\varphi(t)} \leq 0.018 < \phi_{p_1}(m_1) \approx 0.034,\ g^{\rho_1}_{\rho_1\varphi(t)} \leq 0.03 < \phi_{p_2}(m_2) \approx 0.034$,
$f^{\rho_2}_{\gamma\rho_2} \geq 0.239 > \phi_{p_1}(M_1\gamma) \approx 0.009,\ g^{\rho_2}_{\gamma\rho_2} \geq 0.147 > \phi_{p_2}(M_2\gamma) \approx 0.009$,
$f^{\rho_3}_{\rho_3\varphi(t)} \leq 0.026 < \phi_{p_1}(m_1) \approx 0.034,\ g^{\rho_3}_{\rho_3\varphi(t)} \leq 0.011 < \phi_{p_2}(m_2) \approx 0.034$.
Thus with Theorem (4.1), BVP (5.1), (5.2) has at least three positive solutions in K.

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An S-partially contractive mapping with a control function ϕ

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Abstract. In this article, we introduce a ϕ-contraction principle in a partial S-metric space, we show the existence of a fixed point for a self mapping in a partial S-metric space. Also, we show that we have uniqueness only under some specific conditions.

Keywords. Partial S-metric space, Banach contraction principle, Fixed point.

1 Introduction and Preliminaries

Finding a fixed point for a self mapping on different types of metric spaces has been one the main topics of research in pure mathematics. it starts with the Banach contraction principle which was introduced by Banach in the early nineties.
Since the Banach contraction was introduced, many results were found in fixed point theory field in different type of metric spaces, such as [13], [14], [15],[16],[22], [23], [24],[25], [4], [5], [6], [8],[9], [10], [11], [12], [19], [20], [21],[26],[27],[28].

An S- metric space was introduced in [2].

Definition 1. [2] Let X be a nonempty set. An *S-metric space* on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions, for all $x, y, z, a \in X$:

- $S(x, y, z) \geq 0$,
- $S(x, y, z) = 0$ if and only if $x = y = z$,
- $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an *S-metric space.*

In this article, we are interested in partial S-metric space which was introduced in [1]. We recall some definitions of partial metric spaces and state some of their

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properties.

Definition 2. [1] Let X be a nonempty set. A *partial S-metric space* on X is a function $S_p : X^3 \to [0, \infty)$ that satisfies the following conditions, for all $x, y, z, t \in X$:

(P1) $x = y$ if and only if $S_p(x, x, x) = S_p(y, y, y) = S_p(x, x, y)$

(P2) $S_p(x, y, z) \leq S_p(x, x, t) + S_p(y, y, t) + S_p(z, z, t) - S_p(t, t, t)$

(P3) $S_p(x, x, x) \leq S_p(x, y, z)$

(P4) $S_p(x, x, y) = S_p(y, y, x)$.

The pair (X, S_p) is called a partial S-metric space.

We recall some definitions of partial S-metric spaces and state some of their properties.

Definition 3. A sequence $\{x_n\}_{n=0}^{\infty}$ of elements in X is called *Cauchy* if the limit $\lim_{n,m\to\infty} S_p(x_n, x_n, x_m)$ exists and finite. The partial S-metric space (X, S_p) is called *complete* if for each Cauchy sequence $\{x_n\}_{n=0}^{\infty}$ there exists $z \in X$ such that

$$S_p(z, z, z) = \lim_n S_p(z, z, x_n) = \lim_{n,m} S_p(x_n, x_n, x_m).$$

Also, (X, S_p) is a complete partial S-metric space if and only if (X, S_p^s) is a complete S-metric space. A sequence $\{x_n\}_n$ in a partial S-metric space (X, S_p) is called 0-*Cauchy* if $\lim_{n,m\to\infty} S_p(x_n, x_n, x_m) = 0$. We say that (X, S_p) is 0-*complete* if every 0-Cauchy in X converges to a point $x \in X$ such that $S_p(x, x, x) = 0$.

Example 1. *(see [1]) Let $X = \mathbb{Q} \cap [0, \infty)$ with the partial metric $p(x, y, z) = max\{x, y, z\}$. Then (X, S_p) is a 0-complete partial metric space which is not complete.*

Definition 4. Let (X, S_p) be a complete partial S-metric space. Set $\rho_p = \inf\{S_p(x, y, z) : x, y, z \in X\}$ and define the set $X_p = \{x \in X : S_p(x, x, x) = \rho_p\}$.

The following Lemma summarizes the relation between certain comparison functions that usually act as control functions in the studied contractive typed mappings in fixed point theory. For such a summary and fixed point theory for $\phi-$ contractive mappings, see [18].

Lemma 1. *Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function and relative to the function ϕ consider the following conditions:*

- *(i) ϕ is monotone increasing.*
- *(ii) $\phi(t) < t$ for all $t > 0$.*
- *(iii) $\phi(0) = 0$.*
- *(iv) ϕ is right uppersemicontinuous.*
- *(v) ϕ is right continuous.*
- *(vi) $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t \geq 0$.*

Then the following are valid:

- *(1) The conditions (i) and (ii) imply (iii).*
- *(2) The conditions (ii) and (v) imply (iii).*
- *(3) The conditions (i) and (vi) imply (ii).*
- *(4) The conditions (i) and (iv) imply imply (vi).*
- *(5) If ϕ satisfies (i) then (iv) $\Leftrightarrow$ (v).*

2 Main Results

Now, we prove our main result.

Theorem 1. *Let (X, S_p) be a complete partial S-metric space. Suppose $T : X \to X$ is a given self mapping satisfying:*

$$S_p(Tx, Tx, Ty) \leq \max\{\phi(S_p(x, x, y)), S_p(x, x, x), S_p(y, y, y)\}, \tag{1}$$

where ϕ is defined as in Lemma 1. Then:

(1) the set X_p is nonempty;

(2) there is a unique $u \in X_p$ such that $Tu = u$;

Proof. For any $x \in X$, we have $S_p(Tx, Tx, Tx) \leq S_p(x, x, x)$ and hence the sequence $\{S_p(T^n x, T^n x, T^n x)\}_{n \geq 0}$ is a nonincreasing sequence. Now Define

$$M_x := 2[f^{-1}(S_p(x, x, Tx)) + S_p(x, x, x)],$$

where $f(t) = t - \phi(t)$. Notice that $f(0) = 0$ (and hence $f^{-1}(0) = 0$) and $f(t) < t$ for $t > 0$ and hence $f^{-1}(t) > t$ for $t > 0$. Now we prove by induction that

$$S_p(T^n x, T^n x, x) \leq M_x, \ \ \forall n \geq 0. \tag{2}$$

Notice that the inequality (2) is true for $n = 0, 1$ since: $S_p(x, x, x) \leq M_x$ and $S_p(Tx, Tx, x) \leq f^{-1}(S_p(Tx, Tx, x)) \leq M_x$.

Suppose that (2) is true for each $n \leq n_0 - 1$ for some positive integer $n_0 \geq 2$. Then we have

$$\begin{aligned}
S_p(T^{n_0} x, T^{n_0} x, x) &\leq 2S_p(T^{n_0} x, T^{n_0} x, Tx) + S_p(Tx, Tx, x) \\
&\leq 2\max\{\phi(S_p(T^{n_0-1} x, T^{n_0-1} x, x)), S_p(T^{n_0-1} x, T^{n_0-1} x, T^{n_0-1} x), \\
&\quad S_p(x, x, x)\} + S_p(Tx, Tx, x) \\
&\leq 2\max\{\phi(S_p(T^{n_0-1} x, T^{n_0-1} x, x))), S_p(x, x, x)\} + S_p(Tx, Tx, x)
\end{aligned}$$

Therefore, we have two cases.

Case 1:

$$\begin{aligned}
S_p(T^{n_0} x, T^{n_0} x, x) &\leq \phi(S_p(T^{n_0-1} x, T^{n_0-1} x, Tx)) + S_p(Tx, Tx, x) \\
&\leq 2[\phi(f^{-1}(S_p(Tx, Tx, x)) + S_p(x, x, x))] + S_p(Tx, Tx, x) \\
&= 2[f^{-1}(S_p(Tx, Tx, x)) + S_p(x, x, x) - f(f^{-1}(S_p(Tx, Tx, x)) \\
&\quad + S_p(x, x, x))] + S_p(Tx, Tx, x) \\
&\leq M_x - 2f(f^{-1}(S_p(Tx, Tx, x)) + S_p(x, x, x)) + S_p(Tx, Tx, x) \\
&= M_x - S_p(Tx, Tx, x) - S_p(x, x, x) \leq M_x.
\end{aligned}$$

Case 2:

$$\begin{aligned}
S_p(T^{n_0} x, T^{n_0} x, x) &\leq S_p(x, x, x) + S_p(Tx, Tx, x) \\
&\leq S_p(x, x, x) + f^{-1}(S_p(Tx, Tx, x)) = M_x.
\end{aligned}$$

Hence, we obtain (2). Next we prove that the sequence $\{S_p(T^n x, T^n x, T^n x)\}_{n \geq 0}$ is Cauchy. Equivalently, we show that

$$\lim_{n,m \to \infty} S_p(T^n x, T^n x, T^m x) = r_x, \tag{3}$$

where $r_x := \inf_n S_p(T^nx, T^nx, T^nx)$. Its clear that $r_x \leq S_p(T^nx, T^nx, T^nx) \leq S_p(T^nx, T^nx, T^mx)$ for all n, m. Also, given any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S_p(T^{n_0}x, T^{n_0}x, T^{n_0}x) < r_x + \epsilon$ and $\phi^{n_0}(2M_x) < r_x + \epsilon$. Therefore, for any $m, n > 2n_0$ we have

$$\begin{aligned}
r_x &\leq S_p(T^nx, T^nx, T^mx)\\
&\leq \max\{\phi(S_p(T^{n-1}x, T^{n-1}x, T^{m-1}x)), S_p(T^{n-1}x, T^{n-1}x, T^{n-1}x),\\
&\quad S_p(T^{m-1}x, T^{m-1}x, T^{m-1}x)\}\\
&\leq \max\{\phi^2(S_p(T^{n-2}x, T^{n-2}x, T^{m-2}x)), S_p(T^{n-2}x, T^{n-2}x, T^{n-2}x),\\
&\quad S_p(T^{m-2}x, T^{m-2}x, T^{m-2}x)\}\\
&\leq \max\{\phi^{n_0}(S_p(T^{n-n_0}x, T^{n-n_0}x, T^{m-n_0}x)),\\
&\quad S_p(T^{n-n_0}x, T^{n-n_0}x, T^{n-n_0}x), S_p(T^{m-n_0}x, T^{m-n_0}x, T^{m-n_0}x)\}\\
&\leq \max\{\phi^{n_0}(S_p(T^{n-n_0}x, T^{n-n_0}x, x) + S_p(T^{m-n_0}x, T^{m-n_0}x, x)),\\
&\quad S_p(T^{n-n_0}x, T^{n-n_0}x, T^{n-n_0}x), S_p(T^{m-n_0}x, T^{m-n_0}x, T^{m-n_0}x)\}\\
&< \max\{\phi^{n_0}(2M_x), r_x + \epsilon, r_x + \epsilon\}\\
&< r_x + \epsilon.
\end{aligned}$$

Hence, we obtain (3). Since (X, S_p) is a complete partial S-metric space, there exists $z \in X$ such that $S_p(z, z, z) = r_x$. Next, we show that $S_p(z, z, z) = p(Tz, Tz, z)$.

Next, we show that $S_p(z, z, z) = S_p(z, z, Tz) = S_p(Tz, Tz, z)$. For each natural number n we have

$$S_p(z, z, Tz) \leq 2S_p(z, z, z_n) - S_p(z_n, z_n, z_n) + S_p(Tz, Tz, z_n).$$

From the contraction condition of our theorem, we deduce that there exists a subsequence of natural numbers $\{n_l\}$ such that $S_p(Tz, Tz, z_{n_l}) \leq \phi(S_p(z, z, z_{n_l-1}))$, for $l \geq 1$, or $S_p(Tz, Tz, z_{n_l}) \leq S_p(z, z, z)$ for $l \geq 1$, or $S_p(Tz, z, z_{n_l}) \leq S_p(z_{n_l-1}, z_{n_l-1}, z_{n_l-1})$, for $l \geq 1$, in all of these three cases, if we take the limit as l goes toward ∞ we get $S_p(z, z, Tz) \leq S_p(z, z, z)$. But, we know by the property (iv) of the partial S-metric space that $S_p(z, z, z) \leq S_p(z, z, Tz)$. Therefore,

$$S_p(z, z, z) = S_p(z, z, Tz). \tag{4}$$

Now we show that X_p (see Definition 4) is nonempty. For each $k \in \mathbb{N}$ choose $x_k \in X$ with $S_p(x_k, x_k, x_k) < \rho_p + 1/k$, where $x_k = T^kx$. First, we prove that

$$\lim_{m,n\to\infty} S_p(z_n, z_n, z_m) = \rho_p. \tag{5}$$

Given $\epsilon > 0$, take $n_0 := [f^{-1}(3/\epsilon)] + 1$. If $k > n_0$, then

$$\begin{aligned}
\rho_p &\leq S_p(Tz_k, Tz_k, Tz_k) \leq S_p(z_k, z_k, z_k) = r_{x_k} \leq S_p(x_k, x_k, x_k) < \rho_p + 1/k\\
&< \rho_p + 1/n_0 < \rho_p + 1/f^{-1}(3/\epsilon).
\end{aligned}$$

Set $U_k := S_p(z_k, z_k, z_k) - S_p(Tz_k, Tz_k, Tz_k)$. Then $U_k < 1/f^{-1}(3/\epsilon)$ for $k > n_0$. Thus, if $m, n > n_0$ then by (4) and the fact that f (and hence f^{-1}) is increasing, we have

$$\begin{aligned}
S_p(z_n, z_n, z_m) &\leq S_p(z_n, z_n, Tz_n) + S_p(Tz_n, Tz_n, Tz_m) + S_p(Tz_m, Tz_m, z_m) \\
&\quad -S_p(Tz_n, Tz_n, Tz_n) - S_p(Tz_m, Tz_m, Tz_m) \\
&= U_n + U_m + S_p(Tz_n, Tz_n, Tz_m) \\
&< 2/f^{-1}(3/\epsilon) + \max\{\phi(S_p(z_n, z_n, z_m)), S_p(z_n, z_n, z_n), S_p(z_m, z_m, z_m)\} \\
&\leq \max\{f^{-1}\left(2/f^{-1}(3/\epsilon)\right), 3/f^{-1}(3/\epsilon) + \rho_p\} \\
&\leq \max\{f^{-1}(2\epsilon/3)), \rho_p + \epsilon\} \\
&\leq \rho_p + \epsilon + f^{-1}(2\epsilon/3).
\end{aligned}$$

Therefore, if we let $\epsilon \to 0$ we get (5). Since (X, S_p) is a complete partial metric space, there exists $u \in X$ such that $S_p(u, u, u) = \lim_{m,n\to\infty} S_p(z_n, z_n, z_m) = \rho_p$. Consequently, $u \in X_p$ and hence X_p is nonempty.

Now choose an arbitrary $x \in X_p$. Then

$$\rho_p \leq S_p(Tz, Tz, Tz) \leq S_p(Tz, Tz, z) = S_p(z, z, z) = r_x = \rho_p,$$

which, using P2, implies that $Tz = z$. To prove uniqueness of the fixed point we suppose that $u, v \in X_p$ are both fixed points of T. Then

$$\begin{aligned}
\rho_p \leq S_p(u, u, v) = S_p(Tu, Tu, Tv) &\leq \max\{\phi(S_p(u, u, v)), S_p(u, u, u), S_p(v, v, v)\} \\
&\leq \max\{\phi(p(u, v)), \rho_p\}.
\end{aligned}$$

Case 1: $\rho_p \leq S_p(u, u, v) \leq \rho_p \Rightarrow S_p(u, u, v) = \rho_p = S_p(u, u, u) = S_p(v, v, v) \Rightarrow u = v$.

Case 2:

$$\begin{aligned}
& S_p(u, u, v) \leq \phi(S_p(u, u, v)) \\
\Rightarrow\quad & S_p(u, u, v) - \phi(S_p(u, u, v)) \leq 0 \\
\Rightarrow\quad & f(S_p(u, u, v)) \leq 0 \\
\Rightarrow\quad & f(S_p(u, u, v)) = 0 \\
\Rightarrow\quad & S_p(u, u, v) = 0 \\
\Rightarrow\quad & u = v.
\end{aligned}$$

Thus, the fixed point is unique. □

Note that the above theorem does not guarantee uniqueness of the fixed point in X. However, if (1) is replaced by the condition below, we can show uniqueness in X.

In the next result, we change our contraction condition so that we obtain uniqueness of the fixed point in the whole space X.

Theorem 2. *Let (X, S_p) be a complete partial S-metric space. Suppose $T : X \to X$ is a given self mapping satisfying:*

$$S_p(Tx,Tx,Ty) \leq \max\left\{\phi(S_p(x,x,y)), \frac{S_p(x,x,x)+S_p(y,y,y)}{2}\right\}, \tag{6}$$

where $\phi : [0,\infty) \to [0,\infty)$ is as in Theorem 1. Then there is a unique point $z \in X$ such that $Tz = z$. Furthermore, $z \in X_p$.

Proof. Using Theorem 1 we only need to prove uniqueness. Suppose there exists $u, v \in X$ such that $Tu = u$ and $Tv = v$. Now

$$S_p(u,u,v) = S_p(Tu,Tu,Tv) \leq \max\left\{\phi(S_p(u,u,v)), \frac{S_p(u,u,u)+S_p(v,v,v)}{2}\right\}.$$

Case 1:

$$\begin{aligned} & S_p(u,u,v) \leq \phi(S_p(u,u,v)) \\ \Rightarrow\quad & S_p(u,u,v) - \phi(S_p(u,u,v)) \leq 0 \\ \Rightarrow\quad & f(S_p(u,u,v)) \leq 0 \\ \Rightarrow\quad & f(S_p(u,u,v)) = 0 \\ \Rightarrow\quad & S_p(u,u,v) = 0 \\ \Rightarrow\quad & u = v. \end{aligned}$$

Case 2:

$$\begin{aligned} & S_p(u,u,v) \leq \frac{S_p(u,u,u)+S_p(v,v,v)}{2} \\ \Rightarrow\quad & 2S_p(u,u,v) - S_p(u,u,u) - S_p(v,v,v) \leq 0 \\ \Rightarrow\quad & 2S_p(u,u,v) - S_p(u,u,u) - S_p(v,v,v) = 0 \\ \Rightarrow\quad & u = v. \end{aligned}$$

□

As a consequence of Theorem2, we obtain the following Corollary.

Corollary 1. *Let (X, S_p) be a 0-complete partial S-metric space. Suppose $T : X \to X$ is a given self mapping satisfying:*

$$S_p(Tx,Tx,Ty) \leq \phi(S_p(x,x,y)), \tag{7}$$

where $\phi : [0,\infty) \to [0,\infty)$ is an increasing function such that $f(t) = t - \phi(t)$ is increasing with f^{-1} is right continuous at 0. Also assume $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t \geq 0$ (and hence $\phi(0) = 0, \phi(t) < t$ for $t > 0$). Then there is a unique $z \in X$ such that $Tz = z$. Also $S_p(z,z,z) = 0$.

Example 2. *Let* $X = [0,1] \cup [3,4]$. *Define* $S_p : X^3 \to [0,\infty)$, $T : X \to X$ *and* $\phi : [0,\infty) \to [0,\infty)$ *as follows:*

$$S_p(x,y,z) = \max\{x,y,z\}$$
$$T(x) = \begin{cases} \frac{x}{2} & , \quad x \in [0,1] \\ \frac{7}{5} & , \quad x \in [3,4] \end{cases}$$
$$\phi(t) = \frac{t}{1+t}$$

The above definitions satisfy the hypothesis of Theorem 2. In particular, we make the following observations:

- (X,p) *is a complete partial metric space.*
- *We can easily prove by induction that* $\phi^n(t) = \frac{t}{1+nt}$ *which implies that* $\lim_{n\to\infty} \phi^n(t) = 0$.
- T *satisfies condition (6):*

  1) If $\{x,y,z\} \cap [3,4] \neq \emptyset$ *then*

$$\begin{aligned} S_p(Tx,Ty,Tz) &= \max\{Tx,Ty,Tz\} = \frac{7}{5} \\ &\leq \max\left\{\phi(S_p(x,y,z)), \frac{S_p(x,x,x)+S_p(y,y,y)}{2}\right\} \end{aligned}$$

  2) If $\{x,y,z\} \subset [0,1]$ *then*

$$\begin{aligned} S_p(Tx,Ty,Tz) &= \max\{Tx,Ty,Tz\} = \max\left\{\frac{x}{2},\frac{y}{2},\frac{z}{2}\right\} \\ &\leq \max\left\{\phi(S_p(x,y)), \frac{S_p(x,x,x)+S_p(y,y,y)}{2}\right\}. \end{aligned}$$

By Theorem 2, there is a unique fixed point which is $z = 0$.

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Approximation by complex q-Gamma operators in compact disks

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Abstract. In this paper, the order of simultaneous approximation and Voronovskaya type theorems with quantitative estimate for complex q-Gamma operators attached to analytic functions in compact disks are obtained.

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1 Introduction

In recent years, an intensive research has been conducted on polynomials and operators in compact disks, such as [1], [3]-[8].

For a real function of real variable f: $[0, \infty) \to \mathbb{R}$, it is well known that the Gamma operators are given by $G_n(f;x) = \frac{1}{x^n\Gamma(n)} \int_0^\infty f(t/n)t^{n-1}e^{-t/x}dt$, $x \in [0, \infty)$. In 2005, Zeng [9] obtained the approximation properties of G_n defined above, supposed f satisfies exponential growth condition. he studied the approximation properties to the locally bounded functions and the absolutely continuous functions and obtained some good properties in real disks.

In this paper, we introduce complex q-Gamma operators as follows

$$G_{n,q}(f;z) = \frac{1}{z^n\Gamma_q(n)} \int_0^{\infty/A} f\left(\frac{t}{[n]_q}\right) t^{n-1} E_q\left(-\frac{qt}{z}\right) d_q t. \tag{1}$$

We give a suitable exponential growth condition in a parabolic domain for $f(z)$. Let $\mathbb{D}_R$=$\{z \in \mathbb{C}; |z| < R\}$ be with $1 < R < \infty$ and suppose that $f : [R, +\infty) \cup \overline{\mathbb{D}_R} \to \mathbb{C}$ is continuous in $[R, +\infty) \cup \overline{\mathbb{D}_R}$, analytic in $\mathbb{D}_R$, *i.e.* $f(z) = \sum_{k=0}^\infty c_k z^k$, for all $z \in \mathbb{D}_R$, and

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that there exist $M, C, B > 0$ and $A \in \left(\frac{1}{R}, 1\right)$, with the property $|c_k| \leq Mq^{k(k-1)/2}\frac{A^k}{[k]_q!}$, for all $k = 0, 1, ...$, which implies $|f(z)| \leq ME_q(A|z|)$ for all $z \in \mathbb{D}_R$ and $|f(x)| \leq Ce^{Bx}$, for all $x \in [R, +\infty)$.

We recall some concepts of q-calculus. All of the results can be found in [7]. For any fixed real number $0 < q \leq 1$ and each nonnegative integer k, we denote q-integers by $[k]_q$, where

$$[k]_q = \begin{cases} \frac{1-q^k}{1-q}, & q \neq 1; \\ k, & q = 1. \end{cases}$$

Also q-factorial and q-binomial coefficients are defined as follows:

$$[k]_q! = \begin{cases} [k]_q[k-1]_q...[1]_q, & k = 1, 2, ...; \\ 1, & k = 0, \end{cases}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad (n \geq k \geq 0).$$

The q-improper integrals are defined as

$$\int_0^{\infty/A} f(x)d_qx = (1-q)\sum_{-\infty}^{\infty} f\left(\frac{q^n}{A}\right)\frac{q^n}{A}, \quad A > 0,$$

provided the sums converge absolutely.
The $q-$analogs $e_q(x)$ and $E_q(x)$ of the exponential function are given as

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!} = \frac{1}{(1-(1-q)x)_q^{\infty}}, \quad |x| < \frac{1}{1-q}, \quad |q| < 1,$$

$$E_q(x) = \sum_{k=0}^{\infty} q^{k(k-1)/2}\frac{x^k}{[k]_q!} = (1+(1-q)x)_q^{\infty}, \quad |q| < 1,$$

where $(1-x)_q^{\infty} = \prod_{j=0}^{\infty}(1-q^jx)$. It is easily observed that $e_q(x)E_q(-x) = e_q(-x)E_q(x) = 1$. The q-Gamma integral is defined as

$$\Gamma_q(t) = \int_0^{\infty/A} x^{t-1}E_q(-qx)d_qx, \quad t > 0, \tag{2}$$

which satisfies the following functional equations: $\Gamma_q(t+1) = [t]_q\Gamma_q(t), \quad \Gamma_q(1) = 1$.

2 Auxiliary Results

In the sequel, we suppose that $e_k(t) = t^k$, $k = 0, 1, 2, ...$ In order to obtain the main results, we need the following lemmas:

Lemma 2.1. *For $n \in \mathbb{N}$ and $z \in \mathbb{C}$, we have the following identities:*

$$G_{n,q}(e_k;z) = \frac{[n+k-1]_q!}{[n-1]_q![n]_q^k}e_k(z), \tag{3}$$

$$G_{n,q}(e_k;z) = \frac{[n+k-1]_q z}{[n]_q}G_n(e_{k-1};z). \tag{4}$$

Proof. From (1) and (2), we have

$$\begin{aligned} G_{n,q}(e_k;z) &= \frac{1}{z^n\Gamma_q(n)}\int_0^{\infty/A}\left(\frac{t}{[n]_q}\right)^k t^{n-1}E_q\left(-\frac{qt}{z}\right)d_qt \\ &= \frac{z^k}{[n]_q^k\Gamma_q(n)}\int_0^{\infty/A}\left(\frac{t}{z}\right)^{n+k-1}E_q\left(-\frac{qt}{z}\right)d_q\left(\frac{t}{z}\right) \\ &= \frac{\Gamma_q(n+k)z^k}{[n]_q^k[n-1]_q!} = \frac{[n+k-1]_q!}{[n-1]_q![n]_q^k}e_k(z), \end{aligned}$$

so we proved (3), and (4) is easily obtained according to (3). □

Lemma 2.2. *If f is analytic in D_R, $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in D_R$, then for all $n \in \mathbb{N}$ and $1 \le r \le R$, we have*

$$G_{n,q}(f;z) = \sum_{k=0}^{\infty} c_k \cdot G_{n,q}(e_k;z). \tag{5}$$

Proof. By Lemma 2.1, we obtain that $G_{n,q}(e_k;z)$ is a polynomial of degree $\le k$, $k = 0,1,2,...$ for all $z \in \mathbb{C}$. From the hypothesis on f in section 1, it follows that $G_{n,q}(f;z)$ is analytic in D_R (see [2], pp. 1171-1172 and p. 1178). Therefore, it is easy to obtain Lemma 2.2. □

3 Main Results

We start with the following quantitative estimates of the convergence for complex q-Gamma operators attached to an analytic function in a disk of radius $R > 1$ and center 0.

Theorem 3.1. *Let $\mathbb{D}_R$=$\{z \in \mathbb{C}; |z| < R\}$ be with $1 < R < \infty$ and suppose that $f : [R,+\infty)\bigcup\overline{\mathbb{D}_R} \to \mathbb{C}$ is continuous in $[R,+\infty)\cup\overline{\mathbb{D}_R}$, analytic in $\mathbb{D}_R$, i.e. $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$, and $f(z)$ satisfies exponential-type growth condition in the statement of section 1.*

(i) Let $1 \le r < \frac{1}{A}$ be arbitrary fixed. For all $|z| \le r$, $n \ge 2$ ($n \in \mathbb{N}$), we have

$$|G_{n,q}(f;z) - f(z)| \le \frac{L_{q,r,A}}{[n]_q},$$

where $L_{q,r,A} = Mr^2A^2C_{q,r,A}$, $C_{q,r,A}$ is a constant depends only on q, r, A.

(ii) For the simultaneous approximation by complex q-Gamma operators, we have: if $1 \le$

$r \leq r_1 < \frac{1}{A}$ are arbitrary fixed, then for all $|z| \leq r$ and $n, p \in \mathbb{N}$, $n \geq 2$, we have

$$|G_{n,q}^{(p)}(f;z) - f^{(p)}(z)| \leq \frac{L_{q,r_1,A}}{[n]_q} \frac{p! r_1}{(r_1 - r)^{p+1}},$$

where $L_{q,r_1,A}$ is defined at the above point (i).

Proof. (i) Suppose that $|z| \leq r$, by Lemma 2.2, we have $G_{n,q}(f;z) = \sum_{k=0}^{\infty} c_k G_{n,q}(e_k;z)$, so we get

$$|G_{n,q}(f;z) - f(z)| \leq \sum_{k=0}^{\infty} |c_k| \cdot |G_{n,q}(e_k;z) - e_k(z)| = \sum_{k=2}^{\infty} |c_k| \cdot |G_{n,q}(e_k;z) - e_k(z)|,$$

since $G_{n,q}(e_0;z) = e_0(z) = 1$ and $G_{n.q}(e_1;z) = e_1(z) = z$.

By Lemma 2.1, for all $|z| \leq r$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
& |G_{n,q}(e_k;z) - e_k(z)| \\
= & \left| \frac{[n+k-1]_q z}{[n]_q} G_{n,q}(e_{k-1};z) - \frac{[n+k-1]_q z}{[n]_q} e_{k-1}(z) + \frac{[n+k-1]_q z}{[n]_q} e_{k-1}(z) - e_k(z) \right| \\
\leq & \left| \frac{[n+k-1]_q z}{[n]_q} \right| \cdot |G_{n,q}(e_{k-1};z) - e_{k-1}(z)| + |e_k(z)| \cdot \left| \frac{[n+k-1]_q}{[n]_q} - 1 \right| \\
\leq & \frac{[n+k-1]_q r}{[n]_q} |G_{n,q}(e_{k-1};z) - e_{k-1}(z)| + \frac{[k-1]_q r^k}{[n]_q},
\end{aligned}$$

step by step, we get by the above recurrence that

$$\begin{aligned}
& |G_{n,q}(e_k;z) - e_k(z)| \\
\leq & \frac{[n+k-1]_q}{[n]_q} \frac{[n+k-2]_q}{[n]_q} \dots \frac{[n+2]_q}{[n]_q} \frac{q^n r^k}{[n]_q} + \frac{[n+k-1]_q}{[n]_q} \frac{[n+k-2]_q}{[n]_q} \dots \frac{[n+3]_q}{[n]_q} \frac{q^n [2]_q r^k}{[n]_q} \\
& + \frac{[n+k-1]_q}{[n]_q} \frac{[n+k-2]_q}{[n]_q} \dots \frac{[n+4]_q}{[n]_q} \frac{q^n [3]_q r^k}{[n]_q} + \dots + \frac{[n+k-1]_q}{[n]_q} \frac{[n+k-2]_q}{[n]_q} \frac{q^n [k-3]_q r^k}{[n]_q} \\
& + \frac{[n+k-1]_q}{[n]_q} \frac{q^n [k-2]_q r^k}{[n]_q} + \frac{q^n [k-1]_q r^k}{[n]_q} \\
\leq & \frac{[n+k-1]_q}{[n]_q} \frac{[n+k-2]_q}{[n]_q} \dots \frac{[n+2]_q}{[n]_q} r^k q^n \left(\frac{1}{[n]_q} + \frac{[2]_q}{[n]_q} + \dots + \frac{[k-1]_q}{[n]_q} \right) \\
= & \frac{[n+k-1]_q!}{[n+1]_q! [n]_q^{k-2}} \frac{(1 + [2]_q + \dots + [k-1]_q) q^n r^k}{[n]_q} \leq \frac{[n+k-1]_q!}{[n+1]_q! [n]_q^{k-2}} \frac{[k]_q [k-1]_q r^k}{[n]_q},
\end{aligned}$$

for all $|z| \leq r$ and $n \in \mathbb{N}$.

From Lemma 2.2 and the hypothesis on c_k, immediately implies for all $n \geq 2$ and

$|z| \leq r$

$$\begin{aligned}
|G_{n,q}(f;z)-f(z)| &\leq \sum_{k=2}^{\infty}|c_k|\cdot|G_{n,q}(e_k;z)-e_k(z)| \\
&\leq M\sum_{k=2}^{\infty}\frac{[n+k-1]_q!}{[n+1]_q![n]_q^{k-2}}\frac{[k]_q[k-1]_q r^k}{[n]_q}\frac{q^{k(k-1)/2}A^k}{[k]_q!} \\
&\leq \frac{Mr^2A^2}{[n]_q}\sum_{k=2}^{\infty}\begin{bmatrix} n+k-1 \\ k-2 \end{bmatrix}_q\left(\frac{rA}{[n]_q}\right)^{k-2},
\end{aligned}$$

by Heine's binomial formula (see [7]), we have

$$\sum_{k=0}^{\infty}\begin{bmatrix} n+k+1 \\ k \end{bmatrix}_q\left(\frac{rA}{[n]_q}\right)^{k} = \frac{1}{\left(1-\frac{rA}{[n]_q}\right)_q^{n+2}}$$

$$\leq \left\{\begin{array}{ll}
\dfrac{1}{\left(1-\frac{rA}{[n_0]_q}\right)_q^{n_0+2}}, & n=n_0, \\
\dfrac{1}{\left(1-\frac{rA}{[n]_q}\right)_q^{\infty}} = \displaystyle\sum_{j=0}^{\infty}\frac{\left(\frac{rA}{(1-q)[n]_q}\right)^j}{[j]_q!} = e_q\left(\frac{rA}{[n]_q}\right), & n=\infty,
\end{array}\right. \leq C_{q,r,A},$$

where, n_0 is a finite number and $C_{q,r,A}$ is a constant depends only on $q,\ r,\ A$. Thus

$$|G_{n,q}(f;z)-f(z)| \leq \frac{Mr^2A^2C_{q,r,A}}{[n]_q},$$

therefore we have

$$|G_{n,q}(f;z)-f(z)| \leq \frac{L_{q,r,A}}{[n]_q},$$

where $L_{q,r,A} = Mr^2A^2C_{q,r,A}$, for all $1 \leq r < \frac{1}{A}$.

(ii) Denoting by γ the circle of radius $r_1 > r$ and center 0, since for any $|z| \leq r$ and $v \in \gamma$, we have $|v-z| \geq r_1 - r$, by Cauchy's formulas it follows that for all $|z| \leq r$ and $n \in \mathbb{N}$, $n \geq 2$, we have

$$\begin{aligned}
|G_{n,q}^{(p)}(f;z)-f^{(p)}(z)| &= \frac{p!}{2\pi}\left|\int_\gamma \frac{G_{n,q}(f;v)-f(v)}{(v-z)^{p+1}}dv\right| \\
&\leq \frac{L_{q,r_1,A}}{[n]_q}\frac{p!}{2\pi}\frac{2\pi r_1}{(r_1-r)^{p+1}} = \frac{L_{q,r_1,A}}{[n]_q}\frac{p!r_1}{(r_1-r)^{p+1}},
\end{aligned}$$

which proves (ii) and Theorem 3.1. □

Next, we will give Voronovskaya type result in compact disks, for complex q-Gamma operators attached to an analytic function in D_R, $R > 1$ and center 0.

Theorem 3.2. *Suppose that $f:\overline{\mathbb{D}_R}\cup[R,\infty)\to\mathbb{C}$ is continuous and bounded in $\overline{\mathbb{D}_R}\cup[R,\infty)$ and analytic in $\mathbb{D}_R$. Let $1\le r<\frac{1}{A}$ be arbitrary fixed, $n\ge 2$ $(n\in\mathbb{N})$, then we have the following Voronovskaya type result*

$$\left|G_{n,q}(f;z)-f(z)-\frac{z^2f''(z)}{[2]_q[n]_q}\right|\le\frac{J_{q,r,A}}{[n]_q^2},$$

where $J_{q,r,A}=\frac{1}{[2]_q^2}\sum_{k=3}^{\infty}[k-2]_q^2[k-1]_q(rA)^k<\infty$.

Proof. Denoting $E_{q,k,n}(z)=G_{n,q}(e_k;z)-e_k(z)-\frac{q^n[k]_q[k-1]_qe_k(z)}{[2]_q[n]_q}$, since $E_{q,0,n}(z)=E_{q,1,n}(z)=E_{q,2,n}(z)=0$, then we have

$$\left|G_{n,q}(f;z)-f(z)-\frac{z^2f''(z)}{[2]_q[n]_q}\right|\le\sum_{k=3}^{\infty}|c_k|\cdot|E_{q,k,n}(z)|,$$

so, it remains to estimate $E_{q,k,n}(z)$ for $k\ge 3$.

By Lemma 2.1 and simple calculation, we have

$$E_{q,k,n}(z)=\frac{[n+k-1]_qz}{[n]_q}E_{q,k-1,n}(z)+\frac{q^n[k-1]_q[k-2]_q\left([n+k-1]_q-q^2[n]_q\right)}{[2]_q[n]_q^2}z^k,$$

this implies, for all $|z|\le r$, $k\ge 3$, $n\in\mathbb{N}$,

$$|E_{q,k,n}(z)|\le\frac{[n+k-1]_qr}{[n]_q}|E_{q,k-1,n}(z)|+\frac{q^n[k-1]_q^2[k-2]_q}{[2]_q[n]_q^2}r^k,$$

taking in the last inequality, $k=3,4,\ldots$, and reasoning by recurrence, we obtain

$$\begin{aligned}|E_{q,k,n}(z)| &\le \frac{[n+k-1]_q}{[n]_q}\frac{[n+k-2]_q}{[n]_q}\cdots\frac{[n+3]_q}{[n]_q}\frac{q^n\cdot[2]_q^2}{[2]_q[n]_q^2}r^k\\ &\quad+\frac{[n+k-1]_q}{[n]_q}\frac{[n+k-2]_q}{[n]_q}\cdots\frac{[n+4]_q}{[n]_q}\frac{q^n[2]_q\cdot[3]_q^2}{[2]_q[n]_q^2}r^k\\ &\quad+\ldots+\frac{[n+k-1]_q}{[n]_q}\frac{q^n[k-3]_q[k-2]_q^2}{[2]_q[n]_q^2}r^k+\frac{q^n[k-2]_q[k-1]_q^2}{[2]_q[n]_q^2}r^k\\ &\le \frac{[n+k-1]_q}{[n]_q}\frac{[n+k-2]_q}{[n]_q}\cdots\frac{[n+3]_q}{[n]_q}\frac{q^nr^k}{[2]_q[n]_q^2}\sum_{j=3}^{k}[j-2]_q[j-1]_q^2\\ &\le \frac{[n+k-1]_q}{[n]_q}\frac{[n+k-2]_q}{[n]_q}\cdots\frac{[n+3]_q}{[n]_q}\frac{[k-2]_q^2[k-1]_q^2q^nr^k}{[2]_q[n]_q^2},\end{aligned}$$

by the hypothesis on c_k, we have

$$\begin{aligned}&\left|G_{n,q}(f;z)-f(z)-\frac{z^2f''(z)}{[2]_q[n]_q}\right|\le\sum_{k=3}^{\infty}|c_k|\cdot|E_{q,k,n}(z)|\\ \le\ &\frac{1}{[2]_q[n]_q^2}\sum_{k=3}^{\infty}\frac{[n+k-1]_q}{[n]_q[k-1]_q}\frac{[n+k-2]_q}{[n]_q[k-2]_q}\cdots\frac{[n+3]_q}{[3]_q[n]_q}\frac{[k-2]_q^2[k-1]_q^2(rA)^k}{[2]_q[k]_q}\\ \le\ &\frac{1}{[2]_q^2[n]_q^2}\sum_{k=3}^{\infty}[k-2]_q^2[k-1]_q(rA)^k,\end{aligned}$$

for all $|z| \leq r$ and $n \geq 2$ ($n \in \mathbb{N}$). Since the series $\sum_{k=3}^{\infty} u^{k+1}$ and its q-derivative $\sum_{k=3}^{\infty}[k+1]_q u^k$ are uniformly and absolutely convergent in any compact disk included in the open unit disk, therefore, for $1 \leq r < \frac{1}{A}$, we have $\frac{1}{[2]_q^2[n]_q^2}\sum_{k=3}^{\infty}[k-2]_q^2[k-1]_q(rA)^k < \infty$. □

Denoting with $||P_k||_r = max\{|P_k(z)| : |z| \leq r\}$, where $P_k(z)$ is a complex polynomial of degree $\leq$ k. Now we will give the exact order of approximation by complex q-Gamma operators.

Theorem 3.3. *In the hypothesis of Theorem 3.1, if f is not a polynomial of degree ≤ 1 in the case (i), we have*

$$||G_{n,q}(f) - f||_r \geq \frac{1}{[n]_q} U_r(f), \quad n \in \mathbb{N},$$

where the constant $U_r(f)$ depends only on f and r.

Proof. Applying the norm $||\cdot||_r$ to the identity

$$G_{n,q}(f;z) - f(z) = \frac{1}{[n]_q}\left\{\frac{z^2}{[2]_q} f''(z) + \frac{1}{[n]_q}\left[[n]_q^2\left(G_{n,q}(f;z) - f(z) - \frac{z^2}{[2]_q[n]_q} f''(z)\right)\right]\right\},$$

we get

$$||G_{n,q}(f) - f||_r \geq \frac{1}{[n]_q}\left\{\left|\left|\frac{e_2}{[2]_q} f''\right|\right|_r - \frac{1}{[n]_q}\left[[n]_q^2\left|\left|G_{n,q}(f) - f - \frac{e_2}{[2]_q[n]_q} f''\right|\right|_r\right]\right\}.$$

Since f is not a polynomial of degree ≤ 1 in $\mathbb{D}_R$, it follows that $||\frac{e_2}{[2]_q} f''||_r > 0$. Indeed, supposing the contrary it follows that $z^2 f''(z) = 0$ for all $z \in \overline{\mathbb{D}_R}$, therefore we get $f''(z) = 0$ for all $z \in \overline{\mathbb{D}_R}$, by the uniqueness of analytic functions we get $f''(z) = 0$ for all $z \in \overline{\mathbb{D}_R}$, that is f is a linear function in $\mathbb{D}_R$, which is in contradiction with the hypothesis.

Now, by Theorem 3.2, we have

$$\left|G_{n,q}(f;z) - f(z) - \frac{z^2 f''(z)}{[2]_q[n]_q}\right| \leq \frac{1}{[2]^2[n]_q^2}\sum_{k=3}^{\infty}[k-2]_q^2[k-1]_q(rA)^k.$$

Therefore, there exists an index n_0 (depending only on f and r) such that for all $n \geq n_0$, we have

$$\left|\left|\frac{e_2}{[2]_q} f''\right|\right|_r - \frac{1}{[n]_q}\left[[n]_q^2\left|\left|G_{n,q}(f) - f - \frac{e_2}{[2]_q[n]_q} f''\right|\right|_r\right] \geq \frac{1}{[2]_q^2}||e_2 f''||_r,$$

which implies

$$||G_{n,q}(f) - f||_r \geq \frac{1}{[2]_q^2}||e_2 f''||_r,$$

for all $n \geq n_0$.

For $1 \leq n \leq n_0 - 1$, we have

$$||G_{n,q}(f) - f||_r \geq \frac{1}{[n]_q}([n]_q||G_{n,q}(f) - f||_r) = \frac{1}{[n]_q} V_{r,n}(f) > 0,$$

Therefore, finally we obtain

$$||G_{n,q}(f)-f||_r \geq \frac{1}{[n]_q}U_r(f),$$

for all n, with $U_r(f)=min\left\{V_{r,1}(f),V_{r,2}(f),...,V_{r,n_0}(f),\frac{1}{[2]_q^2}||e_2f''||_r\right\}$. □

Combining Theorem 3.1 with Theorem 3.3, we immediately get the following result:

Corollary 3.4. *In the hypothesis of Theorem 3.1 and Theorem 3.3, we have*

$$||G_{n,q}(f)-f||_r \sim \frac{1}{[n]_q}, \quad n\in\mathbb{N}.$$

Theorem 3.5. *In the hypothesis of Theorem 3.1, if $1\leq r\leq r_1<\frac{1}{A}$ are arbitrary fixed and f is not a polynomial of degree $\leq p-1$, then for all $|z|\leq r$ and $n,p\in\mathbb{N}$ $(n\geq 2)$, we have*

$$||G_{n,q}^{(p)}(f)-f^{(p)}||_r \sim \frac{1}{[n]_q}.$$

Proof. Taking into account the upper estimate in case (ii) of Theorem 3.1, it remains to prove the lower estimate only.

Denoting by Γ the circle of radius $r_1>r$ and center 0, since for any $|z|\leq r$ and $v\in\Gamma$, we have $|v-z|\geq r_1-r$, by Cauchy's formulas it follows that for all $|z|\leq r$ and $n\in\mathbb{N}$, we get

$$G_{n,q}^{(p)}(f;z)-f^{(p)}(z)=\frac{p!}{2\pi i}\int_\Gamma \frac{G_{n,q}(f;v)-f(v)}{(v-z)^{p+1}}dv, \tag{6}$$

as we have the identity

$$G_{n,q}(f;z)-f(z)=\frac{1}{[n]_q}\left\{\frac{z^2}{[2]_q}f''(z)+\frac{1}{[n]_q}\left[[n]_q^2\left(G_{n,q}(f;z)-f(z)-\frac{z^2}{[2]_q[n]_q}f''(z)\right)\right]\right\}, \tag{7}$$

applying (7) to (6), we have

$$\begin{aligned}
&G_{n,q}^{(p)}(f;z)-f^{(p)}(z)\\
=\;&\frac{1}{[n]_q}\left\{\frac{p!}{2\pi i}\int_\Gamma \frac{v^2f''(v)}{[2]_q(v-z)^{p+1}}dv+\frac{1}{[n]_q}\frac{p!}{2\pi i}\int_\Gamma \frac{[n]_q^2\left[G_{n,q}(f;v)-f(v)-\frac{v^2}{[2]_q[n]_q}f''(v)\right]}{(v-z)^{p+1}}dv\right\}\\
=\;&\frac{1}{[n]_q}\left\{\left|\left|\left(\frac{e_2f''}{[2]_q}\right)^{(p)}\right|\right|_r+\frac{1}{[n]_q}\frac{p!}{2\pi i}\int_\Gamma \frac{[n]_q^2\left[G_{n,q}(f;v)-f(v)-\frac{v^2}{[2]_q[n]_q}f''(v)\right]}{(v-z)^{p+1}}dv\right\},
\end{aligned}$$

applying the norm $||\cdot||_r$ to the above identity, we have

$$\begin{aligned}
&||G_{n,q}^{(p)}(f)-f^{(p)}||_r\\
\geq\;&\frac{1}{[n]_q}\left\{\left|\left|\left(\frac{e_2f''}{[2]_q}\right)^{(p)}\right|\right|_r-\frac{1}{[n]_q}\left|\left|\frac{p!}{2\pi}\int_\Gamma \frac{[n]_q^2\left[G_{n,q}(f;v)-f(v)-\frac{v^2}{[2]_q[n]_q}f''(v)\right]}{(v-z)^{p+1}}dv\right|\right|_r\right\},
\end{aligned}$$

by using Theorem 3.2, we have

$$\left\|\frac{p!}{2\pi}\int_{\Gamma}\frac{[n]_q^2\left[G_{n,q}(f;v)-f(v)-\frac{v^2}{[2]_q[n]_q}f''(v)\right]}{(v-z)^{p+1}}dv\right\|_r \leq \frac{p!}{2\pi}\frac{2\pi r_1}{(r_1-r)^{p+1}}J_{q,r_1,A}=\frac{J_{q,r_1,A}p!r_1}{(r_1-r)^{p+1}},$$

by the hypothesis on f, we have $\left\|\left(\frac{e_2 f''}{[2]_q}\right)^{(p)}\right\|_r > 0$, reasoning exactly as in the proof of Theorem 3.3, we immediately get the desired result. □

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Value sharing of meromorphic functions of differential polynomials of finite order

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Abstract

In this paper, we shall study the uniqueness problems on meromorphic functions of differential polynomials of finite order sharing a value. Our results improve or generalize many previous results on value sharing of meromorphic functions, such as Fang and Hua, Yang and Hua, Lin and Yi, Zhang, Xu, et al.

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1 Introduction and main results

Let $\mathbb{C}$ denote the complex plane and $f(z)$ be a non-constant meromorphic function on $\mathbb{C}$. We assume the reader is familiar with the standard notion used in the Nevanlinna value distribution theory such as $T(r, f)$, $m(r, f)$, $N(r, f)$ (see [7, 13, 14]), and $S(r, f)$ denotes any quantity that satisfies the condition $S(r, f) = o(T(r, f))$ as $r \to \infty$ outside of a possible exceptional set of finite linear measure. A meromorphic function $a(z)$ is called a small function with respect to $f(z)$, provided that $T(r, a) = S(r, f)$.

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Let $a \in \mathbb{C} \bigcup \{\infty\}$, we say that $f(z)$, $g(z)$ share a CM (counting multiplicities) if $f(z)-a$, $g(z)-a$ have the same zeros with the same multiplicities and we say that $f(z)$, $g(z)$ share a IM (ignoring multiplicities) if we do not consider the multiplicities. $N_k(r, f)$ denotes the truncated counting function bounded by k.

Define the order of f as

$$\sigma(f) = \limsup_{r \longrightarrow \infty} \frac{\log^+ T(r, f)}{\log r},$$

The following well known theorem in value distribution theory was posed by Hayman and settled by several authors almost at the same time [2, 4].

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Theorem A: *Let $f(z)$ be a transcendental meromorphic function, $n \geq 1$ a positive integer. Then $f^n f' = 1$ has infinitely many solutions.*

Fang and Hua [5], Yang and Hua [12] got a unicity theorem respectively corresponding to Theorem A.

Theorem B: *Let f and g be two non-constant entire (meromorphic) functions, $n \geq 6(n \geq 11)$ be a positive integer. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $4(c_1c_2)^{n+1}c^2 = -1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^{n+1} = 1$.*

Note that $f^n(z)f'(z) = \frac{1}{n+1}(f^{n+1}(z))'$, Fang [6] considered the case of kth derivative and proved

Theorem C: *Let f and g be two non-constant entire functions, and let n, k be two positive integers with $n > 2k+4$. If $(f^n(z))^{(k)}$ and $(g^n(z))^{(k)}$ share 1 CM, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$.*

Theorem D: *Let f and g be two non-constant entire functions, and let n, k be two positive integers with $n > 2k+8$. If $(f^n(z)(f(z)-1))^{(k)}$ and $(g^n(z)(g(z)-1))^{(k)}$ share 1 CM, then $f(z) \equiv g(z)$.*
For more results on this field, see [8, 9, 17]. Corresponding to Theorems C and D, It is natural to ask the following question.

Question 1.1. *Does Theorem C or D holds if f and g are meromorphic functions?*

Remark 1.1. Question 1.1 seems to have been solved by Bhoosnurmath and Dyavanal [3], but their proofs contain some gaps that were pointed out by Zhang [15, Annex remarks], Xu et al [10, Remark 2], respectively. The gaps have not been filled as far as we know. Here we give a partial answer to Problem 1.1.

Theorem 1.1. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions with $\sigma(f) < +\infty$. Let n, k be two positive integers with $n > \max\{3k+8, 2(\sigma(f)-1)k\}$. If $[f^n(z)]^{(k)}$ and $[g^n(z)]^{(k)}$ share 1 CM, then one of the following two conclusions holds:*
(1) $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$;
(2) $f = c_3e^{dz}$, $g = c_4e^{-dz}$, where c_3, c_4 and d are constants such that $(-1)^k(c_3c_4)^n(nd)^{2k} = 1$.

Remark 1.2. Theorem 1.1 affirmatively answered Problems 1.1. Namely, Theorem C holds for the case of meromorphic functions of finite order, provided that n is sufficiently large. But unfortunately, Theorems D fails if $f(z)$ and $g(z)$ are meromorphic functions without the condition $\Theta(\infty, f) > 2/n$, even if f and g share ∞ CM. We give the following counterexample.

Example 1.1. *Let*

$$f(z) = \frac{h(z)(1-h^n(z))}{1-h^{n+1}(z)}, \quad g(z) = \frac{1-h^n(z)}{1-h^{n+1}(z)}, \tag{1.1}$$

where n is a positive integer and $h(z)$ is a non-constant meromorphic function.

We deduce from (1.1) that $f^n(f-1) = g^n(g-1)$, thus f and g satisfy the conditions of Theorem D, but $f \not\equiv g$.
Note that

$$T(r, f) = T(r, gh) = nT(r, h) + S(r, f).$$

By the second fundamental theorem, we deduce

$$\overline{N}(r, f) = \sum_{j=1}^{n} \overline{N}(\frac{1}{h - a_j}) \geq (n-2)T(r, h) + S(r, f),$$

where $a_j (\neq 1)$ $(j = 1, 2, \cdots, n)$ are distinct roots of the algebraic equation $h^{n+1} = 1$.
Therefore,

$$\Theta(\infty, f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, f)}{T(r, f)} \leq 2/n.$$

When $n > 3k + 8$, then $\frac{n}{2k} + 1 > \frac{5}{2}$, so from Theorem 1.1 we have

Corollary 1.1. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions with $\sigma(f) < 3$. Let n, k be two positive integers with $n > 3k+8$. If $[f^n(z)]^{(k)}$ and $[g^n(z)]^{(k)}$ share 1 CM, then one of the following two conclusions holds:*
(1) $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$;
(2) $f = c_3e^{dz}$, $g = c_4e^{-dz}$, where c_3, c_4 and d are constants such that $(-1)^k(c_3c_4)^n(nd)^{2k} = 1$.

Consider IM sharing value and we have

Theorem 1.2. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions with $\sigma(f) < +\infty$. Let n, k be two positive integers with $n > \max\{9k + 14, 2(\sigma(f) - 1)k\}$. If $[f^n(z)]^{(k)}$ and $[g^n(z)]^{(k)}$ share 1 IM, then one of the following two conclusions holds:*
(1) $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$;
(2) $f = c_3e^{dz}$, $g = c_4e^{-dz}$, where c_3, c_4 and d are constants such that $(-1)^k(c_3c_4)^n(nd)^{2k} = 1$.

Corollary 1.2. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions with $\sigma(f) < 6$. Let n, k be two positive integers with $n > 9k + 14$. If $[f^n(z)]^{(k)}$ and $[g^n(z)]^{(k)}$ share 1 IM, then one of the following two conclusions holds:*
(1) $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$;
(2) $f = c_3e^{dz}$, $g = c_4e^{-dz}$, where c_3, c_4 and d are constants such that $(-1)^k(c_3c_4)^n(nd)^{2k} = 1$.

2 Preliminary lemmas and a main proposition

Lemma 2.1. *[11] Let $f(z)$ be a non-constant meromorphic function and let $a_0(z), a_1(z), \cdots, a_n(z) (\not\equiv 0)$ be small functions of f. Then*

$$T(r, a_nf^n + a_{n-1}f^{n-1} + \cdots + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.2. *[16] Let $f(z)$ be a non-constant meromorphic function, s, k be two positive integers. Then*

$$N_s(r, \frac{1}{f^{(k)}}) \leq T(r, f^{(k)}) - T(r, f) + N_{s+k}(r, \frac{1}{f}) + S(r, f),$$

$$N_s(r, \frac{1}{f^{(k)}}) \leq k\overline{N}(r, f) + N_{s+k}(r, \frac{1}{f}) + S(r, f).$$

Lemma 2.3. *Let $f(z)$ be a non-constant meromorphic function of finite order, and let k be a positive integer. Suppose that $f^{(k)} \not\equiv 0$, then*

$$N(r, \frac{1}{f^{(k)}}) \leq N(r, \frac{1}{f}) + k\overline{N}(r, f) + O(\log r).$$

Proof. Note that f is of finite order, by Lemma $1.4'$ in [14, P. 21], we have

$$m(r, \frac{f'}{f}) = O(\log r).$$

Now we prove $m(r, \frac{f^{(k)}}{f}) = O(\log r)$ by mathematical induction. Suppose that the conclusion is true for the case of $k = m$, if $k = m + 1$, we have

$$\frac{f^{(m+1)}}{f} = (\frac{f^{(m)}}{f})' + \frac{f^{(m)}}{f}\frac{f'}{f}.$$

Then we get

$$\begin{aligned} m(r, \frac{f^{(m+1)}}{f}) &\leq m(r, (\frac{f^{(m)}}{f})') + m(r, \frac{f^{(m)}}{f}) + m(r, \frac{f'}{f}) + O(1) \\ &= m(r, \frac{(\frac{f^{(m)}}{f})'}{\frac{f^{(m)}}{f}}\frac{f^{(m)}}{f}) + O(\log r) \\ &\leq m(r, \frac{(\frac{f^{(m)}}{f})'}{\frac{f^{(m)}}{f}}) + m(r, \frac{f^{(m)}}{f}) + O(\log r) \\ &= O(\log r). \end{aligned}$$

Moreover, we have

$$m(r, \frac{1}{f}) \leq m(r, \frac{1}{f^{(k)}}) + m(r, \frac{f^{(k)}}{f}) = m(r, \frac{1}{f^{(k)}}) + O(\log r).$$

Hence

$$T(r, f) - N(r, \frac{1}{f}) \leq T(r, f^{(k)}) - N(r, \frac{1}{f^{(k)}}) + O(\log r).$$

That is

$$\begin{aligned} N(r,\frac{1}{f^{(k)}}) &\leq T(r,f^{(k)})-T(r,f)+N(r,\frac{1}{f})+O(\log r) \\ &= m(r,f^{(k)})+N(r,f^{(k)})-T(r,f)+N(r,\frac{1}{f})+O(\log r) \\ &\leq m(r,f)+m(r,\frac{f^{(k)}}{f})+N(r,f)+k\overline{N}(r,f)-T(r,f)+N(r,\frac{1}{f})+O(\log r) \\ &= N(r,\frac{1}{f})+k\overline{N}(r,f)+O(\log r). \end{aligned}$$

This completes the proof of Lemma 2.3.

Lemma 2.4. *[12] Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and n,k be two positive integers, a be a finite nonzero constant. If f and g share a CM, then one of the following cases holds:*
(i) $T(r,f) \leq N_2(r,1/f)+N_2(r,1/g)+N_2(r,f)+N_2(r,g)+S(r,f)+S(r,g)$, the same inequality holding for $T(r,g)$;
(ii) $fg\equiv a^2$; (iii) $f\equiv g$.

Lemma 2.5. *Let $f(z)$ and $g(z)$ be non-constant meromorphic functions, n,k be two positive integers with $n>k+2$, a be a finite nonzero constant. If $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share a IM. Then $T(r,f)=O(T(r,g))$, $T(r,g)=O(T(r,f))$ and $\sigma(f)=\sigma(g)$.*

Proof. Let $F=f^n$. By the second fundamental theorem for small functions, we have

$$T(r,F^{(k)})\leq \overline{N}(r,f)+\overline{N}(r,\frac{1}{F^{(k)}})+\overline{N}(r,\frac{1}{F^{(k)}-a})+S(r,F). \tag{2.1}$$

By (2.1) and Lemma 2.1 and Lemma 2.2 with $s=1$ applied to F, we have

$$\begin{aligned} nT(r,f) &\leq N_{k+1}(r,\frac{1}{F})+\overline{N}(r,\frac{1}{F^{(k)}-a})+\overline{N}(r,f)+S(r,F) \\ &\leq (k+1)\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{[f^n]^{(k)}-a})+\overline{N}(r,f)+S(r.f) \\ &\leq (k+2)T(r,f)+\overline{N}(r,\frac{1}{[g^n]^{(k)}-a})+S(r,f). \end{aligned}$$

Namely,

$$\begin{aligned} (n-k-2)T(r,f) &\leq \overline{N}(r,\frac{1}{[g^n]^{(k)}-a})+S(r,f) \\ &\leq n(k+1)T(r,g)+S(r,f). \end{aligned}$$

Since $n>k+2$, we have $T(r,f)=O(T(r,g))$. Similarly we have $T(r,g)=O(T(r,f))$. Thus $\sigma(f)=\sigma(g)$.
This completes the proof of Lemma 2.5.

By the arguments similar to the proof of Lemma 2.5, we get the following proposition.

Proposition 2.1. *Let f be a transcendental meromorphic function, n, k be two positive integers with $n > k+2$, $a(z)(\not\equiv 0, \infty)$ be a small function of f. Then $[f^n]^{(k)} - a(z)$ has infinitely many zeros.*

Lemma 2.6. *[10] Let f and g be two non-constant meromorphic functions, $k, n > 2k+1$ be two positive integers. If $[f^n]^{(k)} = [g^n]^{(k)}$, then $f = tg$ for a constant t such that $t^n = 1$.*

Lemma 2.7. *Let f, g be two nonconstant meromorphic functions with $\sigma(f) < +\infty$, n, k be two positive integers with $n > \max\{3k+8, 2(\sigma(f)-1)k\}$. If $[f^n]^{(k)}[g^n]^{(k)} = 1$, then $f = c_3 e^{dz}$, $g = c_4 e^{-dz}$, where c_3, c_4 and d are constants such that $(-1)^k (c_3 c_4)^n (nd)^{2k} = 1$.*

Proof. Note that $n > k+2$, $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 1 IM. Then by Lemma 2.5 we get $\sigma(f) = \sigma(g) < +\infty$.

First, we prove

$$f \neq 0, \quad g \neq 0. \tag{2.2}$$

Suppose that z_0 is a zero of f with multiplicity s, then z_0 is a pole of g, say multiplicity t, and z_0 is a zero of $[f^n]^{(k)}$ with multiplicity $ns-k$, a pole of $[g^n]^{(k)}$ with multiplicity $nt+k$, thus we have

$$ns - k = nt + k,$$

namley

$$n(s-t) = 2k. \tag{2.3}$$

Note that $n > 3k+8$ and we get a contradiction from (2.3). Thus f has no zero. Similarly, g has no zero. Thus (2.2) holds.
Next we prove

$$N(r,f) = O(\log r), \quad N(r,g) = O(\log r). \tag{2.4}$$

Rewrite $[f^n]^{(k)}[g^n]^{(k)} = 1$ as

$$[f^n]^{(k)} = \frac{1}{[g^n]^{(k)}}. \tag{2.5}$$

We deduce from (2.5) that

$$N(r, [f^n]^{(k)}) = N(r, \frac{1}{[g^n]^{(k)}}). \tag{2.6}$$

As $N(r, [f^n]^{(k)}) = nN(r,f) + k\overline{N}(r,f)$, this together with (2.2), (2.6) and Lemma 2.3 implies that

$$nN(r,f) + k\overline{N}(r,f) \leq k\overline{N}(r,g) + O(\log r). \tag{2.7}$$

Similarly we get

$$nN(r,g) + k\overline{N}(r,g) \leq k\overline{N}(r,f) + O(\log r). \tag{2.8}$$

Combining (2.7) and (2.8) yields

$$N(r,f)+N(r,g)=O(\log r). \tag{2.9}$$

Thus we obtain (2.4), which means that both f and g have at most finitely many poles. Let

$$f=\frac{e^{h(z)}}{p(z)},\quad g=\frac{e^{h_1(z)}}{q(z)}, \tag{2.10}$$

where $p(z)$ and $q(z)$ are polynomials with $\deg(p(z))=p$, $\deg(q(z))=q$, $h(z)$ and $h_1(z)$ are nonconstant entire functions. By Corollary 1 in [14, P. 65], $h(z)$ and $h_1(z)$ are polynomials with $\deg(h(z))=\deg(h_1(z))=h=\sigma(f)$. Then

$$f^n=\frac{e^{nh(z)}}{p^n(z)},\quad g^n=\frac{e^{nh_1(z)}}{q^n(z)}. \tag{2.11}$$

Let $H(z)=nh(z)$, $P(z)=p^n(z)$, $H_1(z)=nh_1(z)$, $Q(z)=q^n(z)$. By mathematical induction we get that

$$[f^n]^{(k)}=\frac{e^{H(z)}P_k(z)}{P^{k+1}(z)},\quad [g^n]^{(k)}=\frac{e^{H_1(z)}Q_k(z)}{Q^{k+1}(z)}, \tag{2.12}$$

where $P_k(z)$ and $Q_k(z)$ are two polynomials with $\deg(P_k(z))=k(h-1+np)$ and $\deg(Q_k(z))=k(h-1+nq)$. By $[f^n]^{(k)}[g^n]^{(k)}=1$, we have

$$h(z)+h_1(z)\equiv C, \tag{2.13}$$

where C is a constant. Furthermore, we get

$$\deg(P_k(z))+\deg(Q_k(z))=\deg(P^{k+1}(z)Q^{k+1}(z)), \tag{2.14}$$

which implies that

$$2k(h-1)=n(p+q). \tag{2.15}$$

By (2.4), if

$$N(r,f)+N(r,g)\neq 0, \tag{2.16}$$

then $p+q\geq 1$, we deduce from (2.15) that

$$n\leq 2k(h-1)=2k(\sigma(f)-1), \tag{2.17}$$

which contradicts the assumption. Therefore

$$N(r,f)+N(r,g)=0, \tag{2.18}$$

namely both f and g are entire functions and $p=q=0$. From (2.15) we obtain that $h=1$. Thus $h(z)=dz+l_3$, $h_1(z)=-dz+l_4$.
Rewrite f and g as

$$f=c_3e^{dz},\qquad g=c_4e^{-dz},$$

where c_3,c_4 and d are nonzero constants. We deduce that $(-1)^k(c_3c_4)^n(nd)^{2k}=1$.

This completes the proof of Lemma 2.7.

Lemma 2.8. *[1] Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and n, k be two positive integers, a be a finite nonzero constant. If f and g share a IM, then one of the following cases holds:*
(i)$T(r,f) \leq N_2(r,1/f)+N_2(r,1/g)+N_2(r,f)+N_2(r,g)+2\overline{N}(r,1/f)+\overline{N}(r,1/g)+2\overline{N}(r,f)+\overline{N}(r,g)+S(r,f)+S(r,g)$*, the similar inequality holding for* $T(r,g)$*;*
(ii) $fg \equiv a^2$; *(iii)* $f \equiv g$.

3 Proof of Theorem 1.1

Let $F = [f^n]^{(k)}$, $G = [g^n]^{(k)}$, $F^* = f^n$, $G^* = g^n$, then F and G share 1 CM.
Thus by Lemma 2.5, one of the following cases holds:
(i) $T(r,F) \leq N_2(r,1/F) + N_2(r,1/G) + N_2(r,F) + N_2(r,G) + S(r,F) + S(r,G)$, the same inequality holding for $T(r,G)$;
(ii) $FG \equiv 1$; *(iii)* $F \equiv G$.

Case *(i)***.** By Lemma 2.1 and Lemma 2.2 with $s = 2$, we obtain

$$\begin{aligned}
T(r,F^*) &\leq N_{k+2}(r,1/F^*) + N_{k+2}(r,1/G^*) + (k+2)\overline{N}(r,g) + 2\overline{N}(r,f) \\
&\quad +S(r,f) + S(r,g) \\
&\leq (k+2)\overline{N}(r,1/f) + (k+2)\overline{N}(r,1/g) + (k+2)\overline{N}(r,g) + 2\overline{N}(r,f) \\
&\quad +S(r,f) + S(r,g) \\
&\leq (2k+4)T(r,g) + (k+4)T(r,f) + S(r,f) + S(r,g),
\end{aligned}$$

namely

$$nT(r,f) \leq (2k+4)T(r,g) + (k+4)T(r,f) + S(r,f) + S(r,g). \tag{3.1}$$

Similarly we have

$$nT(r,g) \leq (2k+4)T(r,f) + (k+4)T(r,g) + S(r,f) + S(r,g). \tag{3.2}$$

From (3.1) and (3.2) we deduce that

$$(n-3k-8)(T(r,f) + T(r,g) \leq S(r,f) + S(r,g), \tag{3.3}$$

which is a contradiction since $n > 3k + 8$.

Case *(ii)***.** We have $[f^n]^{(k)}[g^n]^{(k)} = 1$. By Lemma 2.7 we get the conclusion (2) of Theorem 1.1.

Case *(iii)***.** We have $[f^n]^{(k)} \equiv [g^n]^{(k)}$. By Lemma 2.6 we get the conclusion (1) of Theorem 1.1.

This completes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

Let $F=[f^n]^{(k)}$, $G=[g^n]^{(k)}$, $F^*=f^n$, $G^*=g^n$, then F and G share 1 IM.
Thus by Lemma 2.8, one of the following cases holds:
(i) $T(r,F) \le N_2(r,1/F)+N_2(r,1/G)+N_2(r,F)+N_2(r,G)+2\overline{N}(r,1/F)+\overline{N}(r,1/G)+2\overline{N}(r,F)+\overline{N}(r,G)+S(r,F)+S(r,G)$, the same inequality holding for $T(r,G)$;
(ii) $FG\equiv 1$; (iii) $F\equiv G$.

Case (i)**.** By Lemma 2.1 and Lemma 2.2 with $s=1,2$, we obtain

$$\begin{aligned} T(r,F^*) &\le N_{k+2}(r,1/F^*)+N_{k+2}(r,1/G^*)+(k+2)\overline{N}(r,g)+2\overline{N}(r,f)\\ &\quad 2(N_{k+1}(r,1/F^*)+k\overline{N}(r,f))+N_{k+1}(r,1/G^*)+k\overline{N}(r,g)\\ &\quad +2\overline{N}(r,f)+\overline{N}(r,g)+S(r,f)+S(r,g)\\ &\le (3k+4)\overline{N}(r,1/f)+(2k+3)\overline{N}(r,1/g)+(2k+4)\overline{N}(r,f)\\ &\quad +(2k+3)\overline{N}(r,g)+S(r,f)+S(r,g)\\ &\le (5k+8)T(r,f)+(4k+6)T(r,g)+S(r,f)+S(r,g), \end{aligned}$$

namely

$$nT(r,f)\le (5k+8)T(r,f)+(4k+6)T(r,g)+S(r,f)+S(r,g). \tag{4.1}$$

Similarly we have

$$nT(r,g)\le (5k+8)T(r,g)+(4k+6)T(r,f)+S(r,f)+S(r,g). \tag{4.2}$$

From (4.1) and (4.2) we deduce that

$$(n-9k-14)(T(r,f)+T(r,g)\le S(r,f)+S(r,g), \tag{4.3}$$

which is a contradiction since $n>9k+14$.

Case (ii)**.** We have $[f^n]^{(k)}[g^n]^{(k)}=1$. By Lemma 2.7 we get the conclusion (2) of Theorem 1.2.

Case (iii)**.** We have $[f^n]^{(k)}\equiv[g^n]^{(k)}$. By Lemma 2.6 we get the conclusion (1) of Theorem 1.2.

This completes the proof of Theorem 1.2.

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Regularized optimization method for determining the space-dependent source in a parabolic equation without iteration☆

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Abstract

In this paper, we consider an inverse problem of identifying a space-dependent source in the parabolic equation which is a classical ill-posed problem. The inverse source problem is formulated into a regularized optimization problem. Then, a non-iterative algorithm based on a sequence well-posed direct problems solved by the finite element method is proposed for solving the optimization problem. In order to obtain a reasonable regularization solution, we utilize the damped Morozov discrepancy principle together with the linear model function method for choosing regularization parameters. Numerical results for one- and two-dimensional examples show that the proposed method is efficient and robust with respect to data noise, especially for reconstructing the discontinuous source functions. Furthermore, the proposed method is successfully used to solve a real example of identifying the magnitude of groundwater pollution source.

Keywords: inverse source problem, parabolic equation, optimization, finite element method, discrepancy principle.

1. Introduction

Inverse source identification problems arise in many branches of applied science and engineering science, which aim to determine the unknown source from some measurable information related to the source. For example, Identification of a pollution source intensity from some given measurements of the pollutant concentrations is crucial to environmental safeguard in watersheds [1]. In this paper, we consider the inverse problem for determining the unknown space-dependent source in a parabolic equation from a final measurement. As we all know, this inverse source problem is ill-posed since small errors inherently presented in the practical measurement can induce enormous and highly oscillatory errors in

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reconstructing the unknown heat source.

The inverse problem of determining an unknown space-dependent source in the parabolic equation has been considered in a few theoretical papers concerned with existence and uniqueness of the solution[2, 3]. Recently, many authors are interested in numerical reconstruction of the space-dependent source in parabolic equations [4, 5, 7, 6, 8, 9, 10, 11, 12]. In [4], the authors transferred the inverse heat source problems to the problems of numerical differentiation for obtaining stable solutions. An effective meshless numerical method and a finite difference approximate method were proposed in [7] and [6], respectively. In [9], a regularization method based on the quasi-reversibility method together with the error estimate was proposed for identifying an unknown space-dependent source in one dimensional standard heat equation. In [10] and [11], two iterative methods were proposed for finding the spacewise dependent source: one is an iterative algorithm based on a sequence of well-posed direct problems; the other is a variational conjugate gradient-type iterative algorithm which also need to solve a sequence of well-posed direct problems at each iteration. The paper [12] is devoted to identify an unknown heat source depending simultaneously on both space and time variables that is transformed into an optimization problem. The aim of this paper is to construct a regularized optimization method, which is a non-iterative method. We firstly formulate the inverse problem of determining the spacewise dependent source into a regularized optimization problem. Then, the optimization problem is reduced to a system of linear algebraic equations based on a sequence well-posed direct problems solved by the finite element method.

This paper is organized as follows. In section 2, the source identification problem is formulated and some properties of the solution of direct problem are given. In section 3, a regularized optimization method is proposed for solve the source identification problem. In section 4, implementations of the regularized optimization method are presented. In section 5, numerical results for one- and two-dimensional examples are given to illustrate the efficiency and stability of the proposed method with respect to data noise. Finally, some conclusions are drawn.

2. Mathematical formulation of the source identification problem

Let Ω be a bounded domain possessing piecewise-smooth boundaries in the Euclidean space $\mathbf{R}^n$, $n \geq 1$. $x = (x_1, x_2, \cdots, x_n)$ denotes an arbitrary point in Ω, and $\partial\Omega$ is used for the boundary of the domain Ω. Let us denote by Q_T a cylinder $\Omega \times (0, T)$ consisting of all points $(x, t) \in \mathbf{R}^{n+1}$ with $x \in \Omega$ and $t \in (0, T)$.

2.1. functional spaces

The space $L_2(\Omega)$ is a Banach space consisting of all square integrable functions on the domain Ω with the norm

$$\|u\|_{2,\Omega} = \left(\int_\Omega |u(x)|^2 dx \right)^{1/2}$$

and the scalar product

$$(u, v) = \int_\Omega u(x)v(x)dx.$$

The Sobolev spaces $W_2^l(\Omega)$, where l is a positive integer, consists of all functions from $L_2(\Omega)$ having all generalized derivatives of the first l orders that are square integrable over Ω. The norm of the space $W_2^l(\Omega)$ is defined by

$$\|u\|_{2,\Omega}^{(l)} = \left(\sum_{k=0}^{l} \sum_{|\alpha|=k} \|D_x^\alpha u\|_{2,\Omega}^2 \right)^{1/2},$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ is a multi-index, and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$,

$$D_x^\alpha u \equiv \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

The space $\overset{0}{W}{}_2^l(\Omega)$ is a subspace of $W_2^l(\Omega)$ in which the set all functions in Ω that are infinite differentiable and have compact support is dense.

The Sobolev space $W_2^{l_1,l_2}(Q_T)$ with positive integers $l_i \geq 0, i = 1, 2$ is defined as a Banach space of all functions u belonging to the space $L_2(Q_T)$ along with their weak x-derivatives of the first l_1 orders and t-derivatives of the first l_2 orders. The norm of the space $W_2^{l_1,l_2}(Q_T)$ is defined by

$$\|u\|_{2,Q_T}^{(l_1,l_2)} = \left(\int_{Q_T} \left(\sum_{k=0}^{l_1} \sum_{|\alpha|=k} |D_x^\alpha u|^2 + \sum_{k=1}^{l_2} |D_t^k u|^2 \right) dxdt \right)^{1/2}.$$

The space $W_{2,0}^{l_1,l_2}(Q_T)$ is a subspace of $W_2^{l_1,l_2}(Q_T)$ in which the set of all smooth functions in Q_T that vanish on the lateral $\partial\Omega \times [0, T]$ is dense.

2.2. The source identification problem

The source identification problem considered in this paper is stated as follows: find the function $u(x, t)$ and the unknown source function $f(x)$ which satisfy the following parabolic equation and boundary conditions

$$\begin{cases} u_t(x,t) = (Lu)(x,t) + f(x), \ (x,t) \in \Omega \times (0,T), \\ u(x,0) = 0, \ x \in \Omega, \\ u(x,t) = 0, \ (x,t) \in \partial\Omega \times [0,T], \end{cases} \tag{2.1}$$

and the final over-specified measurement

$$u(x,T) = g(x), \ x \in \Omega, \tag{2.2}$$

where

$$Lu \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(a_{ij}(x) \frac{\partial u}{\partial x_i} \right) + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u.$$

Moreover, the operator L is supposed to be uniformly elliptic, which means that $a_{ij}(x) = a_{ji}(x)$ and

$$0 < \nu \sum_{i=1}^{n} \zeta_i^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) \zeta_i \zeta_j \leq \mu \sum_{i=1}^{n} \zeta_i^2 \tag{2.3}$$

with positive constants ν and μ, and arbitrary point $\zeta = (\zeta_1, \cdots, \zeta_n) \in \mathbf{R}^n$. Considering the practical engineering applications, we confine the coefficients of the operator L to satisfying that

$$a_{ij}(x), b_i(x), c(x) \in C(\bar{\Omega}),\ \frac{\partial a_{ij}(x)}{\partial x_k} \in C(\bar{\Omega}), k = 1, 2, \cdots, n. \tag{2.4}$$

Remark 1. *As mentioned in the introduction, there are many numerical methods [4, 5, 7, 6, 8, 9, 10, 11, 12, 22] for identifying the space-dependent source $f(x)$ in parabolic equations. However, these methods are mainly concerned and implemented in the one-dimensional or standard parabolic equations. In other words, some of these methods maybe not adapted to the generally parabolic equation in (2.1). Limited to our knowledge, we think the methods proposed in [4, 6, 9, 10, 11, 22] should be adapted and extended to the general n-dimensional inverse problem (2.1), but some new difficulties maybe occur and should be overcome. For example, numerical differential problems of both the second order and the first order, which are both ill-posed, should be computed in [4]. The finite difference approximation applied to reconstructing the source term in [6, 9] should be improved to deal with any n-dimensional domain. We pay more attention to the two iteration methods constructed for the generally problem (2.1) in [10, 11]. In the two iteration methods, the boundary element method with fundamental solutions of parabolic equations is used to solve a sequence of well-posed direct problems. Generally, the fundamental solutions of linear parabolic equations with variable coefficients are very complex, and their existence is also no general results [23]. Compared to these known methods, the regularized optimization method proposed in this paper is very simple for solving the general n-dimensional inverse source problem (2.1), and more suitable for parallel computing which greatly enhance the efficiency of the regularized method.*

2.3. Properties of the direct problem

The direct problem is finding a solution $u(x, t)$ satisfying the problem (2.1) when the coefficients of the operator L and the source $f(x)$ are known. From results of Chapter 1 in [13], we have the following lemma for the direct problem.

Lemma 1. *Let the operator L be uniformly elliptic and its coefficients satisfy (2.4), and let $f(x) \in L_2(\Omega)$. Then the direct problem (2.1) has a solution $u \in W_{2,0}^{2,1}(Q_T)$, this solution is unique and the following estimate is valid:*

$$\|u\|_{2,Q_T}^{(2,1)} \le C_1 \sqrt{T}\, \|f\|_{2,\Omega}, \tag{2.5}$$

where the constant C_1 does not depend on u.

Therefore, given $f(x) \in L_2(\Omega)$, $u(x, T)$ is well defined since $u(x, t) \in W_{2,0}^{2,1}(Q_T)$. Moreover, it is reasonable to assume that the over-specified measurement $g(x)$ satisfies

$$g(x) \in W_2^1(\Omega). \tag{2.6}$$

3. Regularized Optimization method

3.1. Regularized optimization functional

We now consider the inverse source problem (2.1)-(2.2) as the following constrained optimization problem: finding a source function $f(x)$ such that

$$\min_{f\in\Phi} J(f) = \int_\Omega |u(x,T;f) - g(x)|^2 dx + \alpha \int_\Omega |f(x)|^2 dx, \tag{3.1}$$

where α is the regularization parameter and the constrained set is

$$\Phi = \{f(x) \mid |f(x)| \le M,\ f(x) \in L_2(\Omega)\}, \tag{3.2}$$

M is a constant. The solution $u(x,t;f)$ in (3.1) with respect to the source term $f(x)$ is a weak solution of (2.1) which satisfies

$$u(x,0;f) = 0 \tag{3.3}$$

and the variational formulation

$$\int_\Omega u_t \psi dx + \int_\Omega \left(\sum_{i,j=1}^n a_{ij}(x) u_{x_i} \psi_{x_j} - \sum_{i=1}^n b_i(x) u_{x_i} \psi - c(x) u\psi \right) dx = \int_\Omega f(x)\psi dx \tag{3.4}$$

for any $\psi(x) \in \overset{0}{W}{}_2^1$ and a.e. $t \in (0,T)$.

Theorem 1. *There exists a minimizer $\tilde f(x) \in \Phi$ such that*

$$J(\tilde f) = \min_{f\in\Phi} J(f).$$

Proof. From the non-negativeness of the functional $J(f)$, it follows that $J(f)$ has the greatest lower bound $\inf_{f\in\Phi} J(f)$, which means that there exists a minimizing sequence $\{f_m\}$ in Φ such that

$$\inf_{f\in\Phi} J(f) \le J(f_m) \le \inf_{f\in\Phi} J(f) + \frac{1}{m}$$

with the associated weak solution $u_m := u(x,t;f_m)$. Obviously, there exists a constant C_2 such that

$$\|f_m\|_{2,\Omega} \le C_2,$$

where C_2 is independent of m. Thus, we can extract a subsequence, again denoted by $\{f_m\}$, such that f_m converges weakly to $\tilde f$ in Φ due to the closure of Φ.

From Lemma 1, we know that the sequence $\{u_m\}$ is bounded in $W_2^{2,1}(Q_T)$. Hence, we can also extract a subsequence, still denoted by $\{u_m\}$, such that u_m converges weakly to u^*. Therefore, the rest we need to prove that $u^* = u(x,t;\tilde f)$. In order to do this, multiplying both side of the equation

$$\int_\Omega u_{m,t} \psi dx + \int_\Omega \left(\sum_{i,j=1}^n a_{ij}(x) u_{m,x_i} \psi_{x_j} - \sum_{i=1}^n b_i(x) u_{m,x_i} \psi - c(x) u_m\psi \right) dx = \int_\Omega f_m(x)\psi dx \tag{3.5}$$

by any function $\gamma(t) \in C^1[0,T]$ with $\gamma(T) = 0$, then integrating with respect to t on $[0,T]$, we derive that

$$-\int_\Omega u(x,0)\gamma(0)\psi dx - \int_0^T \gamma \int_\Omega u_m \psi dxdt + \int_0^T \gamma \int_\Omega \left(\sum_{i,j=1}^n a_{ij}(x) u_{m,x_i} \psi_{x_j} - \sum_{i=1}^n b_i(x) u_{m,x_i} \psi - c(x) u_m \psi \right) dxdt = \int_0^T \gamma \int_\Omega \tilde{f} \psi dxdt + \int_0^T \gamma \int_\Omega (f_m - \tilde{f}) \psi dxdt.$$

The last term of the above equality converges to zero since f_m converges weakly to $\tilde{f}$. Noting that u_m converges weakly to u^* in $W_2^{2,1}(Q_T)$ and Letting $m \to \infty$, we have

$$-\int_\Omega u(x,0)\gamma(0)\psi dx - \int_0^T \gamma \int_\Omega u^* \psi dxdt + \int_0^T \gamma \int_\Omega \left(\sum_{i,j=1}^n a_{ij}(x) u^*_{x_i} \psi_{x_j} - \sum_{i=1}^n b_i(x) u^*_{x_i} \psi - c(x) u^* \psi \right) dxdt = \int_0^T \gamma \int_\Omega \tilde{f} \psi dxdt. \tag{3.6}$$

Obviously, (3.6) is also true for any $\gamma(t) \in C_0^\infty(0,T)$, this implies that

$$\int_\Omega u_t^* \psi dx + \int_\Omega \left(\sum_{i,j=1}^n a_{ij}(x) u^*_{x_i} \psi_{x_j} - \sum_{i=1}^n b_i(x) u^*_{x_i} \psi - c(x) u^* \psi \right) dx = \int_\Omega \tilde{f}(x) \psi dx$$

for any $\psi(x) \in \overset{0}{W}{}_2^1$ and $u^*(x,0) = 0$. Hence, it follows that $u^* = u(x,t;\tilde{f})$ by the definition of $u(x,t;\tilde{f})$. Then, the weakly lower semi-continuity of $J(f)$ ensures that $\tilde{f}$ is a minimizer of $J(f)$.

3.2. Approximation by the finite element method

In this subsection, we introduce the finite element method for solving the continuous minimization problems (3.1), (3.2) and (3.4). Similarly to that done in [14, 15], we first triangulate the domain Ω with a regular triangulation T^h of simplicial elements, and define S_h to be the continuous piecewise linear finite element space defined over T^h. The space $\overset{0}{S}_h$, in which all functions vanish on the boundary $\partial\Omega$, is a subspace of S_h. Let $\{P_i\}_{i=1}^{M_h}$ be the set of interior nodes, i.e., those that do not lie on the boundary $\partial\Omega$. So, a function in the space $\overset{0}{S}_h$ is uniquely determined by its value at the point P_i, and the set of pyramid functions $\{\phi_j\}_{j=1}^{M_h} \subset \overset{0}{S}_h$, defined by

$$\phi_j(P_i) = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \tag{3.7}$$

forms a basis of $\overset{0}{S}_h$. Obviously, a function $v(x)$ in $\overset{0}{S}_h$ can be extract that $v(x) = \sum_{j=1}^{M_h} v_j \phi_j(x)$, where $v_j = v(P_j)$ is the value of $v(x)$ at P_j. The time interval $[0,T]$ is partitioned into N equal subintervals by using nodal points

$$0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$$

with $t_m = m\Delta t$ and $\Delta t = \frac{T}{N}$. Let $u^m = u(x, t_m)$ for $0 \leq m \leq N$. Then, we define the difference quotient

$$D_t u^m = \frac{u^m - u^{m-1}}{\Delta t}$$

for a given sequence $\{u^m\}_{m=1}^N \subset L^2(\Omega)$.

Let $f(x)$ be extended to the boundary $\partial\Omega$. Then, we define $f_h = \sum_{j=1}^{K_h} f_j \phi_j(x)$ that approximate $f(x) \in L_2(\Omega)$ and project it into the space S_h, where K_h is the numbers of all nodes of T^h, and f_j is the value of $f(x)$ at the j-th node. And now we can formulate the continuous optimal problem (3.1) as the following finite element approximation

$$\min_{f \in S_h \bigcap \Phi} J(f_h) = \int_\Omega |u_h^N(f_h) - g(x)|^2 dx + \alpha \int_\Omega |f_h|^2 dx, \tag{3.8}$$

where $u_h^m(f_h) = \sum_{j=1}^{M_h} u_j^m \phi_j(x)$ for $m = 0, 1, \cdots, N$ satisfies that

$$u_h^0(f_h) = 0 \tag{3.9}$$

and

$$\int_\Omega \psi_h D_t u_h^m(f_h) dx + \int_\Omega \left(\sum_{i,j=1}^n a_{ij}(x)(u_h^m(f_h))_{x_i}(\psi_h)_{x_j} - \sum_{i=1}^n b_i(x)(u_h^m(f_h))_{x_i}\psi_h - c(x)u_h^m(f_h)\psi_h \right) dx = \int_\Omega f_h(x)\psi_h dx \tag{3.10}$$

for any $\psi_h \in \overset{0}{S}_h$.

Theorem 2. *There exists at least a minimizer to the discrete minimization problem (3.8)-(3.10).*

The proof of Theorem 2 follows the same lines as the proof of Theorem 3.1 in [15]. So, we omit it. On the other hand, from results of [14, 15] we can also obtain that the minimizer sequence of the discrete minimization problem corresponding to h and Δt has a subsequence that converges to a minimizer of the continuous problem (3.1)-(3.4) as $h \to 0, \Delta t \to 0$.

4. Implementations of the regularized optimization method

4.1. Regularized least square method

Due to the linearity of the governing equation and the homogenous boundary and initial conditions, we easily see that the problem (2.1) satisfies the principle of superposition in terms of the source function, which is noted in [16, 17] and used to construct inverse methods for recovering the initial function. Here, we also use this principle of superposition

to formulate the finite element approximation (3.8) into a linear algebraic system. Noting that

$$f_h = \sum_{j=1}^{K_h} f_j \phi_j(x), \tag{4.1}$$

we have

$$u_h^N(f_h) = \sum_{j=1}^{K_h} f_j u_h^N(\phi_j(x)), \tag{4.2}$$

where $u_h^N(\phi_j(x))$ is computed by the finite element method proposed in [18] when the spatial domain is one-dimensional; otherwise, for the two-dimensional domain, $u_h^N(\phi_j(x))$ is computed by the functions of PDE Toolbox in Matlab. Therefore, we rewrite the approximation functional $J(f_h)$ as the form of $\tilde{f} = (f_1, f_2, \cdots, f_{K_h})^T$ in the form

$$J(\tilde{f}) = \int_\Omega \left| \sum_{j=1}^{K_h} f_j u_h^N(\phi_j(x)) - g(x) \right|^2 dx + \alpha \int_\Omega \left| \sum_{j=1}^{K_h} f_j \phi_j(x) \right|^2 dx, \tag{4.3}$$

From the necessary condition for minimizing the approximation function $J(\tilde{f})$

$$\frac{\partial J(\tilde{f})}{\partial f_i} = 0, \; i = 1, 2, \cdots, K_h, \tag{4.4}$$

we obtain the following linear algebraic system

$$(A + \alpha G)\tilde{f} = b, \tag{4.5}$$

where $A = (a_{ij})_{K_h \times K_h}$, $G = (g_{ij})_{K_h \times K_h}$, $b = (b_1, b_2, \cdots, b_{K_h})^T$, and

$$a_{ij} = \int_\Omega u_h^N(\phi_i(x)) u_h^N(\phi_j(x)) dx, \; g_{ij} = \int_\Omega \phi_i(x)\phi_j(x) dx, \; b_i = \int_\Omega u_h^N(\phi_i(x)) g(x) dx. \tag{4.6}$$

For a given regularization parameter α, the solution $\tilde{f}^*$ of equation (4.5) is a discrete reconstruction of the unknown source $f(x)$; and $f_h^* = \sum_{j=1}^{K_h} f_j^* \phi_j(x)$ is an approximation of $f(x)$ in the space S_h. Hence, the algorithm for reconstructing the unknown source $f(x)$ from the final over-specified measurement $g(x) = u(x, T)$ is summarized as the following algorithm.

Algorithm 1 : Algorithm with non-iteration for reconstructing the unknown source.

Given the final measurement $g(x) = u(x, T)$ and a regularization parameter α.

Step 1. Solve the direct problem (2.1) for each basis source term $f(x) = \phi_i(x)$ via the finite element method, then obtain $u_h^N(\phi_i(x))$.

Step 2. Compute the Matrices A and G and the vector b by using (4.6).

Step 3. Solve the regularized linear algebraic system (4.5).

Step 4. Reconstruct the source $f(x)$ by using the formulation (4.1).

Remark 2. *The cost of Algorithm 1 is mainly taken in the first step if we run it by serial computing. Fortunately, we note that the first step of Algorithm 1 can be computed in parallel. Therefore, the efficiency of Algorithm 1 will be very high if we run it on a parallel computer system. In addition, we can choose other bases instead of the continuous piecewise linear finite element basis, such as the polynomial basis, and the trigonometric function basis, which can greatly reduce the amount of computation if the number of the basis functions is relatively small.*

4.2. Strategy for choosing regularization parameters

Due to the ill-posedness of the inverse source problem, the regularization parameter α play an important role for reconstructing a reasonable solution. The measurement noises and the round-off errors may be highly amplified due to the choice of an unreasonable regularization parameter and therefore making the inverse solution completely useless. In this study, we employ the damped Morozov discrepancy principle [19, 20] to choose regularization parameters, i.e., choosing regularization parameters α such that

$$\int_\Omega \left| \sum_{j=1}^{K_h} f_j u_h^N(\phi_j(x)) - g^\delta(x) \right|^2 dx + \alpha^\gamma \int_\Omega \left| \sum_{j=1}^{K_h} f_j \phi_j(x) \right|^2 dx = C\delta^2, \tag{4.7}$$

where γ is the damped coefficient, C is a constant, δ is the noise level which meet that $\|g - g^\delta\| \leq \delta$. Here, g is the exact data and g^δ is the measurement data. In order to obtain regularization parameters in a stable and quick way, we adopt the linear model function method proposed in [19, 20] to solve the discrepancy equation (4.7) with $\gamma = 1.5$ and $C = 1.5$.

5. Numerical examples

The solution of the governing equation in (2.1) with nonhomogeneous boundary condition $B(x,t)$ and initial condition $u_0(x)$ is not a linear mapping for the source term $f(x)$. Therefore, we first divide the problem with nonhomogeneous boundary conditions into the following two problems, i.e.,

$$u(x,t;f) = u_1(x,t;f) + u_2(x,t), \tag{5.1}$$

where $u_1(x,t;f)$ satisfies the problem

$$\begin{cases} (u_1)_t(x,t) = (Lu_1)(x,t) + f(x), \ (x,t) \in \Omega \times (0,T), \\ u_1(x,0) = 0, \ x \in \Omega, \\ u_1(x,t) = 0, \ (x,t) \in \partial\Omega \times [0,T], \end{cases} \tag{5.2}$$

and $u_2(x,t)$ is the solution of the following homogeneous equation with nonhomogeneous initial and boundary conditions

$$\begin{cases} (u_2)_t(x,t) = (Lu_2)(x,t), \ (x,t) \in \Omega \times (0,T), \\ u_2(x,0) = u_0(x), \ x \in \Omega, \\ u_2(x,t) = B(x,t), \ (x,t) \in \partial\Omega \times [0,T]. \end{cases} \tag{5.3}$$

Then, By using the data $g(x) - u_2(x,T)$, Algorithm 1 can be implemented for reconstructing the source function $f(x)$.

In all one-dimensional examples of this section, we divide $[0,1]$ into 100 equal subintervals which means that there are 100 elements and 101 nodes; while in all two-dimensional examples, we divide $[0,1] \times [0,1]$ into 50×50 equal sub-rectangles which indicates that the mesh grid has a total of 5000 triangle elements and 2601 nodes. In the computational process, we obtain actually the final data vector $g = \{g(P_i)\}$ at the points of the mesh grid in our simulations, and add a random distributed perturbation to the data vector g with relative noise level $\hat{\delta}$, i.e., $g^\delta = g + \hat{\delta}(2 * \text{rand}(\text{size}(g)) - 1) * g$. The function rand(size(g)) in Matlab generates a random vector whose elements are the standard uniform distribution on the interval (0,1).

In the numerical results listed in Table 1, we report the relative noise levels $\hat{\delta}$, the regularization parameters, the relative error of the inverse solution computed by the formula

$$\text{RelError} = \frac{\|f_h^* - f\|_2}{\|f\|_2}.$$

The comparisons between the exact solutions and the inverse solutions are showed in Figure 1 to Figure 8, respectively.

Table 1. Some numerical results for examples 1-5.

Examples	$\hat{\delta}$	α	RelError
Example 1	0.001	4.2806e-006	3.8244e-003
	0.01	4.1359e-005	1.1608e-002
Example 2	0.001	2.7407e-007	3.6889e-002
	0.01	3.5558e-006	9.3541e-002
Example 3	0.001	1.0017e-007	1.6677e-001
	0.01	3.7607e-006	2.5831e-001
Example 4	0.001	4.7644e-008	1.9297e-002
	0.01	4.9258e-007	5.7354e-002
Example 5	0.001	3.5247e-008	1.7401e-001
	0.01	1.1364e-006	2.6286e-001

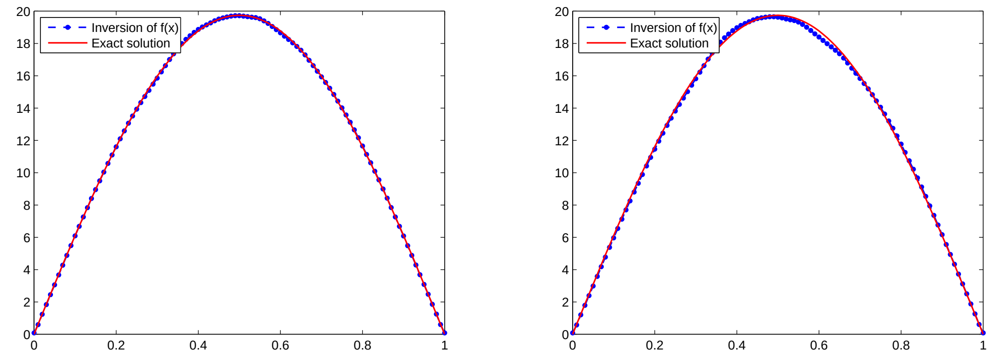
Example 1. We take $\Omega = (0,1), T = 1$, and $Lu = \Delta u = \frac{\partial^2 u}{\partial x^2}$. Let

$$u(x,t) = \left(2 - \exp\left(-\pi^2 t\right)\right)\sin(\pi x),\ (x,t) \in [0,1] \times [0,1].$$

In this case, $f(x) = 2\pi^2 \sin(\pi x)$, $u_0(x) = \sin(\pi x)$, $u(0,t) = u(1,t) = 0$, and the final measurement is given by

$$g(x) = u(x,1) = \left(2 - \exp\left(-\pi^2\right)\right)\sin(\pi x),\ x \in [0,1].$$

The inverse solutions for different noise levels are showed in Figure 1.



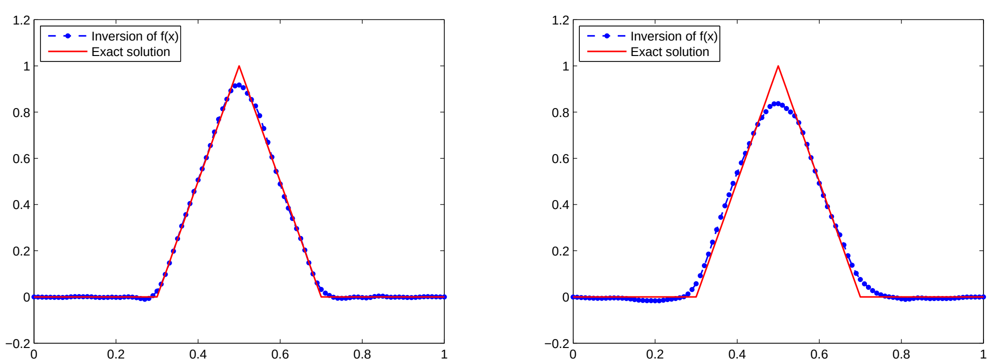
(a) $\hat{\delta} = 0.001$ (b) $\hat{\delta} = 0.01$

Figure 1: The comparison between the exact solution and its inverse solution.

Example 2. Consider a piecewise smooth heat source:

$$f(x) = \begin{cases} 0, & x \in [0, 0.3], \\ 5(x - 0.3), & x \in (0.3, 0.5], \\ -5(x - 0.7), & x \in (0.5, 0.7], \\ 0, & x \in (0.7, 1]. \end{cases}$$

We take $\Omega = (0, 1), T = 1$, $Lu = \Delta u = \frac{\partial^2 u}{\partial x^2}$, $u_0(x) = 0$, $u(0, t) = u(1, t) = 0$. The final over-specified measurement $u(x, T)$ is computed by the finite element method proposed in [18]. Numerical results for different noise levels are showed in Figure 2.



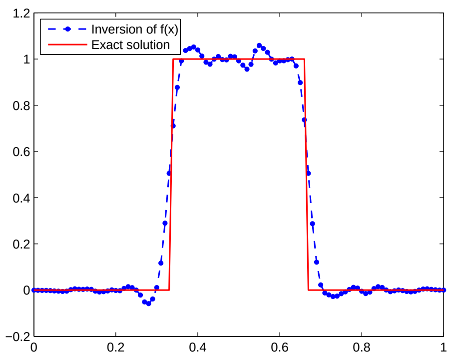
(a) $\hat{\delta} = 0.001$ (b) $\hat{\delta} = 0.01$

Figure 2: The comparison between the exact solution and its inverse solution.

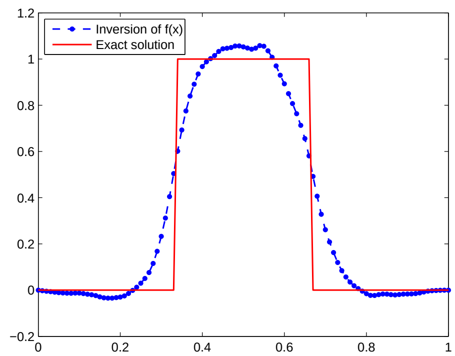
Example 3. Consider a discontinuous source

$$f(x) = \begin{cases} 0, & x \in [0, 1/3), \\ 1, & x \in [1/3, 2/3], \\ 0, & x \in (2/3, 1]. \end{cases}$$

In this example, we also take $\Omega = (0, 1), T = 1$, $Lu = \Delta u = \frac{\partial^2 u}{\partial x^2}$, and the homogenous boundary and initial conditions. Because the source $f(x)$ is a discontinuous function, the direct problem has no analytic solution. So, we obtain the final over-specified measurement $u(x, T)$ by solving the direct problem with the finite element method [18]. The reconstructed solutions for different noise levels are showed in Figure 3.



(a) $\hat{\delta} = 0.001$



(b) $\hat{\delta} = 0.01$

Figure 3: The comparison between the exact solution and its inverse solution ($n = 100$).

Example 4.[6] For this two-dimensional example, we take $\Omega = (0, 1) \times (0, 1), T = 1$, $Lu = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$, the initial value $u_0(x, y) = \sin(\pi x) \sin(\pi y), (x, y) \in \Omega$, and the homogenous boundary conditions. The exact source function is defined by

$$f(x, y) = \exp\left(-\sigma\left[(x - \mu_1)^2 + (y - \mu_2)^2\right]\right).$$

Note that when σ is large enough the above source mimics a Dirac delta distribution $\delta(x - \mu_1, y - \mu_2)$. Here, we take $\sigma = 80$, $\mu_1 = \frac{3}{4}$ and $\mu_2 = \frac{1}{2}$. The final measurement $u(x, y, T)$ is obtained by the functions of PDE Toolbox in Matlab. The exact solution is showed in Figure 4. The reconstructed solutions and their errors are showed in Figure 5 and Figure 6 for $\hat{\delta} = 0.001$ and $\hat{\delta} = 0.01$, respectively.

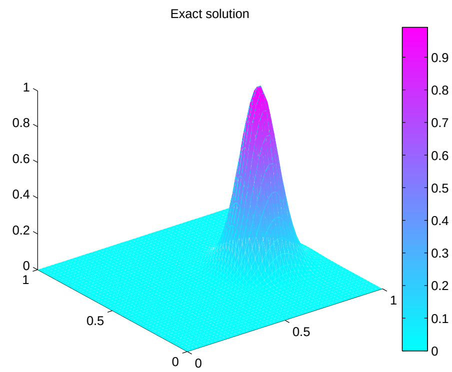
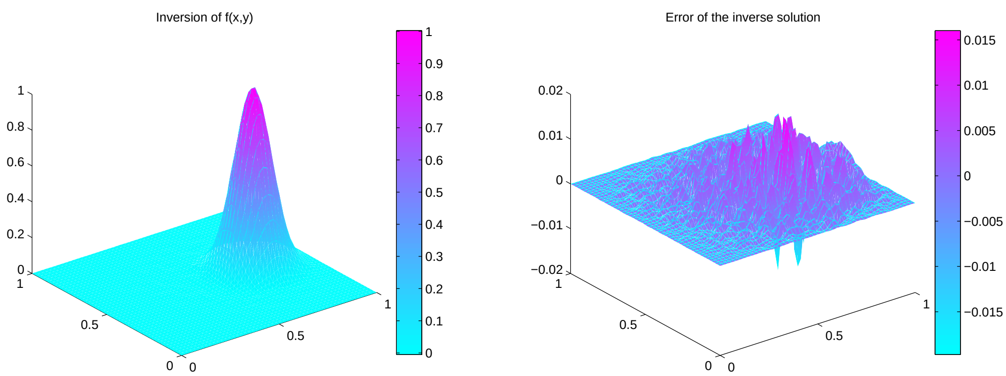


Figure 4: The exact solution of example 4.



(a) Inverse solution

(b) Error of the inverse solution

Figure 5: The inverse solution and its error for $\hat{\delta} = 0.001$.

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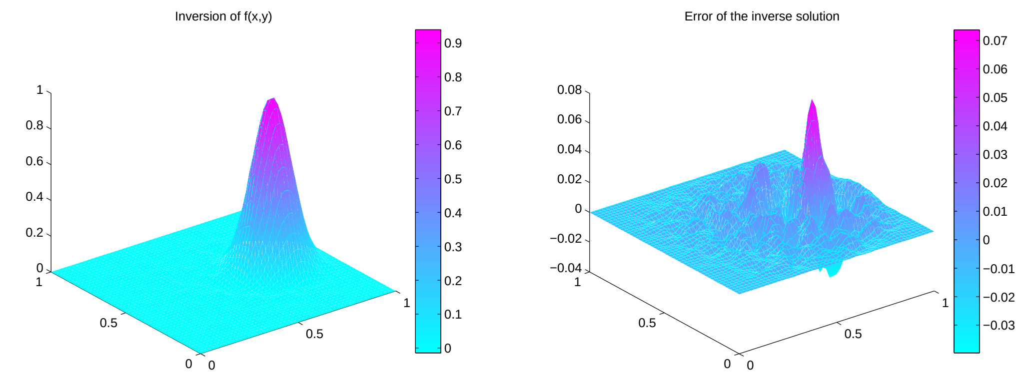


(a) Inverse solution (b) Error of the inverse solution

Figure 6: The inverse solution and its error for $\hat{\delta} = 0.01$.

Example 5. In this two-dimensional example, we consider a discontinuous function

$$f(x,y) = \begin{cases} 0, & (x,y) \in \left\{ (x,y) \middle| 0 < x, y < 1, \sqrt{(x-0.5)^2 + (y-0.5)^2} \geq 0.25 \right\} \\ 1, & (x,y) \in \left\{ (x,y) \middle| \sqrt{(x-0.5)^2 + (y-0.5)^2} < 0.25 \right\}. \end{cases}$$

We also take $\Omega = (0,1) \times (0,1), T = 1,\ Lu = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$, and the homogenous boundary and initial conditions. The final measurement $u(x, y, T)$ is also obtained by the functions of PDE Toolbox in Matlab. The exact solution is showed in Figure 7. The reconstructed solutions and their errors are showed in Figure 8 and Figure 9 for $\hat{\delta} = 0.001$ and $\hat{\delta} = 0.01$, respectively.

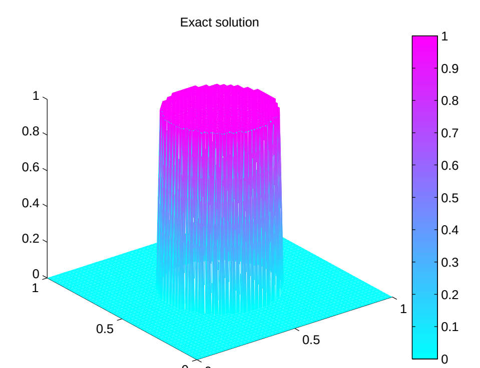
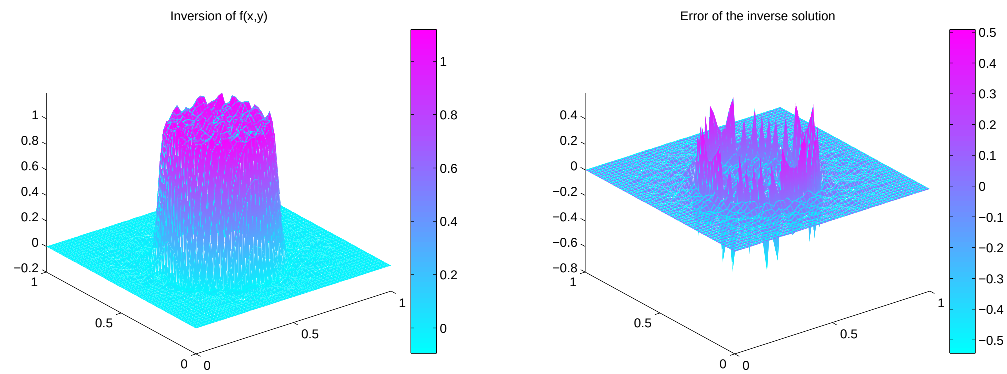


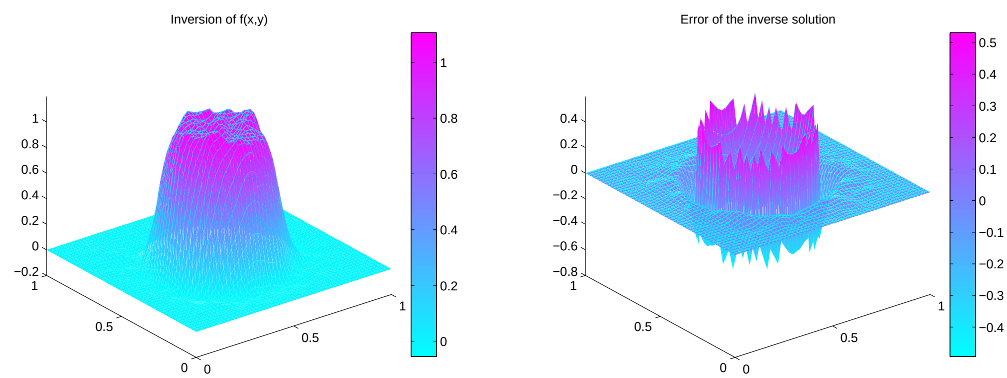
Figure 7: The exact solution of example 5.



(a) Inverse solution

(b) Error of the inverse solution

Figure 8: The inverse solution and its error for $\hat{\delta} = 0.001$.



(a) Inverse solution

(b) Error of the inverse solution

Figure 9: The inverse solution and its error for $\hat{\delta} = 0.01$.

6. Application to a real source determination[21, 22]

This real example is taken from references [21] and [22]. Consider acid contaminant in the groundwater in Fengshui, Zibo of Shandong Province, China. The studied region is a relatively integrated unit of hydrogeology whose area is about 45 km^2. In this region, the groundwater flow accumulated by atmosphere precipitation is gradually pressed when it seeps from the southeast to the north-west until it encountered the coal-seam, and so a strip containing rich groundwater is formed. For this reason, Yuedian and Zhanghua wellsprings were established in 1980s. However, with the excess exploitation of mines, e.g. the exploitation of coal-wells, groundwater pollution has become more and more serious in this region. In particular, acid contaminant of SO_4^{2-} in Zhanghua wellspring becomes higher and higher year after year. Based on the measured concentration data in this

region from 1988 to 1999 along the groundwater flow direction, we are try to determine the average magnitude of acid contaminant seeping into the aquifer every year.

Under some suitable assumptions on the aquifer, choosing the direction of groundwater flow which is from Sijiaofang to Zhanghau as the direction of x axis, and Sijiaofang as zero point, the year of 1988 as initial moment, then this real problem of acid contaminant in the groundwater system can be characterized by the following system:

$$\begin{cases} \frac{\partial u}{\partial t} = a_L v \frac{\partial^2 u}{\partial x^2} - v\frac{\partial u}{\partial x} - \lambda u + \frac{f(x)}{n_e}, \ 0 < x < L, 0 < t < T, \\ u(0,t) = 7.96t + 45.6, 0 \le t \le T, \\ u(L,t) = 1.75t^2 + 331.6, 0 \le t \le T, \\ u(x,0) = 0.0715x + 45.6, 0 \le x \le L, \end{cases} \tag{6.1}$$

where $u = u(x,t)$[mg/l] is the solute concentration at time t and space point x, and the model parameters: $v = 365$[m/y] is the average pore water velocity, $a_L = 1$[m] is the longitude dispersivity, $\lambda = 0.05[\mathrm{y}^{-l}]$ is the attenuation coefficient, $n_e = 0.25$[dimensionless] is the effective porosity and $f(x)$ represents an average magnitude of the pollutants seeping into the aquifer every year. In addition, $L = 4000$[m] denotes the distance from Sijiaofang to Zhanghau, and $T = 11$[y]. The boundary conditions and the initial condition in system (6.1) are obtained by applying data fitting skills from the actually measured data, see [22]. The additional data at the final year of $T = 11$ is also obtained by the similar technique as follows:

$$u(x,T) = 0.1026x + 133.2, 0 \le x \le L. \tag{6.2}$$

The inverse problem considered here is to determine the source magnitude function $f(x)$ in (6.1) from the measured data $u(x,T)$ by the regularized optimization method. In the numerical implementation, we firstly transform this inverse problem into a dimensionless form [21] by setting $U = \frac{u}{45.6}, y = \frac{x}{L}, \tau = \frac{vt}{L}$, then apply Algorithm 1 to solve it by dividing $[0,1]$ with 200 equal subintervals. Firstly, assuming that all of the initial boundary data in the model (6.1) and the additional final data (6.2) are accurate, we reconstruct the source with regularization parameter $\alpha = 0.5 \times 10^{-4}$. Secondly, in the case of the additional final data having random noises, we carry out similar computations with the linear model function method for choosing regularization parameters.

Case 1. Find a solution $f_h^* = \sum\limits_{j=1}^{K_h} f_j^* \phi_j(x)$ in the space S_h. See Figure 10 and Table 2.

Case 2. Find a solution as the form $f_h^* = \sum\limits_{j=0}^{N_p} f_j^* x^j$ in the polynomial function space $\mathbb{P}_{N_p}[x]$. Based on analyses of [22], we only take $N_p = 1$ and $N_p = 2$ to reconstruct the source, respectively. See Table 2.

To show accurateness and reasonableness of the above solutions, we substitute these solutions into the model (6.1) and reconstruct the additional data denoted by $u(x,T;f_h^*)$. Then the residuals $\|u(x,T;f_h^*) - u(x,T)\|_2$ are computed at the 201 nodes and listed in Table 2 as compared with the actually additional final data (6.2).

From Figure 10 and Table 2, we see that the source magnitude function $f(x)$ in the model (6.1) can be determined numerically from the additional final data by the proposed regularized optimization method. We also find from the last column of Table 2 that the

method used here is better than that in paper [22, 21] in the sense of smaller residuals. In addition, Algorithm 1 is very fast in the above second case since only two of three final data for the corresponding basis functions need to be computed.

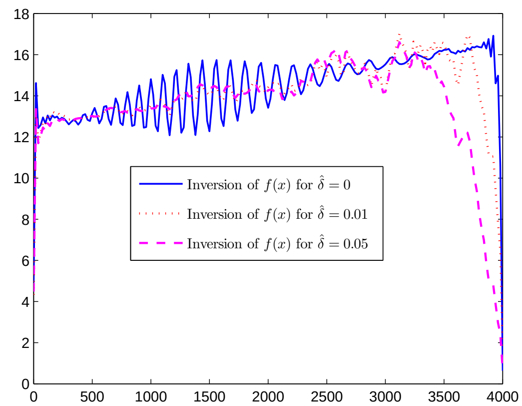


Figure 10: Inverse solutions, where $\hat{\delta}$ is the relative noise level.

Table 2. Numerical results for Example 6.

Cases	Relative noise level	α	$f_h^*(x)$	$\Vert u(x,T;f_h^*) - u(x,T)\Vert_2$
Solutions in the space S_h	0.00	5.0000e-005	See Figure 10(-)	7.0182e-001
	0.01	5.1865e-004	See Figure 10(..)	9.5460e-001
	0.05	3.8324e-003	See Figure 10(- -)	2.6560e+000
Solutions in the space $\mathbb{P}_1[x]$	0.00	5.0000e-005	$12.997+0.0010584x$	2.0369e-001
	0.01	1.0177e-001	$13.052+0.0010055x$	2.7136e-001
	0.05	3.4462e-001	$13.129+0.00090556x$	9.7427e-001
Solutions in the space $\mathbb{P}_2[x]$	0.00	5.0000e-005	$13.001+0.0010514x$ $+0.0000000019274x^2$	2.0386e-001
	0.01	9.4045e-002	$13.343+0.00048213x$ $+0.00000015466x^2$	5.6136e-001
	0.05	3.5021e-001	$13.341+0.00039093x$ $+0.00000017290x^2$	9.1224e-001
Results of [21]	0.00	—	$14.507+0.000016411x$	3.2952e+000
	0.01	4.357e-3	$14.507+0.000015817x$	3.2972e+000
	0.05	1.274e-2	$14.515+0.000011783x$	3.3075e+000

7. Conclusions

In this paper, we mainly study the inverse problem of determining a space-dependent source in the parabolic equation. As we all know, the inverse source problem is a classical ill-posed problem. Basing on a sequence well-posed direct problems solved by the finite element method, we propose a regularized optimization method for solving the inverse

source problem, and use the linear model function method to choose regularization parameters for obtaining a stable solution. The proposed method is a non-iterative method, and can be extended to the parabolic equation with other boundary conditions, even mixed boundary conditions. In addition, we find that the regularization parameter plays an important role in numerically solving the regularized optimization problem. Numerical results for one- and two-dimensional examples show that the proposed method together with regularization parameter chosen strategy is efficient and robust with respect to data noise, especially for reconstructing the discontinuous source functions. Furthermore, the proposed method is successfully used to solve a real example of identifying the magnitude of groundwater pollution source.

In Algorithm 1, Matlab is used to compute the final dada for each basis function, it introduces some errors due to the finite element approximation. Therefore it is desirable to keep this computational errors less than the noise level. Here, we thank the reviewers very much for pointing out this fact. Obviously, the mesh grid is denser the error of the finite element approximation is smaller. Consequently, the computation amount will be increase if Algorithm 1 is run by serial computing. As mentioned above, we can improve it by parallel computing. On the other hand, instead of using the linear finite element basis we approximate the source function $f(x)$ by applying the polynomial function basis such as in the real life example, or the trigonometric function basis. In this case, the number of computing the final data for each basis is independent on the mesh. Thereby, we can improve greatly the efficiency of Algorithm 1 by selecting the number of the basis under some *a priori* information about the source function. Results of our numerical examples show that the proposed regularized optimization method is robust to the error of the finite element approximation. And we will study the error estimation of the proposed method in our future work.

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Knowledge reduction in knowledge bases and its algorithm *

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Abstract: One of rough set theory's strengths is the fact that an unknown target concept can be approximately characterized by existing knowledge structures in a knowledge base. In this paper, we investigate Knowledge reduction in knowledge bases and give its algorithm.

Keywords: Knowledge base; Knowledge reduction; Necessary relation; Algorithm.

1 Introduction

Rough set theory was proposed by Pawlak [10] as a mathematical tool for data reasoning. It may be seen as an extension of classical set theory, has been proved to be an effective approach to deal with intelligent systems characterized by insufficient and incomplete information and has been successfully applied to machine learning, intelligent systems, inductive reasoning, pattern recognition, mereology, image processing, signal analysis, knowledge discovery, decision analysis, expert systems and many other fields [6, 7, 8, 9].

Basic opinion in rough set theory is that knowledge (human intelligence) is the ability to classify elements [1, 4]. That is to say, knowledge is a family of classifications (or partitions) on the universe. Rough set theory mainly consider equivalence relations on the universe, which determine partitions on the universe. One deals with not only a single classification (or partition) on the universe, but also a family of classifications (or partitions) on the universe. This leads to the definition of a knowledge base, which is a important concept in rough set theory.

For a given knowledge base, one of the tasks in data mining and knowledge discovery is to generate new knowledge through known knowledge.

The purpose of this paper is to investigate knowledge reduction in knowledge bases.

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2 Preliminaries

In this section, we recall basic concepts about rough sets and knowledge bases.

Throughout this paper, U denotes a non-empty finite set called the universe, 2^U denotes the set of all subsets of U, $\mathcal{R}^*$ denote the set of all equivalence relations on U. All mappings are assumed to be surjective.

For $\mathcal{R} \subseteq \mathcal{R}^*$, denote $ind(\mathcal{R}) = \bigcap\limits_{R \in \mathcal{R}} R$. Obviously, $ind(\mathcal{R}) \in \mathcal{R}^*$. For $R \in \mathcal{R}^*$, denote $[x]_R = \{y \in U : xRy\}$.

Let $R \in \mathcal{R}^*$. The pair (U, R) is called a Pawlak approximation space. Based on (U, R), one can define the following two rough approximations:

$$\underline{R}(X) = \{x \in U : [x]_R \subseteq X\}, \quad \overline{R}(X) = \{x \in U : [x]_R \cap X \neq \emptyset\}.$$

$\underline{R}(X)$ and $\overline{R}(X)$ are called the Pawlak lower approximation and the Pawlak upper approximation of X, respectively.

The boundary region of X, defined by the difference between these rough approximations, that is $Bnd_R(X) = \overline{R}(X) - \underline{R}(X)$.

A set is rough if its boundary region is not empty; Otherwise, it is crisp. Thus, X is rough if $\underline{R}(X) \neq \overline{R}(X)$.

Definition 2.1 ([13])**.** *A pair $(U, \mathcal{R})$ is called a knowledge base, if $\mathcal{R} \subseteq \mathcal{R}^*$.*

It is well-know that elements in a knowledge base are not of the same importance, some even are redundant. So we often consider knowledge reductions in a knowledge base by deleting unrelated or unimportant elements with the requirement of keeping the ability of classification. This is the meaning of knowledge reduction in knowledge bases.

Definition 2.2 ([13])**.** *Let $(U, \mathcal{R})$ be a knowledge base and $\mathcal{P} \subseteq \mathcal{R}$.*

(1) *$\mathcal{P}$ is called a coordinate subfamily of $\mathcal{R}$, if $ind(\mathcal{P}) = ind(\mathcal{R})$.*

(2) *$R \in \mathcal{P}$ is called independent in $\mathcal{P}$, if $ind(\mathcal{P} - \{R\}) \neq ind(\mathcal{P})$; $\mathcal{P}$ is called a independent subfamily of $\mathcal{R}$, if $\forall\ R \in \mathcal{P}$, R is independent in $\mathcal{P}$.*

(3) *$\mathcal{P}$ is called a knowledge reduction (for short, reduction) of $\mathcal{R}$, if $\mathcal{P}$ is both coordinate and independent.*

In this paper, the set of all coordinate subfamilies (resp., all knowledge reductions) of $\mathcal{R}$ is denoted by $co(\mathcal{R})$ (resp., $red(\mathcal{R})$).

Obviously,

$$\mathcal{P} \in red(\mathcal{R}) \iff \mathcal{P} \in co(\mathcal{R}) \ and\ \forall\ \mathcal{Q} \subset \mathcal{P}, \mathcal{Q} \notin co(\mathcal{R}).$$

3 Knowledge reduction in knowledge bases

Proposition 3.1. *Let $(U, \mathcal{R})$ be a knowledge base. Then there always exist a knowledge reduction of $\mathcal{R}$.*

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Proof. Suppose $\forall\ R \in \mathcal{R}$, $\mathcal{R} - \{R\} \notin co(\mathcal{R})$. Then $\mathcal{R} \in red(\mathcal{R})$.

Suppose $\exists\ R_1 \in \mathcal{R}$, $\mathcal{R} - \{R_1\} \in co(\mathcal{R})$. Then, we consider $\mathcal{R} - \{R_1\}$. Again suppose $\forall\ R \in \mathcal{R} - \{R_1\}$, $(\mathcal{R} - \{R_1\}) - \{R\} \notin co(\mathcal{R})$. Then $\mathcal{R} - \{R_1\} \in red(\mathcal{R})$. Again suppose $\exists\ R_2 \in \mathcal{R} - \{R_1\}$, $(\mathcal{R} - \{R_1\}) - \{R_2\} \in co(\mathcal{R})$. Then, we consider $\mathcal{R} - \{R_1, R_2\}$. Repeat this process. Since $\mathcal{R}$ is finite, we can find a knowledge reduction of $\mathcal{R}$.

Thus, there always exist a knowledge reduction of $\mathcal{R}$. □

Definition 3.2. *Let $(U, \mathcal{R})$ be a knowledge base. Put*

$$\mathcal{D}(x, y) = \{R \in \mathcal{R} | (x, y) \notin R\}\ (x, y \in U).$$

Then

(1) *$\mathcal{D}(x, y)$ is called the discernibility subfamily of $\mathcal{R}$ on x and y.*

(2) *$\mathfrak{D}(\mathcal{R}) = (d_{ij})_{n \times n}$ is called the discernibility matrix of $\mathcal{R}$ where $U = \{x_1, x_2, \cdots, x_n\}$ and $d_{ij} = \mathcal{D}(x_i, x_j)$ $(1 \le i, j \le n)$.*

Example 3.3. *Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. We consider the knowledge base $(U, \mathcal{R})$ where $\mathcal{R} = \{R_1, R_2, R_3, R_4\}$ and*

$U/R_1 = \{\{x_1, x_2, x_5\}, \{x_3, x_4, x_6\}\}$,
$U/R_2 = \{\{x_1, x_6\}, \{x_2, x_3, x_4, x_5\}\}$,
$U/R_3 = \{\{x_1, x_2, x_5, x_6\}, \{x_3, x_4\}\}$,
$U/R_4 = \{\{x_1, x_2, x_5\}, \{x_3, x_4, x_6\}\}$.

We can obtain the discernibility matrix $\mathfrak{D}(\mathcal{R})$ as follows:

$$\begin{pmatrix} \emptyset & \{R_2\} & \mathcal{R} & \mathcal{R} & \{R_2\} & \{R_1, R_4\} \\ \{R_2\} & \emptyset & \{R_1, R_3, R_4\} & \{R_1, R_3, R_4\} & \emptyset & \{R_1, R_2, R_4\} \\ \mathcal{R} & \{R_1, R_3, R_4\} & \emptyset & \emptyset & \{R_1, R_3, R_4\} & \{R_2, R_3\} \\ \mathcal{R} & \{R_1, R_3, R_4\} & \emptyset & \emptyset & \{R_1, R_3, R_4\} & \{R_2, R_3\} \\ \{R_2\} & \emptyset & \{R_1, R_3, R_4\} & \{R_1, R_3, R_4\} & \emptyset & \{R_1, R_2, R_4\} \\ \{R_1, R_4\} & \{R_1, R_2, R_4\} & \{R_2, R_3\} & \{R_2, R_3\} & \{R_1, R_2, R_4\} & \emptyset \end{pmatrix}$$

Proposition 3.4. *Let $(U, \mathcal{R})$ be a knowledge base. The $\forall\ x, y, z \in U$,*

(1) $\mathcal{D}(x, x) = \emptyset$.

(2) $\mathcal{D}(x, y) = \mathcal{D}(y, z)$.

(3) $\mathcal{D}(x, y) \subseteq \mathcal{D}(x, z) \cup \mathcal{D}(z, y)$.

Proof. (1) and (2) are obvious.

(3) Suppose $\mathcal{D}(x, y) \not\subseteq \mathcal{D}(x, z) \cup \mathcal{D}(z, y)$. Then $\mathcal{D}(x, y) - \mathcal{D}(x, z) \cup \mathcal{D}(z, y) \neq \emptyset$. Pick

$$R \in \mathcal{D}(x, y) - \mathcal{D}(x, z) \cup \mathcal{D}(z, y).$$

$R \in \mathcal{D}(x, y)$ implies $(x, y) \notin R$.

Since $R \notin \mathcal{D}(x, z) \cup \mathcal{D}(z, y)$, we have $R \notin \mathcal{D}(x, z)$ and $R \notin \mathcal{D}(z, y)$. Then $(x, z) \in R$ and $(z, y) \in R$. So $(x, y) \in R$. This is a contradiction.

Thus $\mathcal{D}(x, y) \subseteq \mathcal{D}(x, z) \cup \mathcal{D}(z, y)$. □

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Corollary 3.5. *Let $(U, \mathcal{R})$ be a knowledge base. Then d is a distance function on U where*

$$d(x, y) = |\mathcal{D}(x, y)| \ (x, y \in U).$$

Proposition 3.6. *Let $(U, \mathcal{R})$ be a knowledge base. Then*

$$\mathcal{P} \in co(\mathcal{R}) \iff If\ (x, y) \notin ind(\mathcal{R}),\ then\ \mathcal{P} \cap \mathcal{D}(x, y) \neq \emptyset.$$

Proof. (1) "$\Longrightarrow$". Since $\mathcal{P} \in co(\mathcal{R})$, we have $ind(\mathcal{P}) = ind(\mathcal{R})$. Then $(x, y) \notin ind(\mathcal{P})$. So $(x, y) \notin P\ for\ some\ P \in \mathcal{P}$.

$(x, y) \notin P$ implies $P \in \mathcal{D}(x, y)$. Then $P \in \mathcal{P} \cap \mathcal{D}(x, y)$.

Thus $\mathcal{P} \cap \mathcal{D}(x, y) \neq \emptyset$.

"$\Longleftarrow$". Suppose $\mathcal{P} \notin co(\mathcal{R})$. Then $ind(\mathcal{P}) \neq ind(\mathcal{R})$. This implies $ind(\mathcal{P}) - ind(\mathcal{R}) \neq \emptyset$. Pick

$$(x, y) \in ind(\mathcal{P}) - ind(\mathcal{R}).$$

Since $(x, y) \notin ind(\mathcal{R})$, we have $\mathcal{P} \cap \mathcal{D}(x, y) \neq \emptyset$.

Note that $(x, y) \in ind(\mathcal{P})$. Then $\forall\ P \in \mathcal{P}$, $(x, y) \in P$. So $P \notin \mathcal{D}(x, y)$. Thus $\mathcal{P} \cap \mathcal{D}(x, y) = \emptyset$. This is a contradiction.

Thus $\mathcal{P} \in co(\mathcal{R})$. □

The discernibility family can easily determine knowledge reductions.

Theorem 3.7. *Let $(U, \mathcal{R})$ be a knowledge base. Then $\mathcal{P} \in red(\mathcal{R}) \iff$*

(1) *If $(x, y) \notin ind(\mathcal{R})$, then $\mathcal{P} \cap \mathcal{D}(x, y) \neq \emptyset$;*
(2) *$\forall\ R \in \mathcal{P}$, $\exists\ (x_R, y_R) \in ind(\mathcal{R})$, $(\mathcal{P} - \{R\}) \cap \mathcal{D}(x_R, y_R) = \emptyset$.*

Proof. This holds by Proposition 3.6. □

Definition 3.8. *Let $(U, \mathcal{R})$ be a knowledge base. Put*

$$core(\mathcal{R}) = \bigcap_{\mathcal{P} \in red(\mathcal{R})} \mathcal{P}.$$

Then $core(\mathcal{R})$ is called the core of $\mathcal{R}$. Moreover,

(1) *$R \in \mathcal{R}$ is called a necessary relation, if $R \in core(\mathcal{R})$.*
(2) *$R \in \mathcal{R}$ is called a relatively necessary relation, if $R \in \bigcup_{\mathcal{P} \in red(\mathcal{R})} \mathcal{P} - core(\mathcal{R})$.*
(3) *$R \in \mathcal{R}$ is called a absolutely dispensable relation, if $R \in \mathcal{R} - \bigcup_{\mathcal{P} \in red(\mathcal{R})} \mathcal{P}$.*
(4) *$R \in \mathcal{R}$ is called a dispensable relation, if $R \in \mathcal{R} - core(\mathcal{R})$.*

Obviously, R is dispensable $\iff$ R is relatively necessary or absolutely dispensable.

Proposition 3.9. *Let $(U, \mathcal{R})$ be a knowledge base. Then*

$$|red(\mathcal{R})| = 1 \iff core(\mathcal{R}) \in red(\mathcal{R}).$$

Proof. Necessity. This is obvious.

Sufficiency. Denote $red(\mathcal{R}) = \{\mathcal{P}_k : 1 \le k \le n\}$. We only need to prove $n = 1$.

Suppose $n \ge 2$. Since $core(\mathcal{R}) \in red(\mathcal{R})$, we have $core(\mathcal{R}) = \mathcal{P}_i$ for some i. Pick $j \ne i$. Then $\mathcal{P}_i = \bigcap\limits_{k=1}^{n} \mathcal{P}_k \subseteq \mathcal{P}_j$. But $\mathcal{P}_i \ne \mathcal{P}_j$. Thus $\mathcal{P}_i \subset \mathcal{P}_j$. Since $\mathcal{P}_j \in red(\mathcal{R})$, we have $\mathcal{P}_i \notin co(\mathcal{R})$. Then $\mathcal{P}_i \notin red(\mathcal{R})$. This is a contradiction.

Thus $n = 1$. □

Discernibility family can easily determine the core.

Proposition 3.10. *Let $(U, \mathcal{R})$ be a knowledge base. The following are equivalent:*

(1) *R is a necessary relation;*

(2) *R is independent in $\mathcal{R}$;*

(3) *$\exists\, x, y \in U$, $\mathcal{D}(x, y) = \{R\}$.*

Proof. (1) $\Longrightarrow$ (2). Suppose that R is not independent in $\mathcal{R}$. Then

$$ind(\mathcal{R} - \{R\}) = ind(\mathcal{R}).$$

This implies $\mathcal{R} - \{R\} \in co(\mathcal{R})$. Consider $\mathcal{R} - \{R\}$. By Proposition 3.1, $\exists\, \mathcal{P} \subseteq \mathcal{R} - \{R\}$, $\mathcal{P} \in red(\mathcal{R})$.

$\mathcal{P} \subseteq \mathcal{R} - \{R\}$ implies $R \notin \mathcal{P}$. Then R is not a necessary relation. This is a contradiction.

(2) $\Longrightarrow$ (1). Suppose that R is not a necessary relation. Then $\exists\, \mathcal{P} \in red(\mathcal{R})$, $R \notin \mathcal{P}$. So $\mathcal{P} \subseteq \mathcal{R} - \{R\} \subseteq \mathcal{R}$. This implies

$$ind(\mathcal{P}) \supseteq ind(\mathcal{R} - \{R\}) \supseteq ind(\mathcal{R}).$$

By $\mathcal{P} \in red(\mathcal{R})$, $ind(\mathcal{P}) = ind(\mathcal{R})$. Then $ind(\mathcal{R} - \{R\}) = ind(\mathcal{R})$. So R is not independent in $\mathcal{R}$. This is a contradiction.

(2) $\Longrightarrow$ (3). Since R is independent in $\mathcal{R}$, we have $ind(\mathcal{R} - \{R\}) \ne ind(\mathcal{R})$. Then $ind(\mathcal{R} - \{R\}) - ind(\mathcal{R}) \ne \emptyset$. Pick

$$(x, y) \in ind(\mathcal{R} - \{R\}) - ind(\mathcal{R}).$$

Denote $\mathcal{R} = \{R_1, R_2, \dots, R_n\}$. Then $R = R_j$ for some $j \le n$. So

$$(x, y) \in \bigcap_{1 \le i \le n, i \ne j} R_i - \bigcap_{1 \le i \le n} R_i.$$

This implies $(x, y) \notin R_j$ and $(x, y) \in R_i$ $(i \ne j)$.

Thus $\mathcal{D}(x, y) = \{R\}$.

(3) $\Longrightarrow$ (2). Since $\exists\, x, y \in U$, $\mathcal{D}(x, y) = \{R\}$, we have

$$(x, y) \notin R,\ (x, y) \in R^{'}\ (R^{'} \ne R).$$

Then $(x, y) \in ind(\mathcal{R} - \{R\})$. But $(x, y) \notin ind(\mathcal{R})$.

Thus $ind(\mathcal{R} - \{R\}) \ne ind(\mathcal{R})$. Hence R is independent in $\mathcal{R}$. □

Proposition 3.11. *Let* $(U, \mathcal{R})$ *be a knowledge base. Denote*

$$R^\star = \bigcup_{\mathcal{P} \in co(\mathcal{R})} ind(\mathcal{P} - \{R\}).$$

Then the following are equivalent:
(1) R *is a absolutely dispensable relation;*
(2) $\forall\ \mathcal{P} \in co(\mathcal{R})$, $ind(\mathcal{P} - \{R\}) = ind(\mathcal{R})$;
(3) $R^\star = ind(\mathcal{R})$;
(4) $R^\star \subseteq R$.

Proof. (1) $\Longrightarrow$ (2). By Proposition 3.1, $\exists\ \mathcal{Q} \subseteq \mathcal{P}$, $\mathcal{Q} \in red(\mathcal{R})$. Since R is not a necessary relation, we have $R \notin \mathcal{Q}$, which implies $\mathcal{Q} \subseteq \mathcal{R} - \{R\}$. Then

$$\mathcal{Q} \subseteq \mathcal{P} \cap (\mathcal{R} - \{R\}) = \mathcal{P} - \{R\} \subseteq \mathcal{P}.$$

We have

$$ind(\mathcal{Q}) \supseteq ind(\mathcal{P} - \{R\})) \supseteq ind(\mathcal{P}).$$

Note that $\mathcal{P} \in co(\mathcal{R})$ and $\mathcal{Q} \in red(\mathcal{R})$. Then $ind(\mathcal{P}) = ind(\mathcal{R}) = ind(\mathcal{Q})$. Thus

$$ind(\mathcal{P} - \{R\}) = ind(\mathcal{R}).$$

(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are obvious.

(4) $\Longrightarrow$ (1). Suppose $\exists\ \mathcal{P} \in red(\mathcal{R})$, $R \in \mathcal{P}$. Then $\mathcal{P} - \{R\} \subset \mathcal{P}$. Since $\mathcal{P} \in red(\mathcal{R})$, we have $\mathcal{P} - \{R\} \notin co(\mathcal{R})$. Then $ind(\mathcal{P} - \{R\}) - ind(\mathcal{R}) \neq \emptyset$. $\mathcal{P} \in red(\mathcal{R})$ implies $ind(\mathcal{P}) = ind(\mathcal{R})$. Then

$$ind(\mathcal{P} - \{R\}) - ind(\mathcal{P}) \neq \emptyset.$$

Pick $(x, y) \in ind(\mathcal{P} - \{R\}) - ind(\mathcal{P})$. Note that $ind(\mathcal{P}) = ind(\mathcal{P} - \{R\}) \cap R$. Then $(x, y) \notin R$.

Since $\mathcal{P} \in co(\mathcal{R})$ and $R^\star \subseteq R$, we have $ind(\mathcal{P} - \{R\}) \subseteq R$. Then $(x, y) \in R$. This is a contradiction. □

Theorem 3.12. *Let* $(U, \mathcal{R})$ *be a knowledge base. Then*
(1) R *is necessary* $\Leftrightarrow \mathcal{R} - \{R\} \notin co(\mathcal{R})$.
(2) R *is relatively necessary* $\Leftrightarrow \mathcal{R} - \{R\} \in co(\mathcal{R})$ *and* $R^\star \not\subseteq R$.
(3) R *is absolutely dispensable* $\Leftrightarrow R^\star \subseteq R$.
(4) R *is dispensable* $\Leftrightarrow \mathcal{R} - \{R\} \in co(\mathcal{R})$.

Proof. This holds by Proposition 3.10 and Proposition 3.11. □

Example 3.13. *In Example 3.3, we have*
(1) R_2 *is necessary.*
(2) R_1 *and* R_4 *are relatively necessary.*
(3) R_3 *is absolutely dispensable.*
(4) R_1, R_3 *and* R_4 *are dispensable.*

6

4 A algorithm on knowledge reduction

It is more convenient to calculate knowledge reductions and the core in knowledge bases by using the following discernibility function when there are many equivalence relations in knowledge bases.

Below, we give a algorithm on the knowledge reductions with the help of mathematical logic.

"$\bigvee$"(disjunction), "$\bigwedge$"(conjunction), "$\longrightarrow$"(implication), "$\longleftrightarrow$"(biimplication) are propositional connectives in mathematical logic. They are read as "or", "and", "if-then", "if and only if", respectively.

Let $(U, \mathcal{R})$ be a knowledge base. $\forall\ R \in \mathcal{R}$, we specify a Boolean variable "R". If $\mathcal{D}(x, y) = \{R_1, R_2, \cdots, R_k\}$ with $x, y \in U$, then we specify a Boolean function $R_1 \vee R_2 \vee \cdots \vee R_k$.

Denote

$$\bigvee\{R_1, R_2, \cdots, R_k\}\ or\ \bigvee_{i=1}^{k} R_i = R_1 \vee R_2 \vee \cdots \vee R_k,$$

$$\bigwedge\{R_1, R_2, \cdots, R_k\}\ or\ \bigwedge_{i=1}^{k} R_i = R_1 \wedge R_2 \wedge \cdots \wedge R_k.$$

We stipulate that $\vee\ \emptyset = 1$ and $\wedge\ \emptyset = 0$ where 0 and 1 are two Boolean constants.

Definition 4.1. *Let $(U, \mathcal{R})$ be a knowledge base where $U = \{x_1, x_2, \cdots, x_n\}$ and $\mathfrak{D}(\mathcal{R}) = (d_{ij})_{n \times n}$ the discernibility matrix of $\mathcal{R}$. We define the discernibility function $\Delta(\mathcal{R})$ of $\mathcal{R}$ as follows:*

$$\Delta(\mathcal{R}) = \bigwedge(\bigvee d_{ij}).$$

Example 4.2. *In Example 3.3, we have*

$$\triangle(\mathcal{R}) = R_2 \wedge (R_1 \vee R_2 \vee R_3 \vee R_4) \wedge (R_1 \vee R_4) \wedge (R_1 \vee R_3 \vee R_4) \wedge (R_1 \vee R_2 \vee R_4) \wedge (R_2 \vee R_3).$$

Denote

$$L(\mathcal{R}) = \{\bigvee d_{ij} : 1 \leq i, j \leq n\}.$$

We define a binary relation "$\leq$" on $L(\mathcal{R})$ as follows:

$$\bigvee d_{ij} \leq \bigvee d_{kl} \iff d_{ij} \subseteq d_{kl}\ for\ any\ \bigvee d_{ij}, \bigvee d_{kl} \in L(\mathcal{R}).$$

For any $\bigvee d_{ij}, \bigvee d_{kl} \in L(\mathcal{R})$, we denote

$$(\bigvee d_{ij}) \bigsqcup (\bigvee d_{kl}) = \bigvee(d_{ij} \cup d_{kl}), \quad (\bigvee d_{ij}) \bigsqcap (\bigvee d_{kl}) = \bigvee(d_{ij} \cap d_{kl}).$$

Proposition 4.3. $(L(\mathcal{R}),\ \leq)$ *is a poset.*

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Proof. (1) $\bigvee d_{ij} \leq \bigvee d_{ij}$ for any $\bigvee d_{ij} \in L(\mathcal{R})$.

(2) Let $\bigvee d_{ij}, \bigvee d_{kl} \in L(\mathcal{R})$. Suppose $\bigvee d_{ij} \leq \bigvee d_{kl}$ and $\bigvee d_{kl} \leq \bigvee d_{ij}$. Then $d_{ij} \subseteq d_{kl}$ and $d_{kl} \subseteq d_{ij}$. This implies $d_{ij} = d_{kl}$. So $\bigvee d_{ij} = \bigvee d_{kl}$.

(3) Let $\bigvee d_{ij}, \bigvee d_{kl}, \bigvee d_{hv} \in L(\mathcal{R})$. Suppose $\bigvee d_{ij} \leq \bigvee d_{kl}$ and $\bigvee d_{kl} \leq \bigvee d_{hv}$. Then $d_{ij} \subseteq d_{kl}$ and $d_{kl} \subseteq d_{hv}$. This implies $d_{ij} \subseteq d_{hv}$. So $\bigvee d_{ij} \leq \bigvee d_{hv}$.

Thus $(L(\mathcal{R}),\ \leq)$ is a poset. □

Proposition 4.4. *Let $(U, \mathcal{R})$ be a knowledge base where $U = \{x_1, x_2, \cdots, x_n\}$. If $\{d_{ij} : 1 \leq i, j \leq n\}$ is a topology on $\mathcal{R}$, then $(L(\mathcal{R}), \leq, \sqcup, \sqcap)$ is a lattice with top element and bottom element.*

Proof. Denote $\tau = \{d_{ij} : 1 \leq i, j \leq n\}$. By Proposition 4.3, $(L(\mathcal{R}),\ \leq)$ is a poset.

For $\bigvee d_{ij}, \bigvee d_{kl} \in L(\mathcal{R})$, since τ is a topology on $\mathcal{R}$, we have $d_{ij} \cup d_{kl} \in \tau, d_{ij} \cap d_{kl} \in \tau$. This implies

$$(\bigvee d_{ij}) \bigsqcup (\bigvee d_{kl}) = \bigvee (d_{ij} \cup d_{kl}) \in L(\mathcal{R}),$$

$$(\bigvee d_{ij}) \bigsqcap (\bigvee d_{kl}) = \bigvee (d_{ij} \cap d_{kl}) \in L(\mathcal{R}).$$

Obviously, $1_{L(\mathcal{R})} = \vee \mathcal{R}$, $0_{L(\mathcal{R})} = \vee \emptyset$.
Thus $(L(\mathcal{R}), \leq, \bigsqcup, \bigsqcap)$ is a lattice with top element and bottom element. □

Example 4.5. *In Example 3.3, $(\bigvee d_{23}) \bigsqcap (\bigvee d_{63}) = R_3 \notin L(\mathcal{R})$. Then $(L(\mathcal{R}), \leq, \sqcup, \sqcap)$ is not a lattice.*

Definition 4.6. *Let $(U, \mathcal{R})$ be a knowledge base. If $\Delta(\mathcal{R}) = \bigvee\limits_{k=1}^{q} (\bigwedge\limits_{l=1}^{p_k} R_{kl})$, where every $\mathcal{P}_k = \{R_{kl} : l \leq p_k\} \subseteq \mathcal{R}$ has not repetitive elements, then $\bigvee\limits_{k=1}^{q} (\bigwedge\limits_{l=1}^{p_k} R_{kl})$ is called the standard minimum formula of $\Delta(\mathcal{R})$. We denote it by $\Delta^*(\mathcal{R})$. That is,*

$$\Delta^*(\mathcal{R}) = \bigvee_{k=1}^{q} (\bigwedge_{l=1}^{p_k} R_{kl}).$$

Example 4.7. *In Example 3.3, we have*

$R_2 \leq (R_1 \vee R_2 \vee R_3 \vee R_4),\ \ R_2 \leq (R_1 \vee R_2 \vee R_4), R_2 \leq (R_2 \vee R_3),\ \ (R_1 \vee R_4) \leq (R_1 \vee R_3 \vee R_4).$

Obviously,

$$R_2 \wedge (R_1 \vee R_2 \vee R_3 \vee R_4) = R_2, \quad R_2 \wedge (R_1 \vee R_2 \vee R_4) = R_2,$$

$$R_2 \wedge (R_2 \vee R_3) = R_2, \quad (R_1 \vee R_4) \wedge (R_1 \vee R_3 \vee R_4) = R_1 \vee R_4.$$

Then $\Delta(\mathcal{R}) = R_2 \wedge (R_1 \vee R_2 \vee R_3 \vee R_4) \wedge (R_1 \vee R_4) \wedge (R_1 \vee R_3 \vee R_4) \wedge (R_1 \vee R_2 \vee R_4) \wedge (R_2 \vee R_3)$
$= R_2 \wedge (R_1 \vee R_4)$
$= (R_1 \wedge R_2) \vee (R_2 \wedge R_4)$.

Thus $\Delta^*(\mathcal{R}) = (R_1 \wedge R_2) \vee (R_2 \wedge R_4)$.

Theorem 4.8. *Let* $(U, \mathcal{R})$ *be a knowledge base. If* $\Delta^*(\mathcal{R}) = \bigvee_{k=1}^{q} (\bigwedge_{l=1}^{p_k} R_{kl})$ *is the standard minimum formula of* $\Delta(\mathcal{R})$*. Then* $red(\mathcal{R}) = \{\mathcal{P}_k : k \leq q\}$ *where* $\mathcal{P}_k = \{R_{kl} : l \leq p_k\}$.

Proof. (1) Let $\mathcal{P}_{k_0} \in \{\mathcal{P}_k : k \leq q\}$.

(i) Obviously, $\Delta^*(\mathcal{R}) = \bigvee_{k=1}^{q} (\bigwedge_{l=1}^{p_k} R_{kl}) = \bigvee_{k=1}^{q} (\bigwedge \mathcal{P}_k)$. Then $\bigwedge \mathcal{P}_{k_0} \longrightarrow \Delta^*(\mathcal{R})$.
Since $\Delta^*(\mathcal{R}) = \Delta(\mathcal{R}) = \bigwedge(\bigvee d_{ij})$, we have

$$\Delta^*(\mathcal{R}) \Longleftrightarrow \bigvee d_{ij} \ for\ any\ 1 \leq i, j \leq n.$$

Then $\forall\ x, y \in U$, $\bigwedge \mathcal{P}_{k_0} \longrightarrow \bigvee \mathcal{D}(x, y)$.

So $\forall\ (x, y) \notin ind(\mathcal{R})$, $\bigwedge \mathcal{P}_{k_0} \longrightarrow \bigvee \mathcal{D}(x, y)$.

Now $\bigwedge \mathcal{P}_{k_0} \Longleftrightarrow R_{k_0 l}$ for any $l \leq p_{k_0}$ and $\bigvee \mathcal{D}(x, y) \longleftrightarrow R$ for some $R \in \mathcal{D}(x, y)$. Then $\forall\ (x, y) \notin ind(\mathcal{R})$, $R_{k_0 l}$ for any $l \leq p_{k_0} \longrightarrow R$ for some $R \in \mathcal{D}(x, y)$. So $\forall\ (x, y) \notin ind(\mathcal{R})$, there exists $l_0 \leq p_{k_0}$ such that $R = R_{k_0 l_0}$, i.e., $R \in \mathcal{P}_{k_0} \cap \mathcal{D}(x, y)$. Thus $\forall\ (x, y) \notin ind(\mathcal{R})$, $\mathcal{P}_{k_0} \cap \mathcal{D}(x, y) \neq \emptyset$.

By Proposition 3.6, $\mathcal{P}_{k_0} \in co(\mathcal{R})$.

(ii) To prove $\mathcal{P}_{k_0} \in red(\mathcal{R})$, by Theorem 3.7, we only need to show that

$$\forall\ R \in \mathcal{P}_{k_0},\ \exists\ (x_R, y_R) \in ind(\mathcal{R}),\ (\mathcal{P}_{k_0} - \{R\}) \cap \mathcal{D}(x_R, y_R) = \emptyset.$$

Suppose that $\exists\ R_0 \in \mathcal{P}_{k_0}$ such that $(\mathcal{P}_{k_0} - \{R_0\}) \cap \mathcal{D}(x, y) \neq \emptyset$ for any $(x, y) \notin ind(\mathcal{R})$. Pick $R_{xy} \in (\mathcal{P}_{k_0} - \{R_0\}) \cap \mathcal{D}(x, y)$. Then $\bigwedge(\mathcal{P}_{k_0} - \{R_0\}) \longrightarrow R_{xy}$ and $R_{xy} \longrightarrow \bigvee \mathcal{D}(x, y)$. Thus $\forall\ (x, y) \notin ind(\mathcal{R})$, $\bigwedge(\mathcal{P}_{k_0} - \{R_0\}) \longrightarrow \bigvee \mathcal{D}(x, y)$.

$\forall\ (x, y) \in ind(\mathcal{R})$, we have $\mathcal{D}(x, y) = \emptyset$. Then $\bigwedge(\mathcal{P}_{k_0} - \{R_0\}) \longrightarrow \bigvee \mathcal{D}(x, y)$.
It follows that $\forall\ x, y \in U$,

$$\bigwedge(\mathcal{P}_{k_0} - \{R_0\}) \longrightarrow \bigvee \mathcal{D}(x, y).$$

Since $\Delta^*(\mathcal{R})$ contains all true explanations of $\Delta(\mathcal{R})$, we have $\mathcal{P}_{k_0} - \{R_0\} \in \{\mathcal{P}_k : k \leq q\}$. Then
$(\bigwedge \mathcal{P}_{k_0}) \bigvee (\bigwedge(\mathcal{P}_{k_0} - \{R_0\}))$
$= ((\bigwedge(\mathcal{P}_{k_0} - \{R_0\})) \bigwedge \{R_0\}) \bigvee ((\bigwedge(\mathcal{P}_{k_0} - \{R_0\})) \bigwedge 1)$
$= (\bigwedge(\mathcal{P}_{k_0} - \{R_0\})) \bigwedge (\{R_0\} \bigvee 1)$
$= (\bigwedge(\mathcal{P}_{k_0} - \{R_0\})) \bigwedge 1$
$= \bigwedge(\mathcal{P}_{k_0} - \{R_0\})$.
This implies $\mathcal{P}_{k_0} \notin \{\mathcal{P}_k : k \leq q\}$. This is a contradiction.

Thus $\mathcal{P}_{k_0} \in red(\mathcal{R})$. This show $red(\mathcal{R}) \supseteq \{\mathcal{P}_k : k \leq q\}$.

(2) Let $\mathcal{P} \in red(\mathcal{R})$. Then $\mathcal{P} \in co(\mathcal{R})$. By Proposition 3.6, $\mathcal{P} \cap \mathcal{D}(x,y) \neq \emptyset$ for any $(x,y) \notin ind(\mathcal{R})$. Similar to the proof of (1) (ii), we have $\mathcal{P} \in \{\mathcal{P}_k : k \leq q\}$.

Thus $red(\mathcal{R}) \subseteq \{\mathcal{P}_k : k \leq q\}$. Hence $red(\mathcal{R}) = \{\mathcal{P}_k : k \leq q\}$.

□

Algorithms 4.9. *Let $(U, \mathcal{R})$ be a knowledge base. The algorithm of knowledge reductions of $\mathcal{R}$ is shown as follows:*

Input: the knowledge base $(U, \mathcal{R})$;
Output: $red(\mathcal{R})$ and $core(\mathcal{R})$.

Step 1. *Input the knowledge base $(U, \mathcal{R})$;*
Step 2. *Calculate the discernibility matrix $\mathfrak{D}(\mathcal{R})$ of $\mathcal{R}$;*
Step 3. *Give discernibility function $\Delta(\mathcal{R})$ of $\mathcal{R}$;*
Step 4. *Calculate standard minimum formula $\Delta^*(\mathcal{R})$ of $\Delta(\mathcal{R})$;*
Step 5. *Output all knowledge reductions of $\mathcal{R}$ and the core of $\mathcal{R}$.*

Example 4.10. *We consider Example 3.3.*

In Step 1, *we input the knowledge base $(U, \mathcal{R})$.*
In Step 2, *we obtain the discernibility matrix $\mathfrak{D}(\mathcal{R})$.*
In Step 3, *we obtain*

$$\Delta(\mathcal{R}) = R_2 \wedge (R_1 \vee R_2 \vee R_3 \vee R_4) \wedge (R_1 \vee R_4) \wedge (R_1 \vee R_3 \vee R_4) \wedge (R_1 \vee R_2 \vee R_4) \wedge (R_2 \vee R_3).$$

In Step 4, *we obtain $\Delta^*(\mathcal{R}) = (R_1 \wedge R_2) \vee (R_2 \wedge R_4)$.*
In Step 5, *we obtain all knowledge reductions of $\mathcal{R}$: $\{R_1, R_2\}$, $\{R_2, R_4\}$ and $core(\mathcal{R}) = \{R_2\}$.*

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Belief reduction in IVF decision information systems and its algorithm *

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Abstract: Attribute reduction is one of the main problems in the study of information systems. This paper investigates belief reduction in IVF decision information systems by using the generalized D-S theory of evidence and rough set theory.

Keywords: IVF; Decision information system; Similarity relation; Belief reduction.

1 Introduction

Imprecision and uncertainty are two important aspects of incompleteness of information. One theory for the study of insufficient and incomplete information in intelligent systems is rough set theory [3]. Another important method used to deal with uncertainty in information systems is D-S theory of evidence [4]. There are strong connections between these two theory. It has been demonstrated that various belief structures are associated with various approximation spaces such that the different dual pairs of lower and upper approximation operators induced by approximation spaces may be used to interpret the corresponding dual pairs of belief functions induced by belief structures [5, 8, 10]. Based on this observation, D-S theory of evidence may be used to analyze attribute reduction and knowledge acquisition in information systems [7, 9, 12]. In the traditional rough set approach, the values of attributes are assumed to be nominal data, i.e. symbols. In many applications, however, the decision attribute-values can be linguistic terms (i.e. interval value fuzzy sets). The traditional rough set approach would treat these values as symbols, thereby some important information included in these values such as the partial ordering and membership degrees is ignored, which means that the traditional rough set approach cannot effectively deal with interval value fuzzy initial data (e.g. linguistic terms). Thus a new rough set model is needed to deal with such data.

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The purpose of this paper is to investigate belief reduction in IVF decision information systems by using the generalized D-S theory of evidence and rough set theory.

2 Preliminaries

Throughout this paper, "interval-valued fuzzy" denote briefly by " IVF ". U denotes a finite and nonempty set called the universe. 2^U denotes the family of all subsets of U. $F(U)$ denotes the set of all fuzzy sets in U. I denotes $[0,1]$ and $[I]$ denotes $\{[a,b] : a, b \in I \ and \ a \leq b\}$.

2.1 IVF sets

Definition 2.1 ([1]). *$\forall\ a, b \in [I]$, define*

(1) $a = b \Longleftrightarrow a^- = b^-,\ a^+ = b^+$.
(2) $a \leq b \Longleftrightarrow a^- \leq b^-,\ a^+ \leq b^+$; $\quad a < b \Longleftrightarrow a \leq b,\ a \neq b$.
(3) $a^c = [1,1] - a = [1 - a^+, 1 - a^-]$.

Definition 2.2 ([1]). *$\forall\ \{a_i : i \in J\} \subseteq [I]$, define*
(1) $\bigvee\limits_{i\in J} a_i = [\bigvee\limits_{i\in J} a_i^-,\ \bigvee\limits_{i\in J} a_i^+]$.
(2) $\bigwedge\limits_{i\in J} a_i = [\bigwedge\limits_{i\in J} a_i^-,\ \bigwedge\limits_{i\in J} a_i^+]$.

Definition 2.3 ([1]). *A mapping $A : U \to [I]$ is called an IVF set on U. Denote*

$$A(x) = [A^-(x), A^+(x)] \quad (x \in U).$$

Then $A^-(x)$ (resp. $A^+(x)$) is called the lower (resp. upper) degree to which x belongs to A. A^- (resp. A^+) is called the lower (resp. upper) IVF set of A.

The set of all IVF sets in U is denoted by $F^{(i)}(U)$.
Similar to fuzzy sets, the operators $\subseteq, \cap, \cup$ and the complement of IVF sets can be defined.

2.2 IVF decision information systems

Definition 2.4 ([6]). *$(U, A \cup D)$ is called an IVF decision information system, where $U = \{x_0, x_1, \cdots, x_{n-1}\}$ is the universe, A is a condition attribute set and $D = \{d_k \in F^{(i)}(U) : k = 1, 2, \cdots, r\}$ is a decision attribute set.*

Example 2.5 ([6]). *Table 1 gives an IVF decision information system $(U, A \cup D)$ where $U = \{x_0, x_1, \cdots, x_9\}$, $A = \{a_1, a_2, a_3\}$, $D = \{d_1, d_2, d_3\}$.*

Definition 2.6. *Let $(U, A \cup D)$ be an IVF decision information system. Then $B \subseteq A$ determines an equivalence relation as follows:*

$$R_B = \{(x, y) \in U \times U : a(x) = a(y)\ (a \in B)\}.$$

Table 1: An IVF decision information system $(U, A \cup D)$

	a_1	a_2	a_3	d_1	d_2	d_3
x_0	2	1	3	[0.7,0.9]	[0.15,0.2]	[0.4,0.5]
x_1	3	2	1	[0.3,0.5]	[0.5,0.7]	[0.35,0.4]
x_2	2	1	3	[0.7,0.8]	[0.3,0.4]	[0.1,0.2]
x_3	2	2	3	[0.15,0.2]	[0.5,0.8]	[0.2,0.3]
x_4	1	1	4	[0.05,0.1]	[0.2,0.3]	[0.65,0.9]
x_5	1	1	2	[0.1,0.2]	[0.35,0.5]	[1.0,1.0]
x_6	3	2	1	[0.25,0.4]	[1.0,1.0]	[0.3,0.4]
x_7	1	1	4	[0.1,0.2]	[0.25,0.4]	[0.5,0.6]
x_8	2	1	3	[0.45,0.6]	[0.25,0.3]	[0.2,0.3]
x_9	3	2	1	[0.05,0.1]	[0.8,0.9]	[0.05,0.2]

R_B forms a partition $U/R_B = \{[x]_B : x \in U\}$ of U where $[x]_B = \{y \in U : (x, y) \in R_B\}$.

The lower and upper approximations of $X \in F^{(i)}(U)$ with regard to (U, R_B) as follows:

$$\underline{R_B}(X)(x) = \bigwedge_{y \in [x]_B} X(y), \quad \overline{R_B}(X)(x) = \bigvee_{y \in [x]_B} X(y) \ (x \in U).$$

Remark 2.7. *If $y \in [x]_B$, then $[y]_B = [x]_B$. So $\underline{R_B}(X)(y) = \underline{R_B}(X)(x)$ and $\overline{R_B}(X)(y) = \overline{R_B}(X)(x)$.*

Denote $\underline{R_B}(X)([x]_B) = \underline{R_B}(X)(x)$, $\overline{R_B}(X)([x]_B) = \overline{R_B}(X)(x)$ $(*)$

Proposition 2.8. *Let $(U, A \cup D)$ be an IVF decision information system and let $C \subseteq B \subseteq A$. Then $\forall\ X \in F^{(i)}(U)$,*

(1) $\underline{R_B}(\widetilde{U} - X) = \widetilde{U} - \overline{R_B}(X)$.

(2) $\underline{R_C}(X) \subseteq \underline{R_B}(X) \subseteq X \subseteq \overline{R_B}(X) \subseteq \overline{R_C}(X)$.

Proof. (1) $\forall\ x \in U$,

$$\begin{aligned}\underline{R_B}(\widetilde{U} - X)(x) &= \bigwedge_{y \in [x]_B} (\widetilde{U} - X)(y) = \bigwedge_{y \in [x]_B} (\widetilde{U}(y) - X(y)) \\ &= \bigwedge_{y \in [x]_B} \widetilde{U}(y) - \bigvee_{y \in [x]_B} X(y) \\ &= \widetilde{U}(x) - \bigvee_{y \in [x]_B} X(y) = (\widetilde{U} - \overline{R_B}(X))(x).\end{aligned}$$

Then $\underline{R_B}(\widetilde{U} - X) = \widetilde{U} - \overline{R_B}(X)$.

(2) Since $C \subseteq B$, $\forall\ x \in U$, $[x]_C \supseteq [x]_B$. Then

$$\underline{R_C}(X)(x) = \bigwedge_{y \in [x]_C} X(y) \le \bigwedge_{y \in [x]_B} X(y) = \underline{R_B}(X)(x).$$

3

So $\underline{R_C}(X) \subseteq \underline{R_B}(X)$.

Similarly, $\overline{R_B}(X) \subseteq \overline{R_C}(X)$.

$\forall\, x \in U,\ x \in [x]_B$. Then $\underline{R_B}(X)(x) = \bigwedge\limits_{y\in[x]_B} X(y) \leq X(x)$. So $\underline{R_B}(X) \subseteq X$.

Similarly, $X \subseteq \overline{R_B}(X)$.

Hence

$$\underline{R_C}(X) \subseteq \underline{R_B}(X) \subseteq X \subseteq \overline{R_B}(X) \subseteq \overline{R_C}(X).$$

□

3 The generalized D-S theory of evidence

3.1 Necessity IVF measures and possibility IVF measures

Zadeh's theory of possibility [11] is based on the idea that the possibility of an event is determined by its most favorable case only.

$N^{(i)} : F^{(i)}(U) \to [I]$ is called a necessity IVF measure if it satisfies

$$N^{(i)}(\widetilde{\emptyset}) = \overline{0},\ N^{(i)}(\widetilde{U}) = [1,1],\ N^{(i)}(X \cap Y) = N^{(i)}(X) \wedge N^{(i)}(Y).$$

$\Pi^{(i)} : F^{(i)}(U) \to [I]$ is called a possibility IVF measure if it satisfies

$$\Pi^{(i)}(\widetilde{\emptyset}) = \overline{0},\ \Pi^{(i)}(\widetilde{U}) = [1,1],\ \Pi^{(i)}(X \cup Y) = \Pi^{(i)}(X) \vee \Pi^{(i)}Y).$$

It can easily be checked that $N^{(i)} : F^{(i)}(U) \to [I]$ is a necessity IVF measure iff the function Π^i defined by

$$\Pi^{(i)}(X) = [1,1] - N^{(i)}(X)\ (\forall X \in F^{(i)}(U))$$

is a possibility IVF measure.

Proposition 3.1. *Let $A \in 2^U$. For $X \in F^{(i)}(U)$, denote*

$$N_A^{(i)}(X) = \bigwedge_{y\in A} X(y) = [\bigwedge_{y\in A} X^-(y), \bigwedge_{y\in A} X^+(y)],$$

$$\Pi_A^{(i)}(X) = \bigvee_{y\in A} X(y) = [\bigvee_{y\in A} X^-(y), \bigvee_{y\in A} X^+(y)].$$

Then $N_A^{(i)}$ (resp. $\Pi_A^{(i)}$) is a necessity (resp. possibility) IVF measure.

Proof. The proof is obvious. □

3.2 IVF belief functions

Definition 3.2. *Let $(\mathcal{M}, m)$ be a belief structure on U. $\mathrm{Bel}^{(i)} : F^{(i)}(U) \to [I]$ is called a IVF belief function induced by $(\mathcal{M}, m)$ on U, if $\mathrm{Bel}^{(i)}(X) = \sum\limits_{\{Y: Y\in\mathcal{M}\}} m(Y) N_A^{(i)}(X)$.*

It can be prove that $\mathrm{Bel}^{(i)}$ is a IVF belief function iff (i) $\mathrm{Bel}^{(i)}(\widetilde{\emptyset}) = \overline{0}$, (ii) $\mathrm{Bel}^{(i)}(\widetilde{U}) = [1,1]$, (iii) $\mathrm{Bel}^{(i)}(\bigcup\limits_{i=1}^{k} X_i) \geq \sum\limits_{\emptyset\neq J\subseteq\{1,2,\cdots,k\}} (-1)^{|J|+1}\mathrm{Bel}^{(i)}(\bigcap\limits_{i\in J} X_i)$.

4

4 Belief reduction in IVF decision information systems

4.1 The similarity relation R_D

Definition 4.1. *Let $S = (U, A \cup D)$ be a IVF decision information system where $U = \{x_0, x_1, \cdots, x_n\}$, A is a condition attribute set, $D = \{d_1, d_2, \cdots, d_r\}$ is a decision attribute set.*

Denote

$$d_k(x_i) = D_{ik} \ (i = 0, 1, \cdots, n-1, \ k = 1, 2, \cdots, r),$$

For $i, j \in \{0, 1, \cdots, n-1\}$, define

$$R_D(x_i, x_j) = \bigwedge\{[1,1] - D_{ik} \wedge D_{jk}| : k = 1, 2, \cdots, r\}.$$

Obviously, $R_D(x_i, x_i) = [1,1]$, $R_D(x_i, x_j) = R_D(x_j, x_i)$. Then R_D is a similarity relation on U. We can obtain the similar decision class $S_D(x)$:

$$S_D(x)(y) = R_D(x, y) \ (y \in U).$$

Denote

$$S_D(x_i) = \frac{x_0}{S_D(x_i)(x_0)} + \frac{x_1}{S_D(x_i)(x_1)} + \frac{x_2}{S_D(x_i)(x_2)} + \frac{x_3}{S_D(x_i)(x_3)} + \frac{x_4}{S_D(x_i)(x_4)} + \frac{x_5}{S_D(x_i)(x_5)} + \frac{x_6}{S_D(x_i)(x_6)}$$

$$+ \frac{x_7}{S_D(x_i)(x_7)} + \frac{x_8}{S_D(x_i)(x_8)} + \frac{x_9}{S_D(x_i)(x_9)} \quad (i = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9),$$

$$U/R_D = \{S_D(x) : x \in U\}.$$

Example 4.2. *Consider the IVF decision information system $S = (U, A \cup D)$ in Example* 2.5.

$$\begin{aligned} S_D(x_0)(x_1) &= R_D(x_0, x_1) \\ &= ([1,1] - D_{01} \wedge D_{11}) \bigwedge([1,1] - D_{02} \wedge D_{12}) \bigwedge([1,1] - D_{03} \wedge D_{13}) \\ &= ([1,1] - [0.7, 0.9] \wedge [0.3, 0.5]) \bigwedge([1,1] - [0.15, 0.2] \wedge [0.5, 0.7]) \\ &\quad \bigwedge([1,1] - [0.4, 0.5] \wedge [0.35, 0.4]) \\ &= ([1,1] - [0.3, 0.5]) \bigwedge([1,1] - [0.15, 0.2]) \bigwedge([1,1] - [0.35, 0.4]) \\ &= [0.5, 0.7] \bigwedge[0.8, 0.85]) \bigwedge[0.6, 0.65] \\ &= [0.5, 0.65]. \end{aligned}$$

Similarly,

$$S_D(x_0)(x_0) = [1,1], \quad S_D(x_0)(x_2) = [0.2, 0.3], \quad S_D(x_0)(x_3) = [0.7, 0.8],$$

$$S_D(x_0)(x_4) = [0.5, 0.6], \quad S_D(x_0)(x_5) = [0.5, 0.6], \quad S_D(x_0)(x_6) = [0.6, 0.7],$$

$S_D(x_0)(x_7) = [0.5, 0.6], \quad S_D(x_0)(x_8) = [0.4, 0.55], \quad S_D(x_0)(x_9) = [0.8, 0.85].$

Thus

$$S_D(x_0) = \frac{x_0}{[1,1]} + \frac{x_1}{[0.5,0.65]} + \frac{x_2}{[0.2,0.3]} + \frac{x_3}{[0.7,0.8]} + \frac{x_4}{[0.5,0.6]} + \frac{x_5}{[0.5,0.6]} + \frac{x_6}{[0.6,0.7]} + \frac{x_7}{[0.5,0.6]} + \frac{x_8}{[0.4,0.55]} + \frac{x_9}{[0.8,0.85]}.$$

We can also calculate $S_D(x_i)$ $(i = 1, 2, 3, 4, 5, 6, 7, 8, 9)$. *They record in Table 2.*

Table 2: $S_D(x_i)(x_j)$

	x_0	x_1	x_2	x_3	x_4
x_0	[1,1]	[0.5,0.65]	[0.2,0.3]	[0.7,0.8]	[0.5,0.6]
x_1	[0.5,0.65]	[1,1]	[0.5,0.7]	[0.3,0.5]	[0.6,0.65]
x_2	[0.2,0.3]	[0.5,0.7]	[1,1]	[0.6,0.7]	[0.7,0.8]
x_3	[0.7,0.8]	[0.3,0.5]	[0.6,0.7]	[1,1]	[0.7,0.8]
x_4	[0.5,0.6]	[0.6,0.65]	[0.7,0.8]	[0.7,0.8]	[1,1]
x_5	[0.5,0.6]	[0.5,0.65]]	[0.6,0.7]	[0.5,0.65]	[0.1,0.35]
x_6	[0.6,0.7]	[0.3,0.5]	[0.6,0.7]	[0.2,0.5]	[0.6,0.7]
x_7	[0.5,0.6]	[0.6,0.65]	[0.4,0.75]	[0.6,0.75]	[0.4,0.5]
x_8	[0.4,0.55]	[0.5,0.7]	[0.4,0.55]	[0.7,0.75]	[0.7,0.8]
x_9	[0.8,0.85]	[0.3,0.5]	[0.6,0.7]	[0.2,0.5]	[0.7,0.8]
−−	−−	−−	−−	−−	−−
	x_5	x_6	x_7	x_8	x_9
x_0	[0.5,0.6]	[0.6,0.7]	[0.5,0.6]	[0.4,0.55]	[0.8,0.85]
x_1	[0.5,0.65]	[0.3,0.5]	[0.6,0.65]	[0.5,0.7]	[0.3,0.5]
x_2	[0.6,0.7]	[0.6,0.7]	[0.4,0.75]	[0.4,0.55]	[0.6,0.7]
x_3	[0.5,0.65]	[0.2,0.5]	[0.6,0.75]	[0.7,0.75]	[0.2,0.5]
x_4	[0.1,0.35]	[0.6,0.7]	[0.4,0.5]	[0.7,0.8]	[0.7,0.8]
x_5	[1,1]	[0.5,0.65]	[0.4,0.5]	[0.7,0.75]	[0.5,0.65]
x_6	[0.5,0.65]	[1,1]	[0.6,0.7]	[0.6,0.75]	[0.1,0.2]
x_7	[0.4,0.5]	[0.6,0.7]	[1,1]	[0.7,0.75]	[0.6,0.75]
x_8	[0.7,0.75]	[0.6,0.75]	[0.7,0.75]	[1,1]	[0.7,0.75]
x_9	[0.5,0.65]	[0.6,0.75]	[0.6,0.75]	[0.7,0.75]	[1,1]

4.2 Belief reduction

Denote the probability of $X \in F(U)$ by $\mathrm{P}(X)$. In [4], define $\mathrm{P}(X) = \sum\limits_{x\in U} X(x)\mathrm{P}(\{x\})$.

Now we define the probability $\mathrm{P}^{(i)}(X)$ of $X \in F^{(i)}(U)$ by

$$\mathrm{P}^{(i)}(X) = \sum_{x\in U} X(x)\mathrm{P}^{(i)}(\{x\}) = [\sum_{x\in U} X^-(x)\mathrm{P}^{(i)}(\{x\}), \sum_{x\in U} X^+(x)\mathrm{P}^{(i)}(\{x\})].$$

Proposition 4.3. *Let $(U, A \cup D)$ be an IVF decision information system and $B \subseteq A$. For $X \in F^{(i)}(U)$, denote*

$$\mathrm{Bel}_B^{(i)}(X) = \mathrm{P}^{(i)}(\underline{R_B}(X)) = \sum_{x \in U} \underline{R_B}(X)(x)\mathrm{P}(\{x\}).$$

Pick $\mathcal{M} = U/R_B = \{Y_x : x \in U\}$. Define the basic probability assignment m_B by

$$m_B(Y) = \begin{cases} \frac{|Y|}{|U|}, & if\ Y \in \mathcal{M}, \\ 0, & otherwise. \end{cases}$$

Then $\mathrm{Bel}_B^{(i)} : F^{(i)}(U) \to [I]$ is an IVF belief function induced by $(\mathcal{M}, m)$ on U.

Proof.

$$\begin{aligned} \mathrm{Bel}_B^{(i)}(X) &= \mathrm{P}^{(i)}(\underline{R_B}(X)) = \sum_{x \in U} [\underline{R_B}(X)(x)\mathrm{P}(\{x\})] \\ &= \sum_{x \in U} [\mathrm{P}(\{x\})(\bigwedge_{y \in Y_x} X(y))] = \sum_{Y_x \in \mathcal{M}} [\sum_{x \in Y_x} \mathrm{P}(\{x\})(\bigwedge_{y \in Y_x} X(y))] \\ &= \sum_{Y_x \in \mathcal{M}} [\mathrm{P}(Y_x)(\bigwedge_{y \in Y_x} X(y))] = \sum_{Y_x \in \mathcal{M}} m(Y_x)(\bigwedge_{y \in Y_x} X(y)) \\ &= \sum_{\{Y_x : Y_x \in \mathcal{M}\}} m(Y_x) N_{Y_x}^{(i)}(X). \end{aligned} \tag{4.1}$$

Thus $\mathrm{Bel}_B^{(i)}$ is an IVF belief function induced by $(\mathcal{M}, m)$ on U. □

Proposition 4.4. *Let $(U, A \cup D)$ be an IVF decision information system. If $C \subseteq B \subseteq A$ and $X \in F^{(i)}(U)$, then*

$$\mathrm{Bel}_C^{(i)}(X) \leq \mathrm{Bel}_B^{(i)}(X) \leq X.$$

Proof. This holds by Proposition 4.3. □

Definition 4.5. *Let $S = (U, A \cup D)$ be an IVF decision information system.*
(1) *$B \subseteq A$ is called a belief consistent set in S, if*

$$\mathrm{Bel}_B^{(i)}(S_D(x)) = \mathrm{Bel}_A^{(i)}(S_D(x)) \ (x \in U).$$

(2) *If $B \subseteq A$ is a belief consistent set in S and $\forall\ C \subsetneqq B$,*

$$\mathrm{Bel}_C^{(i)}(S_D(x)) \neq \mathrm{Bel}_A^{(i)}(S_D(x)) \ (x \in U),$$

then B is called a belief reduction in S.

Lemma 4.6. *Let $S = (U, A \cup D)$ be an IVF decision information system. Then $B \subseteq A$ is a belief consistent set in S $\Longleftrightarrow$*

$$\sum_{x \in U} \mathrm{Bel}_B^{(i)}(S_D(x)) = \sum_{x \in U} \mathrm{Bel}_A^{(i)}(S_D(x)).$$

Proof. " $\Longrightarrow$ " Suppose that B is a belief consistent set in S. Then

$$\forall\ x \in U,\quad \mathrm{Bel}_B^{(i)}(S_D(x)) = \mathrm{Bel}_A^{(i)}(S_D(x)).$$

Thus

$$\sum_{x\in U} \mathrm{Bel}_B^{(i)}(S_D(x)) = \sum_{x\in U} \mathrm{Bel}_A^{(i)}(S_D(x)).$$

" $\Longleftarrow$ " Suppose that $\sum_{x\in U} \mathrm{Bel}_B^{(i)}(S_D(x)) = \sum_{x\in U} \mathrm{Bel}_A^{(i)}(S_D(x))$.
Then by Proposition 4.4,

$$\forall\ x \in U,\quad \mathrm{Bel}_B^{(i)}(S_D(x)) \le \mathrm{Bel}_A^{(i)}(S_D(x)).$$

This implies that $\forall\ x \in U, \mathrm{Bel}_B^{(i)}(S_D(x)) = \mathrm{Bel}_A^{(i)}(S_D(x))$.
Thus B is a belief consistent set in S. $\square$

Theorem 4.7. *Let $S = (U, A \cup D)$ be an IVF decision information system. Then*

$$B \subseteq A \text{ is a belief reduction in } S \Longleftrightarrow \sum_{x\in U} \mathrm{Bel}_B^{(i)}(S_D(x)) = \sum_{x\in U} \mathrm{Bel}_A^{(i)}(S_D(x))$$

$$\text{and}\ \ \forall\ C \subsetneq B,\ \sum_{x\in U} \mathrm{Bel}_C^{(i)}(S_D(x)) < \sum_{x\in U} \mathrm{Bel}_A^{(i)}(S_D(x)).$$

Proof. " $\Longrightarrow$ " Suppose that B is of belief reduction in S. Then B is a belief consistent set in S. By Lemma 4.6,

$$\sum_{x\in U} \mathrm{Bel}_B^{(i)}(S_D(x)) = \sum_{x\in U} \mathrm{Bel}_A^{(i)}(S_D(x)).$$

By Definition 4.5, $\forall\ C \subsetneq B,\ x \in U,\ \mathrm{Bel}_C^{(i)}(S_D(x)) \neq \mathrm{Bel}_A^{(i)}(S_D(x))$.
By Proposition 4.4, $\forall\ C \subsetneq B,\ x \in U,\ \mathrm{Bel}_C^{(i)}(S_D(x)) \le \mathrm{Bel}_A^{(i)}(S_D(x))$.
Thus

$$\sum_{x\in U} \mathrm{Bel}_C^{(i)}(S_D(x)) < \sum_{x\in U} \mathrm{Bel}_A^{(i)}(S_D(x)).$$

" $\Longleftarrow$ " Suppose that $\sum_{x\in U} \mathrm{Bel}_B^{(i)}(S_D(x)) = \sum_{x\in U} \mathrm{Bel}_A^{(i)}(S_D(x))$

$$\text{and}\ \ \forall\ C \subsetneq B,\ \sum_{x\in U} \mathrm{Bel}_C^{(i)}(S_D(x)) < \sum_{x\in U} \mathrm{Bel}_A^{(i)}(S_D(x)).$$

By Lemma 4.6, B is a belief consistent set in S.
By Proposition 4.4, $\forall\ x \in U, \mathrm{Bel}_C^{(i)}(S_D(x)) \le \mathrm{Bel}_A^{(i)}(S_D(x))$.
Then

$$\mathrm{Bel}_C^{(i)}(S_D(x)) \neq \mathrm{Bel}_A^{(i)}(S_D(x)).$$

Thus B is a belief reduction in S.

$\square$

Algorithms 4.8. *Let $S = (U, A \cup D)$ be an IVF decision information system. The algorithm of belief reduction in S is shown as follows:*

Input: the IVF decision information system S.
Output: All belief reductions in S.

Step 1. *Input the IVF decision information system S;*
Step 2. *Pick $B \subseteq A$;*
Step 3. *Calculate the similar decision class $S_D(x_j)$;*
Step 4. *Calculate* $\mathrm{Bel}_A^{(i)}(S_D(x_j))$ *and* $\mathrm{Bel}_B^{(i)}(S_D(x_j))$*;*
Step 5. *Compare* $\mathrm{Bel}_A^{(i)}(S_D(x_j))$ *and* $\mathrm{Bel}_B^{(i)}(S_D(x_j))$*;*
Step 6. *By Theorem 4.7, B is a belief reduction in S.*

Example 4.9. *Consider the IVF decision information system $S = (U, A \cup D)$ in Example* 4.2.
By (4.1) *and Proposition* 3.1,

$$\begin{aligned}
\mathrm{Bel}_A^{(i)}(S_D(x_1)) &= \sum_{i=1}^{5} m(X_i) \bigwedge_{y \in X_i} S_D(x_1)(y) \\
&= \frac{|X_0|}{|U|} \times \bigwedge_{y \in X_1} S_D(x_0)(y) + \cdots + \frac{|X_5|}{|U|} \times \bigwedge_{y \in X_5} S_D(x_0)(y) \\
&= \frac{3}{10} \times [0.2, 0.3] + \cdots + \frac{1}{10} \times [0.5, 0.6] \\
&= [0.430, 0.525]
\end{aligned}$$

Similarly, we can calculate that

$$\mathrm{Bel}_A^{(i)}(S_D(x_1)) = [0.440, 0.590], \quad \mathrm{Bel}_A^{(i)}(S_D(x_2)) = [0.410, 0.590],$$
$$\mathrm{Bel}_A^{(i)}(S_D(x_3)) = [0.510, 0.675], \quad \mathrm{Bel}_A^{(i)}(S_D(x_4)) = [0.490, 0.590],$$
$$\mathrm{Bel}_A^{(i)}(S_D(x_5)) = [0.470, 0.610], \quad \mathrm{Bel}_A^{(i)}(S_D(x_6)) = [0.400, 0.525],$$
$$\mathrm{Bel}_A^{(i)}(S_D(x_7)) = [0.480, 0.600], \quad \mathrm{Bel}_A^{(i)}(S_D(x_8)) = [0.550, 0.675],$$
$$\mathrm{Bel}_A^{(i)}(S_D(x_9)) = [0.460, 0.625] \quad \textit{and}$$
$$\mathrm{Bel}_B^{(i)}(S_D(x_0)) = [0.430, 0.515], \quad \mathrm{Bel}_B^{(i)}(S_D(x_1)) = [0.420, 0.590],$$
$$\mathrm{Bel}_B^{(i)}(S_D(x_2)) = [0.390, 0.580], \quad \mathrm{Bel}_B^{(i)}(S_D(x_3)) = [0.490, 0.655],$$
$$\mathrm{Bel}_B^{(i)}(S_D(x_4)) = [0.430, 0.560], \quad \mathrm{Bel}_B^{(i)}(S_D(x_5)) = [0.380, 0.545],$$
$$\mathrm{Bel}_B^{(i)}(S_D(x_6)) = [0.380, 0.515], \quad \mathrm{Bel}_B^{(i)}(S_D(x_7)) = [0.480, 0.600],$$
$$\mathrm{Bel}_B^{(i)}(S_D(x_8)) = [0.550, 0.675], \quad \mathrm{Bel}_B^{(i)}(S_D(x_9)) = [0.440, 0.605].$$

By Lemma 4.6, $B = \{a_1, a_2\}$ is not a belief consistent set in S.
Thus $B = \{a_1, a_2\}$ is not a belief reduction in S.

5 Conclusions

In this paper, we have researched belief reduction in IVF decision information systems. In future work, we will investigate knowledge acquisition in IVF decision information systems.

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EXISTENCE AND BIFURCATION OF POSITIVE GLOBAL SOLUTIONS FOR PARAMETERIZED NONHOMOGENEOUS ELLIPTIC EQUATIONS INVOLVING CRITICAL EXPONENTS

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Abstract

We establish extence and bifurcation of positive global solutions for parametrized nonhomogeneous elliptic equations involving critical Sobolev exponents.

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Key words: elliptic problem; critical exponent; multiplicity; bifurcation; parametrized nonhomogeneous problem; positive global solution

1. Introduction

Let $N \geq 3$ and $2^* := 2N/(N-2)$. Let consider a Hilbert space

$$H^1(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$$

with the inner product

$$(u,v) := \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + uv)dx$$

and the corresponding norm

$$||u|| := \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2)dx\right)^{1/2}.$$

By $H^{-1}(\mathbb{R}^N)$, we denote its dual with the dual norm $||\cdot||_*$ and, by $\langle , \rangle$, the pairing of $H^1(\mathbb{R}^N)$ with its dual. We denote by $||\cdot||_p$ the usual norm of $L^p(\mathbb{R}^N)$ for $p \in [1,\infty]$. Let $D^{1,2}(\mathbb{R}^N) := \{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$ be a Hilbert space with the inner product $\int_{\mathbb{R}^N} \nabla u \cdot \nabla v$ and the corresponding norm $||\nabla u||_2$.

In this paper, we are concerned with the multiple existence and bifurcation of positive solutions of the following problem:

$$(P_\mu) \qquad \begin{cases} -\Delta u + u = u^{2^*-1} + \mu f \quad \text{in} \ \ \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N, \ N \geq 3, \end{cases}$$

where $\mu \in \mathbb{R}^+$, $f \in H^{-1}(\mathbb{R}^N)$, $f \geq 0$ and $f \not\equiv 0$ in $\mathbb{R}^N$.

A well-known result for the homoneneous case is that all positive regular solution of

$$-\Delta u = u^{2^*-1}$$

in $\mathbb{R}^N$ are given by

$$\omega_\epsilon := \left(\frac{\epsilon\sqrt{N(N-2)}}{\epsilon^2 + |x|^2}\right)^{(N-2)/2}$$

with $\epsilon > 0$. Every ω_ϵ is a minimizer for the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$. Namely, the Sobolev constant

$$S := \inf_{0 \neq u \in D^{1,2}(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{2/2^*}}$$

is achived by ω_ϵ and

$$(1,1) \qquad ||\nabla \omega_\epsilon||_2^2 = ||\omega_\epsilon||_{2^*}^{2^*} = S^{N/2} (cf.[2,6]).$$

For convenience, we omit "$\mathbb{R}^N$" and "dx" in integration and, throughout this paper, we will use the letter $C > 0$ to denote the natural various contents independent of u.

Our attempt to show multiplicity of positive solutions for problem (P_μ) relies on the Ekeland's variational principle in [13] and the Mountain Pass Theorem in [5]. Since our problem (P_μ) posesses the critical nonlinearity and the embedding

$H^1(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is not compact, in taking the opportunity of variational structure of problem, the (PS) condition is no longer valid and so the Mountain Pass Theorem in [1] could not be applied directly. However, we can use the Mountain Pass Theorem *without* the (PS) condition in [5] to get some $(PS)_c$ sequence of the variational functional for the second solution with $c > 0$.

In the last decade, the existence and properties of solutions of the problem:

$$(P_0) \qquad \begin{cases} -\Delta u + u = g(x,u), u > 0 \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \ N \geq 2 \end{cases}$$

has been stuied by Struss[24], Lions[22, 23], Ding and Ni[12], Cao[7], Zhu[25] and other authors for the case where $g(x,0) = 0$ on $\mathbb{R}^N$ and $g(x,t)$ has a subcritical superlinear growth. On the other hand, the nonhomogeneous problem with $1 < p < 2^* - 1$:

$$(P) \qquad \begin{cases} -\Delta u + u = |u|^{p-2}u + \mu f, u > 0 \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \ N \geq 2, \end{cases}$$

where $\mu \in \mathbb{R}^+$, $f \geq 0$, $f \in L^2(\mathbb{R}^N)$ was studied in [26,11,14,15].

In the critical case $p = 2^*$, the problem is much more difficult than the subcritical case. As we mentioned, the Palais-Smale condition does not hold at some critical levels and the effect of the nonhomogeneous term f to the multiple existence of solutions is delicate. The multiplicity of the solutions of (P_μ) not only depends on the norm of f but also the decay rate of f. In [10], it has shown that if $2 < N < 6$ and $|x|^{N-2}f$ is bounded, then there exists $\mu^* > 0$ such that problem (P_μ) possesses at least two positive solutions with $\mu \in]0, \mu^*[$. In case that $N \geq 6$, there exist $\mu^{**}, \mu_* > 0$ with $\mu_* < \mu^{**}$ such that for each $\mu \in]\mu^{**}, \mu^*[$, problem (P_μ) possesses two positive solutions and for $\mu \in]0, \mu_*[$ problem (P_μ) has a unique positive solution. In [11], the authors also gave similar multiplicity results for subcritical caseas. For critical case, In [18], Hirano and Kim consider the multiplicity of solutions of (P_μ) with $-\Delta + I$ replaced by $-\Delta + \alpha I$, $\alpha > 0$. They assume that $p = 2^*$, $3 \leq N \leq 5$, $f \in L^{2^*/(2^*-1)}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $f \geq 0$ and $f \not\equiv 0$, and $|x|^{N-2}f$ is bounded. It was shown that there exist μ_* and a function $\alpha : (0, \mu_*) \to \mathbb{R}^+$ such that for each $\alpha \in (0, \alpha(\mu))$, problem (P_μ) posesses at least three solutions; the third solution is sign-changing if we assume that there exist exactly two positive solutions. we also refer [21] for critical case. In [19], the effect of the shape of the multiplicity of (P) was investigated when $-\Delta + I$ replaced by $-\epsilon\Delta$, $\epsilon > 0$.

In this paper, we do not assume the decay rate on f but assume only uniform boundedness of f which is independent of solution u and $x \in \mathbb{R}^N$. We study also bifurcation phenomenon and get a bifurcation point of (P_μ). There seems to have some progress on existence result in elliptic equations. We also refer a multiplicity result on parabolic equations for subcritical case in [20, 16] and elliptic with Neumann boundary condition in [17].

We now state our main results:

Proposition 2.3. *Assume $f \in H^{-1}(\mathbb{R}^N)$, $f \geq 0$, $f \not\equiv 0$ in $\mathbb{R}^N$ and $||\mu f||_* \leq C_N^*$, then problem (P_μ) has at least one positive solution u_μ such that*

$$(2.1) \qquad I_\mu(u_\mu) := c_1 = \inf\{I_\mu : u \in \bar{B}_{R_0}\},$$

where $\bar{B}_{R_0} = \{u \in H^1(R^N) : ||u|| \leq R_0\}$ *and* $C_N^* = \frac{1}{2}\left(\frac{4}{N+2}\right)\left(\frac{N}{N+2}\right)^{(N-2)/4} S^{N/4}$.

THEOREM 3.6. *Assume* $f \in H^{-1}(\mathbb{R}^N)$, $f \geq 0$ $f \not\equiv 0$ *in* $\mathbb{R}^N$ *and satisfies* $||\mu f||_* \leq C_N^*$. *Then there exists a positive constant* $\mu^* > 0$ *such that* (P_μ) *possesses at least two positive solutions for* $0 < \mu < \mu^*$, *a unique solution for* $\mu = \mu^*$ *and no positive solution if* $\mu > \mu^*$.

By U_μ, we denote the second solution of (P_μ).

THEOREM 4.5. *(i) The set* $\{U_\mu\}$ *is bounded uniformly in* $H^1(\mathbb{R}^N)$,
(ii) (μ^*, u_{μ^*}) *is a bifurcation point.*

2. Existence of minimal positive solutions

LEMMA 2.1. *The operator* $-\Delta + I$ *has the maximum principle in* $H^1(\mathbb{R}^N)$.

Proof. Let $h \geq 0$ and $-\Delta u + u = h$. Suppose that $u_- \not\equiv 0$, where $u_+(x) = \max\{u(x), 0\}$ and $u_-(x) = \min\{u(x), 0\}$. then $0 < \int |\nabla u_-|^2 + |u_-|^2) = \int h u_- dx$ which leads a contradiction. This completes the proof. ∎

In order to get the existence of positive solutions of (P_μ), we consider the energy functional I_μ of the problem (P_μ) defined by

$$I_\mu(u) := \frac{1}{2}\int (|\nabla u|^2 + |u|^2) - \frac{1}{2^*}\int (u^+)^{2^*} - \mu \int fu, \quad \text{for} \quad u \in H^1(\mathbb{R}^N).$$

First, we study the existence of a local mininum for energy functional I_μ and its properities. We denote

$$C_N^* := \frac{1}{2}\left(\frac{4}{N+2}\right)\left(\frac{N}{N+2}\right)^{(N-2)/4} S^{N/4}. \tag{2,1}$$

LEMMA 2.2. *Assume* $f \in H^{-1}(\mathbb{R}^N)$, $f(x) \geq 0$, $f(x) \not\equiv 0$ *and* $||\mu f||_* \leq C_N^*$, *then there exits a positive constant* $R_0 > 0$ *such that* $I_\mu(u) \geq 0$ *for any* $u \in \partial B_{R_0} = \{u \in H^1(\mathbb{R}^N) : ||u|| = R_0\}$.

Proof. We consider the function $h(t) : [0, +\infty) \to \mathbb{R}^N$ defined by

$$h(t) = \frac{1}{2}t - \frac{1}{2^*}S^{-2^*/2}t^{2^*-1}.$$

Note that $h(0) = 0$, $2^* - 1 > 1$ and $h(t) \to -\infty$ as t $\to \infty$. We can show easly there a unique $t_0 > 0$ achieving the maxinum of $h(t)$ at t_0. Since

$$h'(t_0) = \frac{1}{2} - \frac{2^* - 1}{2^*}S^{-2^*/2}t_0^{2^*-2} = 0,$$

we have

$$t_0 = \left(\frac{2^*}{2(2^* - 1)}\right)^{1/(2^*-2)} S^{2^*/2(2^*-2)}.$$

Hence, we have

$$h(t_0) = \frac{1}{2}\left(\frac{4}{N+2}\right)\left(\frac{N}{N+2}\right)^{(N-2)/4} S^{N/4}. \tag{2,2}$$

Taking $R_0 = t_0$, for all $u \in \partial B_{R_0}$,

$$\begin{aligned} I_\mu(u) &= \frac{1}{2}\int(|\nabla u|^2 + |u|^2) - \frac{1}{2^*}\int (u^+)^{2^*} - \mu\int fu \\ &\geq \frac{1}{2}||u||^2 - \frac{1}{2^*}S^{-2^*/2}||u||^{2^*} - ||\mu f||_*||u|| \\ &= t_0\left[h(t_0) - ||\mu f||_*\right] \end{aligned} \tag{2,3}$$

From $(2,2)$ and $(2,3)$, we have $I_\mu(u)|_{\partial B_{R_0}} \geq 0$. This completes the proof. ■

Proposition 2.3. *Assume $f \in H^{-1}(\mathbb{R}^N)$, $f(x) \geq 0$, $f(x) \not\equiv 0$ in $\mathbb{R}^N$ and $||\mu f||_* \leq C_N^*$, then problem (P_μ) has at least one positive solution u_μ such that*

$$I_\mu(u_\mu) := c_1 = \inf\{I_\mu : u \in \bar{B}_{R_0}\}, \tag{2.4}$$

where $\bar{B}_{R_0} = \{u \in H^1(R^N) : ||u|| \leq R_0\}$.

Proof. By Sobolev inequality, the generalized Hölder and Young's inequality with $\epsilon > 0$, there exists $C_\epsilon > 0$, we have

$$\begin{aligned} I_\mu(u) &= \frac{1}{2}\int(|\nabla u|^2 + |u|^2) - \frac{1}{2^*}\int (u^+)^{2^*} - \mu\int fu \\ &\geq \frac{1}{2}||u||^2 - \frac{1}{2^*}S^{-2^*/2}||u||_{2^*}^{2^*} - ||\mu f||_*||u|| \\ &\geq \left(\frac{1}{2} - \epsilon\right)||u||^2 - \frac{1}{2^*}S^{-2^*/2}||u||^{2^*} - C_\epsilon||\mu f||_*^2. \end{aligned}$$

Taking $\epsilon < \frac{1}{2}$, then, for $R_0 = t_0$ as in Lemma 2,2, we can find a $C_{R_0} > 0$ small enough such that

$$I_\mu(u)|_{\partial B_{R_0}} \geq C_{R_0} \text{ for } ||\mu f||_* \leq C_N^*. \tag{2.5}$$

Since there exists a $\tilde{C}_{R_0} > 0$ such that $|I_\mu(u)| \leq \tilde{C}_{R_0}$ for all $u \in \bar{B}_{R_0}$ and $\bar{B}_{R_0}$ is a complete metric space with respect to the metric $d(u,v) = ||u - v||, u, v \in \bar{B}_{R_0}$, by using the Ekeland's variational principle, from (2.5), we can prove that there exists a sequence $\{u_n\} \subset \bar{B}_{R_0}$ and $u_\mu \in \bar{B}_{R_0}$ such that

$$I_\mu(u_n) \to c_1, \tag{2.6}$$

$$I'_\mu(u_n) \to 0, \tag{2.7}$$

$$\begin{gathered} u_n \to u_\mu \text{ weakly in } H^1(\mathbb{R}^N), \\ u_n \to u_\mu \text{ a.e. in } \mathbb{R}^N, \\ \nabla u_n \to \nabla u_\mu \text{ a.e. in } \mathbb{R}^N \end{gathered} \tag{2.8}$$

and

$$u_n{}^{2^*-1} \to u_\mu{}^{2^*-1} \quad \text{weakly} \quad \text{in} \quad \left(L^{2^*}(\mathbb{R}^N)\right)^* \quad \text{as } n \to \infty.$$

Therefore, u_μ is a weak solution of (P_μ). Hence,

$$\left\langle I'_\mu(u_\mu), \varphi \right\rangle = 0 \ \ \forall \varphi \in H^1(\mathbb{R}^N). \tag{2.9}$$

Moreover, by Lemma 2.1, u_μ is positive on $\mathbb{R}^N$, where I'_μ is the Frèchlet derivative of I_μ.

Next, we are going to prove (2.4). In fact, by the definition of c_1, we know that $I_\mu(u_\mu) \geq c_1$ since $u_\mu \in \bar{B}_{R_0}$, that is,

$$I_\mu(u_\mu) = \frac{1}{2}\int (|\nabla u_\mu|^2 + |u_\mu|^2) - \frac{1}{2^*}\int |u_\mu|^{2^*} - \mu \int f u_\mu \geq c_1 \tag{2.10}$$

By (2.9) and (2.10), we have

$$\left(\frac{1}{2} - \frac{1}{2^*}\right)\int (|\nabla u_\mu|^2 + |u_\mu|^2) - \left(1 - \frac{1}{2^*}\right)\mu \int f u_\mu \geq c_1 \tag{2.11}$$

On the other hand, by (2.6) - (2.8) and Fatou's lemma, we get

$$\begin{aligned} c_1 &= \liminf_n \left(\frac{1}{2} - \frac{1}{2^*}\right)\int (|\nabla u_n|^2 + |u_n|^2) - \limsup_n (1 - \frac{1}{2^*})\mu \int f u_n \\ &\geq \left(\frac{1}{2} - \frac{1}{2^*}\right)\int (|\nabla u_\mu|^2 + |u_\mu|^2) - \left(1 - \frac{1}{2^*}\right)\mu \int f u_\mu. \end{aligned} \tag{2.12}$$

Thus, (2.10) and (2.12) imply (2.4) holds. This completes the proof. ∎

Remark. *(i) $c_1 < 0$, (ii) c_1 is bounded below, (iii) $||u_\mu|| = o(1)$ as $\mu \to 0^+$.*

Indeed: (i) For $t > 0$ and $\varphi > 0$, we have

$$I_\mu(t\varphi) = \frac{t^2}{2}\int (|\nabla\varphi|^2 + |\varphi|^2) - \frac{t^{2^*}}{2^*}\int |\varphi|^{2^*} - t\mu \int f\varphi \leq \frac{t^2}{2}||\varphi||^2 - t\mu \int f\varphi.$$

By taking $t > 0$ sufficiently small, we can see $c_1 < 0$.

(ii) By (2.9) with $\varphi = u_\mu$, and $c_1 = I_\mu(u_\mu)$, we have

$$\begin{aligned} c_1 &= \left(\frac{1}{2} - \frac{1}{2^*}\right)\int (|\nabla u_\mu|^2 + |u_\mu|^2) - \left(1 - \frac{1}{2^*}\right)\mu \int f u_\mu \\ &\geq \left(\frac{1}{2} - \frac{1}{2^*}\right)||u_\mu||^2 - \left(1 - \frac{1}{2^*}\right)||\mu f||_*||u_\mu|| \\ &\geq -\frac{1}{22^*}\left(\frac{(2^*-1)^2}{2^*-2}\right)||\mu f||_*^2 \end{aligned} \tag{2.13}$$

by Young's inequality.

(iii) Since $c_1 < 0$, from (2.13), we see that $||u_\mu|| \to 0$ as $\mu \to 0^+$. Hence, $||u_\mu|| = o(1)$ as $\mu \to 0^+$. We also have that $||u_\mu||$ is uniformly bounded with respect to μ. We will restate results relating to this remark in Proposition 3.4 more precisely.

Proposition 2.4. *Problem (P_μ) possesses at least one minimal positive solution of (P_μ).*

Proof. Let $\mathcal{N}$ be the Nehari manifold

$$\mathcal{N} := \left\{ u \in H^1(\mathbb{R}^N) : \int |\nabla u|^2 + |u|^2 = \int |u|^{2^*} + \int \mu f u \right\} \setminus \{0\} .$$

Note that $||\mu f||_* \ll 1$ for μ small enough and for each $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, there exists a unique $t_u > 0$ such that

$$t_u^2 \int |\nabla u|^2 + |u|^2 - t_u^{2^*} \int |u|^{2^*} - t_u \int \mu f u = 0$$

and $I_\mu(t_u u) > 0$. Then

$$\mathcal{N} = \left\{ t_u u \ : \ u \in H^1(\mathbb{R}^N) \setminus \{0\} \right\}$$

and

$$\mathcal{N} \cong S^{N-1} = \left\{ u \in H^1(\mathbb{R}^N) \ : \ ||u|| = 1 \right\} .$$

Hence,

$$H^1(\mathbb{R}^N) = H_1 \cup H_2 \cup \mathcal{N}, \quad H_1 \cap H_2 = \phi \text{ and } 0 \in H_1,$$

where

$$\begin{aligned} H_1 &= \left\{ tu \ : \ u \in H^1(\mathbb{R}^N) \setminus \{0\} , \ t \in [0, t_u[\right\} \\ H_2 &= \left\{ tu \ : \ u \in H^1(\mathbb{R}^N) \setminus \{0\} , \ t > t_u \right\} . \end{aligned}$$

This implies that for $t > 0$ with $t < t_u, \ tu \in H_1$.

Here, we need to switch our view point, by associating with v a mapping

$$v : [0, \infty[\to H^1(\mathbb{R}^N)$$

defined by

$$(v(t))x = v(x,t), \quad x \in \mathbb{R}^N, t \in [0, \infty[.$$

In other words, we consider v not as a function of x and t together, but rather as a mapping v of t into the space $H^1(\mathbb{R}^N)$ of a function of x.

We have, for any $v_0 \in H_1$, the solution v of the initial value problem:

$$\begin{cases} \dfrac{dv}{dt} - \Delta v + v = & v^{2^*-1} + \mu f(x), \\ v(0) = v_0, \end{cases}$$

converges to u_μ as $t \to \infty$,

Indeed, in the proof of Proposition 2.3, we know that $I_\mu(v(t))$ is decreasing and $\lim_{t\to\infty} I_\mu(v(t)) = I_\mu(u_\mu)$, where $I_\mu(u_\mu)$ is the local minimum.
Since

$$\begin{aligned} I_\mu(v(t)) - I_\mu(v(s)) &= \int_s^t \frac{d}{dt} I_\mu(v(t)) dt \\ &= \int_s^t \left\langle \frac{d}{dt} v, \nabla I_\mu(v(t)) \right\rangle dt \\ &= -\int_t^s \left\| \frac{d}{dt} v \right\|^2 dt, \end{aligned}$$

we have, $\lim_{s,t\to\infty} \left\|\frac{d}{dt}v\right\|^2 = 0$. Thus, $v' \to 0$ a.e. in $\mathbb{R}^N$ as $t \to \infty$ and hence, $\langle I'_\mu(v), \varphi\rangle \to 0$, $\forall\varphi \in C^\infty(\mathbb{R}^N)$. Therefore, we have $v \to u_\mu$ as $t \to \infty$, since $I_\mu(v(t))$ is decreasing and converges to the local minimum $I_\mu(u_\mu)$.
Now, let $v_0 = tu$, where $t \in]0, 1[$ and u is a positive solution. Then $u \in \mathcal{N}$ and $v_0 \in H_1$. Since $v_0 \leq u$ and the solution v converges u_μ as $t \to \infty$, by the order preserving principle, $u_\mu \leq u$. This completes the proof. ∎

Proposition 2.5. *Suppose that $f \in H^{-1}(\mathbb{R}^N)$, $f \geq 0$, $f \not\equiv 0$ in $\mathbb{R}^N$ and $||\mu f||_* \leq C^*_N$. Then there exist $\tilde{\mu} \geq \bar{\mu} > 0$ such that (P_μ) possesses a positive solution for $0 < \mu \leq \bar{\mu}$ and no positive solution for $\mu > \bar{\mu}$.*

Proof. By Proposition 2.3, (P_μ) has a positive solution if $\mu \leq C^*_N/||f||_*$. Suppose (P_μ) has a positive solution for some $\mu = \bar{\mu}$. We show that (P_μ) has a positive solution for any $0 < \mu \leq \bar{\mu}$. For fixed $0 < \mu < \bar{\mu}$, using the Lax-Milgram Theorem, we construct a positive sequence $\{u_n\}$ as following;

Let

$$-\Delta u_1 + u_1 = \mu f$$

and

$$-\Delta u_n + u_n = u_{n-1}^{2^*-1} + \mu f \quad \text{for} \quad n \geq 2. \tag{2.14}$$

Then, by the maximum principle, we have $0 < u_n < u_{n+1} < \cdots < \bar{u}$ for $n \geq 1$. And $||u_1|| \leq \mu||f||_*$ and $||u_1||_{2^*} \leq S^{-1/2}||u_1|| \leq S^{-1/2}\mu||f||_*$. Multiplying (2.14) by u_n, we have $||u_n|| \leq S^{-2^*/2}||\bar{u}||^{2^*-1} + \mu||f||_*$. Therefore, there exists u in $H^1(\mathbb{R}^N)$ such that

$$u_n \to u \text{ weakly in } H^1(\mathbb{R}^N) \text{ as } n \to \infty,$$

$$u_n \to u \text{ a.e. in } \mathbb{R}^N \text{ as } n \to \infty,$$
$$\nabla u_n \to \nabla u \text{ a.e. in } \mathbb{R}^{\mathrm{N}},$$
$$u_n^{2^*-1} \to u^{2^*-1} \text{ weakly in } (L^{2^*}(\mathbb{R}^N))^* \text{ as } n \to \infty.$$

Thus, u is a positive solution of (P_μ).

Next, let u be a positive solution of (P_μ). Then, for any $\epsilon > 0$, multiplying (P_μ) by $\omega_\epsilon^{2^*}$, we have

$$-\Delta u\omega_\epsilon^{2^*} + u\omega_\epsilon^{2^*} = u^{2^*-1}\omega_\epsilon^{2^*} + \mu f(x)\omega_\epsilon^{2^*}. \tag{2.15}$$

Since $2^* > 2$, for any $M > 0$, there exists a constant $C > 0$ such that

$$u^{2^*-1} \geq Mu - C \quad \forall u > 0.$$

Hence, we have, from (2.15),

$$-\int \Delta u\omega_\epsilon^{2^*} + \int u\omega_\epsilon^{2^*} \geq \int \left((Mu - C)\omega_\epsilon^{2^*} + \mu f(x)\omega_\epsilon^{2^*})\right).$$

By Green's formular, we have

$$\int \Delta u\omega_\epsilon^{2^*} = \int u\Delta\omega_\epsilon^{2^*}.$$

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Thus,

$$\mu \int f(x)\omega_\epsilon^{2^*} \leq C \int \omega_\epsilon^{2^*} + \int \left(1 - M - \frac{\Delta\omega_\epsilon^{2^*}}{w_\epsilon^{2^*}}\right) \omega_\epsilon^{2^*} u. \tag{2.16}$$

Since

$$\begin{aligned}\frac{\Delta w_\epsilon^{2^*}}{\omega_\epsilon^{2^*}} = \frac{\Delta(\epsilon + |x|^2)^{-N}}{(\epsilon + |x|^2)^{-N}} &= 2N(N+1)(\epsilon + |x|^2)^{-2} \left(\frac{N+2}{N+1}|x|^2 - \frac{N}{N+1}\epsilon\right) \\ &= 2N(N+1)(\epsilon + 0^2)^{-2} \left(\frac{N+2}{N+1}0^2 - \frac{N}{N+1}\epsilon\right) \\ &= -2N^2\epsilon^{-1},\end{aligned}$$

we get, from (2.16),

$$\mu \int f(x)\omega_\epsilon^{2^*} \leq C \int \omega_\epsilon^{2^*} + \left(2N^2\epsilon^{-1} + 1 - M\right) \int \omega_\epsilon^{2^*} u.$$

If we choose $M = 2N^2\epsilon^{-1} + 1$, then, by (1.1), we have

$$\mu \leq \frac{C \int \omega_\epsilon^{2^*}}{\int f(x)\omega_\epsilon^{2^*}} = \frac{CS^{N/2}}{\int f(x)\omega_\epsilon^{2^*}}.$$

Hence, there exists $\bar{\mu} > 0$ such that

$$\bar{\mu} \leq \tilde{\mu} \doteqdot \inf_{\epsilon>0} \frac{C \int w_\epsilon^{2^*}}{\int f(x)\omega_\epsilon^{2^*}} = \inf_{\epsilon>0} \frac{CS^{N/2}}{\int f(x)\omega_\epsilon^{2^*}}. \tag{2.17}$$

Therefore, if $\mu > \bar{\mu}$, then (P_μ) has no solution and this completes the proof. ∎

3. Multiplicity of positive solutions

From now on, we assume that $f \in H^{-1}(\mathbb{R}^N)$, $f \geq 0$, $f \not\equiv 0$ in $\mathbb{R}^N$and f satisfies $||\mu f||_* \leq C_N^*$.

We set

$$\mu^* := \sup\{\mu \in \mathbb{R}^+ : (P_\mu) \text{ has at least one positive solution in } H^1(\mathbb{R}^N)\}.$$

Then, by Proposition 2.5, we have $0 < \bar{\mu} \leq \mu^* < \infty$.

Remark. The minimal solution u_μ of (P_μ) is increasing with respect to μ. Indeed, suppose $\mu^* > \nu > \mu$. Since

$$-\Delta u_\nu + u_\nu - u_\nu^{2^*-1} - \mu f(x) = (\nu - \mu) f \geq 0,$$

$u_\nu > 0$ is a supersolution of (P_μ). Since $f(x) \geq 0$ and $f(x) \not\equiv 0$, $u \equiv 0$ is a subsolution of (P_μ) for any $\mu > 0$. By the standard barrier method, we can obtain a solution u_μ of (P_μ) such that $0 \leq u_\mu \leq u_\nu$ on $\mathbb{R}^N$. We note that 0 is not a solution of (P_μ), $\nu > \mu$ and u_μ is a minimal solution of (P_μ) because u_μ also can be derived by an iteration scheme with initial value $u_{(0)} = 0$. Therefore, by the maximal principle, $0 < u_\mu < u_\nu$ on $\mathbb{R}^N$ which completes the proof. ∎

Now, consider the corresponding eigenvalue problem:

$$(3.1)_\mu \qquad \begin{cases} -\Delta\varphi + \varphi = \lambda(\mu)(2^* - 1)u_\mu^{2^*-2}\varphi, \\ \quad \varphi \text{ in } H^1(\mathbb{R}^N). \end{cases}$$

Let λ_1 be the first eigenvalue of $(3.1)_\mu$;i.e.,

$$\lambda_1 = \lambda_1(\mu) := \inf\{\int \left(|\nabla\varphi|^2 + |\varphi|^2\right) : \varphi \in H^1(\mathbb{R}^N), (2^* - 1)\int u_\mu^{2^*-2}\varphi^2 dx = 1\}.$$

Then, $0 < \lambda_1 < \infty$ and we can achieve the minimum by some function $\varphi_1 = \varphi_1(\mu) \in H^1(\mathbb{R}^N)$ and $\varphi_1 > 0$ in $\mathbb{R}^N$ if $\mu \in]0, \mu^*[$ $(cf.$ [27]).

We summarize basic properties for $\lambda_1(\mu)$:

LEMMA 3.1. *(i) For $\mu \in]0, \mu^*[$, $\lambda_1(\mu) > 1$,*
(ii) If $0 < \mu < \nu \leq \mu^*$, then $\lambda_1(\nu) < \lambda_1(\mu)$,
(iii) $\lambda_1(\mu) \to +\infty$ as $\mu \to 0^+$.

Proof. (i) For given $0 < \mu < \nu \leq \mu^*$, every solution u_ν of (P_μ) with $\nu \in (\mu, \mu^*)$ is a supersolution of (P_μ). By Taylor expansion, we have

$$\begin{aligned} -\Delta(u_\nu - u_\mu) + u(u_\nu - u_\mu) &= u_\nu^{2^*-1} - u_\mu^{2^*-1} + (\nu - \mu)f \\ &> (2^* - 1)u_\mu^{2^*-2}(u_\nu - u_\mu) \end{aligned}$$

and moreover, we get

$$\begin{aligned} \int \nabla(u_\nu - u_\mu)\nabla\varphi_1 + \int(u_\nu - u_\mu)\varphi_1 &= \int \left(u_\nu^{2^*-1} - u_\mu^{2^*-1}\right)\varphi_1 + \int(\nu - \mu)f\varphi_1 \\ &> (2^* - 1)\int u_\mu^{2^*-2}(u_\nu - u_\mu)\varphi_1. \end{aligned}$$

Therefore, from $(3.1)_\mu$, we have

$$\int \nabla(u_\nu - u_\mu)\nabla\varphi_1 + \int(u_\nu - u_\mu)\varphi_1 = \lambda_1(\mu)(2^* - 1)\int u_\mu^{2^*-2}(u_\nu - u_\mu)\varphi_1,$$

which implies $\lambda_1(\mu) > 1$.

(ii) Since, for $0 < \mu < \nu \leq \mu^*$, $u_\mu < u_\nu$ and

$$\begin{aligned} \lambda_1(\mu)(2^* - 1)\int u_\mu^{2^*-2}\varphi_1(\mu)\varphi_1(\nu) &= \int \nabla\varphi_1(\mu)\nabla\varphi_1(\nu) + \int \varphi_1(\mu)\varphi_1(\nu) \\ &= \lambda_1(\nu)(2^* - 1)\int u_\nu^{2^*-2}\varphi_1(\nu)\varphi_1(\mu), \end{aligned}$$

we have $\lambda_1(\mu) > \lambda_1(\nu)$.

(iii) First, we show that $||u_\mu|| \to 0$ as $\mu \to 0^+$. Multiplying (P_μ) by u_μ, we have,

$$\int \left(|\nabla u_\mu|^2 + |u_\mu|^2\right) = \int u_\mu^{2^*} + \int \mu f u_\mu$$

and hence, for $\epsilon > 0$, we have, by Young's inequality with ϵ,

$$\left(1 - \frac{1}{\lambda_1(2^* - 1)} - \frac{\epsilon}{2}\right)||u_\mu||^2 \leq \frac{\mu^2}{2\epsilon}||f||_*^2 \ \text{ for } \epsilon > 0.$$

Thus, for $\epsilon > 0$ small, we have $||u_\mu||^2 \le C_\epsilon \mu^2$ for some constant $C_\epsilon > 0$, and hence, $||u_\mu|| = o(1)$ as $\mu \to 0^+$. Next, Multiplying $(3.1)_\mu$ by $\varphi_1(\mu)$, we have, by Hölder's inequality, that

$$\begin{aligned}\int \left(|\nabla\varphi_1|^2 + |\varphi_1|^2\right) &= \lambda_1 \cdot (2^* - 1) \int u_\mu^{2^*-2}\varphi_1^2 \\ &\le \lambda_1 \cdot (2^* - 1) \left(\int u_\mu^{2^*}\right)^{(2^*-2)/2^*} \left(\int \varphi_1^{2^*}\right)^{2/2^*} \\ &\le \lambda_1 \cdot (2^* - 1) \left(\int u_\mu^{2^*}\right)^{(2^*-2)/2^*} \left(\int |\nabla\varphi_1|^2 + |\varphi_1|^2\right) \\ &\le \lambda_1 \cdot (2^* - 1) S^{-(2^*-2)/2} ||u_\mu||^{2^*-2} ||\varphi_1||^2\end{aligned}$$

and thus, $S^{(2^*-2)/2} \le \lambda_1 \cdot (2^* - 1)||u_\mu||^{2^*-2}$. Therefore, we have the desired result. This completes the proof. ∎

LEMMA 3.2. *Let u_μ be a positive solution of $(1.3)_\mu$ for which $\lambda_1(\mu) > 1$. Then, for any $g \in H^1(\mathbb{R}^N)$, the problem:*

$$-\Delta w + w = (2^* - 1)u_\mu^{2^*-2} w + g(x), \quad w \in H^1(\mathbb{R}^N) \tag{3.2}$$

has a solution.

Proof. Consider the functional defined by

$$J(w) = \frac{1}{2}\int \left(|\nabla w|^2 + |w|^2\right) - \frac{1}{2}(2^* - 1)\int u_\mu^{2^*-2} w^2 - \int gw, \quad w \in H^1(\mathbb{R}^N).$$

From Hölder's inequality and Young's inequality, we have, for any $\epsilon > 0$,

$$\begin{aligned} J(w) &\ge \left(\frac{1}{2} - \frac{1}{2\lambda_1(\mu)}\right) ||w||^2 - \frac{\epsilon}{2}||w||^2 - \frac{1}{2\epsilon}||g||_*^2 \\ &= \left(\frac{1}{2} - \frac{1}{2\lambda_1(\mu)} - \frac{\epsilon}{2}\right) ||w||^2 - \frac{1}{2\epsilon}||g||_*^2 \end{aligned}$$

and hence, for small $\epsilon > 0$, there exist $C_{1,\epsilon} > 0$ and $C_{2,\epsilon} > 0$ such that

$$J(w) \ge C_{1,\epsilon}||w||^2 - C_{2,\epsilon}||g||_*^2. \tag{3.3}$$

Let $\{w_n\} \subset H^1(\mathbb{R}^N)$ be the minimizing sequence of J. From (3.3), we have $\{w_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Hence, passing subsequence, we may have that there exists $w \in H^1(\mathbb{R}^N)$ such that

$$w_n \to w \text{ weakly in } H^1(\mathbb{R}^N) \text{ as } n \to \infty,$$

$$w_n \to w \text{ a.e. in } \mathbb{R}^N \text{ as } n \to \infty$$

Here, we also note that

$$\nabla w_n \to \nabla w \text{ a.e. in } \mathbb{R}^N \text{ as } n \to \infty.$$

And

$$u_n^{2^*-1} \to \tilde{u}^{2^*-1} \text{ weakly in } (L^{2^*}(\mathbb{R}^N))^* \text{ as } n \to \infty.$$

By Fatou's Lemma
$$||w||^2 \leq \liminf_{n\to\infty} ||w_n||^2.$$
The weak convergence and the fact that $\int u_\mu^{2^*-2} w_n^2 < \infty$ for $n \geq 1$ imply
$$\lim_{n\to\infty} \int g w_n = \int g w, \quad \lim_{n\to\infty} \int u_\mu^{2^*-2} w_n = \int u_\mu^{2^*-2} w$$
and hence,
$$J(w) \leq \lim_{n\to\infty} J(w_n) = d.$$
Then, $J(w) = d$ and w is a minimizer of J. Therefore, w is a critical point of J and w is a solution of (3.2). This completes the proof. ∎

Proposition 3.3. *For $\mu = \mu^*$, the problem (P_μ) has a positive solution u_{μ^*} and $\lambda_1(\mu^*) = 1$. Moreover, the solution u_{μ^*} is unique in $H^1(\mathbb{R}^N)$.*

Proof. For $\mu \in]0, \mu^*[$, multiplying (P_μ) by u_μ, we have, by $(3.1)_\mu$,
$$\begin{aligned}\int \left(|\nabla u_\mu|^2 + |u_\mu|^2\right) &= \int u_\mu^{2^*} + \mu \int f u_\mu \\ &\leq \frac{1}{\lambda_1(\mu)(2^*-1)} \int (|\nabla u_\mu|^2 + |u_\mu|^2) + \mu^* ||f||_* ||u_\mu|| \\ &= \left(\frac{1}{\lambda_1(\mu)(2^*-1)} + \frac{\epsilon\mu^*}{2}\right) ||u_\mu||^2 + \frac{\mu^*}{2\epsilon} ||f||_*^2.\end{aligned}$$
By taking $\epsilon > 0$ small enough, there exists an constant $C_\epsilon > 0$ such that $||u_\mu|| \leq C_\epsilon$ for all $\mu \in]0, \mu^*[$. Then, there exists u_{μ^*} in $H^1(\mathbb{R}^N)$ such that u_μ monotonically increasing to u_{μ^*} as $\mu \to \mu^*$ and $u_\mu \to u_{\mu^*}$ weakly in $H^1(\mathbb{R}^N)$ as $\mu \to \mu^*$. Hence, u_{μ^*} is a positive solution of (P_μ) with $\mu = \mu^*$. We note that $\lambda_1(\mu)$ is a continuous function of $\mu \in]0, \mu^*]$.

Define $F : \mathbb{R}^1 \times H^1(\mathbb{R}^N) \to H^{-1}(\mathbb{R}^N)$ by
$$F(\mu, u) := \Delta u - u + (u^+)^{2^*-1} + \mu f(x).$$
Since $u_\mu \to u_{\mu*}$ weakly as $\mu \to \mu^*$, from Lemma 3.1, $\lambda(\mu^*) \geq 1$. If $\lambda_1(\mu^*) > 1$, then $F_u(\mu^*, u_{\mu^*})\varphi = \Delta\varphi - \varphi + (2^*-1)u_{\mu^*}^{2^*-2}\varphi = 0$ has no nontrivial solution. From Lemma 3.2, $F(\mu^*, u_{\mu^*})$ is an isomorphism of $\mathbb{R}^1 \times H^1(\mathbb{R}^N)$ onto $H^{-1}(\mathbb{R}^N)$, and by the implicitly function theorem to F, we find a neighborhood $]\mu^* - \delta,\ \mu^* + \delta[$ of μ^* such that (P_μ) possesses a positive solution if $\mu \in]\mu^* - \delta,\ \mu^* + \delta[$, which contradicts the definition of μ^*. Therefore, $\lambda_1(\mu^*) = 1$.

Suppose v_{μ^*} is a positive solution of (P_{μ^*}). Then $v_{\mu*} \geq u_{\mu^*}$ since u_{μ^*} is minimal. Let $w = v_{\mu^*} - u_{\mu^*}$. Then, since $\lambda_1(\mu^*) = 1$, we have
$$-\Delta w + w \geq (2^*-1)u_{\mu^*}^{2^*-2} w.$$
Since $\varphi_1 = \varphi_1(\mu^*)$ is the eigenfunction of the problem $(3,1)_{\mu^*}$, we have,
$$(2^*-1)\int u_{\mu^*}^{2^*-2}\varphi_1 w = \int \nabla w \nabla \varphi_1 + \int w\varphi_1 \geq (2^*-1)\int u_{\mu^*}^{2^*-1} w\varphi_1$$
and hence, $w \equiv 0$. This completes the proof. ∎

PROPOSITION 3.4. *The minimal solution u_μ of (P_μ) increasing continuously to u_{μ^*} as $\mu \to \mu^*$ and uniformly bounded in $H^1(\mathbb{R}^N)$ for all $\mu \in]0, \mu^*]$. Moreover, $||u_\mu|| \leq O(\mu^2)$ as $\mu \to 0^+$.*

Proof. It suffices to find the uniform bound of u_μ. Multiplying (P_μ) by u_μ, we have

$$\int \left(|\nabla u_\mu|^2 + |u_\mu|^2\right) = \int u_\mu^{2^*} + \int \mu f u_\mu$$

and hence, for $\epsilon > 0$, we have

$$\left(1 - \frac{1}{\lambda_1(2^* - 1)} - \frac{\epsilon}{2}\right) ||u_\mu||^2 \leq \frac{\mu^2}{2\epsilon} ||f||_*^2 \ \text{ for } \epsilon > 0.$$

Therefore, for $\epsilon > 0$ small, we have $||u_\mu|| \leq C_\epsilon \mu$ for some constant $C_\epsilon > 0$. Next, fix $\tau \in]0, \mu^*]$. If μ increases to τ, then u_μ is increasing up to u_τ and $u_\mu \to u_\tau$ in $H^1(\mathbb{R}^N)$. If it is not the case, then, by multiplying u_μ on (P_μ) again, we have

$$||u_\mu||^2 \leq \left\langle u_\tau^{2^*-1}, u_\mu \right\rangle + \tau \left\langle f, u_\mu \right\rangle$$

and so

$$||u_\mu|| \leq C S^{-(2^*-1)/2} ||u_\tau||^{2^*-1} + \tau ||f||_*$$

for some $C > 0$. Hence, there exists a sequence $\{u_{\mu_j}\}$ in $H^1(\mathbb{R}^N)$ conversing weakly to a solution $\tilde{u}$ of (P_τ) but $\tilde{u} \neq u_\tau$. Since $\{u_{\mu_j}\}$ coverge to $\tilde{u}$ strongly in local L^1 sense, by the maximum principle, we have $u_{\mu_j} \leq \tilde{u} < u_\tau$ which leads a contradiction to the minimality of u_τ. This completes the proof. ∎

REMARK. *From Proposition 3.4 , we have that $\lambda(\mu)$ is a continuously decreasing function from $[0, \mu^*]$ onto $[1, \infty[$ and $||u_\mu|| = o(1)$ as $\mu \to 0^+$.*

Next, we are going to find the second solution. In order to get another positive solution of (P_μ), we consider the following problem:

$$(3.4)_\mu \qquad \begin{cases} -\Delta v + v = (v + u_\mu)^{2^*-1} - u_\mu^{2^*-1} \ \text{ in } \ \mathbb{R}^N, \\ \qquad v \in H^1(\mathbb{R}^N), \ v > 0 \text{ in } \mathbb{R}^N \end{cases}$$

and the corresponding variational functional:

$$J_\mu(v) := \frac{1}{2} \int \left(|\nabla v|^2 + |v|^2\right) - \frac{1}{2^*} \int \left((v^+ + u_\mu)^{2^*} - u_\mu^{2^*} - 2^* u_\mu^{2^*-1} v^+\right)$$

for $v \in H^1(\mathbb{R}^N)$.

Clearly, we can have another positive solution $U_\mu = u_\mu + v_\mu$ if we show the problem $(3.4)_\mu$ possesses a positive solution for $\mu \in]0, \mu^*[$. We look for a critical point of J_μ which is a weak solution of $(3.4)_\mu$ by employing standard argument of the Mountain Pass method without the (PS) condition.

In the proof of the existance second solution, we make use of some arguments in [9, 10, 11].

THEOREM 3.5. *The problem (P_μ) possesses at least two positive solutions for all $\mu \in]0, \mu^*[$.*

Proof.
(i) Let $v \in H^1(\mathbb{R}^N) \setminus \{0\}$, Then, for $\epsilon > 0$, by Young's inequality,

$$\begin{aligned} J_\mu(v) &= \frac{1}{2}\int \left(|\nabla v|^2 + |v|^2\right) dx - \int\int_0^{v^+} \left((u_\mu + s)^{2^*-1} - u_\mu^{2^*-1}\right) ds dx \\ &\geq \frac{1}{2}\left(1 - \frac{1}{\lambda_1}\right)\int \left(|\nabla v|^2 + |v|^2\right) dx \\ &\quad - \int\int_0^{v^+} \left((u_\mu + s)^{2^*-1} - u_\mu^{2^*-1} - (2^*-1)u_\mu^{2^*-2}s\right) ds dx \\ &\geq \frac{1}{2}\left(1 - \frac{1}{\lambda_1}\right)\int \left(|\nabla v|^2 + |v|^2\right) dx - \int\int_0^{v^+} \left(\epsilon u_\mu^{2^*-2}s + C_\epsilon s^{2^*-1}\right) ds dx \\ &\geq \frac{1}{2}\left(1 - \frac{1}{\lambda_1}\right)||v||^2 - \frac{\epsilon}{2}\int u_\mu^{2^*-2}\left(v^+\right)^2 dx - \frac{C_\epsilon}{2^*}\int \left(v^+\right)^{2^*} dx \\ &\geq \frac{1}{2}\left(1 - \frac{1}{\lambda_1} - \frac{\epsilon}{2(2^*-1)\lambda_1}\right)||v||^2 - \frac{C_\epsilon}{2^*}S^{-2^*/2}||v||^{2^*} \end{aligned}$$

for some constant $C_\epsilon > 0$. Hence, for sufficiently small $\epsilon > 0$, there exist $\rho > 0, \delta > 0$ such that

$$J_\mu(v)|_{\partial \tilde{B}_\rho} \geq \delta > 0,$$

where $\tilde{B}_\rho = \{u \in H^1(\mathbb{R}^N) |\ ||u|| \leq \rho\}$.
(ii) Let $v \in H^1(\mathbb{R}^N), v \geq 0$ and $v \not\equiv 0$, then, for $t > 0$, we have

$$\begin{aligned} J_\mu(tv) &= \frac{t^2}{2}\int \left(|\nabla v|^2 + |v|^2\right) dx - \frac{1}{2^*}\int \left((u_\mu + tv)^{2^*} - u_\mu^{2^*} - 2^* u_\mu^{2^*-1} tv\right) dx \\ &\leq \frac{t^2}{2}\int \left(|\nabla v|^2 + |v|^2\right) dx - \frac{t^{2^*}}{2^*}\int |v|^{2^*} dx \\ &\leq \frac{t^2}{2}||v||^2 - \frac{t^{2^*}}{2^*}||v||_{2^*}^{2^*} \end{aligned}$$

Hence, we deduce

$$J_\mu(tv) \to -\infty$$

as $t \to \infty$. Therefore, for each $0 \not\equiv v \in H^1(\mathbb{R}^N)$ with $v \geq 0$, there exists a constant $t_v > 0$ such that $J_\mu(t_v v) \leq 0$ for $t \geq t_v$.

Let $K_1(v) := \frac{1}{2}\int (|\nabla v|^2 + v^2) - \frac{1}{2^*}\int (v^+)^{2^*} - \mu \int fv$.
Because u_μ is the critical point of $K_1(u)$, we can prove that, for $v \in H^1(\mathbb{R}^N)$,

$$J_\mu(v) = K_\mu(v) - K_\mu(0) = K_\mu(v) - K_1(u_\mu). \tag{3,5}$$

where

$$K_\mu(v) := \frac{1}{2}\int \left(|\nabla(v + u_\mu)|^2 + (v + u_\mu\right)^2 - \frac{1}{2^*}\int (v^+ + u_\mu)^{2^*} - \mu \int f(x)(v + u_\mu).$$

(iii) From (ii), there exist small $t_1 > 0$ such that, for $0 < t < t_1$, $J_\mu(t\omega_\epsilon) < \frac{1}{N}S^{N/2}$.

Choose $t_2 > t_1$ such that $J_\mu(t\omega_\epsilon) \le 0$ for all $t \ge t_2$. For $t_1 \le t \le t_2$,

$$\begin{aligned} J_\mu(t\omega_\epsilon) &= \frac{t^2}{2}\int \left(|\nabla\omega|^2 + |\omega_\epsilon|^2\right) dx - \frac{1}{2^*}\int \left((u_\mu + t\omega_\epsilon)^{2^*} - u_\mu^{2^*} - 2^* u_\mu^{2^*-1} t\omega_\epsilon\right) dx \\ &< \frac{t^2}{2}||\omega_\epsilon||^2 - \frac{t^{2^*}}{2^*}||\omega_\epsilon||_{2^*}^{2^*} \\ &= \left(\frac{t^2}{2} - \frac{t^{2^*}}{2^*}\right) S^{N/2} \le \frac{1}{N} S^{N/2}. \end{aligned}$$

(iv) Let

$$\Gamma := \{\gamma \in \mathcal{C}([0,1], H^1); \gamma(0) = 0,\ \gamma(1) = t_2\omega_\epsilon\}$$

and

$$c_\mu = \inf_{\gamma\in\Gamma}\max_{s\in[0,1]} J_\mu(\gamma(s)).$$

Then, we have

$$0 < \alpha \le c_\mu \le \sup_{t\ge 0} J_\mu(t\omega_\epsilon) < \frac{1}{N} S^{N/2}. \tag{3.6}$$

We now applying the Mountain Pass Theorem without Palais-Smale condition in [5] to get a sequence $\{v_n\} \subset H^1(\mathbb{R}^N)$ such that

$$J_\mu(v_n) \to c_\mu, \quad J'_\mu(v_n) \to 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^N). \tag{3.7}$$

Since

$$\begin{aligned} 1 + c_\mu + ||v_n|| + ||u_\mu|| &\ge 1 + c_\mu + ||v_n + u_\mu|| \\ &\ge J_\mu(v_n) - \frac{1}{2^*} J'_\mu(v_n)(v_n^+ + u_\mu) \\ &\ge \left(\frac{1}{2} - \frac{1}{2^*}\right)||v_n||^2 - \frac{2}{2^*}||v_n||||u_\mu|| - \left(1 - \frac{1}{2^*}\right)||u_\mu||_{2^*}^{2^*}, \end{aligned}$$

we see that $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Hence, there exists a subsequence, say again, $\{v_n\}$ such that

$$\begin{gathered} v_n \to v_\mu \text{ weakly in } H^1(\mathbb{R}^N), \\ v_n \to v_\mu \text{ a.e. in } \mathbb{R}^N, \\ \nabla v_n \to \nabla v_\mu \text{ a.e. in } \mathbb{R}^N, \end{gathered}$$

and

$$(v_n + u_\mu)^{2^*-1} - u_\mu^{2^*-1} \to (v^+ + u_\mu)^{2^*-1} - u_\mu^{2^*-1} \text{ weakly in } (L^{2^*}(\mathbb{R}^N))^*.$$

Hence, v_μ is a weak solution of $-\Delta v + v = (v^+ + u_\mu)^{2^*-1} - u_\mu^{2^*-1}$.

Using the maximal principle, we get $v_\mu \ge 0$ in $\mathbb{R}^N$. Set $u_n := v_n + u_\mu$, $u := v_\mu + u_\mu$. Then

$$\begin{gathered} u_n \to u \text{ weakly in } H^1(\mathbb{R}^N), \\ u_n \to u \text{ a.e. in } \mathbb{R}^N, \\ \nabla u_n \to \nabla u \text{ a.e. in } \mathbb{R}^N. \end{gathered}$$

From (3.5),

$$J_\mu(v_n) = K_\mu(v_n) - K_\mu(0) = K_1(u_n) - K_1(u_\mu) \to c_\mu \text{ as } n \to \infty \tag{3.8}$$

and u is a solution of

$$-\Delta u + u = u^{2^*} + \mu f(x). \tag{3.9}$$

Now, we are going to show that $u \not\equiv u_\mu$. In fact, if $u \equiv u_\mu$, i.e., $v_\mu \equiv 0$, then $u_n \not\to u$ strongly in $H^1(\mathbb{R}^N)$ since $J_\mu(0) = 0 < u_\mu$. Let $c_2 := c_\mu + K_1(u_\mu)$. By the Brezis-Lieb Lemma($cf.$ [4]) we have

$$\begin{cases} ||u_n||^2 = ||u_\mu||^2 + ||v_n||^2 + o(1), \\ |u_n^+|^{2^*} = |u_\mu|^{2^*} + |v_n^+|^{2^*} + o(1), \\ \displaystyle\int fu_n = \int fu_\mu + o(1) \text{ as } n \to \infty. \end{cases} \tag{3.10}$$

By (3.8), (3.9), we have

$$\int \left(|\nabla u_n|^2 + u_n^2\right) = \int (u_n^+)^{2^*} + \mu \int f(x)u_n + o(1),$$

$$\int \left(|\nabla u_\mu|^2 + u_\mu^2\right) = \int (u_\mu^+)^{2^*} + \mu \int f(x)u_\mu.$$

Hence,

$$\int \left(|\nabla v_n|^2 + v_n^2\right) = \int (v_n^+)^{2^*} + o(1), \tag{3.11}$$

by substracting the two identities above and by (3.10).
Using (3.8), (3.9), (3.10) and (3.11), we have that, as $n \to \infty$

$$\begin{aligned} c_2 &= c_\mu + K_1(u_\mu) \\ &= J_\mu(v_n) + K_1(u_\mu) + o(1) \\ &= K_1(u_n) + o(1) \\ &= K_1(u_\mu) + \frac{1}{2}\int |\nabla v_n|^2 + v_n^2 - \frac{1}{2^*}\int v_n^{2^*} + o(1) \\ &= K_1(u_\mu) + \left(\frac{1}{2} - \frac{1}{2^*}\right)\int (v_n)^{2^*} + o(1) \\ &= K_1(u_\mu) + \frac{1}{N}\int (v_n)^{2^*} + o(1). \end{aligned}$$

By Sobolev inequality:

$$S||v_n||_{2^*}^2 \le ||v_n||^2 = ||v_n||_{2^*}^{2^*} + o(1),$$

we have $||v_n||_{2^*}^{2^*} \ge S^{N/2}$. Thus,

$$c_2 = c_\mu + K_1(u_\mu) \ge K_1(u_\mu) + \frac{1}{N}S^{N/2}.$$

This leads a contradiction to (3.6). Therefore, we have $v_\mu > 0$. This completes the proof. ∎

Consequently, we have:

THEOREM 3.6. *Assume $f \in H^{-1}(\mathbb{R}^N)$, $f \geq 0$, $f \not\equiv 0$ in $\mathbb{R}^N$ and $||\mu f||_* \leq C_N^*$. Then there exists a positive constant $\mu^* > 0$ such that (P_μ) possesses at least two positive solutions for $0 < \mu < \mu^*$, a unique solution for $\mu = \mu^*$ and no positive solution if $\mu > \mu^*$.*

4. Bifurcation

In order to study bifurcation phenomenon, we consider following eigenvalue problem:

$$(4.1)_\mu \qquad \begin{cases} -\Delta\phi + \phi = \eta(\mu)(2^* - 1)U_\mu^{2^*-2}\phi, \\ \qquad \phi \text{ in } H^1(\mathbb{R}^N). \end{cases}$$

Let η_1 be the first eigenvalue of $(4.1)_\mu$;i.e.,

$$\eta_1 = \eta_1(\mu)\inf\{\int |\nabla\phi|^2 + |\phi|^2; \phi \in H^1(\mathbb{R}^N), \int (2^* - 1)U_\mu^{2^*-2}\phi^2 = 1\}$$

and $\phi_1 > 0$ be the corresponding eigenfunction.

In the proof of the following lemma, we make use of arguments in [3].

LEMMA 4.1. *Let U_μ be a second positive solution of (P_μ) obtained in Theorem 3.5. Then $\eta_1(\mu) < 1$ for $0 < \mu < \mu^*$.*

Proof. Suppose contrary that $\eta_1(\mu) \geq 1$, Let $\psi = U_\mu - u_\mu > 0$. Then ϕ_1 and ψ satisfies

$$(4.2) \qquad \Delta\phi_1 - \phi_1 + (2^* - 1)U_\mu^{2^*-2}\phi_1 \leq 0 \text{ and } \Delta\psi - \psi + (2^* - 1)_\mu^{2^*-2}\psi \geq 0,$$

respectively. Set $\sigma = \psi/\phi_1$;i.e., $\psi = \sigma\phi_1$. Then, by (4.2),

$$(4.3) \qquad \sigma\nabla(\phi_1^2\nabla\sigma) = \psi\nabla\psi - \frac{\psi}{\phi_1}\nabla\phi_1 \geq 0.$$

Let ζ be a C^∞ function on $\mathbb{R}^+$ with $0 \leq \zeta(t) \leq 1$,

$$\zeta(t) := \begin{cases} 1 \text{ for } 0 \leq t \leq 1, \\ 0 \text{ for } t \geq 2. \end{cases}$$

For $R > 0$, set $\zeta_R(t) = \zeta\left(\frac{|x|}{R}\right)$ in $\mathbb{R}^N$. Multiplying (4.3) by ζ_R^2 and intergrating over $\mathbb{R}^N$, we have by Green' theorem,

$$(4.4) \qquad \begin{aligned} \int \zeta_R^2\phi_1^2|\nabla\sigma|^2 &\leq 2\left|\int \phi_1^2\zeta_R\sigma\nabla\sigma\cdot\nabla\zeta_R\right| \\ &\leq 2\left[\int_{R<|x|<2R}\zeta_R^2\phi_1^2|\nabla\sigma|^2\right]^{1/2}\left[\int \phi_1^2\sigma^2|\nabla\zeta_R|^2\right]^{1/2} \\ &\leq C_1\left[\int_{R<|x|<2R}\zeta_R^2\phi_1^2|\nabla\sigma|^2\right]^{1/2}\left[\int_{R<|x|<2R}\psi^{2^*}\right]^{1/2} \\ &\leq C_2\left[\int_{R<|x|<2R}\zeta_R^2\phi_1^2|\nabla\sigma|^2\right]^{1/2} \end{aligned}$$

for some constants C_1 and C_2, which implies

$$\int \zeta_R^2 \phi_1^2 |\nabla\sigma|^2 \leq C_3$$

for some constant $C_3 > 0$.
Letting $R \to \infty$, we see that

$$\int \phi_1^2 |\nabla\sigma|^2 \leq C_3.$$

But then it follows that the last term in (4.4) tends to 0 as $R \to \infty$, so that

$$\int \phi_1^2 |\nabla\sigma|^2 = 0.$$

Therefore, σ is a positive constant and by (4.2), $\phi_1 \equiv \psi = U_\mu - u_\mu$, and thus $U_\mu \equiv u_\mu$, which leads a contradiction. This completes the proof. ∎

LEMMA 4.2. *For $\mu \in]0, \mu^*[$, U_μ decreases contonusely to u_{μ^*} as $\mu \to \mu^*$ in $H^1(\mathbb{R}^N)$. Moreover,*
(i) $U_\mu \to u_{\mu^*}$ as $\mu \to \mu^*$ by the uniqueness of u_{μ^*},
(ii) $\lim_{\mu\to 0^+} ||U_\mu|| = S^{N/4}$.

Proof. First, we note that

$$\begin{aligned}\left(\frac{1}{2} - \frac{1}{2^*}\right) ||U_\mu||^2 &= \frac{1}{2}||U_\mu||^2 - \frac{1}{2^*}\int \left(U_\mu^{2^*} + \mu \int fU_\mu\right)\\ &= \mu\left(1 - \frac{1}{2^*}\right)\int fU_\mu - \mu\int fu_\mu - \mu\int fv_\mu\\ &+ \frac{1}{2}||u_\mu||^2 + \frac{1}{2}||v_\mu||^2 + \int \nabla u_\mu \nabla v_\mu + \int u_\mu v_\mu - \frac{1}{2^*}\int U_\mu^{2^*}\\ &\geq \mu\left(1 - \frac{1}{2^*}\right)\int fU_\mu + J_\mu(v_\mu) + H(u_\mu),\end{aligned}$$

where $H(u) = \frac{1}{2}||u||^2 - \frac{1}{2^*}\int u^{2^*} - \mu\int fu$.

From Hölder's and Young's inequality, for $\epsilon > 0$, we have

$$\left(\frac{2^* - 2}{22^*} - \frac{\epsilon(2^* - 1)}{22^*}\right) ||U_\mu||^2 \leq \frac{2^* - 1}{\epsilon 22^*}\mu^2||f||_*^2 + \frac{1}{N}S^{N/2} + H(u_\mu).$$

Since

$$\begin{aligned}H(u_\mu) &= \left(\frac{1}{2} - \frac{1}{2^*}\right)||u_\mu||^2 - \mu\left(1 - \frac{1}{2^*}\right)\int fu_\mu\\ &\leq \left(\frac{1}{2} - \frac{1}{2^*}\right)||u_{\mu^*}||^2,\end{aligned}$$

$H(u_\mu)$ is uniformly bounded for $\mu \in (0, \mu^*]$. Moreover, by the remark of Proposition 3.4, $H(u_\mu) = o(1)$ as $\mu \to 0^+$. Taking $\epsilon > 0$ small enought, we have $||U_\mu|| \leq$ C for some C > 0. Since $0 < u_\mu \leq U_\mu$, (i) follows from Proposition 3.3 and Proposition 3.4.

For (ii). By (ii) of Lemma 3.1, and (i) and (iii) in the proof of Theorem 3.5, there exists $d > 0$ such that

$$0 < d < J_\mu(v_\mu) = H(U_\mu) - H(u_\mu) < \frac{1}{N}S^{N/2}$$

and thus, since $J'_\mu(U_\mu)U_\mu = 0$,

$$d + H(u_\mu) \le \frac{1}{N}||U_\mu||^2 - \frac{2^*-1}{2^*}\mu \int fU_\mu \le H(u_\mu) + \frac{1}{N}S^{N/2}.$$

Since U_μ is uniformly bounded,

$$d + o(1) \le \frac{1}{N}||U_\mu||^2 \le \frac{1}{N}S^{N/2} + o(1). \tag{4.5}$$

By Sobolev's inequality, $S||U_\mu||^2_{2^*} \le ||U_\mu||^2 = ||U_\mu||^{2^*}_{2^*} + o(1)$. Then $||U_\mu||^{2^*}_{2^*} \ge S^{N/2} + o(1)$ and so $||U_\mu||^2 \ge S^{N/2} + o(1)$. Therefore by (4.5), we have

$$\lim_{\mu\to 0^+} ||U_\mu|| = S^{N/2}.$$

Now, fix $\rho \in]0, \mu^*]$. Suppose μ increase to ρ, then U_μ is decreasing to U_ρ in $H^1(\mathbb{R}^N)$ and we have

$$||U_\mu|| \le S^{-2^*/2}||U_\rho||^{2^*-1} + \rho||f||_*$$

and so, there exists a sequence U_{μ_j} conving weakly to a solution $\tilde{U}$ of (P_μ) in $H^1(\mathbb{R}^N)$ with $\rho = \mu$ but $\tilde{U} \ne U_\rho$. By the maximum principle, we have $U_\rho < \tilde{U} \le U_{\mu^*}$ which contradicts the uniqueness of solutions bigger than u_μ. Therefore, U_μ is decreasing continuously to U_ρ and $U_\mu \to U_\rho$ in $H^1(\mathbb{R}^N)$. This completes the proof. ∎

Lemma 4.3. *Let V be a positive supersolution of (P_μ) bigger than u_μ, then $V \le U_\mu$.*

Proof. Suppose $V > U_\mu$ in $\mathbb{R}^N$, then $W = V - U_\mu$ satisfies

$$(2^*-1)\int U_\mu^{2^*-2}W\phi_1 \le \int \nabla W \cdot \nabla\phi_1 = \eta_1(2^*-1)\int U_\mu^{2^*-2}W\phi_1$$

and thus, $\eta_1(\mu) \ge 1$, which leads a conrradiction. This completes the proof. ∎

Remark. *From Lemma 4.1 and Lemma 4.3, we can see the uniqueness of second solutions which are bigger than the minimal solutions u_μ.*

Now, we state basic properties of the eigenvalue problem $(4.1)_\mu$:

Lemma 4.4. *(i) $1/(2^*-1) < \eta_1(\mu) < 1$ for $0 < \mu < \mu^*$,*
(ii) $\eta_1(\mu) \to 1/(2^*-1) \to 1/(2^*-1)$ as $\mu \to 0^+$,
(iii) $\eta_1(\mu) \to 1$ as $\mu \to \mu^*$.

Proof. (i) Since $\phi_1 > 0$ is an eigenvector corresponding to the the first eigenvalue $\eta_1(\mu)$, we know

$$\eta_1(\mu)(2^*-1)\int U_\mu^{2^*-1}\phi_1 = \int \nabla U_\mu \cdot \nabla\phi_1 = \int U_\mu^{2^*-1}\phi_1 + \mu\int f\phi_1.$$

and so,

$$\eta_1(\mu)\left((2^*-1)-1\right)\int U_\mu^{2^*-1}\phi_1 = \mu\int f\phi_1.$$

Therefore, by Lemma 4.1, $1 > \eta_1(\mu) > \frac{1}{2^*-1}$.

(ii) As $\mu \to 0^+$,

$$\frac{1}{2^*-1} < \eta_1(\mu) \leq \frac{||U_\mu||^2}{(2^*-1)\,||U_\mu||_{2^*}^{2^*}} \leq \frac{S^{N/2}+o(1)}{(2^*-1)\,(S^{N/2}+o(1))} \to \frac{1}{2^*-1}.$$

Thus, $\eta_1(\mu) \to 1/(2^*-1)$ as $\mu \to 0^+$.

(iii) follows from (i) of Lemma 3.1, Proposition 3.3, Lemma 4.1 and (i) of Lemma 4.2. This completes the proof. ∎

In order to show the existence of a bifurcation point, we make use of Theorem 3.2 in [8].

Now, we have:

THEOREM 4.5. *(i) The set $\{U_\mu\}$ is bounded uniformly in $H^1(\mathbb{R}^N)$,*
(ii) (μ^*, u_{μ^*}) is a bifurcation point.

Proof. (i) It follows immediately from the proof of Lemma 4.2.
(ii) For this, define $F : R \times H^1(\mathbb{R}^N) \to H^{-1}(\mathbb{R}^N)$ by

$$F(\mu, u) := \Delta u - u + (u^+)^{2^*-1} + \mu f(x).$$

It is easy to see that $F(\mu, u)$ is differentiable at solution point (μ, u) for $]0, \mu^*[$ and

$$F_u(\mu, u_\mu)w = \Delta w - w + (2^*-1)u_\mu^{2^*-2}w$$

is an ismorphism of $R \times H(\mathbb{R}^N)$ onto $H^{-1}(\mathbb{R}^N)$. Then, by the Implicit Function Theorem, the solution of $F(\mu, u)$ near (μ, u_μ) are given by a single continuous cuver and $u_\mu \to 0$ in $H^1(\mathbb{R}^N)$ as $\mu \to 0$.

We now are going to prove that (μ^*, u_{μ^*}) is a bifurcation point of F. Since $F_u(\mu^*, u_{\mu^*})\phi = 0, \phi \in H^1(\mathbb{R}^N)$ has a solution $\phi_1 > 0$ in $\mathbb{R}^N$, $\mathcal{N}\left(F_u\left(\mu^*, u_{\mu^*}\right)\right) = \text{span}\{\phi_1\}$ is one dimensional and $\text{codim}\mathcal{R}\left(F_u\left(\mu^*, u_{\mu^*}\right)\right) = 1$ by the Fredholm alternative. Suppose there exists $v \in H^1(\mathbb{R}^N)$ satisfying

$$\Delta v - v + (2^*-1)\, u_{\mu^*}^{2^*-2} v = -f(x).$$

Then

$$0 = \int \left(\nabla v \cdot \nabla \phi_1 + v\phi_1 - (2^*-1)\, u_{\mu^*}^{2^*-2} v\phi_1\right) = \int f\phi_1,$$

which is impossible because $0 \not\equiv f \geq 0$. Hence, $F_u\left(\mu^*, u_{\mu^*}\right) \notin \mathcal{R}\left(F_u\left(\mu^*, u_{\mu^*}\right)\right)$. Thus, by Theorem 3.2 in [8], (μ^*, u_{μ^*}) is the bifurcation point near which, the solution of (p_μ) form a curve $(\mu^* + \tau(s), u_{\mu^*} + s\phi_1 + z(s))$ with s near $s = 0$ and $\tau(0) = \tau'(0) = 0, z(0) = z'(0) = 0$. Finally, we will show that $\tau''(0) < 0$ which implies that the bifurcation curve only turns to the left in the $\mu u-$plane. For this, differentiate (P_μ) in s, we have

$$\Delta u_s - u_s + (2^*-1)\, u^{2^*-2} u_s + \tau'(s) f(x) = 0, \tag{4.6}$$

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where $u_s = \phi_1 + z'(s)$. Multiplying $F_u(\mu^*, u_{\mu^*})\phi_1 = 0$ by u_s and (4,6) by ϕ_1, integrating and substracting, we have

$$\begin{aligned}
&\tau'(s)\int f\phi_1 \\
&= (2^*-1)\int \left(u_{\mu^*}^{2^*-2} - (u_{\mu^*} + s\phi_1 + z(s))^{2^*-2}\right)(\phi_1 + z'(s))\phi_1 \\
&= -s(2^*-1)(2^*-2)\int (u_{\mu^*} + \theta(s\phi_1 + z(s)))^{2^*-3}\left(\phi_1 + \frac{z(s)}{s}\right)(\phi_1 + z'(s))\,\phi_1
\end{aligned}$$

for some $\theta(s) \in (0,1)$. Therefore,

$$\tau''(0)\int f\phi_1 = \left(lim_{s\to 0}\frac{\tau'(s)}{s}\right)\int f\phi_1 = -(2^*-1)(2^*-2)\int (u_{\mu^*})^{2^*-3}\phi_1^3$$

and $\tau''(0) < 0$. This completes proof. ∎

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Some inequalities on meromorphic function and its derivative on its Borel direction *

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Abstract

In view of Nevanlinna theory in the angular domain, we establish some inequalities of meromorphic function concerning its derivation in its Borel direction. By applying these inequalities, we also investigate exceptional values of meromorphic functions with infinite order in the Borel direction.

Key words: Infinite order; Borel direction; Exceptional value.
Mathematical Subject Classification (2010): 30D30 30D35.

1 Introduction and main results

It is assumed that the reader is familiar with the basic results and the standard notations of the Nevanlinna theory of meromorphic functions (see [7, 17, 20]). We denote by $\mathbb{C}$ the open complex plane, by $\widehat{\mathbb{C}}(=\mathbb{C}\bigcup\{\infty\})$ the extended complex plane, and by $\Omega(\subset\mathbb{C})$ an angular domain. In addition, the order of meromorphic function f is defined by

$$\rho(f)=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r},$$

and the exponent of convergence of distinct a-points of f is defined by

$$\overline{\rho}(a,f)=\limsup_{r\to\infty}\frac{\log^{+}\overline{N}(r,a,f)}{\log r}.$$

Let f be a meromorophic function of order $\rho(0<\rho<\infty)$, then we say that a is an exceptional value in the sense of Borel (evB for short) for f for the distinct zeros if $\overline{\rho}(a,f)<\rho$.

It is well known that the singular direction of meromorphic function is an interesting topic in the field of complex analysis, such as, Julia direction, Borel direction, T direction, Hayman direction, and so on (see [1, 3, 4, 8, 10, 12, 13, 14]). Moreover, we know that every one singular direction is always responding to exceptional value, such as, the Julia's direction relating with Picard exceptional value and the Borel's direction relating with Borel exceptional value, and so

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on. 2011, Peng and Sun [11] gave some examples on T direction which is a singular direction relating with T exceptional value. In the discussion of the topic of singular direction, we find that the characteristics of meromorphic functions in the angular domain played an important role(see [6, 18, 19, 24, 25]). So, we firstly introduce the characteristics of meromorphic functions in the angular domain as follows [5, 25].

For a meromorphic function f on the angular domain $\Omega(\alpha,\beta)=\{z:\alpha\leq\arg z\leq\beta\}$ and $0<\beta-\alpha\leq 2\pi$. Define

$$A_{\alpha,\beta}(r,f)=\frac{\omega}{\pi}\int_1^r(\frac{1}{t^\omega}-\frac{t^\omega}{r^{2\omega}})\{\log^+|f(te^{i\alpha})|+\log^+|f(te^{i\beta})|\}\frac{dt}{t},$$

$$B_{\alpha,\beta}(r,f)=\frac{2\omega}{\pi r^\omega}\int_\alpha^\beta\log^+|f(re^{i\theta})|\sin\omega(\theta-\alpha)d\theta,$$

$$C_{\alpha,\beta}(r,f)=2\sum_{1<|b_\mu|<r}(\frac{1}{|b_\mu|^\omega}-\frac{|b_\mu|^\omega}{r^{2\omega}})\sin\omega(\theta_\mu-\alpha),$$

$$S_{\alpha,\beta}(r,f)=D_{\alpha,\beta}(r,f)+C_{\alpha,\beta}(r,f),$$

where $D_{\alpha,\beta}(r,f)=A_{\alpha,\beta}(r,f)+B_{\alpha,\beta}(r,f)$, $\omega=\frac{\pi}{\beta-\alpha}$ and $b_\mu=|b_\mu|e^{i\theta_\mu}(\mu=1,2,\cdots)$ are the poles of f on $\Omega(\alpha,\beta)$ counted according to their multiplicities. $S_{\alpha,\beta}(r,f)$ is called the Nevanlinna's angular characteristic, and $C_{\alpha,\beta}(r,f)$ is called the angular counting function of the poles of f on $\Omega(\alpha,\beta)$, and $\overline{C}_{\alpha,\beta}(r,f)$ is the reduced function of $C_{\alpha,\beta}(r,f)$. Similarly, the order of meromorphic function f on $\Omega(\alpha,\beta)$ is defined by

$$\rho_{\alpha,\beta}(f)=\limsup_{r\to\infty}\frac{\log S_{\alpha,\beta}(r,f)}{\log r},$$

and the exponent of convergence of distinct a-points of f on $\Omega(\alpha,\beta)$ is defined by

$$\overline{\rho}_{\alpha,\beta}(a,f)=\limsup_{r\to\infty}\frac{\log^+\overline{C}_{\alpha,\beta}(r,a,f)}{\log r}.$$

Suppose that f is a meromorphic function of order $\rho_{\alpha,\beta}(f)(0<\rho_{\alpha,\beta}(f)<\infty)$, then we say that a is an exceptional value on the angular domain in the sense of Borel (evaB for short) for f for the distinct zeros if $\overline{\rho}_{\alpha,\beta}(a,f)<\rho_{\alpha,\beta}(f)$.

Remark 1.1 *By the second fundamental theorem in the whole complex plane, we know that a meromorophic function f of order $\rho(0<\rho<\infty)$ at most has two evB for the distinct zeros. However, the corresponding conclusion can not hold for meromorphic function f with order $\rho_{\alpha,\beta}(0<\rho_{\alpha,\beta}<\infty)$ on $\Omega(\alpha,\beta)$ since $Q_{\alpha,\beta}(r,f)=O\{\log(rS_{\alpha,\beta}(r,f))\}$ is not valid, as $r\to\infty(r\notin E)$ and E is the set with finite linear measure.*

Thus,*it is an interesting topic to research the exceptional value of meromorphic functions on the angular domain.*

Before stating the our results, we will introduce the definition as follows.

Definition 1.1 *[2]. Let f be a meromorphic function of infinite order, $\rho(r)$ be a real function satisfying the following conditions:*

(i) $\rho(r)$ is continuous, non-decreasing for $r\geq r_0$ and $\rho(r)\to\infty$ as $r\to\infty$;

(ii)

$$\lim_{r\to\infty}\frac{\log U(R)}{\log U(r)}=1,\quad R=r+\frac{r}{\log U(r)},$$

where $U(r)=r^{\rho(r)}$ $(r\geq r_0)$;

(iii)

$$\limsup_{r\to\infty}\frac{\log T(r,f)}{\log U(r)}=1.$$

Then $\rho(r)$ is called infinite order of meromorphic function f. This definition was given by Xiong Qinglai[2].

We will give the definition of Borel direction of meromorphic functions f of infinite order $\rho(r)$ as follows.

Definition 1.2 *[2]. Let f be a meromorphic function of infinite order $\rho(r)$. If for any $\varepsilon(0<\varepsilon<\pi)$, the equality*

$$\limsup_{r\to\infty}\frac{\log n(\Omega(\theta-\varepsilon,\theta+\varepsilon,r),f=a)}{\rho(r)\log r}=1,$$

holds for any complex number $a\in\widehat{\mathbb{C}}$, at most except two exception, where $n(\Omega(\theta-\varepsilon,\theta+\varepsilon,r),f=a)$ is the counting function of zero of the function $f-a$ in the angular domain $\Omega(\theta-\varepsilon,\theta+\varepsilon)$, counting multiplicities. Then the ray $\arg z=\theta$ is called a Borel direction of $\rho(r)$ order of meromorphic function f.

Remark 1.2 *Chuang [2] proved that every meromorphic function f with infinite order $\rho(r)$ has as least one Borel direction of infinite order $\rho(r)$.*

In 2012, Long and Wu [9] studied the uniqueness of meromorphic functions with infinite order sharing some values in the Borel direction. Later, Zhang, Xu and Yi [21] further investigated the uniqueness of meromorphic functions sharing some values in the Borel direction, and improved the results of Long and Wu. In 2013, Zhang [23] also studied the problems of Borel directions of meromorphic functions concerning shared values and obtained that if two meromorphic functions with infinite order share three distinct values, their Borel direction are same. In the same year, Xu, Wu and Tu [15] investigated the relations between exceptional values and Borel direction, and obtained a series of results. In this paper, we mainly further investigate the exceptional values of meromrophic function and its derivation in its Borel direction. Now, we give the main theorem of this paper as follows.

Theorem 1.1 *Let f be a transcendental meromorphic function of infinite order $\rho(r)$ on the whole complex plane, $\arg z=\theta(0\le\theta<2\pi)$ be one Borel direction of $\rho(r)$ order of function f and $\Omega:=\Omega(\theta-\varepsilon,\theta+\varepsilon)$ for any $\varepsilon(0<\varepsilon<\pi)$. Let $a,b(\neq 0)$ be distinct points and k be a positive integer. Then*

$$\begin{aligned}S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\le&\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f)+(k+1)\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a,f)\\&+\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}\left(r,b,f^{(k)}\right)+Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f),\end{aligned}\tag{1}$$

where $Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f)$ is defined as in Lemma 2.2 of Section 2.

In order to prove Theorem 1.1, we will prove the more general form of the inequality of meromorphic function and its derivation in the Borel direction as follows.

Theorem 1.2 *Let f be a transcendental meromorphic function of infinite order $\rho(r)$ on the whole complex plane, $\arg z=\theta(0\le\theta<2\pi)$ be one Borel direction of $\rho(r)$ order of function f and $\Omega:=\Omega(\theta-\varepsilon,\theta+\varepsilon)$ for any $\varepsilon(0<\varepsilon<\pi)$. Let $a_j,b_l(j=1,2,\dots,p;l=1,2,\dots,q)$ be distinct*

complex numbers satisfying $b_l \neq 0$, and m_j, n_l, s be any positive integers. Then

$$\begin{aligned}\Bigg\{pq - \Bigg[\sum_{j=1}^{p} \frac{kq+1}{m_j+1} &+ \sum_{l=1}^{q} \frac{1}{n_l+1} \\ &+ \frac{1}{s+1}\left(1 + k\sum_{l=1}^{q}\frac{1}{n_l+1}\right)\Bigg]\Bigg\} S_{\theta-\varepsilon,\theta+\varepsilon}(r,f) \\ &\leq \frac{s}{s+1}\left(1 + k\sum_{l=1}^{q}\frac{1}{n_l+1}\right)\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f| \leq s) \\ &+ (kq+1)\sum_{j=1}^{p}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f| \leq m_j) \\ &+ \sum_{l=1}^{q}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}\left(r,b_l,f^{(k)}| \leq n_l\right) + Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f), \qquad (2)\end{aligned}$$

where $\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a,f| \leq k)$ is the counting function of distinct a-points of f on Ω whose multiplicities do not exceed k.

Let $p = q = 1$ and $s \to \infty, m_j \to \infty$ and $n_l \to \infty$ in Theorem 1.2, we can get Theorem 1.1 easily.

We also investigate the problem on exceptional value of meromorphic function and its derivation in its Borel direction, by applying the conclusions of Theorems 1.1 and 1.2. To state the theorem, we will introduce the definitions as follows.

Definition 1.3 *Let $\arg z = \theta(0 \leq \theta < 2\pi)$ be one Borel direction of $\rho(r)$ order of function f and k be a positive integer, we call that a is*

(i) an exceptional value in the sense of Borel for f in the Borel direction (evBB for short) for distinct zeros of multiplicity $\leq k$, if $\overline{\rho}_\theta^k(a,f) < 1$;

(ii) an exceptional value in the sense of Borel for f in the Borel direction (evBB for short) for distinct zeros , if $\overline{\rho}_\theta(a,f) < 1$; where

$$\overline{\rho}_\theta^k(a,f) = \limsup_{r\to\infty}\frac{\log^+\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a,f| \leq k)}{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)}, \quad \overline{\rho}_\theta(a,f) = \limsup_{r\to\infty}\frac{\log^+\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a,f)}{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)}.$$

In particular, we say that a is an evBB for f for simple zeros if $k = 1$, a is an evBB for f for simple and double zeros if $k = 2$.

Definition 1.4 *Let $\arg z = \theta(0 \leq \theta < 2\pi)$ be one Borel direction of $\rho(r)$ order of function f and k, l be two positive integers, then we call a an evBB for f^l for distinct zeros of order $\leq k$, if $\overline{\rho}_\theta^k(a,f^{(l)}) < 1$, where*

$$\overline{\rho}_\theta^k(a,f^{(l)}) = \limsup_{r\to\infty}\frac{\log^+\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a,f^{(l)}| \leq k)}{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)}.$$

Theorem 1.3 *Let f be a transcendental meromorphic function of infinite order $\rho(r)$ on the whole complex plane, $\arg z = \theta(0 \leq \theta < 2\pi)$ be one Borel direction of $\rho(r)$ order of function f and $\Omega := \Omega(\theta-\varepsilon,\theta+\varepsilon)$ for any $\varepsilon(0 < \varepsilon < \pi)$. If ∞ is an evBB for f for distinct poles of order $\leq s$, and $a_j(j = 1, 2, \ldots, p)$ are evBB for f for distinct zeros of order $\leq m_j$, and $b_l(\neq 0)(l = 1, 2, \ldots, q)$ are evBB for $f^{(k)}$ for distinct zeros of order $\leq n_l$, where k, p, q, s and all of m_j, n_l are positive integers. Then*

$$\sum_{j=1}^{p}\frac{kq+1}{m_j+1} + \sum_{l=1}^{q}\frac{1}{n_l+1} + \frac{1}{s+1}\left(1 + k\sum_{l=1}^{q}\frac{1}{n_l+1}\right) \geq pq.$$

Let $p = q = 1$ in Theorem 1.3, we can obtain the corollary as follows.

Corollary 1.1 *Let f be a transcendental meromorphic function of infinite order $\rho(r)$ on the whole complex plane, $\arg z = \theta(0 \le \theta < 2\pi)$ be one Borel direction of $\rho(r)$ order of function f and $\Omega := \Omega(\theta - \varepsilon, \theta + \varepsilon)$ for any $\varepsilon(0 < \varepsilon < \pi)$. If ∞ is an evBB for f for distinct poles of order $\le s$, and a is an evBB for f for distinct zeros of order $\le m$, $b(\neq 0)$ is an evBB for $f^{(k)}$ for distinct zeros of order $\le n$, and s, m, n, k are positive integers. Then*

$$\frac{k+1}{m+1} + \frac{1}{n+1} + \frac{n+1+k}{(n+1)(s+1)} \ge 1. \tag{3}$$

Let $s \to \infty$ and $m \to \infty$ in (3), that is, ∞, a are evBB for f for distinct zeros. From Corollary 1.1, we have $\frac{1}{n+1} \ge 1$, which implies $n = 0$. Thus, we can obtain the following corollary.

Corollary 1.2 *Let f be a transcendental meromorphic function of infinite order $\rho(r)$ on the whole complex plane, $\arg z = \theta(0 \le \theta < 2\pi)$ be one Borel direction of $\rho(r)$ order of function f and $\Omega := \Omega(\theta - \varepsilon, \theta + \varepsilon)$ for any $\varepsilon(0 < \varepsilon < \pi)$. If ∞, a are evBB for f for distinct zeros. Then, for all positive integers k and n, we have $\overline{\rho}_\theta^n(b, f^{(k)}) = 1$ for all $b \neq 0, \infty$.*

2 Some Lemmas

To prove our results, we need the following Lemmas.

Lemma 2.1 *(see [6, 16]). Let f be a nonconstant meromorphic function on $\Omega(\alpha, \beta)$. Then for arbitrary complex number a, we have*

$$S_{\alpha,\beta}\left(r, \frac{1}{f-a}\right) = S_{\alpha,\beta}(r, f) + \varepsilon(r, a),$$

where $\varepsilon(r, a) = O(1)$ as $r \to \infty$.

Lemma 2.2 *(see [5, 6, 25]). Suppose that f is a non-constant meromorphic function in one angular domain $\Omega(\alpha, \beta)$ with $0 < \beta - \alpha \le 2\pi$, then for arbitrary q distinct $a_j \in \widehat{\mathbb{C}}(1 \le j \le q)$, we have*

$$\sum_{i=}^{q} D_{\alpha,\beta}(r, \frac{1}{f-a_j}) \le 2S_{\alpha,\beta}(r, f) - C_1(r) + Q_{\alpha,\beta}(r, f),$$

where $C_1(r) = 2C_{\alpha,\beta}(r, f) - C_{\alpha,\beta}(r, f') + C_{\alpha,\beta}(r, \frac{1}{f'})$ and

$$(q-2)S_{\alpha,\beta}(r, f) \le \sum_{j=1}^{q} \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-a_j}\right) + Q_{\alpha,\beta}(r, f),$$

where the term $\overline{C}_{\alpha,\beta}(r, \frac{1}{f-a_j})$ will be replaced by $\overline{C}_{\alpha,\beta}(r, f)$ when some $a_j = \infty$ and

$$\begin{aligned} Q_{\alpha,\beta}(r, f) =& A_{\alpha,\beta}\left(r, \frac{f'}{f}\right) + B_{\alpha,\beta}\left(r, \frac{f'}{f}\right) \\ &+ \sum_{j=1}^{q}\left\{A_{\alpha,\beta}\left(r, \frac{f'}{f-a_j}\right) + B_{\alpha,\beta}\left(r, \frac{f'}{f-a_j}\right)\right\} + O(1). \end{aligned} \tag{4}$$

Lemma 2.3 *(see [6, P138].) Let f be a nonconstant meromorphic function in the whole complex plane $\mathbb{C}$. Given one angular domain on $\Omega(\alpha, \beta)$. Then for any $1 \le r < R$, we have*

$$A_{\alpha,\beta}\left(r, \frac{f'}{f}\right) \le K\left\{\left(\frac{R}{r}\right)^{\omega}\int_1^R \frac{\log^+ T(r, f)}{t^{1+\omega}}dt + \log^+\frac{r}{R-r} + \log\frac{R}{r} + 1\right\},$$

and

$$B_{\alpha,\beta}\left(r,\frac{f'}{f}\right)\leq\frac{4\omega}{r^{\omega}}m\left(r,\frac{f'}{f}\right),$$

where $\omega=\frac{\pi}{\beta-\alpha}$ *and* K *is a positive constant not depending on* r *and* R.

Remark 2.1 *Nevanlinna conjectured that*

$$D_{\alpha,\beta}\left(r,\frac{f'}{f}\right)=A_{\alpha,\beta}\left(r,\frac{f'}{f}\right)+B_{\alpha,\beta}\left(r,\frac{f'}{f}\right)=o\left(S_{\alpha,\beta}\left(r,f\right)\right)\tag{5}$$

when r *tends to* $+\infty$ *outside an exceptional set of finite linear measure, and he proved that* $A_{\alpha,\beta}\left(r,\frac{f'}{f}\right)+B_{\alpha,\beta}\left(r,\frac{f'}{f}\right)=O(1)$ *when the function* f *is meromorphic in* $\mathbb{C}$ *and has finite order. In 1974, Gol'dberg[5] constructed a counter-example to show that (3) is not valid.*

Lemma 2.4 *(see [22, Lemma 4]). Let* f *be a meromorphic function in* $\mathbb{C}$, $\Omega(\alpha,\beta)$ $(0<\beta-\alpha\leq 2\pi)$ *be a closed angular domain, then*

$$Q_{\alpha,\beta}\left(r,f\right)=\begin{cases}O(1), & f\ is\ of\ finite\ order,\\ O(\log U(r)), & f\ is\ of\ infinite\ order,\end{cases}$$

where $Q_{\alpha,\beta}\left(r,f\right)$ *is stated as in (4),* $U(r)=r^{\rho(r)}$, $\rho(r)$ *is the precise order of* $T(r,f)$ *when* f *is of infinite order,* E *is a set of finite linear measure.*

Lemma 2.5 *(see [22, Lemma 5]). Let* f *be a meromorphic function on a closed angular domain* $\Omega(\alpha,\beta)$ *and* $\omega=\frac{\pi}{\beta-\alpha}$, *then for any* $a\in\widehat{\mathbb{C}}$ *and for any* $\varepsilon\in(0,\frac{\beta-\alpha}{2})$,

$$\begin{aligned}
&C_{\alpha,\beta}(r,a,f)\geq 2\omega\sin(\omega\varepsilon)\int_1^r\frac{n(t,\Omega_{\varepsilon},f=a)}{t^{\omega+1}}dt+O(1),\\
&C_{\alpha,\beta}(r,a,f)\geq\frac{4\omega\sin(\omega\varepsilon)}{r^{\omega}}N(r,\Omega_{\varepsilon},f=a)+o(1),\\
&C_{\alpha,\beta}(r,a,f)\leq 4\omega\int_1^r\frac{n(t,\Omega,f=a)}{t^{\omega+1}}dt,\\
&C_{\alpha,\beta}(r,a,f)\leq 2n(r,\Omega,f=a),
\end{aligned}$$

where $\Omega_{\varepsilon}=(\alpha+\varepsilon,\beta-\varepsilon)$.

Remark 2.2 *For the reduced case that each multiple zero of* $f-a$ *in* $\Omega(\alpha,\beta)$ *is counted only once (ignoring multiplicities), Lemma 2.5 still holds, and its proof is similar to the case counting multiplicities.*

Lemma 2.6 *(see [3]). Let* f *be a meromorphic function of infinite order* $\rho(r)$. *Then the ray* $\arg z=\theta$ *is one Borel direction of* $\rho(r)$ *order of meromorphic function* f *if and only if* f *satisfies the equality*

$$\limsup_{r\to\infty}\frac{\log S_{\theta-\epsilon,\theta+\epsilon}(r,f)}{\rho(r)\log r}=1,$$

for any $\epsilon(0<\epsilon<\frac{\pi}{2})$.

3 Proof of Theorem 1.2

Proof: Since f is a meromorphic function of infinite order $\rho(r)$ and $\arg z=\theta(0\leq\theta<2\pi)$ is one Borel direction of $\rho(r)$ order of meromorphic function f, by Lemma 2.6, we can get for any $\varepsilon(0<\varepsilon<\pi)$

$$\limsup_{r\to\infty}\frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)}{\rho(r)\log r}=1.\tag{6}$$

By Lemmas 2.2-2.4, we have

$$D_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{f'}{f})=O(\log U(r)).$$

By Ref. [6, 25], we can get $Q_{\theta-\varepsilon,\theta+\varepsilon}(r,^{(k)})=Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f)$. Thus, we have

$$D_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{f^{(k)}}{f})=O(\log U(r))=Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f). \tag{7}$$

Hence, for any positive integer k, we have

$$\sum_{j=1}^{p}D_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f)\leq D_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f^{(k)})+Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f). \tag{8}$$

By Lemma 2.1, we have

$$S_{\theta-\varepsilon,\theta+\varepsilon}(r,f^{(k)})=D_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f^{(k)})+C_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f^{(k)})+O(1). \tag{9}$$

Then, it follows from (8) and (9) that

$$\sum_{j=1}^{p}D_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f)\leq S_{\theta-\varepsilon,\theta+\varepsilon}(r,f^{(k)})-C_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f^{(k)})+Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f). \tag{10}$$

From (10), we have

$$\begin{aligned}pS_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\leq& S_{\theta-\varepsilon,\theta+\varepsilon}(r,f^{(k)})+\sum_{j=1}^{p}C_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f)-C_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f^{(k)}) \\ &+Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f).\end{aligned} \tag{11}$$

By applying Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}qS_{\theta-\varepsilon,\theta+\varepsilon}(r,f^{(k)})\leq&\sum_{j=1}^{q}C_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f^{(k)})+C_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f^{(k)})\\&+C_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f^{(k)})-C_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f^{(k+1)})\\&-2C_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f^{(k)})+C_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f^{(k+1)})\\&+Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\\ \leq&\sum_{j=1}^{q}C_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f^{(k)})+C_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f^{(k)})\\&+C_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f^{(k+1)})-C_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f^{(k+1)})\\&-C_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f^{(k)})+Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\\ \leq&\sum_{j=1}^{q}C_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f^{(k)})+C_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f^{(k)})\\&+\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f)-C_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f^{(k+1)})+Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f).\end{aligned} \tag{12}$$

It follows from (11) and (12) that

$$\begin{aligned}pqS_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\leq{}& C_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f)+(q-1)\sum_{j=1}^{p}C_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f)\\&-(q-1)C_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f^{(k)})+\sum_{j=1}^{p}C_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f)\\&+\sum_{l=1}^{q}C_{\theta-\varepsilon,\theta+\varepsilon}(r,b_l,f^{(k)})-C_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f^{(k+1)})\\&+Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f).\end{aligned}\tag{13}$$

If z_0 is a zero of $f-a$ of order $j>k$ in $\Omega(\theta-\varepsilon,\theta+\varepsilon)$, then z_0 is a zero of $f^{(k+1)}$ of order $j-(k+1)$ in $\Omega(\theta-\varepsilon,\theta+\varepsilon)$, and if z_0 is a zero of $f^{(k)}-b$ of order m in $\Omega(\theta-\varepsilon,\theta+\varepsilon)$, then z_0 is a zero of $f^{(k+1)}$ of order $m-1$ in $\Omega(\theta-\varepsilon,\theta+\varepsilon)$. Moreover, If z_0 is the zero of $f-a$ of order $>k$ and also zero of $f^{(k)}$ in $\Omega(\theta-\varepsilon,\theta+\varepsilon)$, then z_0 is not zero of $f^{(k)}-b$ in $\Omega(\theta-\varepsilon,\theta+\varepsilon)$ as $b\neq 0$. Thus, we have

$$\begin{aligned}&\sum_{j=1}^{p}C_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f)+\sum_{l=1}^{q}C_{\theta-\varepsilon,\theta+\varepsilon}(r,b_l,f^{(k)})-C_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f^{(k+1)})\\&\qquad\leq\sum_{j=1}^{p}C_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f|\leq k+1)+\sum_{l=1}^{q}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,b_l,f^{(k)}),\end{aligned}\tag{14}$$

$$\sum_{j=1}^{p}C_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f)-C_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f^{(k)})\leq\sum_{j=1}^{p}C_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f|\leq k).$$

Substituting (14) to (13), we get

$$\begin{aligned}pqS_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\leq{}&\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f)+(q-1)\sum_{j=1}^{p}C_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f|\leq k)\\&+\sum_{j=1}^{p}C_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f|\leq k+1)+\sum_{l=1}^{q}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,b_l,f^{(k)})\\&+Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f).\end{aligned}\tag{15}$$

For any positive integer k, we have

$$\begin{aligned}C_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f|\leq k)&\leq k\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f)\\&\leq\frac{k}{m_j+1}\left[m_j\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f|\leq m_j)+C_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f)\right]\\&\leq\frac{k}{m_j+1}\left[m_j\overline{C}(r,a_j,f|\leq m_j)+S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\right]+O(1),\end{aligned}\tag{16}$$

and

$$\begin{aligned}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,b_l,f^{(k)})&\leq\frac{1}{n_l+1}\left[n_l\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,b_j,f^{(k)}|\leq n_l)+S_{\theta-\varepsilon,\theta+\varepsilon}(r,f^{(k)})\right]+O(1)\\\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f)&\leq\frac{1}{s+1}\left[s\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f|\leq s)+S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\right],\end{aligned}\tag{17}$$

and since $S_{\theta-\varepsilon,\theta+\varepsilon}(r,f^{(k)}) \leq S_{\theta-\varepsilon,\theta+\varepsilon}(r,f) + k\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f) + Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f)$, then it follows from (15)-(17) that

$$\begin{aligned}
pqS_{\theta-\varepsilon,\theta+\varepsilon}(r,f) \leq\ & (q-1)\sum_{j=1}^{p}\frac{k}{m_j+1}\left[m_j\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f|\leq m_j) + S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\right]\\
& + \sum_{j=1}^{p}\frac{k+1}{m_j+1}\left[m_j\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f|\leq m_j) + S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\right]\\
& + \sum_{l=1}^{q}\frac{1}{n_l+1}\left[n_l\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,b_l,f^{(k)}|\leq n_l) + S_{\theta-\varepsilon,\theta+\varepsilon}(r,f^{(k)})\right]\\
& + \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f) + Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\\
\leq\ & (q-1)\sum_{j=1}^{p}\frac{km_j}{m_j+1}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f|\leq m_j)\\
& + \sum_{j=1}^{p}\frac{m_j(k+1)}{m_j+1}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f|\leq m_j)\\
& + \sum_{l=1}^{q}\frac{n_l}{n_l+1}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,b_l,f^{(k)}|\leq n_l)\\
& + (1+\sum_{l=1}^{q}\frac{k}{n_l+1})\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f)\\
& + \left(\sum_{j=1}^{p}\frac{kq+1}{m_j+1} + \sum_{j=1}^{q}\frac{1}{n_l+1}\right)S_{\theta-\varepsilon,\theta+\varepsilon}(r,f) + Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\\
\leq\ & (kq+1)\sum_{j=1}^{p}\frac{km_j}{m_j+1}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f|\leq m_j)\\
& + \sum_{l=1}^{q}\frac{n_l}{n_l+1}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,b_l,f^{(k)}|\leq n_l)\\
& + \left(1+\sum_{l=1}^{q}\frac{k}{n_l+1}\right)\frac{s}{s+1}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f|\leq s)
\end{aligned}$$

$$\begin{aligned}
& + \left(\sum_{j=1}^{p}\frac{kq+1}{m_j+1} + \sum_{j=1}^{q}\frac{1}{n_l+1} + (1+\sum_{l=1}^{q}\frac{k}{n_l+1})\frac{1}{s+1}\right)\\
& \times S_{\theta-\varepsilon,\theta+\varepsilon}(r,f) + Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f).
\end{aligned}$$

Since m_j, n_l, k, p, q and s are positive integers, it follows from the above inequality that

$$\begin{aligned} pqS_{\theta-\varepsilon,\theta+\varepsilon}(r,f) \leq & \left(\sum_{j=1}^{p}\frac{kq+1}{m_j+1}+\sum_{j=1}^{q}\frac{1}{n_l+1}+(1+\sum_{l=1}^{q}\frac{k}{n_l+1})\frac{1}{s+1}\right)S_{\theta-\varepsilon,\theta+\varepsilon}(r,f) \\ & +(kq+1)\sum_{j=1}^{p}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f|\leq m_j) \\ & +\sum_{l=1}^{q}\frac{n_l}{n_l+1}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,b_l,f^{(k)}|\leq n_l) \\ & +\left(1+\sum_{l=1}^{q}\frac{k}{n_l+1}\right)\frac{s}{s+1}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f|\leq s) \\ & +Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f). \end{aligned} \tag{18}$$

Thus, from (18), we can prove (2) easily.

Therefore, this completes the proof of Theorem 1.2.

□

4 The proof of Theorem 1.3

Proof: Since f is a meromorphic function of infinite order $\rho(r)$ and $\arg z=\theta(0\leq\theta<2\pi)$ is one Borel direction of $\rho(r)$ order of meromorphic function f, by Lemma 2.6, we can get for any $\varepsilon(0<\varepsilon<\pi)$

$$\limsup_{r\to\infty}\frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)}{\rho(r)\log r}=1. \tag{19}$$

Since ∞ is an evBB for f for distinct poles of order $\leq s$, and $a_j(j=1,2,\ldots,p)$ are evBB for f for distinct zeros of order $\leq m_j$, and $b_l(\neq 0)(l=1,2,\ldots,q)$ are evBB for $f^{(k)}$ for distinct zeros of order $\leq n_l$, from Definition 1.3 and (19), we have that there exists a number $\eta(0<\eta<1)$ such that for sufficiently large r,

$$\overline{C}(r,\infty,f|\leq s)\leq (U(r))^{\eta}, \tag{20}$$

$$\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f|\leq m_j)<(U(r))^{\eta}, j=1,2,\ldots,p, \tag{21}$$

$$\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,b_l,f^{(k)}|\leq n_l)<(U(r))^{\eta}, l=1,2,\ldots,q. \tag{22}$$

Set

$$\Lambda:=\sum_{j=1}^{p}\frac{kq+1}{m_j+1}+\sum_{j=1}^{q}\frac{1}{n_l+1}+(1+\sum_{l=1}^{q}\frac{k}{n_l+1})\frac{1}{s+1}.$$

From Theorem 1.2, we have

$$\begin{aligned} (pq-\Lambda)S_{\theta-\varepsilon,\theta+\varepsilon}(r,f) \leq & (kq+1)\sum_{j=1}^{p}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f|\leq m_j) \\ & +\sum_{l=1}^{q}\frac{n_l}{n_l+1}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,b_l,f^{(k)}|\leq n_l) \\ & +\left(1+\sum_{l=1}^{q}\frac{k}{n_l+1}\right)\frac{s}{s+1}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f|\leq s) \\ & +Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f). \end{aligned} \tag{23}$$

From (20)-(23), for sufficiently large r, it follows that

$$(pq-\Lambda)S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\leq O((U(r))^{\eta})+Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f). \tag{24}$$

Since $\eta<1$, from (19) and (24) for sufficiently large r, we can get $pq-\Lambda\leq 0$, that is,

$$\sum_{j=1}^{p}\frac{kq+1}{m_j+1}+\sum_{l=1}^{q}\frac{1}{n_l+1}+\frac{1}{s+1}\left(1+k\sum_{l=1}^{q}\frac{1}{n_l+1}\right)\geq pq.$$

Thus, this completes the proof of Theorem 1.3.

□

5 Remarks

From the procedure of proofs of Theorems 1.1 and 1.2, we find that the conclusions of Theorems 1.1 and 1.2 can still hold for transcendental meromorphic function f with finite order $\rho(0<\rho<\infty)$ on the whole complex plane.

Thus, it is a natural question to ask: Does the conclusion of Theorem 1.3 still holds when f is a transcendental meromorphic function with finite order $\rho(0<\rho<\infty)$ on the whole complex plane?

In fact, we can not give a positive answer to the above question. Now, we will give a simple procedure to prove this assertion as follows.

Firstly, similar to Definitions 1.3 and 1.4, we can get some definitions of exception values of meromorphic function with finite order in the Borel direction, if $\overline{\rho}_{\theta}^{k}(a,f)<\rho,\overline{\rho}_{\theta}(a,f)<\rho,\overline{\rho}_{\theta}^{k}(a,f^{(l)})<\rho$, when $\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)$ is replaced by $\log r$. Thus, from the definition of Borel direction, (20)-(23) can be replaced by

$$\overline{C}(r,\infty,f|\leq s)\leq r^{\eta'}, \tag{25}$$

$$\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f|\leq m_j)<r^{\eta'},j=1,2,\ldots,p, \tag{26}$$

$$\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,b_l,f^{(k)}|\leq n_l)<r^{\eta'},l=1,2,\ldots,q. \tag{27}$$

where $\eta'<\rho$ and r is sufficiently large, and (24) can be replaced by

$$(pq-\Lambda)\,S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\leq O\left(r^{\eta'}\right)+Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f). \tag{28}$$

However, by Lemmas 2.1-2.5, we can not be sure to derive a contradiction from (28). Therefore, Theorem 1.3 may not be true when f is of finite order.

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Second order duality for multiobjective optimization problems

Meraj Ali Khan, Falleh R. Al-Solamy

Abstract

In this paper, we first introduce a new class of generalized convex functions, called second order (F, α, ρ, d)-V-convex functions and then discuss appropriate duality results for second order Mangasarian type, Mond-Weir type and general Mond-Weir type multiobjective duals.

Keywords: Multiobjective optimization; Second order generalized convex functions; Weak efficiency; Duality

Mathematics Subject Classification 2000: 90C29, 90C30, 90C46, 49N15

1. Introduction

Optimization theory is one of the most lively and exciting branch in modern mathematics, in which the importance of convexity is well known. But the notion of convexity does no longer suffice for many mathematical models used in decision sciences, economics, management sciences, stochastics, applied mathematics and engineering. Therefore, various generalizations of convex functions have been provided for the validity of results to larger classes of optimization problems. The generalization of convex functions was originally proposed by Hanson [7], which were named as invex functions by Craven [4], and η-convex functions by Kaul and Kaur [10]. In [9], Jeyakumar and Mond introduced V-invexity and its generalization for vector functions. More specifically, Preda [16] introduced the concept of (F, ρ)-convexity, an extension of F-convexity defined by Hanson and Mond [8] and ρ-convexity given by Vial [17]. Recently, Agarwal et al. [1] introduced a new class of generalized V-type I functions for a multiobjective problem and discussed sufficiency and duality results.

Second order duality was first introduced by Mangasarian [11] for a scalar programming problem. Mond [13] reproved second order duality results of Mangasarian [11] under simpler assumptions, and showed that the second order dual has computational advantages over the first order dual. Zhang and Mond [19] extended the class of (F, ρ)-convex functions to second order (F, ρ)-convex functions and discussed duality results for Mangasarian type, Mond-Weir type and general Mond-Weir type multiobjective duals. Aghezzaf [2] introduced new classes of generalized second order (F, ρ)-convexity for vector-valued functions and established various duality results for mixed type vector dual. In [6], Hachimi and Aghezzaf proposed a new class of generalized second order type I vector-valued functions for multiobjective programming problem and obtained mixed type duality theorems. Ahmad and Husain [3]

defined a class of generalized second order (F, α, ρ, d)-convex functions and established duality results for Mond-Weir type multiobjective dual. Gulati and Agarwal [5] established Huard type converse duality theorems for second-order scalar and multiobjective dual problems showing certain inconsistencies in the earlier work of Yang et al. [18] and Mond and Zhang [15].

Being inspired by the excellent work of Mond and Zhang [15], Zhang and Mond [19] and Ahmad and Husain [3], we introduce the concept of second order $(\mathcal{F}, \alpha, \rho, d)$-V-convex function and its generalizations, which includes most of the introduced classes of generalized convex functions. To characterize the introduced definitions, an example of second order (F, α, ρ, d)-V-convex function is given. Weak, strong and strict converse duality theorems are proved for second order Mangasarian type, Mond-Weir type and general Mond-Weir type multiobjective duals. These results extend the results appeared in [3, 15, 16, 19].

2. Notations and preliminaries

The following conventions for vectors in $\mathbb{R}^n$ will be followed: $x \geqq y \Leftrightarrow x_i \geqq y_i,\ i = 1, 2, \ldots, n$; $x \geq y \Leftrightarrow x \geqq y$, and there exists at least one i such that $x_i > y_i$; $x > y \Leftrightarrow x_i > y_i,\ i = 1, 2, \ldots, n$. The index sets are $K = \{1, 2, \ldots, k\}$ and $M = \{1, 2, \ldots, m\}$.

Consider the following nonlinear multiobjective programming problem:

(P) Minimize $f(x) = [f_1(x), f_2(x), \ldots, f_k(x)]$

subject to $x \in S = \{x \in X : \ g(x) \leqq 0\}$,

where $X \subseteq \mathbb{R}^n$ is a nonempty open set and the functions $f = (f_1, f_2, \ldots, f_k) : X \to \mathbb{R}^k$ and $g = (g_1, g_2, \ldots, g_m) : X \to \mathbb{R}^m$ are twice differentiable at $\bar{x} \in X$.

Definition 1. A point $\bar{x} \in S$ is said to be a weakly efficient solution of (P), if there exists no other $x \in S$ such that

$$f(x) < f(\bar{x}).$$

The following definitions are due to Mond and Zhang [15]:

Definition 2. Function $f : X \to \mathbb{R}^k$ is said to be second order V-invex at $\bar{x} \in X$, if there exist functions $\eta : X \times X \to \mathbb{R}^n$ and $\alpha_i : X \times X \to \mathbb{R}_+ \setminus \{0\},\ i \in K$ such that

$$f_i(x) - f_i(\bar{x}) + \frac{1}{2}p^T \nabla^2 f_i(\bar{x})p \geqq \alpha_i(x, \bar{x})[\nabla f_i(\bar{x}) + \nabla^2 f_i(\bar{x})p]\eta(x, \bar{x})$$

holds for all $p \in \mathbb{R}^n$ and for all $x \in X$.

Definition 3. Function $f : X \to \mathbb{R}^k$ is said to be second order V-quasiinvex at $\bar{x} \in X$, if there exist functions $\eta : X \times X \to \mathbb{R}^n$ and $\gamma_i : X \times X \to \mathbb{R}_+ \setminus \{0\},\ i \in K$ such that

$$\sum_{i=1}^{k} \gamma_i(x,\bar{x}) f_i(x) \leqq \sum_{i=1}^{k} \gamma_i(x,\bar{x})[f_i(\bar{x}) - \tfrac{1}{2} p^T \nabla^2 f_i(\bar{x}) p]$$
$$\Rightarrow \quad \sum_{i=1}^{k} [\nabla f_i(\bar{x}) + \nabla^2 f_i(\bar{x}) p] \eta(x,\bar{x}) \leqq 0$$

holds for all $p \in \mathbb{R}^n$ and for all $x \in X$.

Definition 4. Function $f : X \to \mathbb{R}^k$ is said to be second order V-pseudoinvex at $\bar{x} \in X$, if there exist functions $\eta : X \times X \to \mathbb{R}^n$ and $\beta_i : X \times X \to \mathbb{R}_+ \setminus \{0\},\ i \in K$ such that

$$\sum_{i=1}^{k} [\nabla f_i(\bar{x}) + \nabla^2 f_i(\bar{x}) p] \eta(x,\bar{x}) \geqq 0 \Rightarrow \sum_{i=1}^{k} \beta_i(x,\bar{x}) f_i(x) \geqq \sum_{i=1}^{k} \beta_i(x,\bar{x})[f_i(\bar{x}) - \tfrac{1}{2} p^T \nabla^2 f_i(\bar{x}) p]$$

holds for all $p \in \mathbb{R}^n$ and for all $x \in X$.

Definition 5. A functional $F : X \times X \times \mathbb{R}^n \to \mathbb{R}$ is said to be sublinear in its third argument, if for any $x, \bar{x} \in X$,

(i) $F(x,\bar{x}; a_1 + a_2) \leqq F(x,\bar{x}; a_1) + F(x,\bar{x}; a_2) \ \ \forall \ \ a_1, a_2 \in \mathbb{R}^n$,

(ii) $F(x,\bar{x}; \alpha a) = \alpha F(x,\bar{x}; a) \ \ \forall\ \alpha \in R, \alpha \geqq 0$ and $\forall\ a \in \mathbb{R}^n$.

The following definitions of second order (F,ρ)-convexity and its generalization were introduced by Zhang and Mond [19]. Let F be a functional sublinear in its third argument, $\phi : X \to \mathbb{R}$ be twice differentiable at $\bar{x} \in X$, $d : X \times X \to \mathbb{R}$ be a metric and $\rho \in \mathbb{R}$.

Definition 6. Function $\phi : X \to \mathbb{R}$ is said to be second order (F,ρ)-convex at $\bar{x} \in X$, if

$$\phi(x) - \phi(\bar{x}) + \frac{1}{2} p^T \nabla^2 \phi(\bar{x}) p \geqq F(x,\bar{x}; \nabla\phi(\bar{x}) + \nabla^2\phi(\bar{x}) p) + \rho d(x,\bar{x})$$

holds for all $p \in \mathbb{R}^n$ and for all $x \in X$.

Definition 7. Function $\phi : X \to \mathbb{R}$ is said to be second order (F,ρ)-quasiconvex at $\bar{x} \in X$, if

$$\phi(x) \leqq \phi(\bar{x}) - \frac{1}{2} p^T \nabla^2 \phi(\bar{x}) p \Rightarrow F(x,\bar{x}; \nabla\phi(\bar{x}) + \nabla^2\phi(\bar{x}) p) \leqq -\rho d(x,\bar{x})$$

holds for all $p \in \mathbb{R}^n$ and for all $x \in X$.

Definition 8. Function $\phi : X \to \mathbb{R}$ is said to be second order (F, ρ)-pseudoconvex at $\bar{x} \in X$, if

$$F(x, \bar{x}; \nabla\phi(\bar{x}) + \nabla^2\phi(\bar{x})p) \geqq -\rho d(x, \bar{x}) \Rightarrow \phi(x) \geqq \phi(\bar{x}) - \frac{1}{2}p^T\nabla^2\phi(\bar{x})p$$

holds for all $p \in \mathbb{R}^n$ and for all $x \in X$.

Finally, in view of Definitions 2-8 and the concept of generalized second order (F, α, ρ, d)-convex functions [3], we propose our definitions of second order (F, α, ρ, d)-V-convex function and its generalizations as follows:

Definition 9. Function $f : X \to \mathbb{R}^k$ is said to be (strictly) second order (F, α, ρ, d)-V-convex at $\bar{x} \in X$, if there exist functions $\alpha_i : X \times X \to \mathbb{R}_+ \setminus \{0\}, i \in K,\ d : X \times X \to \mathbb{R}$ and $\rho = (\rho_1, \rho_2, \ldots, \rho_k) \in \mathbb{R}^k$ such that

$$f_i(x) - f_i(\bar{x}) + \frac{1}{2}p^T\nabla^2 f_i(\bar{x})p\ (>) \geqq F(x, \bar{x}; \alpha_i(x, \bar{x})(\nabla f_i(\bar{x}) + \nabla^2 f_i(\bar{x})p)) + \rho_i d^2(x, \bar{x})$$

holds for all $p \in \mathbb{R}^n$ and for all $x \in X$.

Remark 1.

(*i*) For $k = 1$ and $\alpha_i(x, \bar{x}) = 1$, the above definition becomes that of (strictly) second order (F, ρ)-convex function introduced by Zhang and Mond [19].

(*ii*) If $\rho_i = 0,\ i \in K$ and $F(x, \bar{x}; a) = a^T\eta(x, \bar{x})$ for a certain mapping $\eta : X \times X \to \mathbb{R}^n$, the inequality reduces to that of (strictly) second order V-invex function introduced by Mond and Zhang [15].

(*iii*) If $\alpha_i(x, \bar{x}) = \alpha(x, \bar{x}),\ i \in K$, then we get the definition of (strictly) second order (F, α, ρ, d)-convex function given by Ahmad and Husain [3].

Following example includes earlierly studied classes as special cases of second order (F, α, ρ, d)-V-convex function.

Example 1. Consider the function $f = (f_1, f_2, f_3) : X \to \mathbb{R}^3$, where $X = \mathbb{R}$ such that

$$f_1(x) = (x + 2)^2, \quad f_2(x) = 2 - x^2, \quad f_3(x) = -x^2 - 2x.$$

The feasible region is $S = \{x \in X : x \geqq 2\}$.

Let $F(x, \bar{x}; a) = \frac{a}{12}(x^2 + \bar{x}^2 - 4);\ \alpha_1(x, \bar{x}) = 2;\ \alpha_2(x, \bar{x}) = 4;\ \alpha_3(x, \bar{x}) = 12;$
$\rho_1 = -1;\ \rho_2 = 1;\ \rho_3 = -1;\ d(x, \bar{x}) = |x - \bar{x} + 2|;\ p = 2;\ \bar{x} = 2.$

It can be seen that $f = (f_1, f_2, f_3)$ is second order (F, α, ρ, d)-V-convex for all $x \in X$,

$$f_1(x) - f_1(\bar{x}) + \tfrac{1}{2}p^T\nabla^2 f_1(\bar{x})p = x^2 + 4x - 8$$

$$\geqq F(x, \bar{x}; \alpha_1(x, \bar{x})(\nabla f_1(\bar{x}) + \nabla^2 f_1(\bar{x})p)) + \rho_1 d^2(x, \bar{x}) = x^2, \qquad (I)$$

$$f_2(x) - f_2(\bar{x}) + \tfrac{1}{2}p^T\nabla^2 f_2(\bar{x})p = -\ x^2$$

$$\geqq F(x, \bar{x}; \alpha_2(x, \bar{x})(\nabla f_2(\bar{x}) + \nabla^2 f_2(\bar{x})p)) + \rho_2 d^2(x, \bar{x}) = -\frac{5}{3}x^2, \qquad (II)$$

$$f_3(x) - f_3(\bar{x}) + \tfrac{1}{2}p^T\nabla^2 f_3(\bar{x})p = -x^2 - 2x + 4$$

$$\geqq F(x, \bar{x}; \alpha_3(x, \bar{x})(\nabla f_3(\bar{x}) + \nabla^2 f_3(\bar{x})p)) + \rho_3 d^2(x, \bar{x}) = -11x^2. \qquad (III)$$

The above inequalities show that $f = (f_1, f_2, f_3)$ is second order (F, α, ρ, d)-V-convex for all $p \in R$ at $\bar{x}$.

If $\alpha_1(x, \bar{x}) = \alpha_2(x, \bar{x}) = \alpha_3(x, \bar{x}) = 2$, then Inequality (II) does not hold. If $\alpha_1(x, \bar{x}) = \alpha_2(x, \bar{x}) = \alpha_3(x, \bar{x}) = 4$, then Inequality (I) is not satisfied. Similarly, if $\alpha_1(x, \bar{x}) = \alpha_2(x, \bar{x}) = \alpha_3(x, \bar{x}) = 12$, then Inequality (I) is not satisfied. Hence, $f = (f_1, f_2, f_3)$ is not second order (F, α, ρ, d)-convex [3] for all $p \in \mathbb{R}$ at $\bar{x}$.

Let $\alpha_1(x, \bar{x}) = \alpha_2(x, \bar{x}) = \alpha_3(x, \bar{x}) = 1$. Then Inequality (II) does not hold. Therefore, $f = (f_1, f_2, f_3)$ is not second order (F, ρ)-convex [19] for all $p \in \mathbb{R}$ at $\bar{x}$.

Let $\rho_1 = \rho_2 = \rho_3 = 0$. Then Inequalities (I) and (II) are not satisfied. Hence $f = (f_1, f_2, f_3)$ is not second order V-invex [15] for all $p \in \mathbb{R}$ at $\bar{x}$.

Definition 10. Function $f : X \to \mathbb{R}^k$ is said to be (strictly) second order $(F, \tilde{\alpha}, \tilde{\rho}, d)$-V-quasiconvex at $\bar{x} \in X$, if there exist functions $\tilde{\alpha}_i : X \times X \to \mathbb{R}_+ \setminus \{0\}, i \in K,\ d : X \times X \to \mathbb{R}$ and $\tilde{\rho} \in \mathbb{R}$ such that

$$\sum_{i=1}^{k} \tilde{\alpha}_i(x, \bar{x}) f_i(x) \leqq \sum_{i=1}^{k} \tilde{\alpha}_i(x, \bar{x}) f_i(\bar{x}) - \frac{1}{2}p^T\nabla^2 \sum_{i=1}^{k} \tilde{\alpha}_i(x, \bar{x}) f_i(\bar{x})p$$

$$\Rightarrow F(x, \bar{x}; \sum_{i=1}^{k}(\nabla f_i(\bar{x}) + \nabla^2 f_i(\bar{x})p)) + \tilde{\rho} d^2(x, \bar{x})\ (<) \leqq\ 0$$

holds for all $p \in \mathbb{R}^n$ and for all $x \in X$.

Definition 11. Function $f : X \to \mathbb{R}^k$ is said to be (strictly) second order $(F, \bar{\alpha}, \bar{\rho}, d)$-V-pseudoconvex at $\bar{x} \in X$, if there exist functions $\bar{\alpha}_i : X \times X \to \mathbb{R}_+ \setminus \{0\}, i \in K,\ d : X \times X \to \mathbb{R}$ and $\bar{\rho} \in \mathbb{R}$ such that

$$F(x, \bar{x}; \sum_{i=1}^{k}(\nabla f_i(\bar{x}) + \nabla^2 f_i(\bar{x})p)) + \bar{\rho} d^2(x, \bar{x}) \geqq 0$$

$$\Rightarrow \sum_{i=1}^{k} \bar{\alpha}_i(x,\bar{x}) f_i(x)\ (>) \geqq \sum_{i=1}^{k} \bar{\alpha}_i(x,\bar{x}) f_i(\bar{x}) - \tfrac{1}{2} p^T \nabla^2 \sum_{i=1}^{k} \bar{\alpha}_i(x,\bar{x}) f_i(\bar{x}) p$$

holds for all $p \in \mathbb{R}^n$ and for all $x \in X$.

Remark 2. By using the sublinearity of F, one can see from the above definitions that a second order (F,α,ρ,d)-V-convex function is both second order $(F,\bar{\alpha},\bar{\rho},d)$-V-pseudoconvex (with $\bar{\alpha}_i = \dfrac{1}{\alpha_i}$, $i \in K$ and $\bar{\rho} = \sum_{i=1}^{k} \dfrac{1}{\alpha_i(x,\bar{x})}\rho_i$) and second order $(F,\tilde{\alpha},\tilde{\rho},d)$-V-quasiconvex (with $\tilde{\alpha}_i = \dfrac{1}{\alpha_i}$, $i \in K$ and $\tilde{\rho} = \sum_{i=1}^{k} \dfrac{1}{\alpha_i(x,\bar{x})}\rho_i$). Obviously, the converse is not necessarily true.

Following Kuhn-Tucker theorem will be needed in the sequel:

Proposition 1 [12]. Let $\bar{x}$ be a weakly efficient solution of (P) at which the Kuhn-Tucker constraint qualification is satisfied. Then there exist $\lambda \in \mathbb{R}^k$ and $u \in \mathbb{R}^m$ such that

$$\sum_{i=1}^{k} \nabla \lambda_i f_i(\bar{x}) + \sum_{j=1}^{m} \nabla u_j g_j(\bar{x}) = 0,$$

$$\sum_{j=1}^{m} u_j g_j(\bar{x}) = 0,$$

$$\lambda \geqq 0,\ \sum_{i=1}^{k} \lambda_i = 1,\ u \geqq 0.$$

3. Mangasarian type duality

In this section, we consider the following second order Mangasarian type dual for (P) and discuss duality results.

(SD) Maximize $\big(f_1(y) + u^T g(y) - \tfrac{1}{2} p^T \nabla^2 (f_1(y) + u^T g(y)) p,$

$$\ldots, f_k(y) + u^T g(y) - \tfrac{1}{2} p^T \nabla^2 (f_k(y) + u^T g(y)) p\big)$$

subject to

$$\sum_{i=1}^{k} (\nabla \lambda_i f_i(y) + \nabla^2 \lambda_i f_i(y) p) + \sum_{j=1}^{m} (\nabla u_j g_j(y) + \nabla^2 u_j g_j(y) p) = 0, \tag{1}$$

$$\lambda \geqq 0, \tag{2}$$

$$\sum_{i=1}^{k} \lambda_i = 1, \tag{3}$$

$$u \geqq 0. \tag{4}$$

Let Q be the set of all feasible solutions of (SD).

Theorem 1 (Weak duality)**.** Suppose that for all $x \in S$ and $(y, u, \lambda, p) \in Q$,

(i) f is second order (F, α, ρ, d)-V-convex at y, and g is second order $(F, \hat{\alpha}, \hat{\rho}, d)$-V-convex at y;

(ii) $\sum_{i=1}^{k} \dfrac{\lambda_i}{\alpha_i(x,y)} = 1$ and $\hat{\alpha}_j(x,y) = 1,\ j \in M$; and

(iii) $\sum_{i=1}^{k} \dfrac{\lambda_i \rho_i}{\alpha_i(x,y)} + \sum_{j=1}^{m} u_j \hat{\rho}_j \geqq 0.$

Then

$$f_i(x) \nless f_i(y) + \sum_{j=1}^{m} u_j g_j(y) - \frac{1}{2} p^T \nabla^2 (f_i(y) + \sum_{j=1}^{m} u_j g_j(y)) p,\ i \in K. \tag{5}$$

Proof. Suppose contrary to the result that (5) cannot hold, i.e.,

$$f_i(x) < f_i(y) + \sum_{j=1}^{m} u_j g_j(y) - \frac{1}{2} p^T \nabla^2 (f_i(y) + \sum_{j=1}^{m} u_j g_j(y)) p,\ i \in K,$$

which on using (2), (3), $\alpha_i(x,y) > 0,\ i \in K$ and hypothesis (ii) becomes

$$\sum_{i=1}^{k} \frac{\lambda_i f_i(x)}{\alpha_i(x,y)} < \sum_{i=1}^{k} \frac{\lambda_i f_i(y)}{\alpha_i(x,y)} + \sum_{j=1}^{m} u_j g_j(y) - \frac{1}{2} p^T \nabla^2 \left\{ \sum_{i=1}^{k} \frac{\lambda_i f_i(y)}{\alpha_i(x,y)} + \sum_{j=1}^{m} u_j g_j(y) \right\} p,$$

or

$$\sum_{i=1}^{k} \frac{\lambda_i f_i(x)}{\alpha_i(x,y)} - \sum_{i=1}^{k} \frac{\lambda_i f_i(y)}{\alpha_i(x,y)} - \sum_{j=1}^{m} u_j g_j(y) + \frac{1}{2} p^T \nabla^2 \left\{ \sum_{i=1}^{k} \frac{\lambda_i f_i(y)}{\alpha_i(x,y)} + \sum_{j=1}^{m} u_j g_j(y) \right\} p < 0. \tag{6}$$

According to hypothesis (i), it follows that

$$f_i(x) - f_i(y) + \frac{1}{2} p^T \nabla^2 f_i(y) p \geqq F(x, y; \alpha_i(x,y)(\nabla f_i(y) + \nabla^2 f_i(y) p)) + \rho_i d^2(x,y)$$

and

$$g_j(x) - g_j(y) + \frac{1}{2} p^T \nabla^2 g_j(y) p \geqq F(x, y; \hat{\alpha}_j(x,y)(\nabla g_j(y) + \nabla^2 g_j(y) p)) + \hat{\rho}_j d^2(x,y).$$

On multiplying the first inequality by $\dfrac{\lambda_i}{\alpha_i(x,y)} \geqq 0,\ i \in K$ and second by $u_j \geqq 0$, with $\hat{\alpha}_j(x,y) = 1,\ j \in M$, then summing over i and j respectively, and on using the sublinearity of F, we have

$$\sum_{i=1}^{k} \frac{\lambda_i f_i(x)}{\alpha_i(x,y)} + \sum_{j=1}^{m} u_j g_j(x) - \sum_{i=1}^{k} \frac{\lambda_i f_i(y)}{\alpha_i(x,y)} - \sum_{j=1}^{m} u_j g_j(y)$$

$$+ \frac{1}{2} p^T \nabla^2 \left\{ \sum_{i=1}^{k} \frac{\lambda_i f_i(y)}{\alpha_i(x,y)} + \sum_{j=1}^{m} u_j g_j(y) \right\} p$$

$$\geqq F\left(x, y; \sum_{i=1}^{k} (\nabla \lambda_i f_i(y) + \nabla^2 \lambda_i f_i(y) p)\right) + F\left(x, y; \sum_{j=1}^{m} (\nabla u_j g_j(y) + \nabla^2 u_j g_j(y) p)\right)$$

$$+ \sum_{i=1}^{k} \frac{\lambda_i \rho_i d^2(x,y)}{\alpha_i(x,y)} + \sum_{j=1}^{m} u_j \hat{\rho}_j d^2(x,y). \quad (7)$$

The relations (1), (7) and the sublinearity of F yield

$$\sum_{i=1}^{k} \frac{\lambda_i f_i(x)}{\alpha_i(x,y)} + \sum_{j=1}^{m} u_j g_j(x) - \sum_{i=1}^{k} \frac{\lambda_i f_i(y)}{\alpha_i(x,y)} - \sum_{j=1}^{m} u_j g_j(y)$$

$$+ \frac{1}{2} p^T \nabla^2 \left\{ \sum_{i=1}^{k} \frac{\lambda_i f_i(y)}{\alpha_i(x,y)} + \sum_{j=1}^{m} u_j g_j(y) \right\} p$$

$$\geqq \left(\sum_{i=1}^{k} \frac{\lambda_i \rho_i}{\alpha_i(x,y)} + \sum_{j=1}^{m} u_j \hat{\rho}_j \right) d^2(x,y),$$

which by virtue of hypothesis (iii) gives

$$\sum_{i=1}^{k} \frac{\lambda_i f_i(x)}{\alpha_i(x,y)} + \sum_{j=1}^{m} u_j g_j(x) - \sum_{i=1}^{k} \frac{\lambda_i f_i(y)}{\alpha_i(x,y)} - \sum_{j=1}^{m} u_j g_j(y)$$

$$+ \frac{1}{2} p^T \nabla^2 \left\{ \sum_{i=1}^{k} \frac{\lambda_i f_i(y)}{\alpha_i(x,y)} + \sum_{j=1}^{m} u_j g_j(y) \right\} p \geqq 0.$$

By $u \geqq 0$ and $g(x) \leqq 0$, it follows that

$$\sum_{i=1}^{k} \frac{\lambda_i f_i(x)}{\alpha_i(x,y)} - \sum_{i=1}^{k} \frac{\lambda_i f_i(y)}{\alpha_i(x,y)} - \sum_{j=1}^{m} u_j g_j(y) + \frac{1}{2} p^T \nabla^2 \left\{ \sum_{i=1}^{k} \frac{\lambda_i f_i(y)}{\alpha_i(x,y)} + \sum_{j=1}^{m} u_j g_j(y) \right\} p \geqq 0,$$

a contradiction to (6). This completes the proof. □

Theorem 2 (Strong duality)**.** Let $\bar{x}$ be a weakly efficient solution of (P) at which the Kuhn-Tucker constraint qualification is satisfied. Then there exist $\bar{\lambda} \in \mathbb{R}^k$, $\bar{u} \in \mathbb{R}^m$ and $\bar{p} \in \mathbb{R}^n$ such that $(\bar{x}, \bar{u}, \bar{\lambda}, \bar{p} = 0) \in Q$ and the corresponding objective values of

(P) and (SD) are equal. If, in addition, the hypotheses of weak duality (Theorem 1) hold, then $(\bar{x}, \bar{u}, \bar{\lambda}, \bar{p} = 0)$ is a weakly efficient solution of (SD).

Proof. Since $\bar{x}$ is a weakly efficient solution of (P) at which the Kuhn-Tucker constraint qualification is satisfied, from Proposition 1, there exist $\bar{\lambda} \in \mathbb{R}^k$ and $\bar{u} \in \mathbb{R}^m$ such that

$$\sum_{i=1}^{k} \nabla \bar{\lambda}_i f_i(\bar{x}) + \sum_{j=1}^{m} \nabla \bar{u}_j g_j(\bar{x}) = 0,$$

$$\sum_{j=1}^{m} \bar{u}_j g_j(\bar{x}) = 0,$$

$$\bar{\lambda} \geqq 0, \ \sum_{i=1}^{k} \bar{\lambda}_i = 1, \ \bar{u} \geqq 0.$$

Therefore, $(\bar{x}, \bar{u}, \bar{\lambda}, \bar{p} = 0) \in Q$ and the corresponding objective values of (P) and (SD) are equal. Weak efficiency of $(\bar{x}, \bar{u}, \bar{\lambda}, \bar{p} = 0)$ thus follows from weak duality (Theorem 1). □

Theorem 3 (Strict converse duality)**.** Let $\bar{x} \in S$ and $(\bar{y}, \bar{u}, \bar{\lambda}, \bar{p}) \in Q$ such that

(*i*) $\sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{x}) \leqq \sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{y}) + \sum_{j=1}^{m} \bar{u}_j g_j(\bar{y}) - \frac{1}{2}\bar{p}^T \nabla^2 \left\{ \sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{y}) + \sum_{j=1}^{m} \bar{u}_j g_j(\bar{y}) \right\} \bar{p};$

(*ii*) f is strictly second order (F, α, ρ, d)-V-convex at $\bar{y}$ with $\alpha_i(\bar{x}, \bar{y}) = 1, \ i \in K$ and g is second order $(F, \hat{\alpha}, \hat{\rho}, d)$-V-convex at $\bar{y}$ with $\hat{\alpha}_j(\bar{x}, \bar{y}) = 1, \ j \in M$; and

(*iii*) $\sum_{i=1}^{k} \bar{\lambda}_i \rho_i + \sum_{j=1}^{m} \bar{u}_j \hat{\rho}_j \geqq 0.$

Then $\bar{x} = \bar{y}$.

Proof. We assume that $\bar{x} \neq \bar{y}$, and exhibit a contradiction. Using (2)-(4), hypothesis (*ii*), and the sublinearity of F, we obtain

$$\sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{x}) - \sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{y}) + \frac{1}{2}\bar{p}^T \nabla^2 \sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{y}) \bar{p} > F\left(\bar{x}, \bar{y}; \sum_{i=1}^{k} (\nabla \bar{\lambda}_i f_i(\bar{y}) + \nabla^2 \bar{\lambda}_i f_i(\bar{y}) \bar{p})\right) + \sum_{i=1}^{k} \bar{\lambda}_i \rho_i d^2(\bar{x}, \bar{y})$$

and

$$\sum_{j=1}^{m} \bar{u}_j g_j(\bar{x}) - \sum_{j=1}^{m} \bar{u}_j g_j(\bar{y}) + \frac{1}{2}\bar{p}^T \nabla^2 \sum_{j=1}^{m} \bar{u}_j g_j(\bar{y}) \bar{p} \geqq F\left(\bar{x}, \bar{y}; \sum_{j=1}^{m} (\nabla \bar{u}_j g_j(\bar{y}) + \nabla^2 \bar{u}_j g_j(\bar{y}) \bar{p})\right)$$

$$+\sum_{j=1}^{m}\bar{u}_j\hat{\rho}_j d^2(\bar{x},\bar{y}).$$

Adding these inequalities, we get

$$\sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{x})+\sum_{j=1}^{m}\bar{u}_j g_j(\bar{x})-\sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{y})-\sum_{j=1}^{m}\bar{u}_j g_j(\bar{y})+\frac{1}{2}\bar{p}^T\nabla^2\left\{\sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{y})+\sum_{j=1}^{m}\bar{u}_j g_j(\bar{y})\right\}\bar{p}$$

$$> F\left(\bar{x},\bar{y};\sum_{i=1}^{k}(\nabla\bar{\lambda}_i f_i(\bar{y})+\nabla^2\bar{\lambda}_i f_i(\bar{y})\bar{p})\right)+F\left(\bar{x},\bar{y};\sum_{j=1}^{m}(\nabla\bar{u}_j g_j(\bar{y})+\nabla^2\bar{u}_j g_j(\bar{y})\bar{p})\right)$$

$$+\sum_{i=1}^{k}\bar{\lambda}_i\rho_i d^2(\bar{x},\bar{y})+\sum_{j=1}^{m}\bar{u}_j\hat{\rho}_j d^2(\bar{x},\bar{y})$$

$$\geqq F\left(\bar{x},\bar{y};\sum_{i=1}^{k}(\nabla\bar{\lambda}_i f_i(\bar{y})+\nabla^2\bar{\lambda}_i f_i(\bar{y})\bar{p})+\sum_{j=1}^{m}(\nabla\bar{u}_j g_j(\bar{y})+\nabla^2\bar{u}_j g_j(\bar{y})\bar{p})\right)$$

$$+\sum_{i=1}^{k}\bar{\lambda}_i\rho_i d^2(\bar{x},\bar{y})+\sum_{j=1}^{m}\bar{u}_j\hat{\rho}_j d^2(\bar{x},\bar{y})\ \text{(by the sublinearity of } F\text{)},$$

which on using (1) and $F(\bar{x},\bar{y};0)=0$ gives

$$\sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{x})+\sum_{j=1}^{m}\bar{u}_j g_j(\bar{x})-\sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{y})-\sum_{j=1}^{m}\bar{u}_j g_j(\bar{y})+\frac{1}{2}\bar{p}^T\nabla^2\left\{\sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{y})+\sum_{j=1}^{m}\bar{u}_j g_j(\bar{y})\right\}\bar{p}$$

$$>\left(\sum_{i=1}^{k}\bar{\lambda}_i\rho_i+\sum_{j=1}^{m}\bar{u}_j\hat{\rho}_j\right)d^2(\bar{x},\bar{y}).$$

This inequality along with hypothesis (iii), $\bar{u}\geqq 0$ and $g(\bar{x})\leqq 0$ yields

$$\sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{x})-\sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{y})-\sum_{j=1}^{m}\bar{u}_j g_j(\bar{y})+\frac{1}{2}\bar{p}^T\nabla^2\left\{\sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{y})+\sum_{j=1}^{m}\bar{u}_j g_j(\bar{y})\right\}\bar{p}>0,$$

a contradiction to hypothesis (i). Hence, $\bar{x}=\bar{y}$. □

4. Mond-Weir type duality

In this section, we present the following Mond-Weir [13] type dual associated to (P):

(MD) Maximize $(f_1(y)-\frac{1}{2}p^T\nabla^2 f_1(y)p,\ldots,f_k(y)-\frac{1}{2}p^T\nabla^2 f_k(y)p)$

subject to

$$\sum_{i=1}^{k}(\nabla\lambda_i f_i(y)+\nabla^2\lambda_i f_i(y)p)+\sum_{j=1}^{m}(\nabla u_j g_j(y)+\nabla^2 u_j g_j(y)p)=0, \tag{8}$$

$$u_j g_j(y)-\frac{1}{2}p^T\nabla^2 u_j g_j(y)p \geqq 0,\ j\in M, \tag{9}$$

$$\lambda \geqq 0, \tag{10}$$

$$\sum_{i=1}^{k}\lambda_i=1, \tag{11}$$

$$u \geqq 0. \tag{12}$$

Let U be the set of all feasible solutions of (MD). In this section and in Section 5, f^λ denotes the vector $(\lambda_1 f_1, \lambda_2 f_2, \ldots, \lambda_k f_k)$ and g^u denotes the vector $(u_1 g_1, u_2 g_2, \ldots, u_m g_m)$.

Theorem 4 (Weak duality)**.** Suppose that for all $x\in S$ and $(y,u,\lambda,p)\in U$,

(*i*) f^λ is second order $(F,\bar{\alpha},\bar{\rho},d)$-V-pseudoconvex at y, and g^u is second order $(F,\tilde{\alpha},\tilde{\rho},d)$-V-quasiconvex at y; and

(*ii*) $\bar{\rho}+\tilde{\rho}\geqq 0$.

Then

$$f_i(x)\nleq f_i(y)-\frac{1}{2}p^T\nabla^2 f_i(y)p,\ i\in K. \tag{13}$$

Proof. Since $x\in S$ and $(y,u,\lambda,p)\in U$, we have

$$u_j g_j(x)\leqq 0\leqq u_j g_j(y)-\frac{1}{2}p^T\nabla^2 u_j g_j(y)p,\ j\in M.$$

As $\tilde{\alpha}_j(x,y)>0,\ j\in M$, we get

$$\sum_{j=1}^{m}\tilde{\alpha}_j(x,y)u_j g_j(x)\leqq \sum_{j=1}^{m}\tilde{\alpha}_j(x,y)u_j g_j(y)-\frac{1}{2}p^T\nabla^2\sum_{j=1}^{m}\tilde{\alpha}_j(x,y)u_j g_j(y)p.$$

Using second order $(F,\tilde{\alpha},\tilde{\rho},d)$-V-quasiconvexity of g^u at y, we obtain

$$F\left(x,y;\sum_{j=1}^{m}(\nabla u_j g_j(y)+\nabla^2 u_j g_j(y)p)\right)+\tilde{\rho}d^2(x,y)\leqq 0. \tag{14}$$

The equation (8) along with the sublinearity of F gives

$$F\left(x, y; \sum_{i=1}^{k}(\nabla\lambda_i f_i(y) + \nabla^2\lambda_i f_i(y)p)\right) + F\left(x, y; \sum_{j=1}^{m}(\nabla u_j g_j(y) + \nabla^2 u_j g_j(y)p)\right)$$

$$\geqq F\left(x, y; \sum_{i=1}^{k}(\nabla\lambda_i f_i(y) + \nabla^2\lambda_i f_i(y)p) + \sum_{j=1}^{m}(\nabla u_j g_j(y) + \nabla^2 u_j g_j(y)p)\right) = 0. \quad (15)$$

Inequalities (14), (15) and hypothesis (ii) imply

$$F\left(x, y; \sum_{i=1}^{k}(\nabla\lambda_i f_i(y) + \nabla^2\lambda_i f_i(y)p)\right) + \bar{\rho}d^2(x, y) \geqq 0,$$

which by second order $(F, \bar{\alpha}, \bar{\rho}, d)$-V-pseudoconvexity of f^{λ} at y yields

$$\sum_{i=1}^{k}\bar{\alpha}_i(x, y)\lambda_i f_i(x) \geqq \sum_{i=1}^{k}\bar{\alpha}_i(x, y)\lambda_i f_i(y) - \frac{1}{2}p^T\nabla^2\sum_{i=1}^{k}\bar{\alpha}_i(x, y)\lambda_i f_i(y)p. \quad (16)$$

Now suppose contrary to (13), i.e.,

$$f_i(x) < f_i(y) - \frac{1}{2}p^T\nabla^2 f_i(y)p, \ i \in K.$$

Using $\lambda \geqq 0$, $\sum_{i=1}^{k}\lambda_i = 1$, and $\bar{\alpha}_i(x, y) > 0$, $i \in K$, we get

$$\sum_{i=1}^{k}\bar{\alpha}_i(x, y)\lambda_i f_i(x) < \sum_{i=1}^{k}\bar{\alpha}_i(x, y)\lambda_i f_i(y) - \frac{1}{2}p^T\nabla^2\sum_{i=1}^{k}\bar{\alpha}_i(x, y)\lambda_i f_i(y)p,$$

a contradiction to (16). Hence the theorem. □

The proof of the following weak duality theorem is similar to that of Theorem 4, and hence is omitted.

Theorem 5 (Weak duality)**.** Suppose that for all $x \in S$ and $(y, u, \lambda, p) \in U$,

(i) f^{λ} is second order $(F, \tilde{\alpha}, \tilde{\rho}, d)$-V-quasiconvex at y, and g^u is strictly second order $(F, \bar{\alpha}, \bar{\rho}, d)$-V-pseudoconvex at y; and

(ii) $\tilde{\rho} + \bar{\rho} \geqq 0$.

Then

$$f_i(x) \nless f_i(y) - \frac{1}{2}p^T\nabla^2 f_i(y)p, \ i \in K.$$

Since the proof of the strong duality theorem follows on the similar lines of Theorem 2, we just state the theorem but omit the details of the proof.

Theorem 6 (Strong duality). Let $\bar{x}$ be a weakly efficient solution of (P) at which the Kuhn-Tucker constraint qualification is satisfied. Then there exist $\bar{\lambda} \in \mathbb{R}^k$, $\bar{u} \in \mathbb{R}^m$ and $\bar{p} \in \mathbb{R}^n$ such that $(\bar{x}, \bar{u}, \bar{\lambda}, \bar{p} = 0) \in U$ and the corresponding objective values of (P) and (MD) are equal. If, in addition, the hypotheses of weak duality (Theorem 4 or 5) hold, then $(\bar{x}, \bar{u}, \bar{\lambda}, \bar{p} = 0)$ is a weakly efficient solution of (MD).

Theorem 7 (Strict converse duality). Let $\bar{x} \in S$ and $(\bar{y}, \bar{u}, \bar{\lambda}, \bar{p}) \in U$ such that

(i) $\sum\limits_{i=1}^{k} \bar{\lambda}_i f_i(\bar{x}) \leqq \sum\limits_{i=1}^{k} \bar{\lambda}_i f_i(\bar{y}) - \frac{1}{2}\bar{p}^T \nabla^2 \sum\limits_{i=1}^{k} \bar{\lambda}_i f_i(\bar{y})\bar{p}$;

(ii) $f^{\bar{\lambda}}$ is strictly second order $(F, \bar{\alpha}, \bar{\rho}, d)$-V-pseudoconvex at $\bar{y}$ with $\bar{\alpha}_i(\bar{x}, \bar{y}) = \delta(\bar{x}, \bar{y}),\ i \in K$ and $g^{\bar{u}}$ is second order $(F, \tilde{\alpha}, \tilde{\rho}, d)$-V-quasiconvex at $\bar{y}$; and

(iii) $\bar{\rho} + \tilde{\rho} \geqq 0$.

Then $\bar{x} = \bar{y}$.

Proof. We assume that $\bar{x} \neq \bar{y}$ and exhibit a contradiction. Since $\bar{x} \in S$ and $(\bar{y}, \bar{u}, \bar{\lambda}, \bar{p}) \in U$, we have

$$\bar{u}_j g_j(\bar{x}) \leqq 0 \leqq \bar{u}_j g_j(\bar{y}) - \frac{1}{2}\bar{p}^T \nabla^2 \bar{u}_j g_j(\bar{y})\bar{p},\ j \in M.$$

By $\tilde{\alpha}_j(\bar{x}, \bar{y}) > 0,\ j \in M$, it follows that

$$\sum_{j=1}^{m} \tilde{\alpha}_j(\bar{x}, \bar{y})\bar{u}_j g_j(\bar{x}) \leqq \sum_{j=1}^{m} \tilde{\alpha}_j(\bar{x}, \bar{y})\bar{u}_j g_j(\bar{y}) - \tfrac{1}{2}\bar{p}^T \nabla^2 \sum_{j=1}^{m} \tilde{\alpha}_j(\bar{x}, \bar{y})\bar{u}_j g_j(\bar{y})\bar{p}.$$

On using second order $(F, \tilde{\alpha}, \tilde{\rho}, d)$-V-quasiconvexity of $g^{\bar{u}}$ at $\bar{y}$, we get

$$F(\bar{x}, \bar{y}; \sum_{j=1}^{m}(\nabla \bar{u}_j g_j(\bar{y}) + \nabla^2 \bar{u}_j g_j(\bar{y})\bar{p})) + \tilde{\rho} d^2(\bar{x}, \bar{y}) \leqq 0. \tag{17}$$

Now from (8), (17), hypothesis (iii) and the sublinearity of F, we obtain

$$F(\bar{x}, \bar{y}; \sum_{i=1}^{k}(\nabla \bar{\lambda}_i f_i(\bar{y}) + \nabla^2 \bar{\lambda}_i f_i(\bar{y})\bar{p})) + \bar{\rho} d^2(\bar{x}, \bar{y}) \geqq 0.$$

The strict second order $(F, \bar{\alpha}, \bar{\rho}, d)$-V-pseudoconvexity of $f^{\bar{\lambda}}$ at $\bar{y}$ yields

$$\sum_{i=1}^{k} \bar{\alpha}_i(\bar{x}, \bar{y})\bar{\lambda}_i f_i(\bar{x}) > \sum_{i=1}^{k} \bar{\alpha}_i(\bar{x}, \bar{y})\bar{\lambda}_i f_i(\bar{y}) - \tfrac{1}{2}\bar{p}^T \nabla^2 \sum_{i=1}^{k} \bar{\alpha}_i(\bar{x}, \bar{y})\bar{\lambda}_i f_i(\bar{y})\bar{p}.$$

Since $\bar{\alpha}_i(\bar{x}, \bar{y}) = \delta(\bar{x}, \bar{y}),\ i \in K$, we have

$$\sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{x}) > \sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{y}) - \frac{1}{2}\bar{p}^T \nabla^2 \sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{y})\bar{p},$$

a contradiction to hypothesis (i). Hence, $\bar{x} = \bar{y}$. □

5. General Mond-Weir type duality

For (P), we present the following second order general Mond-Weir type dual:

(GD) Maximize
$$\Bigg(f_1(y) + \sum_{j\in J_\circ} u_j g_j(y) - \tfrac{1}{2}p^T\nabla^2(f_1(y) + \sum_{j\in J_\circ} u_j g_j(y))p,$$
$$\ldots, f_k(y) + \sum_{j\in J_\circ} u_j g_j(y) - \tfrac{1}{2}p^T\nabla^2(f_k(y) + \sum_{j\in J_\circ} u_j g_j(y))p\Bigg)$$

subject to

$$\sum_{i=1}^{k}(\nabla\lambda_i f_i(y) + \nabla^2\lambda_i f_i(y)p) + \sum_{j=1}^{m}(\nabla u_j g_j(y) + \nabla^2 u_j g_j(y)p) = 0, \tag{18}$$

$$u_j g_j(y) - \frac{1}{2}p^T\nabla^2 u_j g_j(y)p \geqq 0,\ j \in J_\beta,\ \beta = 1, 2, \ldots, r, \tag{19}$$

$$\lambda \geqq 0, \tag{20}$$

$$\sum_{i=1}^{k}\lambda_i = 1, \tag{21}$$

$$u \geqq 0, \tag{22}$$

where $J_\beta \subseteq M,\ \beta = 0, 1, 2, \ldots, r$ with $\bigcup_{\beta=0}^{r} J_\beta = M$ and $J_\beta \cap J_\gamma = \emptyset$, if $\beta \neq \gamma$.

Remark 3. Let $J_\beta = \emptyset$. Then the dual (GD) reduces to Mangasarian type dual considered in Section 3. If $J_\circ = \emptyset$, then (GD) becomes Mond-Weir type dual discussed in Section 4.

Let Y be the set of all feasible solutions of (GD).

Theorem 8 (Weak duality)**.** Suppose that for all $x \in S$ and $(y, u, \lambda, p) \in Y$,

(i) $(\lambda_i f_i + u_{J_\circ} g_{J_\circ})_{i\in K}$ is second order $(F, \bar{\alpha}, \bar{\rho}, d)$-V-pseudoconvex at y, and $(u_j g_j)_{j\in J_\beta}$, $\beta = 1, 2, \ldots, r$ is second order $(F, \tilde{\alpha}, \tilde{\rho}, d)$-V-quasiconvex at y; and

(ii) $\bar{\rho} + \sum_{\beta=1}^{r} \tilde{\rho}_\beta \geqq 0$.

Then

$$f_i(x) \nleq f_i(y) + \sum_{j\in J_\circ} u_j g_j(y) - \frac{1}{2}p^T\nabla^2\{f_i(y) + \sum_{j\in J_\circ} u_j g_j(y)\}p,\ i\in K. \tag{23}$$

Proof. Since $x \in S$ and $(y,u,\lambda,p)\in Y$, we have

$$u_j g_j(x) \leqq 0 \leqq u_j g_j(y) - \frac{1}{2}p^T\nabla^2 u_j g_j(y)p,\ j\in J_\beta,\ \beta = 1,2,\ldots,r.$$

As $\tilde{\alpha}_j(x,y) > 0,\ j\in J_\beta$, we get

$$\sum_{j\in J_\beta}\tilde{\alpha}_j(x,y)u_j g_j(x) \leqq \sum_{j\in J_\beta}\tilde{\alpha}_j(x,y)u_j g_j(y) - \tfrac{1}{2}p^T\nabla^2\sum_{j\in J_\beta}\tilde{\alpha}_j(x,y)u_j g_j(y)p,\ \beta=1,2,\ldots,r.$$

The second order $(F,\tilde{\alpha},\tilde{\rho},d)$-V-quasiconvexity of $(u_j g_j)_{j\in J_\beta}, \beta=1,2,\ldots,r$ at y implies

$$F\left(x,y;\sum_{j\in J_\beta}(\nabla u_j g_j(y) + \nabla^2 u_j g_j(y)p)\right) + \tilde{\rho}_\beta d^2(x,y) \leqq 0,\ \beta=1,2,\ldots,r. \tag{24}$$

Inequality (24) along with (18), hypothesis (ii) and the sublinearity of F yields

$$F\left(x,y;\sum_{i=1}^k(\nabla\lambda_i f_i(y) + \nabla^2\lambda_i f_i(y)p) + \sum_{j\in J_\circ}(\nabla u_j g_j(y) + \nabla^2 u_j g_j(y)p)\right) + \bar{\rho}d^2(x,y) \geqq 0.$$

On using second order $(F,\bar{\alpha},\bar{\rho},d)$-V-pseudoconvexity of $(\lambda_i f_i + u_{J_\circ}g_{J_\circ})_{i\in K}$ at y, we obtain

$$\sum_{i=1}^k\bar{\alpha}_i(x,y)\left(\lambda_i f_i(x) + \sum_{j\in J_\circ}u_j g_j(x)\right) \geqq \sum_{i=1}^k\bar{\alpha}_i(x,y)\left(\lambda_i f_i(y) + \sum_{j\in J_\circ}u_j g_j(y)\right) - \tfrac{1}{2}p^T\nabla^2\left(\sum_{i=1}^k\bar{\alpha}_i(x,y)(\lambda_i f_i(y) + \sum_{j\in J_\circ}u_j g_j(y))\right)p. \tag{25}$$

Now, suppose contrary to the result that (23) cannot hold, then by $u \geqq 0$ and $g(x)\leqq 0$, it follows that

$$f_i(x) + \sum_{j\in J_\circ}u_j g_j(x) < f_i(y) + \sum_{j\in J_\circ}u_j g_j(y) - \tfrac{1}{2}p^T\nabla^2\{f_i(y) + \sum_{j\in J_\circ}u_j g_j(y)\}p,\ i\in K.$$

Using (20), (21), $\bar{\alpha}_i(x,y) > 0,\ i\in K$ and summing over i, we get

$$\sum_{i=1}^k\bar{\alpha}_i(x,y)(\lambda_i f_i(x) + \sum_{j\in J_\circ}u_j g_j(x)) < \sum_{i=1}^k\bar{\alpha}_i(x,y)(\lambda_i f_i(y) + \sum_{j\in J_\circ}u_j g_j(y))$$

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$$-\tfrac{1}{2}p^T\nabla^2\{\sum_{i=1}^{k}\bar{\alpha}_i(x,y)(\lambda_i f_i(y)+\sum_{j\in J_\circ}u_j g_j(y))\}p,$$

a contradiction to (25). This completes the proof. $\square$

The proof of the following strong duality theorem follows on the lines of Theorem 2 and hence being omitted.

Theorem 9 (Strong duality)**.** Let $\bar{x}$ be a weakly efficient solution of (P) at which the Kuhn-Tucker constraint qualification is satisfied. Then there exist $\bar{\lambda}\in\mathbb{R}^k$, $\bar{u}\in\mathbb{R}^m$ and $\bar{p}\in\mathbb{R}^n$ such that $(\bar{x},\bar{u},\bar{\lambda},\bar{p}=0)\in Y$ and the corresponding objective values of (P) and (GD) are equal. If, in addition, the hypotheses of weak duality (Theorem 8) hold, then $(\bar{x},\bar{u},\bar{\lambda},\bar{p}=0)$ is a weakly efficient solution of (GD).

Theorem 10 (Strict converse duality)**.** Let $\bar{x}\in S$ and $(\bar{y},\bar{u},\bar{\lambda},\bar{p})\in Y$ such that

(*i*) $\sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{x})\leqq\sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{y})+\sum_{j\in J_\circ}\bar{u}_j g_j(\bar{y})-\tfrac{1}{2}\bar{p}^T\nabla^2(f_i(\bar{y})+\sum_{j\in J_\circ}\bar{u}_j g_j(\bar{y}))\bar{p}$;

(*ii*) $(\bar{\lambda}_i f_i+\bar{u}_{J_\circ}^T g_{J_\circ})_{i\in K}$ is strictly second order $(F,\bar{\alpha},\bar{\rho},d)$-V-pseudoconvex at $\bar{y}$ with $\bar{\alpha}_i(\bar{x},\bar{y})=\delta(\bar{x},\bar{y}),\ i\in K$ and $(\bar{u}_j g_j)_{j\in J_\beta},\ \beta=1,2,\ldots,r$ is second order $(F,\tilde{\alpha},\tilde{\rho},d)$-V-quasiconvex at $\bar{y}$; and

(*iii*) $\bar{\rho}+\sum_{\beta=1}^{r}\tilde{\rho}_\beta\geqq 0.$

Then $\bar{x}=\bar{y}$.

Proof. We assume that $\bar{x}\neq\bar{y}$ and exhibit a contradiction. Since $\bar{x}\in S$ and $(\bar{y},\bar{u},\bar{\lambda},\bar{p})\in Y$, we have

$$\bar{u}_j g_j(\bar{x})\leqq 0\leqq\bar{u}_j g_j(\bar{y})-\frac{1}{2}\bar{p}^T\nabla^2\bar{u}_j g_j(\bar{y})\bar{p},\ j\in J_\beta,\ \beta=1,2,\ldots,r.$$

As $\tilde{\alpha}_j(\bar{x},\bar{y})>0,\ j\in J_\beta$, it follows that

$$\sum_{j\in J_\beta}\tilde{\alpha}_j(\bar{x},\bar{y})\bar{u}_j g_j(\bar{x})\leqq\sum_{j\in J_\beta}\tilde{\alpha}_j(\bar{x},\bar{y})\bar{u}_j g_j(\bar{y})-\tfrac{1}{2}\bar{p}^T\nabla^2\sum_{j\in J_\beta}\tilde{\alpha}_j(\bar{x},\bar{y})\bar{u}_j g_j(\bar{y})\bar{p},\ \beta=1,2,\ldots,r.$$

The second order $(F,\tilde{\alpha},\tilde{\rho},d)$-V-quasiconvexity of $(\bar{u}_j g_j)_{j\in J_\beta},\ \beta=1,2,\ldots,r$ at $\bar{y}$ gives

$$F\left(\bar{x},\bar{y};\sum_{j\in J_\beta}(\nabla\bar{u}_j g_j(\bar{y})+\nabla^2\bar{u}_j g_j(\bar{y})\bar{p})\right)+\tilde{\rho}_\beta d^2(\bar{x},\bar{y})\leqq 0. \tag{26}$$

The inequality (26) along with (18), hypothesis (*iii*) and the sublinearity of F yields

$$F\left(\bar{x},\bar{y};\sum_{i=1}^{k}(\nabla\bar{\lambda}_i f_i(\bar{y})+\nabla^2\bar{\lambda}_i f_i(\bar{y})\bar{p})+\sum_{j\in J_\circ}(\nabla\bar{u}_j g_j(\bar{y})+\nabla^2\bar{u}_j g_j(\bar{y})\bar{p})\right)+\bar{\rho}d^2(\bar{x},\bar{y})\geqq 0.$$

On using strict second order $(F, \bar{\alpha}, \bar{\rho}, d)$-V-pseudoconvexity of $(\bar{\lambda}_i f_i + \bar{u}_{J_\circ}^T g_{J_\circ})_{i\in K}$ at $\bar{y}$, we obtain

$$\sum_{i=1}^{k} \bar{\alpha}_i(\bar{x}, \bar{y}) \left(\bar{\lambda}_i f_i(\bar{x}) + \sum_{j\in J_\circ} \bar{u}_j g_j(\bar{x})\right) > \sum_{i=1}^{k} \bar{\alpha}_i(\bar{x}, \bar{y}) \left(\bar{\lambda}_i f_i(\bar{y}) + \sum_{j\in J_\circ} \bar{u}_j g_j(\bar{y})\right) - \frac{1}{2}\bar{p}^T \nabla^2 \left(\sum_{i=1}^{k} \bar{\alpha}_i(\bar{x}, \bar{y})(\bar{\lambda}_i f_i(\bar{y}) + \sum_{j\in J_\circ} \bar{u}_j g_j(\bar{y}))\right) \bar{p},$$

which by the feasibility of $\bar{x}$ for (P) gives

$$\sum_{i=1}^{k} \bar{\alpha}_i(\bar{x}, \bar{y})\bar{\lambda}_i f_i(\bar{x}) > \sum_{i=1}^{k} \bar{\alpha}_i(\bar{x}, \bar{y}) \left(\bar{\lambda}_i f_i(\bar{y}) + \sum_{j\in J_\circ} \bar{u}_j g_j(\bar{y})\right) - \frac{1}{2}\bar{p}^T \nabla^2 \left(\sum_{i=1}^{k} \bar{\alpha}_i(\bar{x}, \bar{y})(\bar{\lambda}_i f_i(\bar{y}) + \sum_{j\in J_\circ} \bar{u}_j g_j(\bar{y}))\right) \bar{p}.$$

Since $\bar{\alpha}_i(\bar{x}, \bar{y}) = \delta(\bar{x}, \bar{y}),\ i \in K$, we obtain

$$\sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{x}) > \sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{y}) + \sum_{j\in J_\circ} \bar{u}_j g_j(\bar{y}) - \frac{1}{2}\bar{p}^T \nabla^2 \left(\sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{y}) + \sum_{j\in J_\circ} \bar{u}_j g_j(\bar{y})\right) \bar{p},$$

a contradiction to hypothesis (i). Hence, $\bar{x} = \bar{y}$. □

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ON A FIFTH-ORDER DIFFERENCE EQUATION

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ABSTRACT. We investigate the following difference equation

$$x_n = \frac{x_{n-3}x_{n-4}x_{n-5}}{x_{n-1}x_{n-2}(a_n + b_n x_{n-3}x_{n-4}x_{n-5})}, \quad n \in \mathbb{N}_0,$$

where $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ are two real sequences and the initial values $x_{-5}, \dots, x_{-1}$ are real numbers. The case when the sequences $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ are constant is thoroughly studied. Our results considerably extend some results in the recent literature.

1. INTRODUCTION

There has been a great recent interest in nonlinear difference equations and systems of difference equations (see, for example, [1]-[6], [8]-[14], [18]-[43] and the references therein), and, among them, some renewed interest in the difference equations and systems which can be solved in closed form (see, for example, [1]-[4], [6], [8], [19], [22], [23], [26], [27], [29]-[37], [39]-[43] and the related references therein). For some classical methods for solving difference equations and systems see, for example, [7], [16] and [17]. Many of the papers in the theory deal with difference equations and systems which can be regarded as perturbations of solvable ones (see, for example, [25] and [38]), so that their solutions are frequently compared with the solutions of the solvable ones, or are connected with some other solvable equations as it is the case in [21], [25] and [38]. This fact also shows the importance of solvable difference equations and systems.

Among the papers in the area there are some which present formulas of some particular difference equations and/or systems of difference equations which are almost always proved by induction, but do not give any theoretical explanation related to the presented formulas and how the equations/systems can be solved. Paper [22] by S. Stević, in which a natural explanation is given for the formula presented in [8], motivated numerous authors to re-attract their interest in difference equations which can be solved in closed form. Various other explanations and extensions of some results in the literature can be also found in papers [26], [36] and [41].

It is said that the difference equation

$$x_n = f(x_{n-1}, \dots, x_{n-k}), \quad n \in \mathbb{N}_0,$$

where $k \in \mathbb{N}$, is solvable in closed form if every solution can be written in terms of the initial values $x_{-k}, \dots, x_{-1}$ and index n only.

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Paper [43] is one of the papers of above mentioned type. Namely, formulas for solutions of the next four difference equations

$$x_{n+1} = \frac{x_{n-2}x_{n-3}x_{n-4}}{x_n x_{n-1}(1 + x_{n-2}x_{n-3}x_{n-4})}, \quad n \in \mathbb{N}_0, \tag{1}$$

$$x_{n+1} = \frac{x_{n-2}x_{n-3}x_{n-4}}{x_n x_{n-1}(1 - x_{n-2}x_{n-3}x_{n-4})}, \quad n \in \mathbb{N}_0, \tag{2}$$

$$x_{n+1} = \frac{x_{n-2}x_{n-3}x_{n-4}}{x_n x_{n-1}(-1 + x_{n-2}x_{n-3}x_{n-4})}, \quad n \in \mathbb{N}_0, \tag{3}$$

$$x_{n+1} = \frac{x_{n-2}x_{n-3}x_{n-4}}{x_n x_{n-1}(-1 - x_{n-2}x_{n-3}x_{n-4})}, \quad n \in \mathbb{N}_0, \tag{4}$$

are presented in [43] and for some of them are given sketches of the inductive proofs, but there are no theoretical explanations for the formulas.

A natural problem is to extend the results in [43] and give theoretical explanations for formulas presented therein.

Here, we will study the next difference equation

$$x_n = \frac{x_{n-3}x_{n-4}x_{n-5}}{x_{n-1}x_{n-2}(a_n + b_n x_{n-3}x_{n-4}x_{n-5})}, \quad n \in \mathbb{N}_0, \tag{5}$$

where $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ are real sequences and the initial values $x_{-5}, \dots, x_{-1}$ are real numbers, which is a natural extension of equations (1)-(4) (we shifted the indices backward for one, since the equation in this form as well as the result might look clearer).

To deal with equation (5) we essentially use the idea in [22], later exploited in numerous papers, where a suitable change of variables is used so that the equation therein is transformed into a solvable difference equation (see, for example, [1], [2], [4], [19], [27], [29]-[31], [33]-[37], [39]-[42]).

Solution $(x_n)_{n\geq -s}$, of the difference equation

$$x_n = f(x_{n-1}, \dots, x_{n-s}), \quad n \in \mathbb{N}_0, \tag{6}$$

where $f : \mathbb{R}^s \to \mathbb{R}$, $s \in \mathbb{N}$, is called eventually periodic with period p, if there is an $n_1 \geq -s$ such that

$$x_{n+p} = x_n, \quad \text{for} \quad n \geq n_1.$$

It is called periodic with period p, if $n_1 = -s$. For some results in this area see, e.g. [5, 9, 10, 11, 12, 13, 14, 15, 18, 20, 24, 28] and the references therein.

Throughout the paper we use the following standard conventions

$$\sum_{j=k}^{l} a_j = 0, \quad \text{when} \quad k > l,$$

and

$$\prod_{j=k}^{k-1} b_j = 1,$$

where k and l are integers.

2. Formulas for well-defined solutions of equation (5)

Assume that $(x_n)_{n\geq -5}$ is a solution of equation (5). If $x_{-5} = 0$ or $x_{-4} = 0$ or $x_{-3} = 0$ and $x_{-2} \neq 0 \neq x_{-1}$, then from (5) we see that x_0 is or not defined (if $a_0 = 0$) or $x_0 = 0$, and consequently x_1 is not defined. If $x_{-2} = 0$ or $x_{-1} = 0$, then

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from (5) we see that x_0 is not defined. This means that if one of the initial values x_{-j}, $j \in \{1, \dots, 5\}$ is equal to zero, then such a solution is not defined.

Now assume that $x_{n_0} = 0$ for some $n_0 \in \mathbb{N}_0$, that x_i are defined when $-5 \le i \le n_0$, and that n_0 is the smallest index for which a member of the solution is equal to zero. Then, from (5) we see that $x_{n_0-3} = 0$ or $x_{n_0-4} = 0$ or $x_{n_0-5} = 0$, which along with (5) would imply respectively that x_{n_0-2} if $n_0 \ge 2$ is not defined, or x_{n_0-3} if $n_0 \ge 3$ is not defined, or x_{n_0-4} if $n_0 \ge 4$ is not defined, which would be a contradiction with the fact that x_n are defined for $-5 \le n \le n_0$.

If $n_0 = 2$, then we have that $x_{-1} = 0$ or $x_{-2} = 0$ or $x_{-3} = 0$, if $n_0 = 1$, then we have that $x_{-2} = 0$ or $x_{-3} = 0$ or $x_{-4} = 0$, while if $n_0 = 0$, then we have that $x_{-3} = 0$ or $x_{-4} = 0$ or $x_{-5} = 0$. So, in these three cases we have that at least one of the initial values is equal to zero, and consequently by previous considerations such solutions are not defined.

If $n_0 = 3$, then from (5) we have that $x_0 = 0$ or $x_{-1} = 0$ or $x_{-2} = 0$. If $x_{-1} = 0$ or $x_{-2} = 0$, then x_0 is not defined, while the case $x_0 = 0$ has been previously considered.

If $n_0 = 4$, then from (5) we have that $x_1 = 0$ or $x_0 = 0$ or $x_{-1} = 0$. If $x_{-1} = 0$, then x_0 is not defined, while the cases $x_1 = 0$ or $x_0 = 0$ have been previously considered. Thus, according to all above mentioned such solutions are not defined.

Hence of some interest are solutions for which

$$x_{-j} \ne 0, \quad j \in \{1, \dots, 5\},$$

since for them it must be

$$x_n \ne 0, \qquad n \ge -5. \tag{7}$$

Now assume that $(x_n)_{n \ge -5}$ is a well-defined solution of equation (5). By previous considerations we have that (7) holds, so that for every well-defined solution we can use the following change of variables

$$y_n = \frac{1}{x_n x_{n-1} x_{n-2}}, \quad n \ge -3, \tag{8}$$

which transforms equation (5) into the following linear third-order difference equation

$$y_n = a_n y_{n-3} + b_n, \quad n \in \mathbb{N}_0. \tag{9}$$

Now note that every integer $n \ge -3$ can be written in the following form $n = 3m + i$, for some $m \ge -1$ and $i \in \{0, 1, 2\}$. Hence, equation (9) can be written in the next form

$$y_{3m+i} = a_{3m+i} y_{3(m-1)+i} + b_{3m+i}, \quad m \in \mathbb{N}_0, \tag{10}$$

where $i \in \{0, 1, 2\}$.

This means that the sequences $(y_{3m+i})_{m \ge -1}$, $i \in \{0, 1, 2\}$, are solutions of the following three linear first order difference equations

$$z_m = a_{3m+i} z_{m-1} + b_{3m+i}, \quad m \in \mathbb{N}_0, \tag{11}$$

$i \in \{0, 1, 2\}$.

The linear first order difference equation is solved in closed form and by using well-known formula for its solution we have that

$$y_{3m+i} = y_{i-3} \prod_{j=0}^{m} a_{3j+i} + \sum_{l=0}^{m} b_{3l+i} \prod_{j=l+1}^{m} a_{3j+i}, \quad m \in \mathbb{N}_0, \tag{12}$$

$i \in \{0,1,2\}$. This formula can be easily obtained, for example, if we multiply the next equalities

$$y_{3l+i} = a_{3l+i} y_{3(l-1)+i} + b_{3l+i},$$

by $\prod_{j=l+1}^{m} a_{3j+i}$, $l = 0,1,\dots,m$, and sum up such obtained equalities ([17]).

Now we find formulas for well-defined solutions of equation (5). From (8) with $n = 3m+i$, we have that

$$\begin{aligned} x_{3m+i} &= \frac{1}{y_{3m+i}x_{3m+i-1}x_{3m+i-2}} \\ &= \frac{x_{3m+i-3}}{y_{3m+i}x_{3m+i-1}x_{3m+i-2}x_{3m+i-3}} = \frac{y_{3m+i-1}}{y_{3m+i}} x_{3(m-1)+i}. \end{aligned} \tag{13}$$

By repeating use of (13) we obtain

$$x_{3m} = x_{-3} \prod_{s=0}^{m} \frac{y_{3s-1}}{y_{3s}}, \quad m \geq -1, \tag{14}$$

and

$$x_{3m+i} = x_{i-6} \prod_{s=-1}^{m} \frac{y_{3s+i-1}}{y_{3s+i}}, \quad m \geq -1, \tag{15}$$

for $i \in \{1,2\}$.

Using formula (12) in (14) and (15) we obtain formulas for general solution of equation (5)

$$\begin{aligned} x_{3m} &= x_{-3} \prod_{s=0}^{m} \frac{y_{3(s-1)+2}}{y_{3s}} \\ &= x_{-3} \prod_{s=0}^{m} \frac{y_{-1}\prod_{j=0}^{s-1} a_{3j+2} + \sum_{l=0}^{s-1} b_{3l+2} \prod_{j=l+1}^{s-1} a_{3j+2}}{y_{-3}\prod_{j=0}^{s} a_{3j} + \sum_{l=0}^{s} b_{3l} \prod_{j=l+1}^{s} a_{3j}} \\ &= x_{-3} \prod_{s=0}^{m} \frac{(x_{-1}x_{-2}x_{-3})^{-1}\prod_{j=0}^{s-1} a_{3j+2} + \sum_{l=0}^{s-1} b_{3l+2} \prod_{j=l+1}^{s-1} a_{3j+2}}{(x_{-3}x_{-4}x_{-5})^{-1}\prod_{j=0}^{s} a_{3j} + \sum_{l=0}^{s} b_{3l} \prod_{j=l+1}^{s} a_{3j}}, \end{aligned} \tag{16}$$

$m \geq -1$, and

$$\begin{aligned} x_{3m+i} &= x_{i-6} \prod_{s=-1}^{m} \frac{y_{3s+i-1}}{y_{3s+i}} \\ &= x_{i-6} \prod_{s=-1}^{m} \frac{y_{i-4}\prod_{j=0}^{s} a_{3j+i-1} + \sum_{l=0}^{s} b_{3l+i-1} \prod_{j=l+1}^{s} a_{3j+i-1}}{y_{i-3}\prod_{j=0}^{s} a_{3j+i} + \sum_{l=0}^{s} b_{3l+i} \prod_{j=l+1}^{s} a_{3j+i}} \\ &= x_{i-6} \prod_{s=-1}^{m} \frac{(x_{i-4}x_{i-5}x_{i-6})^{-1}\prod_{j=0}^{s} a_{3j+i-1} + \sum_{l=0}^{s} b_{3l+i-1} \prod_{j=l+1}^{s} a_{3j+i-1}}{(x_{i-3}x_{i-4}x_{i-5})^{-1}\prod_{j=0}^{s} a_{3j+i} + \sum_{l=0}^{s} b_{3l+i} \prod_{j=l+1}^{s} a_{3j+i}}, \end{aligned} \tag{17}$$

for $m \geq -1$ and for $i \in \{1,2\}$.

3. Case when sequences a_n and b_n are constant

In this section we consider the case when the sequences $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ are constant, that is, when

$$a_n = a, \qquad b_n = b, \qquad n \in \mathbb{N}_0.$$

In this case equation (5) becomes

$$x_n = \frac{x_{n-3}x_{n-4}x_{n-5}}{x_{n-1}x_{n-2}(a + bx_{n-3}x_{n-4}x_{n-5})}, \quad n \in \mathbb{N}_0. \tag{18}$$

By using formulas (16) and (17) in this case we obtain

$$\begin{aligned} x_{3m} &= x_{-3}\prod_{s=0}^{m} \frac{(x_{-1}x_{-2}x_{-3})^{-1}\prod_{j=0}^{s-1} a + \sum_{l=0}^{s-1} b\prod_{j=l+1}^{s-1} a}{(x_{-3}x_{-4}x_{-5})^{-1}\prod_{j=0}^{s} a + \sum_{l=0}^{s} b\prod_{j=l+1}^{s} a} \\ &= x_{-3}\prod_{s=0}^{m} \frac{(x_{-1}x_{-2}x_{-3})^{-1}a^s + b\sum_{l=0}^{s-1} a^{s-l-1}}{(x_{-3}x_{-4}x_{-5})^{-1}a^{s+1} + b\sum_{l=0}^{s} a^{s-l}}, \end{aligned} \tag{19}$$

$m \geq -1$, and

$$\begin{aligned} x_{3m+i} &= x_{i-6}\prod_{s=-1}^{m} \frac{(x_{i-4}x_{i-5}x_{i-6})^{-1}\prod_{j=0}^{s} a + \sum_{l=0}^{s} b\prod_{j=l+1}^{s} a}{(x_{i-3}x_{i-4}x_{i-5})^{-1}\prod_{j=0}^{s} a + \sum_{l=0}^{s} b\prod_{j=l+1}^{s} a} \\ &= x_{i-6}\prod_{s=-1}^{m} \frac{(x_{i-4}x_{i-5}x_{i-6})^{-1}a^{s+1} + b\sum_{l=0}^{s} a^{s-l}}{(x_{i-3}x_{i-4}x_{i-5})^{-1}a^{s+1} + b\sum_{l=0}^{s} a^{s-l}}, \end{aligned} \tag{20}$$

for $m \geq -1$ and for $i \in \{1, 2\}$.

We have now two cases.

3.1. **Case $a \neq 1$.** In this case formulas (19) and (20) become

$$\begin{aligned} x_{3m} &= x_{-3}\prod_{s=0}^{m} \frac{(x_{-1}x_{-2}x_{-3})^{-1}a^s + b\sum_{l=0}^{s-1} a^{s-l-1}}{(x_{-3}x_{-4}x_{-5})^{-1}a^{s+1} + b\sum_{l=0}^{s} a^{s-l}} \\ &= x_{-3}\prod_{s=0}^{m} \frac{(x_{-1}x_{-2}x_{-3})^{-1}a^s(1-a) + b(1-a^s)}{(x_{-3}x_{-4}x_{-5})^{-1}a^{s+1}(1-a) + b(1-a^{s+1})} \\ &= x_{-3}\prod_{s=0}^{m} \frac{a^s((1-a)(x_{-1}x_{-2}x_{-3})^{-1} - b) + b}{a^{s+1}((1-a)(x_{-3}x_{-4}x_{-5})^{-1} - b) + b}, \end{aligned} \tag{21}$$

$m \geq -1$, and

$$\begin{aligned} x_{3m+i} &= x_{i-6}\prod_{s=-1}^{m} \frac{(x_{i-4}x_{i-5}x_{i-6})^{-1}a^{s+1} + b\sum_{l=0}^{s} a^{s-l}}{(x_{i-3}x_{i-4}x_{i-5})^{-1}a^{s+1} + b\sum_{l=0}^{s} a^{s-l}} \\ &= x_{i-6}\prod_{s=-1}^{m} \frac{(x_{i-4}x_{i-5}x_{i-6})^{-1}a^{s+1}(1-a) + b(1-a^{s+1})}{(x_{i-3}x_{i-4}x_{i-5})^{-1}a^{s+1}(1-a) + b(1-a^{s+1})} \\ &= x_{i-6}\prod_{s=-1}^{m} \frac{a^{s+1}((1-a)(x_{i-4}x_{i-5}x_{i-6})^{-1} - b) + b}{a^{s+1}((1-a)(x_{i-3}x_{i-4}x_{i-5})^{-1} - b) + b}, \end{aligned} \tag{22}$$

for $m \geq -1$ and for $i \in \{1, 2\}$.

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3.2. **Case** $a = 1$**.** In this case formulas (19) and (20) become

$$x_{3m} = x_{-3} \prod_{s=0}^{m} \frac{(x_{-1}x_{-2}x_{-3})^{-1} + bs}{(x_{-3}x_{-4}x_{-5})^{-1} + b(s+1)}, \tag{23}$$

$m \geq -1$, and

$$x_{3m+i} = x_{i-6} \prod_{s=-1}^{m} \frac{(x_{i-4}x_{i-5}x_{i-6})^{-1} + b(s+1)}{(x_{i-3}x_{i-4}x_{i-5})^{-1} + b(s+1)}, \tag{24}$$

for $m \geq -1$ and for $i \in \{1, 2\}$.

3.3. **Asymptotic behavior of solutions of equation** (18)**.** Here we study the asymptotic behavior of well-defined solutions of equation (18). Prior to stating and proving our results we introduce some quantities which will be used in the statements of the results.

Set

$$L_0 = \frac{(1-a)(x_{-1}x_{-2}x_{-3})^{-1} - b}{a((1-a)(x_{-3}x_{-4}x_{-5})^{-1} - b)},$$

$$L_i = \frac{(1-a)(x_{i-4}x_{i-5}x_{i-6})^{-1} - b}{(1-a)(x_{i-3}x_{i-4}x_{i-5})^{-1} - b},$$

where $i = 1, 2$, and set

$$K_0 = \frac{1}{x_{-1}x_{-2}x_{-3}b} - \frac{1}{x_{-3}x_{-4}x_{-5}b} - 1,$$

$$K_i = \frac{1}{b}\left(\frac{1}{x_{i-4}x_{i-5}x_{i-6}} - \frac{1}{x_{i-3}x_{i-4}x_{i-5}}\right),$$

where $i = 1, 2$.

Our first result considers the case $|a| > 1$, $b \neq 0$.

Theorem 1. *Assume that* $|a| > 1$, $b \neq 0$, *and* $(x_n)_{n \geq -5}$ *is a well-defined solution of equation* (18). *Then the following statements are true.*

(a) If $|L_0| < 1$, *then* $x_{3m} \to 0$ *as* $m \to +\infty$.
(b) If $|L_0| > 1$, *then* $|x_{3m}| \to +\infty$ *as* $m \to +\infty$.
(c) If $L_0 = 1$, *then the sequence* $(x_{3m})_{m \geq -1}$ *is constant.*
(d) If $L_0 = -1$, *then the sequences* $(x_{6m})_{m \in \mathbb{N}_0}$ *and* $(x_{6m+3})_{m \geq -1}$ *are convergent.*
(e) If $|L_i| < 1$, *for some* $i \in \{1, 2\}$, *then* $x_{3m+i} \to 0$ *as* $m \to +\infty$.
(f) If $|L_i| > 1$, *for some* $i \in \{1, 2\}$, *then* $|x_{3m+i}| \to +\infty$ *as* $m \to +\infty$.
(g) If $L_i = 1$, *for some* $i \in \{1, 2\}$, *then the sequence* $(x_{3m+i})_{m \geq -2}$ *is constant.*
(h) If $L_i = -1$, *for some* $i \in \{1, 2\}$, *then the sequences* $(x_{6m+i})_{m \geq -1}$ *and* $(x_{6m+3+i})_{m \geq -1}$ *are convergent.*

Proof. (a), (b) Let

$$p_s = \frac{a^s((1-a)(x_{-1}x_{-2}x_{-3})^{-1} - b) + b}{a^{s+1}((1-a)(x_{-3}x_{-4}x_{-5})^{-1} - b) + b}. \tag{25}$$

Then we have

$$\lim_{s \to +\infty} p_s = \lim_{s \to +\infty} \frac{((1-a)(x_{-1}x_{-2}x_{-3})^{-1} - b)/a + \frac{b}{a^{s+1}}}{(1-a)(x_{-3}x_{-4}x_{-5})^{-1} - b + \frac{b}{a^{s+1}}} = L_0. \tag{26}$$

From (21) and (26), these two statements easily follow.

(c) In this case, we have that

$$p_s = \frac{((1-a)(x_{-1}x_{-2}x_{-3})^{-1} - b)/a + \frac{b}{a^{s+1}}}{(1-a)(x_{-3}x_{-4}x_{-5})^{-1} - b + \frac{b}{a^{s+1}}} = 1, \quad s \in \mathbb{N}_0. \tag{27}$$

From (21) and (27) the result easily follows.

(d) Since $L_0 = -1$, and by using the asymptotic relation

$$\frac{1}{1+x} = 1 - x + O(x^2), \tag{28}$$

when x is close to the origin, we have that

$$\begin{aligned} p_s &= -\frac{(1-a)(x_{-3}x_{-4}x_{-5})^{-1} - b - \frac{b}{a^{s+1}}}{(1-a)(x_{-3}x_{-4}x_{-5})^{-1} - b + \frac{b}{a^{s+1}}} \\ &= -\left(1 - \frac{b}{a^{s+1}((1-a)(x_{-3}x_{-4}x_{-5})^{-1} - b)}\right) \\ &\quad \times \left(1 - \frac{b}{a^{s+1}((1-a)(x_{-3}x_{-4}x_{-5})^{-1} - b)} + O\Big(\frac{1}{a^{2s}}\Big)\right) \\ &= -\left(1 + O\Big(\frac{1}{a^s}\Big)\right), \end{aligned} \tag{29}$$

for large enough s. From (21), (29), the assumption $|a| > 1$, and by a known criterion for the convergence of products the result easily follows.

(e), (f) Let

$$q_s^i = \frac{a^{s+1}((1-a)(x_{i-4}x_{i-5}x_{i-6})^{-1} - b) + b}{a^{s+1}((1-a)(x_{i-3}x_{i-4}x_{i-5})^{-1} - b) + b}, \quad i = 1, 2. \tag{30}$$

Then we have

$$\lim_{s\to+\infty} q_s^i = \lim_{s\to+\infty} \frac{(1-a)(x_{i-4}x_{i-5}x_{i-6})^{-1} - b + \frac{b}{a^{s+1}}}{(1-a)(x_{i-3}x_{i-4}x_{i-5})^{-1} - b + \frac{b}{a^{s+1}}} = L_i, \quad i = 1, 2. \tag{31}$$

From (22) and (31), these two statements easily follow.

(g) In this case we have that

$$q_s^i = \frac{(1-a)(x_{i-4}x_{i-5}x_{i-6})^{-1} - b + \frac{b}{a^{s+1}}}{(1-a)(x_{i-3}x_{i-4}x_{i-5})^{-1} - b + \frac{b}{a^{s+1}}} = 1, \quad s \in \mathbb{N}_0. \tag{32}$$

From (22) and (32) the result easily follows.

(h) Since $L_i = -1$ and by using (28), we have that

$$\begin{aligned} q_s^i &= -\frac{(1-a)(x_{i-3}x_{i-4}x_{i-5})^{-1} - b - \frac{b}{a^{s+1}}}{(1-a)(x_{i-3}x_{i-4}x_{i-5})^{-1} - b + \frac{b}{a^{s+1}}} \\ &= -\left(1 - \frac{b}{a^{s+1}((1-a)(x_{i-3}x_{i-4}x_{i-5})^{-1} - b)}\right) \\ &\quad \times \left(1 - \frac{b}{a^{s+1}((1-a)(x_{i-3}x_{i-4}x_{i-5})^{-1} - b)} + O\Big(\frac{1}{a^{2s}}\Big)\right) \\ &= -\left(1 + O\Big(\frac{1}{a^s}\Big)\right), \end{aligned} \tag{33}$$

for large enough s. From (22), (33), the assumption $|a| > 1$, and by a known criterion for the convergence of products the result easily follows. □

Now we consider the case $|a| < 1$, $b \neq 0$.

Theorem 2. *Assume that $|a| < 1$, $b \neq 0$, and $(x_n)_{n \geq -5}$ is a well-defined solution of equation* (18). *Then the sequences $(x_{3m+i})_{m \geq -1}$, $i \in \{0, 1, 2\}$, are convergent.*

Proof. From (25) and (28), we have

$$\begin{aligned} p_s =& \frac{1 + a^s((1-a)(x_{-1}x_{-2}x_{-3})^{-1} - b)/b}{1 + a^{s+1}((1-a)(x_{-3}x_{-4}x_{-5})^{-1} - b)/b} \\ =& (1 + a^s((1-a)(x_{-1}x_{-2}x_{-3})^{-1} - b)/b) \\ & \times (1 - a^{s+1}((1-a)(x_{-3}x_{-4}x_{-5})^{-1} - b)/b + O(a^{2s})) \\ =& 1 + O(a^s), \end{aligned} \tag{34}$$

for large enough s, while from (30) and by using (28), we have

$$\begin{aligned} q_s^i =& \frac{1 + a^{s+1}((1-a)(x_{i-4}x_{i-5}x_{i-6})^{-1} - b)/b}{1 + a^{s+1}((1-a)(x_{i-3}x_{i-4}x_{i-5})^{-1} - b)/b} \\ =& (1 + a^{s+1}((1-a)(x_{i-4}x_{i-5}x_{i-6})^{-1} - b)/b) \\ & \times (1 - a^{s+1}((1-a)(x_{i-3}x_{i-4}x_{i-5})^{-1} - b)/b + O(a^{2s})) \\ =& 1 + O(a^s), \end{aligned} \tag{35}$$

for large enough s and $i \in \{1, 2\}$. From (21), (22), (34), (35), the assumption $|a| < 1$, and by a known criterion for the convergence of products we have that the sequences $(x_{3m+i})_{m \geq -1}$, $i \in \{0, 1, 2\}$, are convergent. $\square$

Now we consider the case $a = 1$, $b \neq 0$.

Theorem 3. *Assume that $a = 1$, $b \neq 0$, and $(x_n)_{n \geq -5}$ is a well-defined solution of equation* (18). *Then the following statements are true.*

(a) If $a = 1$ and $K_0 < 0$, then $x_{3m} \to 0$ as $m \to +\infty$.
(b) If $a = 1$ and $K_0 > 0$, then $|x_{3m}| \to +\infty$ as $m \to +\infty$.
(c) If $a = 1$ and $K_0 = 0$, then the sequence $(x_{3m})_{m \geq -1}$ is constant.
(d) If $a = 1$ and $K_i < 0$, for some $i \in \{1, 2\}$, then $x_{3m+i} \to 0$ as $m \to +\infty$.
(e) If $a = 1$ and $K_i > 0$, for some $i \in \{1, 2\}$, then $|x_{3m+i}| \to +\infty$ as $m \to +\infty$.
(f) If $a = 1$ and $K_i = 0$, for some $i \in \{1, 2\}$, then the sequence $(x_{3m+i})_{m \geq -1}$ is constant.

Proof. (a), (b) Let

$$r_s = \frac{(x_{-1}x_{-2}x_{-3})^{-1} + bs}{(x_{-3}x_{-4}x_{-5})^{-1} + b + bs}. \tag{36}$$

From (36) and by using (28) we have that

$$\begin{aligned} r_s =& \frac{1 + \frac{1}{bx_{-1}x_{-2}x_{-3}s}}{1 + \frac{(bx_{-3}x_{-4}x_{-5})^{-1} + 1}{s}} \\ =& \Big(1 + \frac{1}{bx_{-1}x_{-2}x_{-3}s}\Big)\Big(1 - \frac{(bx_{-3}x_{-4}x_{-5})^{-1} + 1}{s} + O\Big(\frac{1}{s^2}\Big)\Big) \\ =& 1 + \frac{K_0}{s} + O\Big(\frac{1}{s^2}\Big), \end{aligned} \tag{37}$$

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for large enough s. From (23) and (37), and known criteria for the convergence of products, these two statements easily follow.

(c) Note that in this case $r_s = 1$, $s \in \mathbb{N}_0$, from which the result follows.

(d), (e) Let

$$r_s^i = \frac{(x_{i-4}x_{i-5}x_{i-6})^{-1} + b + bs}{(x_{i-3}x_{i-4}x_{i-5})^{-1} + b + bs}, \quad i = 1, 2. \tag{38}$$

From (38) and by using (28) we have that

$$\begin{aligned} r_s^i =& \frac{1 + \frac{(bx_{i-4}x_{i-5}x_{i-6})^{-1}+1}{s}}{1 + \frac{(bx_{i-3}x_{i-4}x_{i-5})^{-1}+1}{s}} \\ =& \Big(1 + \frac{(bx_{i-4}x_{i-5}x_{i-6})^{-1} + 1}{s}\Big)\Big(1 - \frac{(bx_{i-3}x_{i-4}x_{i-5})^{-1} + 1}{s} + O\Big(\frac{1}{s^2}\Big)\Big) \\ =& 1 + \frac{K_i}{s} + O\Big(\frac{1}{s^2}\Big), \end{aligned} \tag{39}$$

for large enough s and $i \in \{1, 2\}$. From (24) and (39), and known criteria for the convergence of products, these two statements easily follow.

(f) Note that in this case $r_s^i = 1$, $i = 1, 2$, $s \in \mathbb{N}_0$, from which the result follows. □

If $a = -1$ and $b \neq 0$, then from (21) and (22), we have that

$$x_{3m} = x_{-3} \prod_{s=0}^{m} \frac{(-1)^s(2(x_{-1}x_{-2}x_{-3})^{-1} - b) + b}{(-1)^{s+1}(2(x_{-3}x_{-4}x_{-5})^{-1} - b) + b}, \quad m \geq -1, \tag{40}$$

and

$$x_{3m+i} = x_{i-6} \prod_{s=-1}^{m} \frac{(-1)^{s+1}(2(x_{i-4}x_{i-5}x_{i-6})^{-1} - b) + b}{(-1)^{s+1}(2(x_{i-3}x_{i-4}x_{i-5})^{-1} - b) + b}, \quad m \geq -1, \tag{41}$$

for some $i \in \{1, 2\}$.

Hence, by using formulas (40) and (41) we have that

$$x_{6m} = x_{-3} \frac{(x_{-1}x_{-2}x_{-3})^{-1}}{b - (x_{-3}x_{-4}x_{-5})^{-1}} \left(\frac{(x_{-1}x_{-2}x_{-3})^{-1}(b - (x_{-1}x_{-2}x_{-3})^{-1})}{(x_{-3}x_{-4}x_{-5})^{-1}(b - (x_{-3}x_{-4}x_{-5})^{-1})}\right)^m, \tag{42}$$

$$x_{6m+3} = x_{-3} \left(\frac{(x_{-1}x_{-2}x_{-3})^{-1}(b - (x_{-1}x_{-2}x_{-3})^{-1})}{(x_{-3}x_{-4}x_{-5})^{-1}(b - (x_{-3}x_{-4}x_{-5})^{-1})}\right)^{m+1}, \tag{43}$$

$$x_{6m+i} = x_{i-6} \left(\frac{(x_{i-4}x_{i-5}x_{i-6})^{-1}(b - (x_{i-4}x_{i-5}x_{i-6})^{-1})}{(x_{i-3}x_{i-4}x_{i-5})^{-1}(b - (x_{i-3}x_{i-4}x_{i-5})^{-1})}\right)^{m+1}, \tag{44}$$

$$x_{6m+3+i} = x_{i-3} \left(\frac{(x_{i-4}x_{i-5}x_{i-6})^{-1}(b - (x_{i-4}x_{i-5}x_{i-6})^{-1})}{(x_{i-3}x_{i-4}x_{i-5})^{-1}(b - (x_{i-3}x_{i-4}x_{i-5})^{-1})}\right)^{m+1}, \tag{45}$$

for $i \in \{1, 2\}$.

From (42)-(45) it is not difficult to describe the asymptotic behavior of well-defined solutions of equation (18) for the case $a = -1$, in terms of the quantities

$$N_0 := \frac{(x_{-1}x_{-2}x_{-3})^{-1}(b - (x_{-1}x_{-2}x_{-3})^{-1})}{(x_{-3}x_{-4}x_{-5})^{-1}(b - (x_{-3}x_{-4}x_{-5})^{-1})}$$

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and

$$N_i := \frac{(x_{i-4}x_{i-5}x_{i-6})^{-1}(b - (x_{i-4}x_{i-5}x_{i-6})^{-1})}{(x_{i-3}x_{i-4}x_{i-5})^{-1}(b - (x_{i-3}x_{i-4}x_{i-5})^{-1})}, \quad i = 1, 2.$$

Namely, it is easy to see that the following result holds.

Theorem 4. *Assume that $a = -1$, $b \neq 0$, and $(x_n)_{n \geq -5}$ is a well-defined solution of equation* (18). *Then the following statements are true.*

(a) If $|N_i| < 1$ for some $i \in \{0, 1, 2\}$, then $x_{6m+3j+i} \to 0$, $j = 0, 1$, as $m \to +\infty$.
(b) If $|N_i| > 1$ for some $i \in \{0, 1, 2\}$, then $|x_{6m+3j+i}| \to +\infty$, $j = 0, 1$, as $m \to +\infty$.
(c) If $N_i = 1$ for some $i \in \{0, 1, 2\}$, then the sequences $(x_{6m+3j+i})_{m \geq -1}$, $j = 0, 1$, are constant.
(d) If $N_i = -1$ for some $i \in \{0, 1, 2\}$, then the sequences $(x_{6m+3j+i})_{m \geq -1}$, $j = 0, 1$, are two-periodic.

Now we consider the case $a \neq 0$, $b = 0$. In this case equation (18) becomes

$$x_n = \frac{x_{n-3}x_{n-4}x_{n-5}}{x_{n-1}x_{n-2}a}, \quad n \in \mathbb{N}_0, \tag{46}$$

and formulas (21)-(24) also hold. Hence for $a \in \mathbb{R} \setminus \{0\}$, we have

$$x_{3m} = x_{-3}\left(\frac{x_{-4}x_{-5}}{ax_{-1}x_{-2}}\right)^{m+1}, \tag{47}$$

$m \geq -1$, and

$$x_{3m+i} = x_{i-3}\left(\frac{x_{i-3}}{x_{i-6}}\right)^{m+1}, \tag{48}$$

for $m \geq -1$ and for $i \in \{1, 2\}$.

Let

$$L_3 := \frac{x_{-4}x_{-5}}{ax_{-1}x_{-2}} \quad \text{and} \quad L_{3+i} := \frac{x_{i-3}}{x_{i-6}}, \quad i \in \{1, 2\}.$$

By using formulas (47) and (48) it is easy to see that the following result holds. We omit the proof.

Theorem 5. *Assume that $a \neq 0$, $b = 0$, and $(x_n)_{n \geq -5}$ is a well-defined solution of equation* (18). *Then the following statements are true.*

(a) If $|L_{3+i}| < 1$, for some $i \in \{0, 1, 2\}$, then $x_{3m+i} \to 0$ as $m \to +\infty$.
(b) If $|L_{3+i}| > 1$, for some $i \in \{0, 1, 2\}$, then $|x_{3m+i}| \to \infty$ as $m \to +\infty$.
(c) If $L_{3+i} = 1$, for some $i \in \{0, 1, 2\}$, then the sequence $(x_{3m+i})_{m \geq -2}$ is constant.
(d) If $L_{3+i} = -1$, for some $i \in \{0, 1, 2\}$, then the sequence $(x_{3m+i})_{m \geq -2}$ is two-periodic.

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4. Domain of undefinable solutions of equation (5)

In Section 2 we proved that solutions of equation (5) for which is $x_{-j} = 0$ for some $j \in \{1, 2, 3, 4, 5\}$ are not defined. The set of all such initial values is characterized here.

Definition 1. ([32]) Consider the difference equation

$$x_n = f(x_{n-1}, \dots, x_{n-s}, n), \quad n \in \mathbb{N}_0, \tag{49}$$

where $s \in \mathbb{N}$, and $x_{-i} \in \mathbb{R}$, $i = \overline{1, s}$. The string of numbers $x_{-s}, \dots, x_{-1}, x_0, \dots, x_{n_0}$ where $n_0 \geq -1$, is called an *undefined solution* of equation (49) if

$$x_j = f(x_{j-1}, \dots, x_{j-s}, j)$$

for $0 \leq j < n_0 + 1$, and x_{n_0+1} is not defined number, that is, the quantity $f(x_{n_0}, \dots, x_{n_0-s+1}, n_0 + 1)$ is not defined.

The set of all initial values $x_{-s}, \dots, x_{-1}$ which generate undefined solutions of equation (49) is called *domain of undefinable solutions* of the equation.

The next result characterizes the domain of undefinable solutions of equation (5) for the case $a_n \neq 0$, $b_n \neq 0$, $n \in \mathbb{N}_0$.

Theorem 6. *Assume that $a_n \neq 0$, $b_n \neq 0$, $n \in \mathbb{N}_0$. Then the domain of undefinable solutions of equation (5) is the following set*

$$\mathcal{U} = \bigcup_{m \in \mathbb{N}_0} \bigcup_{i=0}^{2} \Big\{ (x_{-5}, \dots, x_{-1}) \in \mathbb{R}^5 : x_{i-3} x_{i-4} x_{i-5} = \frac{1}{c_m}, \ \textit{when} \ c_m := -\sum_{j=0}^{m} \frac{b_{3j+i}}{a_{3j+i}} \prod_{l=0}^{j-1} \frac{1}{a_{3l+i}} \neq 0 \Big\}$$
$$\bigcup \bigcup_{j=1}^{5} \Big\{ (x_{-5}, \dots, x_{-1}) \in \mathbb{R}^5 : x_{-j} = 0 \Big\}. \tag{50}$$

Proof. As we have already mentioned the set

$$\bigcup_{j=1}^{5} \Big\{ (x_{-5}, \dots, x_{-1}) \in \mathbb{R}^5 : x_{-j} = 0 \Big\},$$

belongs to the domain of undefinable solutions of equation (5).

Now we will consider the case when $x_{-j} \neq 0$, $j = \overline{1, 5}$ (i.e. $x_n \neq 0$ for every $n \geq -5$). Such a solution $(x_n)_{n \geq -5}$ is not defined if

$$x_{n-3} x_{n-4} x_{n-5} = -\frac{a_n}{b_n} \tag{51}$$

for some $n \in \mathbb{N}_0$.

Since the change of variables (8) implies that equation (5) is transformed to the equations in (10), this along with (51) implies that the solution is not defined if

$$y_{3(m-1)+i} = -\frac{b_{3m+i}}{a_{3m+i}} \tag{52}$$

for some $m \in \mathbb{N}_0$ and $i \in \{0, 1, 2\}$.

Set

$$f_{3m+i}(t) := a_{3m+i} t + b_{3m+i}, \quad m \in \mathbb{N}_0, \ i \in \{0, 1, 2\}.$$

Then $f_{3m+i}^{-1}(t) = (t - b_{3m+i})/a_{3m+i}$, $m \in \mathbb{N}_0$, $i \in \{0, 1, 2\}$, so that

$$f_{3m+i}^{-1}(0) = -\frac{b_{3m+i}}{a_{3m+i}}, \quad m \in \mathbb{N}_0, \ i \in \{0, 1, 2\}. \tag{53}$$

Now write equations in (10) as

$$y_{3m+i} = f_{3m+i}(y_{3(m-1)+i}), \quad m \in \mathbb{N}_0,$$

for $i \in \{0, 1, 2\}$.

Then, we have

$$y_{3m+i} = f_{3m+i} \circ f_{3(m-1)+i} \circ \cdots \circ f_i(y_{i-3}), \quad m \in \mathbb{N}_0,\ i \in \{0, 1, 2\}. \tag{54}$$

From (53) and (54) we have that (52) holds for some $m \in \mathbb{N}_0$, $i \in \{0, 1, 2\}$, if and only if

$$y_{i-3} = f_i^{-1} \circ \cdots \circ f_{3m+i}^{-1}(0),$$

that is,

$$y_{i-3} = -\sum_{j=0}^{m} \frac{b_{3j+i}}{a_{3j+i}} \prod_{l=0}^{j-1} \frac{1}{a_{3l+i}},$$

for some $m \in \mathbb{N}_0$ and $i \in \{0, 1, 2\}$, which along with the relations

$$y_{i-3} = \frac{1}{x_{i-3}x_{i-4}x_{i-5}}, \quad i \in \{0, 1, 2\},$$

implies that the first union in (50) belongs to the domain of undefinable solutions too, as desired. □

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MODIFIED THREE-STEP ITERATIVE SCHEMES FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN UNIFORMLY CONVEX METRIC SPACES

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Abstract. We prove the existence of the common fixed point for three asymptotically nonexpensive mappings defined on a $\mathcal{A}$-uniformly convex metric space. A three-step iterative scheme is constructed which converges to the common fixed point. We also generalize the results of several authors.

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1. Introduction

It is well known that the parallelogram law distinguishes the Hilbert spaces from the general Banach spaces. Recently many authors have introduced the idea for solving problems in Banach spaces by establishing identities and inequalities analogous to the parallelogram law (see for example [9, 20]).

In 1965, the Banach contraction principle was extended to nonexpansive mappings by Browder [3], Goehde [7] and Kirk [10]. In [10], Kirk proved that

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there exists a k-Lipschitzian map which has no fixed point. Goebel and Kirk [6] introduced the notion of asymptotically nonexpansive mappings and obtained a generalization of the results obtained in [3, 7, 10]. Afterwards Takahahashi [20] introduced the notion of convexity in metric spaces. Subsequently, Ćirić [5], Guay et al. [8], Shimizu and Takahashi [15] and many other authors have studied fixed point theorems on convex metric spaces. Shimizu and Takahashi [16] introduced the concept of uniform convexity in convex metric spaces and studied its properties.

Definition 1.1. ([18]) Let (X, d) be a metric space. A mapping $W : X \times X \times [0, 1] \to X$ is said to be a *convex structure* on X if for each $(x, y, \lambda) \in X \times X \times [0, 1]$ and $u \in X$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y).$$

The metric space X together with W is called a *convex metric space*.

Definition 1.2. Let X be a convex metric space. A nonempty subset A of X is said to be *convex* if $W(x, y, \lambda) \in A$ whenever $(x, y, \lambda) \in A \times A \times [0, 1]$.

Takahashi [18] has shown that open spheres $B(x, r) = \{y \in X : d(x, y) < r\}$ and closed spheres $B(x, r) = \{y \in X : d(x, y) \leq r\}$ are convex. All normed spaces and their convex subsets are convex metric spaces. But there are many examples of convex metric spaces which are not embedded in any normed space (see Takahashi [18]).

Recently, Beg [1] introduced and studied the notion of 2-uniformly convex metric spaces.

Definition 1.3. ([1]) A convex metric space X is said to have *property* (B) if it satisfies $d(W(x, a, \alpha), W(y, a, \alpha)) = \alpha d(x, y)$. Taking $x = a$, property (B) implies $\alpha d(x, y) = W(y, a, \alpha)$.

Definition 1.4. ([1]) A convex complete metric space X is said to be *uniformly convex* if for all $x, y, a \in X$,

$$\begin{aligned}&[d\left(a, W\left(x, y, 1/2\right)\right)]^2\\ &\leq \frac{1}{2}\left(1 - \delta\left(\frac{d(x, y)}{\max\left\{d(a, x), d(a, y)\right\}}\right)\right)\left([d(a, x)]^2 + [d(a, y)]^2\right),\end{aligned}$$

where the function δ is a strictly increasing function on the set of strictly positive numbers and $\delta(0) = 0$.

Remark 1.5. ([1]) Uniformly convex Banach spaces are uniformly convex metric spaces.

Definition 1.6. ([1]) A uniformly convex metric space X is said to be *2-uniformly convex* if there exists a constant $c > 0$ such that $\delta(\epsilon) \geq c\epsilon^2$.

Definition 1.7. (1) Let A be a nonempty subset of a metric space X. A mapping $T : A \to A$ is said to be *asymptotically nonexpansive* [6] if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\sum_{n=1}^{\infty}(k_n - 1) < +\infty$ such that

$$d(T^n x, T^n y) \leq k_n d(x, y) \quad \text{for all } x, y \in A,\ n \geq 1.$$

(2) T is said to be *uniformly L-Lipschitzian* with a Lipschitzian constant $L \geq 1$, i.e., there exists a constant $L \geq 1$ such that

$$d(T^n x, T^n y) \leq L d(x, y) \quad \text{for all } x, y \in A,\ n \geq 1.$$

This is a class of mapping introduced by Goebel and Kirk [6], where it is shown that if A is a nonempty bounded closed convex subset of a uniformly convex Banach space and $T : A \to A$ is asymptotically nonexpansive, then T has a fixed point and, moreover, the set $F(T)$ of fixed points of T is closed and convex.

Remark 1.8. As an application of the Lagrange mean value theorem, we can see that

$$t^q - 1 \leq q t^q (t - 1)$$

for $t \geq 1$ and $q > 1$. This together with $\sum_{n=1}^{\infty}(k_n - 1) < +\infty$ implies that $\sum_{n=1}^{\infty}(k_n^q - 1) < +\infty$.

Theorem 1.9. ([16, Theorem 1]) *If (X, d) is uniformly convex complete metric space then every decreasing sequence of nonempty closed bounded convex subsets of X has nonempty intersection.*

Definition 1.10. Let (X, d) be a metric space and Y be a topological space. A mapping $T : X \to X$ is said to be *completely continuous* if the image of each bounded set in X is contained in a compact subset of Y.

In [1, 2], Beg proved the following remarkable results.

Theorem 1.11. *Let (X, d) be a uniformly convex metric space having property (B). Then X is 2-unformly convex if and only if there exists a number $c > 0$ such that*

$$2\left[d\left(a, W\left(x, y, 1/2\right)\right)\right]^2 + c\left[d(x, y)\right]^2 \leq \left[d(a, x)\right]^2 + \left[d(a, y)\right]^2 \tag{1.1}$$

for all $a, x, y \in X$.

Theorem 1.12. *Let A be a nonempty closed bounded convex subset of a uniformly convex complete metric space (X, d) and $T : A \to A$ be an asymptotically nonexpansive mapping. Then T has a fixed point.*

Theorem 1.13. *Let (X, d) be a convex metric space and A be a nonempty convex subset of X. Let $L > 0$ and $T : A \to A$ be uniformly L-Lipschitzian. For $x_1 \in A$. Define*

$$y_n = W(T^n x_n, x_n, 1/2), \quad x_{n+1} = W(T^n y_n, x_n, 1/2)$$

and set $c_n = d(T^n x_n, x_n)$ for all $n \in \mathbb{N}$. Then

$$d(x_n, Tx_n) \leq c_n + c_{n-1}(L + 3L^2 + 2L^3)$$

for all $n \in \mathbb{N}$.

Theorem 1.14. *Let (X, d) be a 2-uniformly convex metric space having property (B), A be a nonempty closed bounded convex subset of X and $T : A \to A$ be asymptotically nonexpansive with sequence $\{k_n\} \in [1, +\infty)^{\mathbb{N}}$ and $\sum_{n=1}^{\infty} (k_n^2 - 1) < +\infty$. Let $x_1 \in A$ and $x_{n+1} = W(T^n x_n, x_n, 1/2)$ for all $n \in \mathbb{N}$.*
Then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Theorem 1.15. *Let (X, d) be 2-uniformly convex metric space having property (B), A be a nonempty closed bounded convex subset of X and $T : A \to A$ be completely continuous asymptotically nonexpansive mapping with sequence $\{k_n\} \in [1, +\infty)^{\mathbb{N}}$ and $\sum_{n=1}^{\infty} (k_n^2 - 1) < +\infty$.*

Let $x_1 \in A$ and $x_{n+1} = W(T^n x_n, x_n, \frac{1}{7}2)$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges to some fixed point of T.

In [12], Rafiq introduced the notion of $\mathcal{A}$-uniformly convex metric space defined as follows:

Definition 1.16. A convex complete metric space X is said to be $\mathcal{A}$-*uniformly convex* if for all $x, y, a \in X$,

$$\begin{aligned}[d\left(a, W\left(x, y, \lambda\right)\right)]^2 \leq \max\{\lambda, 1 - \lambda\} &\left(1 - \delta\left(\frac{d(x, y)}{\max\left\{d(a, x), d(a, y)\right\}}\right)\right) \\ &\times \left([d(a, x)]^2 + [d(a, y)]^2\right),\end{aligned} \tag{1.2}$$

where the function δ is a strictly increasing function on the set of strictly positive numbers and $\delta(0) = 0$.

Remark 1.17. 1. The inequality (1.2) can be easily proved in a Hilbert space H using the well known identity [9]

$$\|\lambda x+(1-\lambda)y\|^2=\lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)\|x-y\|^2$$

for all $x,y\in H$.

2. For $\lambda=1/2$ in (1.2), we get the inequality (1.1).
3. Uniformly convex Banach spaces are $\mathcal{A}$-uniformly convex metric spaces.

Definition 1.18. ([12]) The $\mathcal{A}$-uniformly convex metric space X is said to be $(2,\mathcal{A})$-*uniformly convex* if there exists a constant $c>0$ such that $\delta(\epsilon)\geq c\epsilon^2$.

The purpose of this paper is to generalize the results of [2, 4, 6, 11, 13, 14, 17, 19, 21] and construct a three-step iterative scheme, convergent to the common fixed point, for three asymptotically nonexpansive mappings defined on a $\mathcal{A}$-uniformly convex metric space.

2. Main Results

In the sequel, we will need the following results.

The following lemma is now well known.

Lemma 2.1. *Let $\{a_n\}$ and $\{b_n\}$ be sequences of non-negative real numbers such that $a_{n+1}\leq(1+b_n)a_n$ for all $n\geq 1$ and $\sum_{n=1}^{\infty}b_n<\infty$. Then $\lim_{n\to\infty}a_n$ exists.*

Theorem 2.2. ([2, 12]) *Let A be a nonempty closed convex subset of a uniformly convex complete metric space (X,d) and $T:A\to A$ be an asymptotically nonexpansive mapping. Then the set $F(T)$ of fixed points of T is closed and convex.*

Theorem 2.3. ([12]) *Let (X,d) be a $\mathcal{A}$-uniformly convex metric space. Then X is $(2,A)$-uniformly convex if and only if there exists a number $c>0$ such that for all a, x, y in X and $\lambda\in[0,1]$,*

$$\begin{aligned}&[d(a,W(x,y,\lambda))]^2\\&\leq\max\{\lambda,1-\lambda\}\left[[d(a,x)]^2+[d(a,y)]^2-c\,[d(x,y)]^2\right].\end{aligned}\tag{2.1}$$

Theorem 2.4. *Let (X,d) be a $(2,\mathcal{A})$-uniformly convex metric space, A be a nonempty closed convex subset of X and $T,S,H:A\to A$ be asymptotically nonexpansive with sequence $\{k_n\}\in[1,+\infty)^{\mathbb{N}}$ and $\sum_{n=1}^{\infty}(k_n-1)<+\infty$. Let $x_1\in A$ and $x_{n+1}=W(T^ny_n,x_n,\alpha_n)$, $y_n=W(S^nz_n,x_n,\beta_n)$, $z_n=$*

$W(H^n x_n, x_n, \beta_n)$ for all $n \in \mathbb{N}$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are the real sequences in $[0,1]$ satisfying $\alpha_n, \beta_n, \gamma_n \in [\delta, 1-\delta]$ for some $\delta \in (0,1)$. Then $\lim_{n\to\infty} d(T^n y_n, x_n) = 0 = \lim_{n\to\infty} d(S^n z_n, x_n) = \lim_{n\to\infty} d(H^n x_n, x_n)$.

Proof. Since T, S and H are asymptotically nonexpensive, so each mapping possesses a fixed point $p \in A$ by Theorem 1.12. Let $p \in F(T) \cap F(S) \cap F(H)$.

Claim. $\{x_n\}$ is bounded.

For this claim, we compute as follows:

$$\begin{aligned} d(p, x_{n+1}) &= d(p, W(T^n y_n, x_n, \alpha_n)) \\ &\le \alpha_n d(p, T^n y_n) + (1-\alpha_n) d(p, x_n) \\ &= \alpha_n d(T^n p, T^n y_n) + (1-\alpha_n) d(p, x_n) \\ &\le \alpha_n k_n d(p, y_n) + (1-\alpha_n) d(p, x_n), \end{aligned} \tag{2.2}$$

$$\begin{aligned} d(p, y_n) &= d(p, W(S^n z_n, x_n, \beta_n)) \\ &\le \beta_n d(p, S^n z_n) + (1-\beta_n) d(p, x_n) \\ &= \beta_n d(S^n p, S^n z_n) + (1-\beta_n) d(p, x_n) \\ &\le \beta_n k_n d(p, z_n) + (1-\beta_n) d(p, x_n), \end{aligned} \tag{2.3}$$

$$\begin{aligned} d(p, z_n) &= d(p, W(H^n x_n, x_n, \beta_n)) \\ &\le \beta_n d(p, H^n x_n) + (1-\beta_n) d(p, x_n) \\ &= \beta_n k_n d(H^n p, H^n x_n) + (1-\beta_n) d(p, x_n) \\ &\le \beta_n k_n d(p, x_n) + (1-\beta_n) d(p, x_n) \\ &= [1 + (k_n - 1)\beta_n] d(p, x_n) \\ &\le k_n d(p, x_n). \end{aligned} \tag{2.4}$$

Substituting (2.4) in (2.3) gives

$$\begin{aligned} d(p, y_n) &\le \left[1 + \left(k_n^2 - 1\right) \beta_n\right] d(p, x_n) \\ &\le k_n^2 d(p, x_n). \end{aligned} \tag{2.5}$$

From (2.5) in (2.2), we get

$$\begin{aligned} d(p, x_{n+1}) &\le \left[1 + \left(k_n^3 - 1\right) \alpha_n\right] d(p, x_n) \\ &\le \left[1 + \left(k_n^3 - 1\right)\right] d(p, x_n), \end{aligned}$$

which implies that $\lim_{n\to\infty} d(p, x_n)$ exists and $\{x_n\}$ is bounded.

Let $M{=}\sup_{n\geq 0} d(p,x_n)$. Now with the help of (2.1), we have

$$\begin{aligned}
&[d\left(p,x_{n+1}\right)]^2\\
&=[d(p,W\left(T^ny_n,x_n,\alpha_n\right))]^2\\
&\leq \max\{\alpha_n,1-\alpha_n\}\left[\,[d\left(p,T^ny_n\right)]^2+[d(p,x_n)]^2-c\,[d\left(T^ny_n,x_n\right)]^2\,\right]\\
&\leq [d\left(p,T^ny_n\right)]^2+[d(p,x_n)]^2-\max\{\alpha_n,1-\alpha_n\}c\,[d\left(T^ny_n,x_n\right)]^2\\
&=[d\left(T^np,T^ny_n\right)]^2+[d(p,x_n)]^2-\max\{\alpha_n,1-\alpha_n\}c\,[d\left(T^ny_n,x_n\right)]^2\\
&\leq k_n^2[d\left(p,y_n\right)]^2+[d(p,x_n)]^2-\max\{\alpha_n,1-\alpha_n\}c\,[d\left(T^ny_n,x_n\right)]^2,
\end{aligned}\tag{2.6}$$

$$\begin{aligned}
&[d\left(p,y_n\right)]^2\\
&=[d(p,W\left(S^nz_n,x_n,\beta_n\right))]^2\\
&\leq \max\{\beta_n,1-\beta_n\}\left[\,[d\left(p,S^nz_n\right)]^2+[d(p,x_n)]^2-c\,[d\left(S^nz_n,x_n\right)]^2\,\right]\\
&\leq [d\left(p,S^nz_n\right)]^2+[d(p,x_n)]^2-\max\{\beta_n,1-\beta_n\}c\,[d\left(S^nz_n,x_n\right)]^2\\
&=[d\left(S^np,S^nz_n\right)]^2+[d(p,x_n)]^2-\max\{\beta_n,1-\beta_n\}c\,[d\left(S^nz_n,x_n\right)]^2\\
&\leq k_n^2[d\left(p,z_n\right)]^2+[d(p,x_n)]^2-\max\{\beta_n,1-\beta_n\}c\,[d\left(S^nz_n,x_n\right)]^2,
\end{aligned}\tag{2.7}$$

$$\begin{aligned}
&[d\left(p,z_n\right)]^2\\
&=[d(p,W\left(H^nx_n,x_n,\gamma_n\right))]^2\\
&\leq \max\{\gamma_n,1-\gamma_n\}\left[[d\left(p,H^nx_n\right)]^2+[d(p,x_n)]^2-c\,[d\left(H^nx_n,x_n\right)]^2\right]\\
&\leq [d\left(p,H^nx_n\right)]^2+[d(p,x_n)]^2-\max\{\gamma_n,1-\gamma_n\}c\,[d\left(H^nx_n,x_n\right)]^2\\
&=[d\left(H^np,H^nx_n\right)]^2+[d(p,x_n)]^2-\max\{\gamma_n,1-\gamma_n\}c\,[d\left(H^nx_n,x_n\right)]^2\\
&\leq k_n^2[d\left(p,x_n\right)]^2+[d(p,x_n)]^2-\max\{\gamma_n,1-\gamma_n\}c\,[d\left(H^nx_n,x_n\right)]^2\\
&=(1+k_n^2)\,[d(p,x_n)]^2-\max\{\gamma_n,1-\gamma_n\}c\,[d\left(H^nx_n,x_n\right)]^2.
\end{aligned}\tag{2.8}$$

Substituting (2.8) in (2.7), we get

$$\begin{aligned}
[d\left(p,y_n\right)]^2\leq{}&(1+k_n^2+k_n^4)\,[d(p,x_n)]^2\\
&-k_n^2\max\{\gamma_n,1-\gamma_n\}c\,[d\left(H^nx_n,x_n\right)]^2\\
&-\max\{\beta_n,1-\beta_n\}c\,[d\left(S^nz_n,x_n\right)]^2,
\end{aligned}\tag{2.9}$$

and by (2.6), we obtain

$$\begin{aligned}
&[d(p, x_{n+1})]^2 \\
&\le (1 + k_n^2 + k_n^4 + k_n^6)\, [d(p, x_n)]^2 - \max\{\alpha_n, 1-\alpha_n\} c\, [d(T^n y_n, x_n)]^2 \\
&\quad - k_n^2 \max\{\beta_n, 1-\beta_n\} c\, [d(S^n z_n, x_n)]^2 \\
&\quad - k_n^4 \max\{\gamma_n, 1-\gamma_n\} c\, [d(H^n x_n, x_n)]^2 \\
&\le [1 + (k_n^7 - 1)]\, [d(p, x_n)]^2 - \max\{\alpha_n, 1-\alpha_n\} c\, [d(T^n y_n, x_n)]^2 \\
&\quad - k_n^2 \max\{\beta_n, 1-\beta_n\} c\, [d(S^n z_n, x_n)]^2 \\
&\quad - k_n^4 \max\{\gamma_n, 1-\gamma_n\} c\, [d(H^n x_n, x_n)]^2 .
\end{aligned} \tag{2.10}$$

With the help of condition $\alpha_n, \beta_n, \gamma_n \in [\delta, 1-\delta]$ for some $\delta \in (0,1)$, it can be easily seen that

$$\max\{\alpha_n, 1-\alpha_n\}, \quad \max\{\beta_n, 1-\beta_n\}, \quad \max\{\gamma_n, 1-\gamma_n\} \ge \delta. \tag{2.11}$$

Using (2.11) in (2.10) and by $k_n \ge 1$, we obtain

$$\begin{aligned}
[d(p, x_{n+1})]^2 \le{}& [1 + (k_n^7 - 1)]\, [d(p, x_n)]^2 \\
& - \delta c\Big([d(T^n y_n, x_n)]^2 + [d(S^n z_n, x_n)]^2 + [d(H^n x_n, x_n)]^2 \Big),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\delta c \left([d(T^n y_n, x_n)]^2 + [d(S^n z_n, x_n)]^2 + [d(H^n x_n, x_n)]^2 \right) \\
&\le [d(p, x_n)]^2 - [d(p, x_{n+1})]^2 + M^2 (k_n^7 - 1).
\end{aligned}$$

Thus

$$\begin{aligned}
&\delta c \left(\sum_{j=1}^{m} \left[d(T^j y_j, x_j)\right]^2 + \sum_{j=1}^{m} \left[d(H^j x_j, x_j)\right]^2 + \sum_{j=1}^{m} \left[d(S^j z_j, x_j)\right]^2 \right) \\
&\le \frac{M^2}{2} \sum_{j=1}^{m} \left(k_j^7 - 1\right) + \sum_{j=1}^{m} \Big([d(p, x_j)]^2 - [d(p, x_{j+1})]^2 \Big).
\end{aligned}$$

Hence

$$\sum_{j=1}^{\infty} \left[d(T^j y_j, x_j)\right]^2 < +\infty,$$

$$\sum_{j=1}^{\infty} \left[d(S^j z_j, x_j)\right]^2 < +\infty,$$

and

$$\sum_{j=1}^{\infty} \left[d(H^j x_j, x_j)\right]^2 < +\infty.$$

It implies that

$$\lim_{n\to\infty} d(T^n y_n, x_n) = 0,$$
$$\lim_{n\to\infty} d(S^n z_n, x_n) = 0,$$

and

$$\lim_{n\to\infty} d(H^n x_n, x_n) = 0.$$

This completes the proof. □

Theorem 2.5. *Let (X, d) be a $(2, \mathcal{A})$-uniformly convex metric space, A be a nonempty closed convex subset of X and $T, S, H : A \to A$ be asymptotically nonexpansive with sequence $\{k_n\} \in [1, +\infty)^{\mathbb{N}}$ and $\sum_{n=1} \infty (k_n - 1) < +\infty$. Further let H be completely continuous and T and S are continuous. Let $x_1 \in A$ and $x_{n+1} = W(T^n y_n, x_n, \alpha_n)$, $y_n = W(S^n z_n, x_n, \beta_n)$, $z_n = W(H^n x_n, x_n, \beta_n)$ for all $n \in \mathbb{N}$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are the real sequences in $[0, 1]$ satisfying $\alpha_n, \beta_n, \gamma_n \in [\delta, 1-\delta]$ for some $\delta \in (0, 1)$. Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge to the common fixed point of T, S and H.*

Proof. Consider

$$\begin{aligned}
d(x_{n+1}, H^n x_{n+1}) &\le d(x_{n+1}, x_n) + d(x_n, H^n x_n) + d(H^n x_n, H^n x_{n+1}) \\
&\le d(x_{n+1}, x_n) + d(x_n, H^n x_n) + k_n d(x_n, x_{n+1}) \\
&= (1 + k_n) d(x_{n+1}, x_n) + d(x_n, H^n x_n) \\
&= (1 + k_n) d(W(T^n y_n, x_n, 1/2), x_n) + d(x_n, H^n x_n) \\
&= (1 + k_n)\, \alpha_n d(T^n y_n, x_n) + d(x_n, H^n x_n) \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Thus

$$\begin{aligned}
d(x_{n+1}, H x_{n+1}) &\le d(x_{n+1}, H^{n+1} x_{n+1}) + d(H^{n+1} x_{n+1}, H x_{n+1}) \\
&\le d(x_{n+1}, H^{n+1} x_{n+1}) + k_1 d(x_{n+1}, H^n x_{n+1}) \\
&\to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} d(x_n, H x_n) = 0.$$

Since H is completely continuous and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{Hx_{n_k}\}$ converges.

Therefore from $\lim_{n\to\infty} d(x_n, Hx_n) = 0$, $\{x_{n_k}\}$ converges. Let $\lim_{n\to\infty} x_{n_k} = p$. It follows from the continuity of H and $\lim_{n\to\infty} d(x_n, Hx_n) = 0$ that $p = Hp$. We know that $\lim_{n\to\infty} d(p, x_n)$ exists. But $\lim_{n\to\infty} d(p, x_{n_k}) = 0$. This implies $\lim_{n\to\infty} d(p, x_n) = 0$, i.e., $\lim_{n\to\infty} x_n = p$. Since

$$\begin{aligned} d(x_n, z_n) &= d(x_n, W(H^n x_n, x_n, \gamma_n)) = \gamma_n d(x_n, H^n x_n) \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned}$$

and

$$\begin{aligned} d(x_n, y_n) &= d(x_n, W(S^n z_n, x_n, \beta_n)) = \beta_n d(x_n, S^n z_n) \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned}$$

so $\lim_{n\to\infty} z_n = p = \lim_{n\to\infty} y_n$.

The following estimates hold:

$$\begin{aligned} d(x_n, x_{n+1}) &= d(x_n, W(T^n y_n, x_n, \alpha_n)) = \alpha_n d(x_n, T^n y_n) \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned}$$

$$\begin{aligned} d(y_n, y_{n-1}) &\le d(y_n, x_n) + d(x_n, x_{n-1}) + d(x_{n-1}, y_{n-1}) \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned}$$

$$\begin{aligned} d(z_n, z_{n-1}) &\le d(z_n, x_n) + d(x_n, x_{n-1}) + d(x_{n-1}, z_{n-1}) \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned}$$

$$\begin{aligned} d(z_n, S^n z_n) &\le d(z_n, x_n) + d(x_n, S^n z_n) \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned}$$

$$\begin{aligned} d(y_n, T^n y_n) &\le d(y_n, x_n) + d(x_n, T^n y_n) \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned}$$

$$\begin{aligned} & d(z_n, S^{n-1} z_n) \\ & \le d(z_n, z_{n-1}) + d(z_{n-1}, S^{n-1} z_{n-1}) + d(S^{n-1} z_{n-1}, S^{n-1} z_n) \\ & \le d(z_n, z_{n-1}) + d(z_{n-1}, S^{n-1} z_{n-1}) + k_{n-1} d(z_{n-1}, z_n) \\ & = (1 + k_{n-1}) d(z_n, z_{n-1}) + d(z_{n-1}, S^{n-1} z_{n-1}) \\ & \to 0 \quad \text{as } n \to \infty, \end{aligned} \tag{2.12}$$

$$\begin{aligned} d(z_n, Sz_n) &\leq d(z_n, S^n z_n) + d(S^n z_n, Sz_n) \\ &\leq d(z_n, S^n z_n) + k_1 d(S^{n-1} z_n, z_n) \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned} \tag{2.13}$$

$$\begin{aligned} &d(y_n, T^{n-1} y_n) \\ &\leq d(y_n, y_{n-1}) + d(y_{n-1}, T^{n-1} y_{n-1}) + d(T^{n-1} y_{n-1}, T^{n-1} y_n) \\ &\leq d(y_n, y_{n-1}) + d(y_{n-1}, T^{n-1} y_{n-1}) + k_{n-1} d(y_n, y_{n-1}) \\ &= (1 + k_{n-1}) d(y_n, y_{n-1}) + d(y_{n-1}, T^{n-1} y_{n-1}) \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned} \tag{2.14}$$

$$\begin{aligned} d(y_n, Ty_n) &\leq d(y_n, T^n y_n) + d(T^n y_n, Ty_n) \\ &\leq d(y_n, T^n y_n) + k_1 d(T^{n-1} y_n, y_n) \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{2.15}$$

Now according to the continuity of T and S and by using (2.15) and (2.13), we obtain $Tp = p = Sp$. Thus p is the common fixed point of T, S and H. This completes the proof. □

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ON IDENTITIES BETWEEN SUMS OF EULER NUMBERS AND GENOCCHI NUMBERS OF HIGHER ORDER

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Abstract. In this paper we consider differential equations which are closely related to the generating functions of Euler numbers. By using the same method of Kim's calculation in Kim [24,25], we derive identities involving Euler numbers arising from differential equations. In particular, we derive some new identities between the sums of Euler numbers and Genocchi numbers of higher order.

1. Introduction

We consider the Euler numbers defined by the generating function as follows(see [4,8,17,21-25]):

$$E(t) = \frac{2}{e^t+1} = \sum_{k=0}^{\infty} E_k \frac{t^k}{k!} \tag{1}$$

and the Genocchi numbers defined by the generating function as follows(see [2,3,5,7,9-16,18,20,26-30,32,33,35,36]):

$$G(t) = \frac{2t}{e^t+1} = \sum_{k=0}^{\infty} G_k \frac{t^k}{k!}. \tag{2}$$

Kim(2012) derived some new identities between the sums of products of Frobenius-Euler polynomials and Frobenius-Euler polynomials of higher order (see[1,6,31,34,37,38]).

In this paper we derive differential equations which are closely related to the generating function of Euler numbers. By using these differential equations, we derive some identities between the sums of products of Euler numbers and Euler numbers of higher order. In particular, we obtain some identities between the sums of Euler numbers and Genocchi numbers of higher order.

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2. Computation of sums of the products of Euler numbers

In this section we assume that

$$F = F(t) = \frac{1}{e^t + 1} \quad \text{and} \quad F^N = \underbrace{F \times \cdots \times F}_{N-\text{times}} \tag{3}$$

for $N \in \mathbb{N}$. Thus, by (3), we get

$$\begin{aligned} F(t)^{(1)} = \frac{dF(t)}{dt} &= \frac{-e^t}{(e^t+1)^2} \\ &= \frac{-1-e^{-t}}{(e^t+1)^2} + \frac{1}{(e^t+1)^2} \\ &= -F + F^2. \end{aligned} \tag{4}$$

Let us consider the derivative of (4) with respect to t as follows:

$$2FF^{(1)} = F^{(1)} + F^{(2)}. \tag{5}$$

Thus, by (5) and (3), we get

$$2!F^3 - 2F^2 = F^{(1)} + F^{(2)}. \tag{6}$$

From (6), we note that

$$\begin{aligned} 2!F^3 &= 2F^2 + F^{(1)} + F^{(2)} \\ &= 2(F + F^{(1)}) + F^{(1)} + F^{(2)} \\ &= 2F + 3F^{(1)} + F^{(2)}, \end{aligned} \tag{7}$$

where $F^{(2)} = \frac{d^2F}{dt^2}$. Thus, by the derivative of (5) with respect to t, we get

$$2!3F^2F^{(1)} = 2F^{(1)} + 3F^{(2)} + F^{(3)}. \tag{8}$$

By (8), we see that

$$2!3F^2(-F + F^2) = 2F^{(1)} + 3F^{(2)} + F^{(3)} \tag{9}$$

and

$$F^{(1)} = -F + F^2. \tag{10}$$

By (9), we see that

$$\begin{aligned} 3!(-1)^3F^4 &= 6F^3 + 2F^{(1)} + 3F^{(2)} + F^{(3)} \\ &= 6(F + \frac{3}{2}F^{(1)} + \frac{1}{2}F^{(2)}) + 2F^{(1)} + 3F^{(2)} + F^{(3)} \\ &= 6F + 11F^{(1)} + 6F^{(2)} + F^{(3)}. \end{aligned} \tag{11}$$

Continuing this process, we get

$$(N-1)!F^N = \sum_{k=0}^{N-1} a_K(N)F^{(k)} \tag{12}$$

where $F^{(k)} = \frac{d^kF}{dt^k}$ and $N \in \mathbb{N}$.

Now we try to find the coefficients $a_k(N)$ in (12). By (12), we differentiate the both sides of (12) as follows:

$$(N-1)!NF^{N-1}F^{(1)} = \sum_{k=0}^{N-1} a_k(N)F^{(k+1)}$$

$$= \sum_{k=1}^{N} a_{k-1}(N)F^{(k)}. \tag{13}$$

By (7), we get

$$\begin{aligned}(N-1)!NF^{N-1}F^{(1)} &= N!F^N(-F+F^2) \\ &= N!(-F^N+F^{N+1}) \\ &= -N!F^N + N!F^{N+1}. \end{aligned}\tag{14}$$

By (13) and (14), we get

$$\begin{aligned}N!F^{N+1} &= N!F^N + \sum_{k=1}^{N} a_{k-1}(N)F^{(k)} \\ &= N(N-1)!F^N + \sum_{k=1}^{N} a_{k-1}(N)F^{(k)} \\ &= N\sum_{k=0}^{N-1} a_k(N)F^{(k)} + \sum_{k=1}^{N} a_{k-1}(N)F^{(k)}. \end{aligned}\tag{15}$$

In (12), replacing N by $N+1$, we have

$$N!F^{N+1} = \sum_{k=0}^{N} a_k(N+1)F^{(k)}. \tag{16}$$

By (15) and (16), we get

$$\sum_{k=0}^{N} a_k(N+1)F^{(k)} = N\sum_{k=0}^{N-1} a_k(N)F^{(k)} + \sum_{k=1}^{N} a_{k-1}(N)F^{(k)}. \tag{17}$$

By comparing coefficients on the both sides of (17), we have the followings:

$$a_0(N+1) = Na_0(N) \text{ and } a_N(N+1) = a_{N-1}(N). \tag{18}$$

For $1 \le k \le n-1$, we have

$$a_k(N+1) = Na_k(N) = a_{k-1}(N) \tag{19}$$

where $a_k(N) = 0$ for $k \ge N$ or $k < 0$. From (19), we note that

$$\begin{aligned}a_0(N+1) &= Na_0(N) = N(N-1)a_0(N-1) \\ &= \cdots = N(N-1)\cdots 2a_0(2). \end{aligned}\tag{20}$$

By comparing coefficients on the both sides of (15) with $N = 1$,

$$\begin{aligned}F + F^{(1)} &= 1!F^2 = \sum_{k=0}^{1} a_k(2)F^{(k)} \\ &= a_0(2)F + a_1(2)F^{(1)}. \end{aligned}\tag{21}$$

Thus, by (21), we get

$$a_0(2) = 1 \text{ and } a_1(2) = 1. \tag{22}$$

Finally, we derive the values of $a_k(N)$ in (12) from (19).

Let us consider the following two variable function with variables t, s:

$$g(t,s) = \sum_{N\ge 1}\sum_{0\le k\le N-1} a_k(N)\frac{t^N s^k}{N!}, \text{ where } |t| < 1. \tag{23}$$

Then, Kim [25] derived the followings:

$$g(t,s)=\sum_{N\geq 1}\sum_{0\leq k\leq N-1}\frac{N!}{(k+1)!}\sum_{l_1+\cdots+l_{k+1}=N}\frac{1}{l_1\cdots l_{k+1}}\frac{t^N}{N1}s^k. \tag{24}$$

By (23) and (24), we get

$$a_k(N)=\frac{N!}{(k+1)!}\sum_{l_1+\cdots+l_{k+1}=N}\frac{1}{l_1\cdots l_{k+1}}. \tag{25}$$

Therefore, by (12) and (25), we obtain the following theorem.

Theorem 2.1. *For $n\in\mathbb{N}$, let us consider the following differential equation with respect to t:*

$$F^N(t)=N\sum_{k=0}^{N-1}\frac{1}{(k+1)!}\sum_{l_1+\cdots+l_{k+1}=N}\frac{1}{l_1\cdots l_{k+1}}F^{(k)}(t) \tag{26}$$

where $F^{(k)}(t)=\frac{d^kF(t)}{dt^k}$ and $F^N(t)=\underbrace{F(t)\times\cdots\times F(t)}_{N-times}$. Them $F(t)=\frac{1}{e^t+1}$ is a solution of (26).

We assume that

$$\underbrace{\left(\frac{2}{e^t+1}\right)\cdots\left(\frac{2}{e^t+1}\right)}_{N-\text{times}}=\sum_{n=0}^{\infty}E_n^{(N)}\frac{t^n}{n!} \tag{27}$$

where $E_n^{(N)}$ are called the n-th Euler numbers of order N. By (3) and (27), we get

$$\begin{aligned}F^N(t) &= \underbrace{\left(\frac{1}{e^t+1}\right)\cdots\left(\frac{1}{e^t+1}\right)}_{N-\text{times}}\\ &= \frac{1}{2^N}\underbrace{\left(\frac{2}{e^t+1}\right)\cdots\left(\frac{2}{e^t+1}\right)}_{N-\text{times}}\\ &= \frac{1}{2^N}\sum_{n=0}^{\infty}E_n^{(N)}\frac{t^n}{n!}\end{aligned} \tag{28}$$

and

$$\begin{aligned}F(t) &= \frac{1}{2^N}\left(\frac{2}{e^t+1}\right)\\ &= \frac{1}{2}\sum_{l=0}^{\infty}E_l\frac{t^l}{l!}.\end{aligned} \tag{29}$$

From (29), we note that

$$\begin{aligned}F(t^{(k)}) &= \frac{d^kF(t)}{dt^k}\\ &= \frac{1}{2}\sum_{l=0}^{\infty}E_{l+k}\frac{t^l}{l!}.\end{aligned} \tag{30}$$

Therefore, by (26), (29), and (30), we obtain the following theorem.

Theorem 2.2. *For $N \in \mathbb{N}$, $n \in \mathbb{Z}_+$, we have*

$$E_n^{(N)} = 2^N N \sum_{k=0}^{N-1} \frac{1}{(k+1)!} \sum_{l_1+\cdots+l_{k+1}=N} \frac{E_{n+k}}{l_1 \cdots l_{k+1}}. \tag{31}$$

From (28), we can derive the following equation:

$$\begin{aligned} \sum_{n=0}^{\infty} E_n^{(N)} \frac{t^n}{n!} &= \underbrace{\left(\frac{2}{e^t+1}\right) \cdots \left(\frac{2}{e^t+1}\right)}_{N-\text{times}} \\ &= \underbrace{\left(\sum_{l_1=0}^{\infty} E_{l_1} \frac{t^{l_1}}{l_1}\right) \cdots \left(\sum_{l_N=0}^{\infty} E_{l_N} \frac{t^{l_N}}{l_N}\right)}_{N-\text{times}} \\ &= \sum_{n=0}^{\infty} \sum_{l_1+\cdots+l_N=n} \left(\frac{E_{l_1} \cdots E_{l_N} n!}{l_1! \cdots l_N!}\right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l_1+\cdots+l_N=n} \binom{n}{l_1 \cdots l_N} E_{l_1} \cdots E_{l_N} \right) \frac{t^n}{n!}. \end{aligned} \tag{32}$$

Therefore, by (31) and (32), we obtain the following corollary.

Corollary 2.3. *For $N \in \mathbb{N}$, $n \in \mathbb{Z}_+$, we have*

$$\sum_{l_1+\cdots+l_N=n} \binom{n}{l_1 \cdots l_N} E_{l_1} \cdots E_{l_N} = 2^N N \sum_{k=0}^{N-1} \frac{1}{(k+1)!} \sum_{l_1+\cdots+l_{k+1}=N} \frac{E_{n+k}}{l_1 \cdots l_{k+1}}. \tag{33}$$

3. Identities between sums of Euler numbers and Genocchi numbers of higher order

In this section we assume that

$$G^N = G^N(t) \underbrace{\left(\frac{2t}{e^t+1}\right) \cdots \left(\frac{2t}{e^t+1}\right)}_{N-\text{times}} = \sum_{n=0}^{\infty} G_n^{(N)} \frac{t^n}{n!} \tag{34}$$

where $G_n^{(N)}$ are called the n-th Genocchi numbers of order N. We note that

$$\frac{d^k}{dt^k} \left(\frac{2t}{e^t+1}\right) |_{t=0} = G_k \tag{35}$$

for $k \in \mathbb{N}$. By (35), we obtain the following equation:

$$G_k = \frac{d^k}{dt^k} \left(\frac{2t}{e^t+1}\right) |_{t=0}$$

$$\begin{aligned} &= \sum_{l=0}^{k} \binom{k}{l} (2t)^l F^{(k-l)} \\ &= F^{(k)} + 2kF^{(k-1)}. \end{aligned} \tag{36}$$

By (30) and (36), we get

$$\begin{aligned} G &= \sum_{k=0}^{\infty} G_k \frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty} F^{(k)} \frac{t^k}{k!} + 2\sum_{k=0}^{\infty} kF^{(k-1)} \frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} E_{l+k} \frac{t^l}{l!}\frac{t^k}{k!} + 2\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} kE_{l+k-1}\frac{t^l}{l!}\frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty}\left[\sum_{l=0}^{\infty} (E_{l+k} + 2kE_{l+k-1}) \frac{t^l}{l!}\right]\frac{t^k}{k!}. \end{aligned} \tag{37}$$

Therefore, by (37), we obtain the following theorem which is the identities between the sums of Euler numbers and Genocchi numbers.

Theorem 3.1. *For $k \in \mathbb{Z}_+$, we have*

$$G_k = \sum_{l=0}^{\infty} (E_{l+k} + 2kE_{l+k-1}) \frac{t^l}{l!}. \tag{38}$$

From (38), we easily see that

$$\begin{aligned} G_k^{(N)} &= \left(F^{(k)} + 2kF^{(k-1)}\right)^{(N)} \\ &= F^{(N+k)} + 2kF^{(N+k-1)}. \end{aligned} \tag{39}$$

By (39), we get

$$\begin{aligned} G^N &= \sum_{k=0}^{\infty} G_k^{(N)} \frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty} \left(F^{(N+k)} + 2kF^{(N+k-1)}\right)\frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty}\left(\sum_{l=0}^{\infty} E_{N+l+k}\frac{t^l}{l!}\right)\frac{t^k}{k!} + 2\sum_{k=0}^{\infty}\left(\sum_{l=0}^{\infty} E_{N+l+k-1}\frac{t^l}{l!}\right)\frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty}\left(\sum_{l=0}^{\infty} E_{N+l+k} + 2kE_{N+l+k-1}\frac{t^l}{l!}\right)\frac{t^k}{k!}. \end{aligned} \tag{40}$$

Therefore, by (40), we obtain the following theorem which is the identities between the sums of Euler numbers and Genocchi numbers of higher order.

Theorem 3.2. *For $N \in \mathbb{N}$, $k \in \mathbb{Z}_+$, we have*

$$G_k^{(N)} = \sum_{l=0}^{\infty} (E_{l+k} + 2kE_{l+k-1}) \frac{t^l}{l!}. \tag{41}$$

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An algorithm for multi-attribute decision making based on soft rough sets *

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Abstract: Based on soft rough sets, some new concepts such as soft decision systems, soft relative positive regions, relative reduction in soft decision systems and conditional significance relative to decision partition soft sets are proposed. The multi-attribute decision rule in soft decision systems is presented. An algorithm of multi-attribute decision making based on soft rough sets is given.

Keywords: Soft rough set; Partition soft set; Soft decision system; Relative reduction; Decision making; Decision rule

1 Introduction

In 1999, Molodtsov [9] initiated soft set theory as a new mathematical tool for dealing with uncertainties which classical mathematical tools cannot handle. Recently, there has been a rapid growth of interest in soft set theory. Many efforts have been devoted to further generalizations and extensions of soft sets. Recently there has been a rapid growth of interest in soft set theory and its applications. Many efforts have been devoted to further generalizations and extensions of soft sets. Maji et al. [11] defined fuzzy soft sets, combining soft sets with fuzzy sets. Maji et al. [12] reported a detailed theoretical study on soft sets, with emphasis on the algebraic operations. Jiang et al. [7] extended soft sets with description logics. Aktas et al. [1] initiated the notion of soft groups, extending fuzzy groups. Feng et al. [2, 5] investigated the relationships among soft sets, rough sets and fuzzy sets.

Applications of soft set theory in decision making problems was initiated in [10]. To address fuzzy soft set based decision making problems, Roy et al. [14] presented a novel method of object recognition from an imprecise multi-observer data. Using level soft sets, Feng et al. [3] proposed an adjustable approach to fuzzy soft set based decision making. This approach was further investigated in [4, 8]. Although soft sets have been applied by several authors to the study of

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decision making under uncertainty, it seems that soft set based group decision making has not been discussed yet in the literature. Thus the present study can be seen as a first attempt toward the possible application of soft rough approximations in multi-attribute decision making problems under uncertainty.

The purpose of this paper is to give a method for multi-attribute decision making applying soft rough sets.

2 Preliminaries

Throughout this paper, U denotes an initial universe, E denotes the set of all possible attributes or parameters, 2^U denotes the family of all subsets of U and $|\,.\,|$ is the cardinality of a set. We only consider the case where both U and E are nonempty finite sets.

2.1 Soft sets

Definition 2.1 ([9]). *Let $A \subseteq E$. A pair (f, A) is called a soft set over U, if f is a mapping defined by $f : A \to 2^U$.*

In other words, a soft set over U is a parameterized family of subsets of the U. For $e \in A$, $f(e)$ may be considered as the set of e-approximate elements of (f, A).

Definition 2.2 ([12]). *Let (f, A) and (g, B) be two soft sets over U.*

(1) (f, A) is called a soft subset of (g, B), if $A \subseteq D$ and $f(e) = g(e)$ for each $e \in A$. We denote it by $(f, A) \mathrel{\tilde{\subset}} (g, B)$.

(2) (f, A) and (g, B) are called soft equal, if $A = B$ and $f(e) = g(e)$ for each $e \in A$. We denote it by $(f, A) = (g, B)$.

Obviously, $(f, A) = (g, B)$ if and only if $(f, A) \mathrel{\tilde{\subset}} (g, B)$ and $(f, A) \mathrel{\tilde{\supset}} (g, B)$.

Definition 2.3 ([12]). *Let (f, A), (g, B) and (h, C) be soft sets over U.*

(1) (h, C) is called the intersection of (f, A) and (g, B), if $C = A \cap B$ and $h(e) = f(e) \cap g(e)$ for each $e \in C$. We denote (h, C) by $(f, A) \tilde{\cap} (g, B)$.

(2) (h, C) is called the union of (f, A) and (g, B), if $C = A \cup B$ and

$$h(e) = \begin{cases} f(e), & e \in A - B, \\ f(e) \cup g(e), & e \in A \cap B, \\ g(e), & e \in B - A. \end{cases}$$

We denote (h, C) by $(f, A) \mathrel{\tilde{\cup}} (g, B)$.

Definition 2.4 ([12]). *Let (f, A) and (g, B) be two soft sets over U. (f, A) AND (g, B) denoted by $(f, A) \wedge (g, B)$ is defined by $(f, A) \wedge (g, B) = (h, A \times B)$, where $h(a, b) = f(a) \cap g(b)$ for each $(a, b) \in A \times B$.*

Definition 2.5 ([5]). *A soft set (f, A) over U is called a partition soft set if $\{f(e) |\ e \in A\}$ forms a partition of U.*

Definition 2.6 ([6]). *Let $A \subseteq E$. Let (f, A) be soft sets over U. (f, A) is called a bijective soft set, if f is a mapping $f : A \to 2^U$ such that*

(1) $\bigcup_{e \in A} f(e) = U$.

(2) *For $e_i, e_j \in A$ and $e_i \neq e_j$, $f(e_i) \cap f(e_j) = \emptyset$.*

In other words, suppose $\mathcal{B} = \{f(e_i)|\ e_i \in A, \ \ \leq 1 \leq i \leq n\} \subseteq 2^U$. From Definition 2.7, the mapping $f : A \to 2^U$ can be transformed to the mapping $f : A \to \mathcal{B}$, which is a bijective function, namely, for every $X \in \mathcal{B}$, there is exactly one attribute $e \in A$ such that $f(e) = X$ and no unmapped element remains in both A and $\mathcal{B}$.

Proposition 2.7. *Let (f, A) be a bijective soft set U and let (g, B) be a null soft set over U. $(h, C) = (f, A) \tilde{\cup} (g, B)$ is a bijective soft set.*

Proposition 2.8. *Let (f, A) and (g, B) be two bijective soft sets over U. Then $(h, A \times B) = (f, A) \wedge (g, B)$ is also a bijective soft set.*

2.2 Soft rough sets

Definition 2.9 ([5]). *Let (f, A) be a soft set over U and $X \subseteq U$. Then the pair $P = (U, (f, A))$ is called a soft approximation space. Based on the soft approximation space P, we define the following two operations*

$$\underline{apr}_P X = \{x \in U|\ \exists\ e \in A \ s.t.\ x \in f(e) \subseteq X\},$$

$$\overline{apr}_P X = \{x \in U|\ \exists\ e \in A \ s.t.\ x \in f(e), f(e) \cap X \neq \emptyset\}.$$

$\underline{apr}_P(X)$ and $\overline{apr}_P(X)$ are called the soft P-lower approximation and the soft P-upper approximation of X, respectively. In general, we refer to the pair $(\underline{apr}_P(X), \overline{apr}_P(X))$ as the soft rough set of X with respect to P. Moreover, the sets

$Pos_P(X) = \underline{apr}_P(X)$, $Neg_P(X) = U - \overline{apr}_P(X)$,

$Bnd_P(X) = \overline{apr}_P(X) - \underline{apr}_P(X)$

are called the soft P-positive region, the soft P-negative region and the soft P-boundary region of X, respectively. X is said to be soft a soft P-definable set if $\underline{apr}_P(X) = \overline{apr}_P(X)$; otherwise, X is called a soft P-rough set.

From the analogy with Pawlak rough sets, we also have the following interpretation of above concepts.

(1) $x \in Pos_P(X) = \underline{apr}_P(X)$ means that x surely belongs to X with respect to P;

(2) $x \in \overline{apr}_P(X)$ means that x possibly belongs to X with respect to P;

(3) $x \in Neg_P(X)$ means that x surely does not belong to X with respect to P.

Clearly, $\underline{apr}_P(X)$ and $\overline{apr}_P(X)$ can be expressed equivalently as:

$\underline{apr}_P(X) = \cup_{e \in A}\{f(e)|\ f(e) \subseteq X\}$, $\overline{apr}_P(X) = \cup_{e \in A}\{f(e)|\ f(e) \cap X \neq \emptyset\}$.

3 Soft decision systems

It is worth nothing that information systems and soft sets are closely related. Given a soft set (f, A) over U. (f, A) could induce an information system in a natural way. In the section, we mainly discuss soft decision systems.

Let $(f_i, C_i)(i = 1, 2, \cdots, n)$ be bijective soft sets over U where $C_i \cap C_j = \emptyset$ for $i \neq j$. Denote

$$(f, C) = \widetilde{\cup}_{i=1}^{n}(f_i, C_i), \quad (\varphi, K) = \bigwedge_{i=1}^{n}(f_i, C_i).$$

where $C = \bigcup_{i=1}^{n} C_i$ and $K = C_1 \times C_2 \times \ldots \times C_n$.

3.1 Soft relative positive regions

Definition 3.1. *Let (f, A) and (g, B) be two soft sets over U. Then the soft positive region of (f, A) to (g, B) is defined as follows*

$$Pos_{(f,A)}\ (g, B) = \bigcup_{b \in B} \underline{apr}_P g(b) = \bigcup_{b \in B} \{x \in U | \ \exists\ e \in A\ s.t.\ x \in f(e) \subseteq g(b)\},$$

where $P = (U, (f, A))$.

Example 3.2. *Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ be a common universe, $A = \{e_1, e_2, e_3, e_4\}$ and $B = \{b_1, b_2\}$ be two attribute sets. Suppose that (f, A) and (g, B) are two soft sets over U.*
The mapping of (f, A) is given below:
$f(e_1) = \{x_1, x_2\}, \quad f(e_2) = \{x_4, x_5, x_6\}, \quad f(e_3) = \{x_3, x_7\}$.
The mapping of (g, B) is given below:
$g(b_1) = \{x_1, x_2, x_3\}, \quad g(b_2) = \{x_4, x_5, x_6, x_7\}$.
Then $\underline{apr}_{(U,(f,A))}\ g(b_1) = \{x_1, x_2\}, \quad \underline{apr}_{(U,(f,A))}\ g(b_2) = \{x_4, x_5, x_6\}$.
So $Pos_{(f,A)}\ (g, B) = \{x_1, x_2, x_4, x_5, x_6\}$.

Definition 3.3. *Let $(f_i, C_i)(i = 1, 2, \cdots, n)$ be bijective soft sets over U where $C_i \cap C_j = \emptyset$ for $i \neq j$. Let (g, D) be a partition soft set over U where $C \cap D = \emptyset$. Then the triple $(U, (f, C), (g, D))$ is called a soft decision system, (f, C) is called the condition bijective soft set and (g, D) is called the decision partition soft set.*

Accordingly, in a soft decision system $(U, (f, C), (g, D))$, we have

$$Pos_{(\varphi,K)}\ (g, D) = \bigcup_{d \in D} \underline{apr}_P g(d) = \bigcup_{d \in D} \{x \in U | \ \exists\ e \in K\ s.t.\ x \in \varphi(e) \subseteq g(d)\},$$

where $K = C_1 \times C_2 \times \ldots \times C_n$ and $P = (U, (\varphi, K))$. We call it soft relative positive regions of soft decision systems.

For a given soft decision system, we always consider $Pos_{(\varphi,K)}\ (g, D) \neq \emptyset$.

Example 3.4. *Suppose that* $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ *is a common universe, which is a set of six shops.* $C = \bigcup_{i=1}^{3} C_i$ *denotes the attribute set where* C_1 *stands for the empowerment of sales personnel,* C_2 *stands for the perceived quality of goods, and* C_3 *stands for the high traffic location, respectively. The value sets of these attributes are* $C_1 = \{high,\ medium,\ low\}$, $C_2 = \{good,\ average\}$ *and* $C_3 = \{no,\ yes\}$, *respectively. And* $D = \{profit, loss\}$ *describes shop profit or loss. Suppose that the six shops are characterized by the condition bijective soft set* $\widetilde{\cup}_{i=1}^{3}(f_i, C_i)$, *and the management benefit of shop is characterized by the decision partition soft set* (g, D).

The mapping of each bijective soft set over U *is defined as follows:*
$f_1(high) = \{x_1, x_6\}$, $f_1(medium) = \{x_2, x_3, x_5\}$, $f_1(low) = \{x_4\}$,
$f_2(good) = \{x_1, x_2, x_3\}$, $f_2(average) = \{x_4, x_5, x_6\}$,
$f_3(no) = \{x_1, x_2, x_3, x_4\}$, $f_3(yes) = \{x_5, x_6\}$.

The mapping of the decision partition soft set over U *is defined as follows:*
$g(profit) = \{x_1, x_3, x_6\}$, $g(loss) = \{x_2, x_4, x_5\}$.

Then we can view each bijective soft set (f_i, C_i) *as a collection of approximations as follows:*
$(f_1, C_1) = \{high = \{x_1, x_6\}, medium = \{x_2, x_3, x_5\}, low = \{x_4\}\}$,
$(f_2, C_2) = \{good = \{x_1, x_2, x_3\}, average = \{x_4, x_5, x_6\}\}$,
$(f_3, C_3) = \{no = \{x_1, x_2, x_3, x_4\}, yes = \{x_5, x_6\}\}$.

Similarly, $(g, D) = \{profit = \{x_1, x_3, x_6\}, loss = \{x_2, x_4, x_5\}\}$.

Denote

$$(f, C) = \widetilde{\cup}_{i=1}^{3}(f_i, C_i), \qquad (\varphi, K) = \bigwedge_{i=1}^{3}(f_i, C_i),$$

where $C = \bigcup_{i=1}^{3} C_i$ *and* $K = C_1 \times C_2 \times C_3$.

Let $e_i \in K$, *then*
$e_1 =$*high and good and no,* $e_2 =$*medium and good and no,*
$e_3 =$*low and average and no,* $e_4 =$*medium and average and yes,*
$e_5 =$*high and average and yes.*
$\varphi(e_1) = \{x_1\}$, $\varphi(e_2) = \{x_2, x_3\}$, $\varphi(e_3) = \{x_4\}$,
$\varphi(e_4) = \{x_5\}$, $\varphi(e_5) = \{x_6\}$.

Besides, we have the tabular form of (φ, K) *given in Table 2.*

Table 1: Tabular representation of (φ, K)

	e_1	e_2	e_3	e_4	e_5
x_1	1	0	0	0	0
x_2	0	1	0	0	0
x_3	0	1	0	0	0
x_4	0	0	1	0	0
x_5	0	0	0	1	0
x_6	0	0	0	0	1

So $(U, (f, C), (g, D))$ *is a soft decision system on how to choose profitable shops. Thus*

$$\underline{apr}_{(U,(f,C))}\, g(profit) = \{x_1, x_6\}, \quad \underline{apr}_{(U,(f,C))}\, g(loss) = \{x_4, x_5\}.$$

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Therefore $Pos_{(\varphi,K)}\ (g,D)=\{x_1,x_4,x_5,x_6\}$.

3.2 Relative reduction in soft decision systems

Definition 3.5. *Let* $(U,(f,C),(g,D))$ *be a soft decision system and let* $1\le j\le n$. *Then*

(1) (f_j,C_j) *is called a soft dispensable set of* (f,C) *relative to* (g,D), *if* $Pos_{(\varphi,K)}\ (g,D)=Pos_{(\psi,Q)}\ (g,D)$, *where* $(\psi,Q)=\bigwedge\limits_{i=1,i\ne j}^{n}(f_i,C_i)$. *Otherwise,* (f_j,C_j) *is called a soft indispensable set of* (f,C) *relative to* (g,D).

(2) (f,C) *is called a soft independent set relative to* (g,D), *if every soft bijective set* (f_i,C_i) *of* (f,C) *is a soft indispensable set relative to* (g,D). *Otherwise,* (f,C) *is called a soft dependent set relative to* (g,D).

(3) *The unit set of all the soft indispensable set of* (f,C) *relative to* (g,D) *is called the core of* (f,C) *relative to* (g,D), *denoted by* $core((f,C),(g,D))$.

Definition 3.6. *Let* $(U,(f,C),(g,D))$ *be a soft decision system. Let* $k=1,2,\dots,m$ *and* $1\le j_k\le n$, *denote*

$$(f',C')=\widetilde{\cup}_{k=1}^{m}(f_{j_k},C_{j_k}) \text{ and } (\varphi',K')=\bigwedge_{k=1}^{m}(f_{j_k},C_{j_k}).$$

(f',C') *is called a relative reduction in* $(U,(f,C),(g,D))$, *if*
(1) $Pos_{(\varphi,K)}\ (g,D)=Pos_{(\varphi',K')}\ (g,D)$,
(2) (f',C') *is a soft independent set relative to* (g,D).

Example 3.7. *In Example 3.4, denote*

$(\varphi_1,K_1)=(f_1,C_1)\wedge(f_2,C_2)$, $(\varphi_2,K_2)=(f_1,C_1)\wedge(f_3,C_3)$,
$(\varphi_3,K_3)=(f_2,C_2)\wedge(f_3,C_3)$.

We have

$Pos_{(\varphi_1,K_1)}\ (g,D)=Pos_{(\varphi_2,K_2)}\ (g,D)=Pos_{(\varphi,K)}\ (g,D)=\{x_1,x_4,x_5,x_6\}$,
$Pos_{(\varphi_3,K_3)}\ (g,D)=\{x_4\}$.

But

$Pos_{(f_1,C_1)}\ (g,D)=\{x_1,x_4,x_6\}\ne Pos_{(\varphi,K)}\ (g,D)$,
$Pos_{(f_3,C_3)}\ (g,D)=\emptyset\ne Pos_{(\varphi,K)}\ (g,D)$.

So

$(f_1,C_1)\cup(f_2,C_2)$ *and* $(f_1,C_1)\cup(f_3,C_3)$ *are both relative reductions in* $(U,(f,C),(g,D))$.

3.3 Dependent degree of decision partition soft sets

Definition 3.8. *Let* (f,A) *and* (g,B) *be two soft sets over* U. (f,A) *is said to depend on* (g,B) *to a degree* k $(0\le k\le 1)$, *denoted* $(f,A)\Rightarrow_k\ (g,B)$, *if*

$$k=\gamma((f,A),(g,B))=\frac{\mid Pos_{(f,A)}\ (g,B)\mid}{\mid U\mid}.$$

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Accordingly, in a soft decision system $(U,(f,C),(g,D))$, we have

$$k=\gamma((\varphi,K),\ (g,D))=\frac{\mid Pos_{(\varphi,K)}\ (g,D)\mid}{\mid U\mid}.$$

We call it the dependent degree of decision partition soft sets upon condition bijective soft sets. It characters a degree of condition bijective soft sets in classifying decision partition soft sets. Obviously, we have $0\le k\le 1$.

If $k=1$, then (g,D) is completely dependent on (f,C).

If $k=0$, then (g,D) is completely independent on (f,C).

Example 3.9. *In example 3.4, the dependent degree of the decision partition soft set (g,D) upon the condition bijective soft set $(f,C)=\widetilde{\cup}_{i=1}^{3}(f_i,C_i)$:*

$$k=\gamma(\bigwedge_{i=1}^{3}(f_i,C_i),(g,D))=\frac{|\{x_1,x_4,x_5,x_6\}|}{|U|}=\frac{4}{6}=\frac{2}{3}$$

Proposition 3.10. *Let $(U,(f,C),(g,D))$ be a soft decision system. Let $m,n\in N$ and $m<n$. Then*

$$\gamma(\bigwedge_{i=1}^{m}(f_i,C_i),(g,D))\le\gamma(\bigwedge_{i=1}^{n}(f_i,C_i),(g,D)).$$

Proof. Since we have

$$\begin{aligned}\gamma((\varphi,K),(g,D)) &= \frac{\mid Pos_{(\varphi,K)}(g,D)\mid}{\mid U\mid}=\frac{\mid\bigcup_{d\in D}\underline{apr}_{P}g(d)\mid}{\mid U\mid}\\ &= \frac{\mid\bigcup_{d\in D}\{x\in U\mid\ \exists\ e\in K\ s.t.\ x\in\varphi(e)\subseteq g(d)\}\mid}{\mid U\mid},\end{aligned}$$

$$\begin{aligned}\gamma((\varphi',K'),(g,D)) &= \frac{\mid Pos_{(\varphi',K')}(g,D)\mid}{\mid U\mid}=\frac{\mid\bigcup_{d\in D}\underline{apr}_{P'}g(d)\mid}{\mid U\mid}\\ &= \frac{\mid\bigcup_{d\in D}\{x\in U\mid\ \exists\ e\in K'\ s.t.\ x\in\varphi'(e)\subseteq g(d)\}\mid}{\mid U\mid},\end{aligned}$$

where $K=\bigwedge_{i=1}^{n}C_i,\ K'=\bigwedge_{i=1}^{m}C_i,\ P=(U,(f,C))\ and\ P'=(U,(f',C'))$.

By Definition 2.6, for any $(c_1,c_2,\cdots,c_n)\in C_1\times C_2\times\cdots\times C_n$, we have

$$\varphi(c_1,c_2,\cdots,c_n)=f_1(c_1)\cap f_2(c_2)\cap\cdots\cap f_m(c_m)\cap\cdots\cap f_n(c_n).$$

Moreover, for any $(c_1,c_2,\cdots,c_m)\in C_1\times C_2\times\cdots\times C_m$, we also have

$$\varphi'(c_1,c_2,\cdots,c_m)=f_1(c_1)\cap f_2(c_2)\cap\cdots\cap f_m(c_m).$$

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For $m, n \in N$ and $m < n$, $\underline{apr}_{P'} g(d) \subseteq \underline{apr}_P g(d)$.
So

$$\bigcup_{d\in D} \underline{apr}_{P'} g(d) \subseteq \bigcup_{d\in D} \underline{apr}_P g(d).$$

Hence

$$\gamma(\bigwedge_{i=1}^{m}(f_i, C_i), (g, D))\} \leq \gamma(\bigwedge_{i=1}^{n}(f_i, C_i), (g, D)).$$

□

In other words, condition bijective soft sets can explain the most detailed classification of decision partition soft sets. And deleting some condition bijective soft sets can lose some information about decision partition soft sets. Thus, more information (more condition bijective soft sets) can result in bigger dependent degree of decision partition soft sets.

3.4 Conditional significance relative to decision partition soft sets

Definition 3.11. *Let $(U, (f, C), (g, D))$ be a soft decision system and $1 \leq j \leq n$. The conditional significance of (f_j, C_j) in (f, C) relative to (g, D) is denoted and defined as follows*

$$s((f_j, C_j), (f, C), (g, D)) = \gamma(\bigwedge_{i=1}^{n}(f_i, C_i), (g, D)) \;-\; \gamma(\bigwedge_{i=1, i\neq j}^{n}(f_i, C_i), (g, D)).$$

This definition indicates the decrease of the dependent degree of decision partition soft sets when deleting one bijective soft set (f_j, C_j) from (f, C). The following results are easily obtained from the above definitions.

Proposition 3.12. *Let $(U, (f, C), (g, D))$ be a soft decision system and $1 \leq j \leq n$.*

(1) $0 \leq s((f_j, C_j), (f, C), (g, D)) \leq 1$,
(2)(f_j, C_j) *is a soft indispensable set of* (f, C) *to* (g, D) *if and only if*

$$s((f_j, C_j), (f, C), (g, D)) > 0,$$

(3) $core((f, C), (g, D)) = \widetilde{\cup}\{(f_j, C_j) \mid s((f_j, C_j), (f, C), (g, D)) > 0, j = 1, 2, \cdots, n\}$.

Theorem 3.13. *Let $(U, (f, C), (g, D))$ be a soft decision system. Let $k = 1, 2, \ldots, m$ and $1 \leq j_k \leq n$, denote*

$$(f', C') = \widetilde{\cup}_{k=1}^{m}(f_{j_k}, C_{j_k}) \text{ and } (\varphi', K') = \bigwedge_{k=1}^{m}(f_{j_k}, C_{j_k}),$$

where $C' = \bigcup_{j_k=1}^{m} C_{j_k}$ *and* $K' = C_{j1} \times C_{j2} \times \ldots \times C_{jm}$.
If $\gamma((\varphi', K'), (g, D)) = \gamma((\varphi, K), (g, D))$ *and* $s((f_j, C_j), (f', C'), (g, D)) > 0$, *then* (f', C') *is a relative reduction of* $(U, (f, C), (g, D))$.

3.5 The multi-attribute decision rule in soft decision systems

Definition 3.14. *Let $(U,(f,C),(g,D))$ be a soft decision system. Let $e \in K = C_1 \times C_2 \times \ldots \times C_n,\ d \in D$. The soft rough membership function of $\varphi(e)$ relative to $g(d)$ is denoted and defined as follows*

$$\xi(\varphi(e), g(d)) = \frac{|\varphi(e) \cap g(d)|}{|\varphi(e)|}.$$

Definition 3.15. *Let $(U,(f,C),(g,D))$ be soft decision system. Let $k = 1,2,\ldots,m$ and $1 \le j_k \le n$, denote*

$$(f', C') = \widetilde{\cup}_{k=1}^{m}(f_{j_k}, C_{j_k}) \text{ and } (\varphi', K') = \bigwedge_{k=1}^{m}(f_{j_k}, C_{j_k}),$$

where $C = \bigcup\limits_{j_k=1}^{m} C_{j_k}$ and $K' = C_{j1} \times C_{j2} \times \ldots \times C_{jm}$. Let (f', C') be a relative reduction of $(U,(f,C),(g,D))$. We call

$$\text{If } e,\ \text{then } d\ (\xi(\varphi'(e), g(d)))$$

the multi-attribute decision rule by induced (f', C') in $(U,(f,C),(g,D))$, where $e \in K',\ d \in D$ and $\xi(\varphi'(e), g(d))$ denotes the soft rough membership function of $\varphi'(e)$ relative to $g(d)$, which expresses the support degree of rules.

4 An algorithm for multi-attribute decision making based on soft rough sets

Based on above definitions and results, we will give an algorithm for the multi-attribute decision rule.

Algorithms:

Step 1. Construct a soft decision system $(U,(f,C),(g,D))$.

Step 2. Calculate the dependent degree of (g,D) upon $\bigwedge\limits_{i=1, i\neq j}^{n}(f_i, C_i)$ $(j = 0,1,2,\ldots,n)$.

Step 3. Calculate each conditional significance of (f_j, C_j) in (f,C) relative to (g,D) by Definition 3.11.

Step 4. Find core$((f,C),(g,D))$ by Proposition 3.12.

Step 5. Find relative reductions in $(U,(f,C),(g,D))$ by Theorem 3.13.

(1) If $\gamma(\text{core}((f,C),(g,D)),(g,D)) = \gamma((f,C),(g,D))$, then core$((f,C),(g,D))$ is a relative reduction in $(U,(f,C),(g,D))$. In this case, the process stops.

Otherwise, it continues (2).

(2) Denote

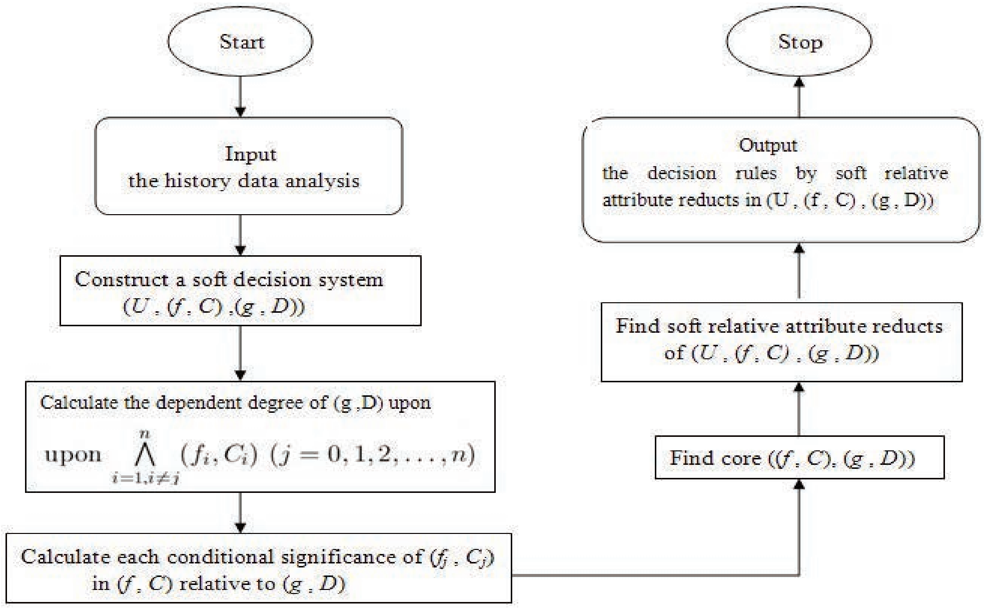
core$((f,C),(g,D)) = \bigwedge\limits_{k=1}^{m}(f_{j_k}, C_{j_k})$, where $k = 1,2,\ldots,m$ and $1 \le j_k \le n$.

(a) Calculate the conditional significance of each bijective soft set (f_i, C_i) $(i \neq j_k)$ about $\widetilde{\cup}_{k=1}^{m}(f_{j_k}, C_{j_k})$ relative to (g, D) by Definition 3.11.

(b) Select (f_i, C_i) with maximal conditional significance one by one. If there are many soft sets with the same maximal significant, we choose the attribute set containing the most elements. So core$((f, C), (g, D))\widetilde{\cup}(f_i, C_i)$ is a relative reduction in $(U, (f, C), (g, D))$.

Step 6. Obtain decision rules by relative reductions in the soft decision system $(U, (f, C), (g, D))$.(Fig.1)

An algorithm



5 Conclusions

This method is based on cases of library history data analysis, then we can find the useful information. The multi-attribute decision rule and the support degree of rules provides scientific objective basis. This method reduces the search domain and hence does a more efficient retrieval than the existing methods. Therefore, the new evaluation method can help users to decide the component adapter scheme and reduce pressure and subjectivity in the component reuse process adapter decision-making.

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Fixed point results for modular ultrametric spaces

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Abstract

In this study, we define the notion of modular ultrametric space. We present a fixed point theorem in modular spherically complete ultrametric space, and prove coincidence point theorem for three self maps in a modular spherically complete ultrametric space.

1 Introduction

Fixed point theory is a developing field of mathematics with various applications to engineering, applied mathematics, some disciplines of sciences, etc. Fixed point theorems play a key role in this theory. Under certain conditions, we get some results related to a self map on any set, which allows one or more fixed points by means of them.

Ultrametric space is a kind of metric space but it has the strong triangle inequality, i.e.,

$$d(x,y) < \max\{d(x,z), d(z,y)\}.$$

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This metric is also known non-Archimedean metric. The notion of ultrametric is utilized outside mathematics. For example, ultrametric distances are tools of taxonomy and phylogenetic tree. The notion of ultrametric space was introduced by Van Rooij [32]. Gajic [13] proved a fixed point theorem for a class of generalized contractive mapping on ultrametric space. Rao et al. [30] introduced two coincidence point theorems for three and four self maps in a spherically complete ultrametric space. Some fixed point results on ultrametric spaces were given by Kirk and Shahzad [17]. There are also some studies in [10, 21].

Modular space was appeared by Nakano [24] in 1950. Many authors [19, 20, 25, 26, 27, 28, 29] gave some remarks on modular spaces. The concept of a modular metric space more general than a metric space was presented by Chistyakov [6]. He also developed the theory of modular metric spaces in [7, 8]. Chaipunya et al. [5] showed the existence of fixed point and uniqueness of quasi-contractive mappings in modular metric spaces. Azadifar et al. [4] proved the existence and uniqueness of a common fixed point of compatible mappings of integral type in modular metric spaces. Hussain and Salimi [14] investigated the existence of fixed points of generalized α-admissible modular contractive mappings in modular metric spaces. Kilinc and Alaca [15] defined (ε, k)-uniformly locally contractive mappings and η-chainable concept and proved a fixed point theorem for these concepts in complete modular metric spaces. Many studies were done in [1, 2, 3, 9, 11, 12, 16, 18, 22, 31, 33, 34].

In this paper, we first introduce the notion of modular ultrametric space. We give some fixed point theorems in a modular spherically complete ultrametric space.

2 Preliminaries

Definition 2.1. [27]. A modular on a real linear space X is a functional $\rho : X \longrightarrow [0, \infty]$ satisfying the following statements:

(A1) $\rho(0) = 0$;

(A2) If $x \in X$ and $\rho(\alpha x) = 0$ for all positive real numbers α, then $x = 0$;

(A3) $\rho(-x) = \rho(x)$ for all $x \in X$;

(A4) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $x, y \in X$.

Let X be a nonempty set and $\lambda \in (0, \infty)$. We indicate that the function

$$\omega : (0, \infty) \times X \times X \to [0, \infty]$$

is denoted by $\omega_\lambda(x, y) = \omega(\lambda, x, y)$ for all $\lambda > 0$ and $x, y \in X$.

Definition 2.2. [7]. Let X be a nonempty set. The function

$$\omega : (0, \infty) \times X \times X \to [0, \infty]$$

is called a metric modular on X if, for all $x, y, z \in X$, the following conditions hold:

(i) $\omega_\lambda(x, y) = 0$ for all $\lambda > 0 \;\Leftrightarrow\; x = y$;

(ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ for all $\lambda > 0$;

(iii) $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$ for all $\lambda, \mu > 0$.

Let's recall the definitions of two sets X_ω and X^*_ω [7]:

$$X_\omega \equiv X_\omega(x_0) = \{x \in X : \omega_\lambda(x, x_0) \to 0 \ \text{ as } \ \lambda \to \infty\}$$

and

$$X^*_\omega \equiv X^*_\omega(x_0) = \{x \in X : \exists\lambda = \lambda(x) > 0 \ \text{ such that } \ \omega_\lambda(x, x_0) < \infty\}.$$

Definition 2.3. [32]. Let (X, d) be a metric space. If the metric d satisfies the strong triangle inequality

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}$$

for all $x, y, z \in X$, then it is called *ultrametric* on X. The pair (X, d) is said to be *ultrametric space*.

3 Modular ultrametric spaces

In this section, we first give some new definitions.

Definition 3.1. Let (X, ω) be a modular metric space. If ω satisfies the strong triangle inequality

$$\omega_\lambda(x, y) \leq \max\{\omega_\lambda(x, z), \omega_\lambda(z, y)\}$$

for all $x, y, z \in X$, then it is called a *modular ultrametric* on X.

Definition 3.2. Let (X, ω) be a modular ultrametric space. For $r > 0$ and $x \in X_\omega$, we define the open sphere $B_\omega(x, r)$ and the closed sphere $B_\omega[x, r]$ with centre x and radius r as follows:

$$\begin{aligned} B_\omega(x, r) =& \{y \in X_\omega : \omega_\lambda(x, y) < r\} \\ B_\omega[x, r] =& \{y \in X_\omega : \omega_\lambda(x, y) \leq r\}. \end{aligned}$$

Definition 3.3. The modular ultrametric space X^*_ω is called a *modular spherically complete ultrametric space* if every nest of balls has a nonempty intersection.

Theorem 3.4. *Let X^*_ω be a modular spherically complete ultrametric space. Assume that there exists an element $x = x(\lambda) \in X^*_\omega$ such that $\omega_\lambda(x, Tx) < \infty$. If $T : X^*_\omega \to X^*_\omega$ is a map such that for every $x, y \in X^*_\omega$, $x \neq y$,*

$$\omega_\lambda(Tx, Ty) < \max\{\omega_\lambda(x, Tx), \omega_\lambda(x, y), \omega_\lambda(y, Ty)\}, \tag{3.1}$$

then T has a unique fixed point.

Proof. Let $B_a = B_\omega[a, \omega_\lambda(a, Ta)]$ be the closed sphere centered at a with the radius $\omega_\lambda(a, Ta)$ and let $\mathcal{A}$ be the collection of these spheres for all $a \in X^*_\omega$.

It is clear that the relation

$$B_a \leq B_b \quad \Leftrightarrow \quad B_b \subseteq B_a$$

is a partial order on $\mathcal{A}$.

Now we pay attention to a totally ordered subfamily $\mathcal{A}_1$ of $\mathcal{A}$. Since X^*_ω is modular spherically complete, we have

$$\bigcap_{B_a \in \mathcal{A}_1} B_a = B \neq \emptyset.$$

Let $b \in B$, $b_a \in \mathcal{A}_1$ and $x \in B_b$. Then we get the following:

$$\omega_\lambda(x,b) \le \omega_\lambda(b,Tb) \le \max\{\omega_\lambda(b,a), \omega_\lambda(a,Ta), \omega_\lambda(Ta,Tb)\}$$

$$= \max\{\omega_\lambda(a,Ta), \omega_\lambda(Ta,Tb)\}. \tag{3.2}$$

The case $\omega_\lambda(Ta,Tb) \le \omega_\lambda(a,Ta)$ implies that

$$\omega_\lambda(x,b) \le \omega_\lambda(a,Ta).$$

In case of $\omega_\lambda(Ta,Tb) > \omega_\lambda(a,Ta)$, it follows from (3.2) that

$$\begin{aligned}\omega_\lambda(x,b) \le \omega_\lambda(b,Tb) \le & \omega_\lambda(Ta,Tb) < \max\{\omega_\lambda(a,Ta), \omega_\lambda(a,b), \omega_\lambda(b,Tb)\} \\ = & \max\{\omega_\lambda(a,Ta), \omega_\lambda(b,Tb)\}.\end{aligned}$$

By $\omega_\lambda(b,Tb) \le \omega_\lambda(a,Ta)$, we have

$$\omega_\lambda(x,b) \le \omega_\lambda(a,Ta)$$

and $\omega_\lambda(b,Tb) > \omega_\lambda(a,Ta)$ shows that $\omega_\lambda(b,Tb) < \omega_\lambda(b,Tb)$, which is a contradiction. So $\omega_\lambda(x,b) \le \omega_\lambda(a,Ta)$ for $x \in B_b$.

Since we have

$$\omega_\lambda(x,a) \le \omega_\lambda(a,Ta),$$

$x \in B_a$ and $B_b \subseteq B_a$ for any $B_a \in \mathcal{A}_1$. Thus B_b is the upper bound for the family $\mathcal{A}$. From Zorn's Lemma, we conclude that $\mathcal{A}$ has a maximal element B_z where $z \in X_\omega^*$.

Now we prove $z = Tz$. Suppose that $z \neq Tz$. The inequality (3.1) implies that

$$\omega_\lambda(Tz, T(Tz)) < \omega_\lambda(z, Tz).$$

If $y \in B_{Tz}$, then $\omega_\lambda(y,Tz) \le \omega_\lambda(Tz,T(Tz)) < \omega_\lambda(z,Tz)$. In this case, we get

$$\omega_\lambda(y,z) \le \max\{\omega_\lambda(y,Tz), \omega_\lambda(Tz,z)\} = \omega_\lambda(Tz,z),$$

i.e., $y \in B_z$ and $B_{Tz} \subseteq B_z$. Moreover, $z \notin B_{Tz}$ since

$$\omega_\lambda(z,Tz) > \omega_\lambda(Tz, T(Tz)).$$

As a consequence, we have $B_{Tz} \subsetneq B_z$ but it contradicts to the maximality of B_z. Hence we have $z = Tz$.

It only remains to show the uniqueness. For this purpose, we take u as a different fixed point. For $u \neq z$, we have

$$\omega_\lambda(z,u) = \omega_\lambda(Tz,Tu) < \max\{\omega_\lambda(Tz,z), \omega_\lambda(z,u), \omega_\lambda(u,Tu)\} = \omega_\lambda(z,u)$$

which is a contradiction. This completes the proof. □

Theorem 3.5. *Let X_ω^* be a modular ultrametric space, and let $f, S, T : X_\omega^* \to X_\omega^*$ be maps satisfying*

(1) $f(X_\omega^*)$ is modular spherically complete,

(2) $\omega_\lambda(Sx,Ty) < \max\{\omega_\lambda(fx,fy),\omega_\lambda(fx,Sx),\omega_\lambda(fy,Ty)\}$ *for* $x,y\in X_\omega^*$, $x\neq y$,

(3) $fS=Sf,\ fT=Tf,\ ST=TS$,

(4) $S(X_\omega^*)\subseteq f(X_\omega^*),\ T(X_\omega^*)\subseteq f(X_\omega^*)$.

Then either $fw=Sw$ *or* $fw=Tw$ *for some* $w\in X_\omega^*$.

Proof. For $a\in X_\omega^*$, let $B_a=[fa,\max\{\omega_\lambda(fa,Sa),\omega_\lambda(fa,Ta)\}]$ denote the closed sphere centered at fa with the radius $\max\{\omega_\lambda(fa,Sa),\omega_\lambda(fa,Ta)\}$. Let A be the collection of all the spheres for all $a\in f(X_\omega^*)$. We state that the relation $B_a\leq B_b$ iff $B_b\subseteq B_a$ is a partial order on A. For a totally ordered subfamily A_1 of A, since $f(X_\omega^*)$ is modular spherically complete, we have

$$\bigcap_{B_a\in A_1} B_a = B\neq\emptyset.$$

Let $fb\in B$ where $b\in f(X_\omega^*)$ and $B_a\in A_1$. Then we have $fb\in B_a$ and so

$$\omega_\lambda(fb,fa)\leq\max\{\omega_\lambda(fa,Sa),\omega_\lambda(fa,Ta)\}. \tag{3.3}$$

If $a=b$, then $B_a=B_b$. Assume that $a\neq b$ and $x\in B_b$. It follows from the condition (2) and (3.3) that

$$\begin{aligned}\omega_\lambda(x,fb)&\leq\max\{\omega_\lambda(fb,Sb),\omega_\lambda(fb,Tb)\}\\&\leq\max\{\omega_\lambda(fb,fa),\omega_\lambda(fa,Ta),\omega_\lambda(Ta,Sb),\\&\qquad\omega_\lambda(fb,fa),\omega_\lambda(fa,Sa),\omega_\lambda(Sa,Tb)\}\\&<\max\{\omega_\lambda(fb,fa),\omega_\lambda(fa,Ta),\omega_\lambda(fa,Sa),\\&\qquad\max\{\omega_\lambda(fb,fa),\omega_\lambda(fb,Sb),\omega_\lambda(fa,Ta)\},\\&\qquad\max\{\omega_\lambda(fa,fb),\omega_\lambda(fa,Sa),\omega_\lambda(fb,Tb)\}\}\\&=\max\{\omega_\lambda(fa,Sa),\omega_\lambda(fa,Ta)\}.\end{aligned}$$

Thus

$$\omega_\lambda(x,fb)<\max\{\omega_\lambda(fa,Sa),\omega_\lambda(fa,Ta)\}. \tag{3.4}$$

From (3.3) and (3.4), we get

$$\begin{aligned}\omega_\lambda(x,fa)&\leq\max\{\omega_\lambda(x,fb),\omega_\lambda(fb,fa)\}\\&\leq\max\{\omega_\lambda(fa,Sa),\omega_\lambda(fa,Ta)\}.\end{aligned}$$

Therefore, $x\in B_a$. We have also $B_b\subseteq B_a$ for any $B_a\in A_1$ and B_b is an upper bound in A for the family A_1. By Zorn's Lemma, there is a maximal element, denoted by B_z, in A, where $z\in F(X_\omega^*)$. There exists an element $w\in X_\omega^*$ such that $z=fw$.

Suppose $fw\neq Sw$ and $fw\neq Tw$. Since $fS=Sf$,

$$\begin{aligned}\omega_\lambda(Sfw,TSw)&<\max\{\omega_\lambda(f^2w,fSw),\omega_\lambda(f^2w,Sfw),\omega_\lambda(fSw,TSw)\}\\&=\omega_\lambda(f^2w,fSw).\end{aligned}\tag{3.5}$$

Since $fT=Tf$,

$$\begin{aligned}\omega_\lambda(STw,Tfw)&<\max\{\omega_\lambda(fTw,f^2w),\omega_\lambda(fTw,STw),\omega_\lambda(f^2w,Tfw)\}\\&=\omega_\lambda(f^2w,fTw)\end{aligned}\tag{3.6}$$

Since $ST = TS$, it follows from (3.5) and (3.6) that

$$\begin{aligned} \omega_\lambda(Sfw, S^2w) &\le \max\{\omega_\lambda(Sfw, TSw), \omega_\lambda(TSw, Tfw), \omega_\lambda(Tfw, S^2w)\} \\ &< \max\{\omega_\lambda(f^2w, fSw), \omega_\lambda(f^2w, fTw), \\ &\qquad \max\{\omega_\lambda(fSw, f^2w), \omega_\lambda(fSw, S^2w), \omega_\lambda(f^2w, Tfw)\}\} \\ &= \max\{\omega_\lambda(f^2w, fSw), \omega_\lambda(f^2w, fTw)\}. \end{aligned} \tag{3.7}$$

From (3.5) and (3.7), we have

$$\begin{aligned} &\max\{\omega_\lambda(Sfw, TSw), \omega_\lambda(Sfw, S^2w)\} \\ &\qquad < \max\{\omega_\lambda(f^2w, fSw), \omega_\lambda(f^2w, fTw)\}. \end{aligned} \tag{3.8}$$

By (3.5) and (3.6),

$$\begin{aligned} \omega_\lambda(Tfw, T^2w) &\le \max\{\omega_\lambda(Tfw, TSw), \omega_\lambda(TSw, Sfw), \omega_\lambda(Sfw, T^2w)\} \\ &< \max\{\omega_\lambda(f^2w, fTw), \omega_\lambda(f^2w, fSw), \\ &\qquad \max\{\omega_\lambda(f^2w, fTw), \omega_\lambda(f^2w, Sfw), \omega_\lambda(fTw, T^2w)\}\} \\ &= \max\{\omega_\lambda(f^2w, fTw), \omega_\lambda(f^2w, fSw)\}. \end{aligned} \tag{3.9}$$

From (3.6) and (3.9), we have

$$\begin{aligned} &\max\{\omega_\lambda(STw, Tfw), \omega_\lambda(Tfw, T^2w)\} \\ &\qquad < \max\{\omega_\lambda(f^2w, fTw), \omega_\lambda(f^2w, fSw)\}. \end{aligned} \tag{3.10}$$

If $\max\{\omega_\lambda(f^2w, fTw), \omega_\lambda(f^2w, fSw)\} = \omega_\lambda(f^2w, fSw)$, then from (3.8), we have

$$\max\{\omega_\lambda(Sfw, TSw), \omega_\lambda(Sfw, S^2w)\} < \omega_\lambda(f^2w, fSw)$$

which gives $f^2w \notin B_{Sw}$. Hence $fz \notin B_{Sw}$. But $fz \in B_z$. Hence $B_z \nsubseteq B_{Sw}$. It is a contradiction to the maximality of B_z in A, since $Sw \in S(X_\omega^*) \subseteq f(X_\omega^*)$. If

$$\max\{\omega_\lambda(f^2w, fTw), \omega_\lambda(f^2w, fSw)\} = \omega_\lambda(f^2w, fTw),$$

then from (3.10), $\max\{\omega_\lambda(STw, Tfw), \omega_\lambda(Tfw, T^2w)\} < \omega_\lambda(f^2w, fTw)$ which gives $f^2w \notin B_{Tw}$. Hence $fz \notin B_{Tw}$. Since $fz \in B_z$, we get $B_z \nsubseteq B_{Tw}$. It contradicts to the maximality of B_z in A, since $Tw \in T(X_\omega^*) \subseteq f(X_\omega^*)$. As a result, either $fw = Sw$ or $fw = Tw$. □

Proposition 3.6. *Let X_ω^* be a modular spherically complete ultrametric space and let*

$$f, T : X_\omega^* \to X_\omega^*$$

be maps satisfying $T(X_\omega^*) \subseteq f(X_\omega^*)$ and

$$\omega_\lambda(Tx, Ty) < \max\{\omega_\lambda(fx, fy), \omega_\lambda(fx, Tx), \omega_\lambda(fy, Ty)\} \tag{3.11}$$

for all $x, y \in X_\omega^*$, with $x \neq y$. Then there exists $z \in X_\omega^*$ such that $fz = Tz$. Moreover, if f and T are coincidentally commuting at z, then z is a unique common fixed point of f and T.

Proof. Let $B_a = [fa, \omega_\lambda(fa, Ta)]$ represent the closed sphere centered at fa with radius $\omega_\lambda(fa, Ta)$ and let A be the collection of these spheres for all $a \in X$. By the same reasoning as in Theorem 3.5, we conclude that A has a maximal element B_z for an element $z \in X_\omega^*$.

Let's assume $fz \neq Tz$. Since $Tz \in T(X) \subseteq f(X)$, there exists $w \in X_\omega^*$ such that $Tz = fw$. It is clear that $w \neq z$. From (3.11), we have

$$\begin{aligned}\omega_\lambda(fw, Tw) =& \omega_\lambda(Tz, Tw)\\ <& \max\{\omega_\lambda(fz, fw), \omega_\lambda(fz, Tz), \omega_\lambda(fw, Tw)\}\\ =& \omega_\lambda(fz, fw).\end{aligned}$$

Thus $fz \notin B_w$ and $B_z \nsubseteq B_w$. This contradicts to the maximality of B_z. So $fz = Tz$.

On the other hand, we suppose that f and T are coincidentally commuting at z. Then

$$f^2z = f(fz) = fTz = Tfz = T(Tz) = T^2z.$$

Suppose $fz \neq z$. By (3.11), we conclude that

$$\begin{aligned}\omega_\lambda(Tfz, Tz) <& \max\{\omega_\lambda(f^2z, fz), \omega_\lambda(f^2z, Tfz), \omega_\lambda(fz, Tz)\}\\ =& \omega_\lambda(Tfz, Tz),\end{aligned}$$

which is a contradiction. Thus $z = fz = Tz$.

Now we show the uniqueness. Let u be a different fixed point. For $u \neq z$, we have

$$\begin{aligned}\omega_\lambda(z, u) &= \omega_\lambda(Tz, Tu) < \max\{\omega_\lambda(fz, fu), \omega_\lambda(fz, Tz), \omega_\lambda(fu, Tu)\}\\ &= \omega_\lambda(z, u),\end{aligned}$$

which is a contradiction. As a consequence, we have the required result. □

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On the backward difference scheme for a class of SIRS epidemic models with nonlinear incidence

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Abstract. In this paper, we construct a backward difference scheme for a class of SIRS epidemic models with nonlinear incidence rate $\beta f(S)g(I)$ and vaccination in susceptible. The dynamical properties of the scheme are investigated. By using the inductive method and the linearization method of difference equations, the positivity and the boundedness of solutions, the existence and local stability of equilibria are obtained. By constructing new discrete type Lyapunov functions, under the conditions which functions $f(S)$ and $g(I)$ satisfy assumptions $(H_1)-(H_3)$, the global stability of the equilibria is obtained. That is, the disease-free equilibrium is globally asymptotically stable if basic reproduction number $\mathscr{R}_0 \leq 1$, and the endemic equilibrium is globally asymptotically stable if $\mathscr{R}_0 > 1$.

Keywords: discrete SIRS epidemic model; backward difference scheme; nonlinear incidence; local and global stability; discrete Lyapunov function.

1. Introduction

As we well known, for some practical purposes, especially the numerical computing, it is often necessary to discretize the continuous-time model to a corresponding discrete difference scheme, that is discrete dynamical model.

In recent years, aim at the continuous-time SIR and SIRS epidemic models, the various discrete dynamical models are constructed, and the dynamical properties of these models are studied in many articles, for example, see [1-22] and the reference

therein. Many important results have been established. These results focus on: the computation of the basic reproduction number, the local and global stability of the disease-free equilibrium and endemic equilibrium, the permanence, persistence and extinction of the disease, the bifurcation and chaos phenomena, etc.

Particularly, we see that, in [1,2], the authors studied a class of discrete SIRS epidemic models with time delays and bilinear incidence derived from corresponding continuous models by applying the nonstandard finite difference scheme (See [23-26]), and the sufficient conditions on the global asymptotic stability of the disease-free equilibrium and the permanence of the disease are established. In [3], the authors studied a discrete SIRS epidemic model with bilinear incidence derived from corresponding continuous model by applying the backward difference scheme, and the sufficient and necessary conditions on the global asymptotic stability of the disease-free equilibrium and endemic equilibrium are established. In [4], the authors discussed a class of discrete SIRS epidemic models with general nonlinear incidence derived from corresponding continuous model by applying the forward d-ifference scheme, and the sufficient conditions for the existence and local stability of the disease-free equilibrium and endemic equilibrium are obtained. In [5], the authors discussed a class of discrete SIRS epidemic models with standard incidence discretized from corresponding continuous model by applying the forward difference scheme, and the sufficient condition for the global stability of the endemic equilibrium is established.

However, from above articles we easily see that the studies on the backward difference scheme for SIRS epidemic models with nonlinear incidence are few. In this paper, we construct a backward difference scheme for a class of continuous-time SIRS epidemic models with nonlinear incidence $\beta f(S)g(I)$ and vaccination in susceptible. We will study the dynamical properties, especially the global asymptotic stability of the disease-free equilibrium and endemic equilibrium for this discrete model. Firstly, the basic properties of the model, such as, the positivity and the boundedness of solutions, the existence and local stability of equilibria are discussed by using the inductive method and the linearization method of difference equations. Further, by constructing new discrete type Lyapunov functions which is different from those given in [3] and using the theory of stability of difference equations, we will establish the global asymptotic stability of equilibria under the assumptions $(H_1)-(H_3)$ (see

Section 2). That is, when assumptions $(H_1)-(H_3)$ hold, the disease-free equilibrium is globally asymptotically stable if and only if basic reproduction number $\mathscr{R}_0 \leq 1$, and the endemic equilibrium is globally asymptotically stable if and only if $\mathscr{R}_0 > 1$.

The organization of this paper is as follows. In the second section we firstly introduce a backward difference scheme, that is discrete dynamical model, for SIRS epidemic models with nonlinear incidence, and further give some basic assumptions. In the third section the results on the positivity and boundedness of solutions, the existence and local stability of equilibria for the model are stated and proved. In the fourth section we will state and prove the theorems on the global asymptotic stability of the disease-free equilibrium and endemic equilibrium for the model. Lastly, in the fifth section we will give a conclusion.

2. Model description

We consider the following continuous SIRS epidemic model with nonlinear incidence and vaccination in susceptible

$$\begin{aligned} \frac{dS}{dt} &= A - \beta f(S)g(I) - d_1 S - \eta S + \delta R, \\ \frac{dI}{dt} &= \beta f(S)g(I) - d_2 I - \gamma I, \\ \frac{dR}{dt} &= \eta S + \gamma I - d_3 R - \delta R, \end{aligned} \tag{1}$$

where $S(t), I(t)$ and $R(t)$ denote the numbers of susceptible, infected and recovered individuals at time t, respectively. A is the recruitment rate of the total population, d_1, d_2, and d_3 represent the death rate of susceptible, infected and recovered individuals, respectively. Particularly, death rate d_2 includes the natural death rate and the disease-related death rate of the infected individuals. δ is the rate at which recovered individuals lose immunity and return to the susceptible class. γ is the natural recovery rate of the infective individuals, β is the proportionality constant. $f(S)$ and $g(I)$ are continuous functions defined on $[0, \infty)$. The transmission of the infection is governed by a nonlinear incidence rate $\beta f(S)g(I)$. In this paper, we always assume that δ is nonnegative constant, and $A, d_1, d_2, d_3, \beta, \gamma$ are positive constants.

Now, we use the backward difference scheme to discretize model (1). Let $h > 0$ be the time step size. Since

$$\frac{dS(t)}{dt} = \lim_{h\to 0} \frac{S(t+h) - S(t)}{h}, \quad \frac{dI(t)}{dt} = \lim_{h\to 0} \frac{I(t+h) - I(t)}{h},$$

$$\frac{dR(t)}{dt} = \lim_{h\to 0} \frac{R(t+h) - R(t)}{h}.$$

$$\lim_{h\to 0}(A - \beta f(S(t+h))g(I(t+h)) - (d_1+\eta)S(t+h) + \delta R(t+h))$$
$$= A - \beta f(S(t))g(I(t)) - (d_1+\eta)S(t) + \delta R(t),$$
$$\lim_{h\to 0}(\beta f(S(t+h))g(I(t+h)) - (d_2+\gamma)I(t+h))$$
$$= \beta f(S(t))g(I(t)) - (d_2+\gamma)I(t)$$

and

$$\lim_{h\to 0}(\eta S(t+h) + \gamma I(t+h) - (d_3+\delta)R(t+h))$$
$$= \eta S(t) + \gamma I(t) - (d_3+\delta)R(t),$$

we can assume from model (1) for any $h > 0$

$$\begin{aligned} \frac{S(t+h)-S(t)}{h} &= A - \beta f(S(t+h))g(I(t+h)) \\ &\quad -(d_1+\eta)S(t+h) + \delta R(t+h), \\ \frac{I(t+h)-I(t)}{h} &= \beta f(S(t+h))g(I(t+h)) - (d_2+\gamma)I(t+h), \\ \frac{R(t+h)-R(t)}{h} &= \eta S(t+h) + \gamma I(t+h) - (d_3+\delta)R(t+h). \end{aligned} \tag{2}$$

Denote $t = n$, $t+h = n+1$, $S(t) = S_n$, $I(t) = I_n$, $R(t) = R_n$, $S(t+h) = S_{n+1}$, $I(t+h) = I_{n+1}$ and $R(t+h) = R_{n+1}$, then from (2) we further obtain the following discrete SIRS epidemic model with nonlinear incidence and vaccination in susceptible

$$\begin{aligned} S_{n+1} - S_n &= h[A - \beta f(S_{n+1})g(I_{n+1}) - (d_1+\eta)S_{n+1} + \delta R_{n+1}], \\ I_{n+1} - I_n &= h[\beta f(S_{n+1})g(I_{n+1}) - (d_2+\gamma)I_{n+1}], \\ R_{n+1} - R_n &= h[\eta S_{n+1} + \gamma I_{n+1} - (d_3+\delta)R_{n+1}]. \end{aligned} \tag{3}$$

In this paper, our main aim namely is to investigate the dynamical properties of model (3). The initial condition for model (3) is given in the following form

$$S_0 > 0,\ I_0 > 0,\ R_0 \geq 0. \tag{4}$$

For model (3) we firstly introduce the following assumption.

(H_1) Functions $f(S)$ and $g(I)$ are continuously differentiable and monotone increasing on R, $f(0) = g(0) = 0$, $\frac{I}{g(I)}$ is monotone increasing on $(0, +\infty)$ and $g'(0) > 0$.

Remark 1. It is obvious that assumption (H_1) is basic for model (3). In fact, when $f(S) = S^p$ (where $0 < p \leq 1$) or $f(S) = \frac{S}{1+\alpha S}$ and $g(I) = \frac{I}{1+\omega I}$, then assumption (H_1) naturally holds.

Remark 2. If function $g(I)$ satisfies that second order derivative $g''(I)$ exists and $g''(I) \leq 0$ for all $I \in [0, \infty)$, then we can easily prove that $\frac{I}{g(I)}$ is monotone increasing on $I \in (0, +\infty)$.

Define a function $F(u, v)$ as follows. For any $u, v \in R$, if $u \neq v$ then $F(u, v) = \frac{f(u)-f(v)}{u-v}$, and if $u = v$ then $F(u, u) = f'(u)$. In order to obtain the global asymptotic stability of equilibria of model (3), we need to further introduce the following assumption.

(H_2) $(d_1 + d_3)\gamma - (d_2 + d_3)\eta > 0$.

(H_3) There are positive constants K_1 and K_3 such that

$$4d_1[d_3 + k_4(d_3 + \delta)] > (d_1 + d_3 - k_4\eta - K_1\delta F(u, v))^2$$

and

$$4K_1(d_1 + \eta)d_2F(u, v) > (d_1 + d_2 - K_3\beta F(u, v))^2$$

for any $0 < u, v \leq \frac{A}{d}$ with $u \neq v$, where $k_4 = \frac{d_2+d_3}{\gamma}$ and $d = \min\{d_1, d_2, d_3\}$.

Remark 3. When $f(S) \equiv S$, we have $F(u, v) \equiv 1$. Choosing positive constants $K_1 = \frac{d_1+d_3-k_4\eta}{\delta}$ and $K_2 = \frac{d_1+d_2}{\beta}$, then assumption (H_3) naturally holds.

Remark 4. When $f(S) = \frac{S}{1+\omega S}$, we have that there is a $\xi = \xi(u, v) \in (0, \frac{A}{d})$ such that $F(u, v) = \frac{1}{(1+\omega\xi)^2}$. Obviously,

$$\frac{1}{(1 + \omega\frac{A}{d})^2} \leq F(u, v) = \frac{1}{(1 + \omega\xi)^2} \leq 1$$

for all $u, v \in (0, \frac{A}{d}]$ with $u \neq v$. Choose positive constants

$$K_1 = \frac{2(d_1 + d_3 - k_4\eta)(1 + \omega\frac{A}{d})^2}{\delta(1 + (1 + \omega\frac{A}{d})^2)}, \quad K_3 = \frac{2(d_1 + d_2)(1 + \omega\frac{A}{d})^2}{\beta(1 + (1 + \omega\frac{A}{d})^2)},$$

then we can easily obtain that

$$(d_1 + d_3 - k_4\eta - K_1\delta F(u, v))^2 \leq (1 - \frac{2(1 + \omega\frac{A}{d})^2}{1 + (1 + \omega\frac{A}{d})^2})^2(d_1 + d_3 - k_4\eta)^2,$$

$$(d_1 + d_2 - K_3\beta F(u, v))^2 \leq (1 - \frac{2(1 + \omega\frac{A}{d})^2}{1 + (1 + \omega\frac{A}{d})^2})^2(d_1 + d_2)^2$$

and

$$4K_1(d_1 + \eta)d_2F(u, v) \geq 8(d_1 + \eta)d_2\frac{d_1 + d_3 - k_4\eta}{\delta(1 + (1 + \omega\frac{A}{d})^2)}$$

for all $u, v \in (0, \frac{A}{d}]$ with $u \neq v$.

Therefore, for $f(S)=\frac{S}{1+\omega S}$, if we assume that the following conditions hold

$$4d_1(d_3+k_4(d_3+\delta))>(1-\frac{2(1+\omega\frac{A}{d})^2}{1+(1+\omega\frac{A}{d})^2})^2(d_1+d_3-k_4\eta)^2 \tag{5}$$

and

$$8(d_1+\eta)d_2\frac{d_1+d_3-k_4\eta}{\delta(1+(1+\omega\frac{A}{d})^2)}>(1-\frac{2(1+\omega\frac{A}{d})^2}{1+(1+\omega\frac{A}{d})^2})^2(d_1+d_2)^2, \tag{6}$$

then it can be easily proved that assumption (H_3) holds.

Particularly, when $\omega=0$, that is $f(S)\equiv S$, we easily see that conditions (5) and (6) naturally hold.

3. Basic properties

Firstly, on the existence of positive solutions with initial condition (4) and the boundedness of all solutions of model (3), we have the following results.

Theorem 1. *Suppose that (H_1) holds. Then model (3) has a unique positive solution (S_n,I_n,R_n) for all $n\geq 0$ with initial condition (4), and*

$$\limsup_{n\to\infty}(S_n+I_n+R_n)\leq\frac{A}{d},$$

where $d=\min\{d_1,d_2,d_3\}$.

Proof: When $n=0$, from model (3) we have

$$\begin{aligned}(1+h(d_1+\eta))S_1&=S_0+h[A+\delta R_1-\beta f(S_1)g(I_1)],\\(1+h(d_2+\gamma))I_1&=I_0+h\beta f(S_1)g(I_1),\\(1+h(d_3+\delta))R_1&=R_0+h[\eta S_1+\gamma I_1].\end{aligned} \tag{7}$$

Solving S_1 from (7), we obtain

$$\begin{aligned}S_1=&\frac{1}{1+h(d_1+\eta)}[S_0+h(A+\delta R_1)+I_0-(1+h(d_2+\gamma))I_1]\\=&\frac{1}{1+h(d_1+\eta)}[S_0+I_0+h(A+\frac{\delta R_0}{1+h(d_3+\delta)})\\&-(1+h(d_2+\gamma)-\frac{h^2\gamma\delta}{1+h(d_3+\delta)})I_1+\frac{h^2\delta\eta}{1+h(d_3+\delta)}S_1].\end{aligned}$$

Therefore, (7) is equivalent to

$$\begin{aligned}S_1=&\ a^{-1}\frac{1}{1+h(d_1+\eta)}[S_0+I_0+h(A+\frac{\delta R_0}{1+h(d_3+\delta)}\\&-(1+h(d_2+\gamma)-\frac{h^2\gamma\delta}{1+h(d_3+\delta)})I_1],\\I_1=&\frac{I_0+h\beta f(S_1)g(I_1)}{1+h(d_2+\gamma)},\quad R_1=\frac{R_0+h\gamma I_1}{1+h(d_3+\delta)},\end{aligned} \tag{8}$$

where

$$a = 1 - \frac{h^2\delta\eta}{(1+h(d_1+\eta))(1+h(d_3+\delta))}.$$

Obviously, $a > 0$, $1 + h(d_2+\gamma) - \frac{h^2\gamma\delta}{1+h(d_3+\delta)} > 0$. Let

$$\bar{I}_1 = \frac{S_0 + I_0 + h(A + \frac{\delta R_0}{1+h(d_3+\delta)})}{1 + h(d_2+\gamma) - \frac{h^2\gamma\delta}{1+h(d_3+\delta)}},$$

then from (8) we have $S_1 > 0$ when $0 < I_1 < \bar{I}_1$, $S_1 < 0$ when $I_1 > \bar{I}_1$, and $S_1 = 0$ when $I_1 = \bar{I}_1$.

Let

$$\Psi^*(I_1) \triangleq \frac{I_1}{g(I_1)} - \frac{1}{1+h(d_2+\gamma)}(\frac{I_0}{g(I_1)} + h\beta f(S_1))$$

with

$$\begin{aligned} S_1 = \ & a^{-1}\frac{1}{1+h(d_1+\eta)}[S_0 + I_0 + h(A + \frac{\delta R_0}{1+h(d_3+\delta)}) \\ & -(1+h(d_2+\gamma) - \frac{h^2\gamma\delta}{1+h(d_3+\delta)})I_1]. \end{aligned}$$

Then, from (8) we have $\Psi^*(I_1) = 0$. Under assumption (H_1), we obtain that $\Psi^*(I_1)$ is monotonically increasing for $I_1 > 0$ and $\lim_{I_1\to 0}\Psi^*(I_1) = -\infty$. On the other hand, when $I_1 = \bar{I}_1$ we have $f(S_1) = f(0) = 0$ and hence,

$$\begin{aligned} \Psi^*(\bar{I}_1) = \ & \frac{\bar{I}_1}{g(\bar{I}_1)} - \frac{1}{1+h(d_2+\gamma)}\frac{I_0}{g(\bar{I}_1)} \\ = \ & \frac{1}{g(\bar{I}_1)}(\bar{I}_1 - \frac{I_0}{1+h(d_2+\gamma)}) > 0. \end{aligned}$$

Therefore, $\Psi^*(I_1) = 0$ has a unique positive solution $y^* \in (0, \bar{I}_1)$. That is,

$$y^* = \frac{1}{1+h(d_2+\gamma)}(I_0 - h\beta f(S_1)g(y^*)).$$

Now, we show that y^* is the unique solution of $\Psi^*(I_1) = 0$ on $(0, \infty)$. Otherwise, there is a $y' \in [\bar{I}_1, \infty)$ such that $\Psi^*(y') = 0$. Since $y' \geq \bar{I}_1$, we have that $S_1 \leq 0$ when $I_1 = y'$. From (H_1), we have $f(S) \leq 0$ for any $S \leq 0$. Hence, from $\Psi^*(y') = 0$ we further have $y' \leq \frac{I_0}{1+h(d_2+\gamma)}$. On the other hand, since $\bar{I}_1 > \frac{I_0}{1+h(d_2+\gamma)}$, we obtain $y' > \frac{I_0}{1+h(d_2+\gamma)}$, which leads to a contradiction.

Therefore, we have $I_1 = y^* > 0$. Again from (8), we further also have $S_1 > 0$ and $R_1 > 0$. This shows that from (7) we can obtain a unique positive solution (S_1, I_1, R_1).

When $n = 1$, a similarly argument as in above, we can obtain a unique positive solution (S_2, I_2, R_2) satisfying model (3) at $n = 1$. By using the induction, we finally obtain a unique positive solution (S_n, I_n, R_n) for all $n > 0$ satisfying model (3).

Let $N_n = S_n + I_n + R_n$, then from model (3) we have

$$N_n = N_{n-1} + h(A - d_1 S_n - d_2 I_n - d_3 R_n) \le N_{n-1} + h(A - dN_n).$$

Hence,

$$N_n \le \frac{hA + N_{n-1}}{1 + hd}.$$

By using iteration method, we obtain

$$\begin{aligned} N_n \le \; & \frac{hA}{1+hd} + \frac{hA}{(1+hd)^2} + \cdots + \frac{hA}{(1+hd)^n} + \frac{N_0}{(1+hd)^n} \\ = \; & \frac{A}{d}[1 - \frac{1}{(1+hd)^n}] + \frac{N_0}{(1+hd)^n}. \end{aligned}$$

Therefore, it holds that $\limsup_{n\to+\infty} N_n \le \frac{A}{d}$. This completes the proof.

The basic reproduction number for model (3) can be defined by

$$\mathscr{R}_0 = \frac{\beta f(S^0) g'(0)}{d_2 + \gamma},$$

where $S^0 = \frac{A(d_3+\delta)}{d_1(d_3+\delta)+d_3\eta}$. On the existence of equilibria of model (3), we have the following result.

Theorem 2. *Suppose that (H_1) holds.*

(1). If $\mathscr{R}_0 \le 1$, then model (3) has only a unique disease-free equilibrium $E^0 = (S^0, 0, R^0)$, where S^0 is given in the above and $R^0 = \frac{\eta A}{d_1(d_3+\delta)+d_3\eta}$.

(2). If $\mathscr{R}_0 > 1$, then model (3) has a unique endemic equilibrium $E^* = (S^*, I^*, R^*)$, except for disease-free equilibrium E^0.

Proof: We know that any equilibrium $E = (S, I, R)$ of model (3) satisfies the following equation

$$\begin{aligned} & A - \beta f(S) g(I) - (d_1 + \eta) S + \delta R = 0, \\ & \beta f(S) g(I) - (d_2 + \gamma) I = 0, \\ & \eta S + \gamma I - (d_3 + \delta) R = 0. \end{aligned} \tag{9}$$

Firstly, when $I = 0$, we have

$$A - (d_1 + \eta) S + \delta R = 0, \quad \eta S - (d_3 + \delta) R = 0.$$

From this, we directly obtain disease-free equilibrium $E^0 = (S^0, 0, R^0)$.

Second, when $I > 0$, from equation (9), we obtain

$$R = \frac{\eta S + \gamma I}{\delta + d_3}, \quad S = S^0 - \frac{d_2(\delta + d_3) + d_3\gamma}{d_1(\delta + d_3) + d_3\eta} I.$$

Substituting S into the second equation of (9) and we have

$$\beta f(S^0 - \frac{d_2(\delta + d_3) + d_3\gamma}{d_1(\delta + d_3) + d_3\eta} I) g(I) - (d_2 + \gamma) I = 0.$$

Let

$$H(I) = \beta f(S^0 - \frac{d_2(\delta + d_3) + d_3\gamma}{d_1(\delta + d_3) + d_3\eta} I) \frac{g(I)}{I} - (d_2 + \gamma).$$

By assumption (H_1), $H(I)$ is strictly monotone decreasing on $(0, +\infty)$ and satisfies

$$\lim_{I \to 0^+} H(I) = \beta f(S^0) g'(0) - (d_2 + \gamma) = (d_2 + \gamma)(\mathscr{R}_0 - 1)$$

and we also have $H(\bar{I}) = -(d_2 + \gamma) < 0$, where $\bar{I} = \frac{S^0(d_1(d_3+\delta)+d_3\eta)}{d_2(d_3+\delta)+d_3\gamma}$.

When $\mathscr{R}_0 \leq 1$, we have $\lim_{I \to 0^+} H(I) \leq 0$. Consequently, there is not any $I^* > 0$ such that $H(I^*) = 0$. Therefore, model (3) only has a unique disease-free equilibrium E_0.

When $\mathscr{R}_0 > 1$, we have $\lim_{I \to 0^+} H(I) > 0$. Therefore, there exists a unique $I^* \in (0, \bar{I})$ such that $H(I^*) = 0$. Furthermore, we have $S^* = S^0 - \frac{d_2(\delta+d_3)+d_3\gamma}{d_1(\delta+d_3)+d_3\eta} I^* > 0$ and $R^* = \frac{\eta S^* + \gamma I^*}{\delta + d_3} > 0$. This implies that model (3) has a unique endemic equilibrium $E^* = (S^*, I^*, R^*)$. This completes the proof.

Further, on the local stability of equilibria of model (3), we have the following result.

Theorem 3. *Suppose that (H_1) holds.*

(1). When $\mathscr{R}_0 < 1$, then disease-free equilibrium E^0 of model (3) is locally asymptotically stable.

(2). When $\mathscr{R}_0 > 1$, then disease-free equilibrium E^0 of model (3) is unstable, and endemic equilibrium E^* is locally asymptotically stable.

Proof: Calculating the linearization system of model (3) at equilibrium E^0, we have

$$\begin{aligned} u_{n+1} - u_n = & \ h[\beta f(S^0) g'(0) v_{n+1} - (d_1 + \eta) u_{n+1} + \delta w_{n+1}], \\ v_{n+1} - v_n = & \ h[\beta f(S^0) g'(0) v_{n+1} - (d_2 + \gamma) v_{n+1}], \\ w_{n+1} - w_n = & \ h[\eta u_{n+1} + \gamma v_{n+1} - (d_3 + \delta) w_{n+1}]. \end{aligned} \tag{10}$$

From the second equation of system (10), we have

$$v_{n+1} = \frac{v_n}{1 + h[d_2 + \gamma - \beta f(S^0) g'(0)]}. \tag{11}$$

When $\mathcal{R}_0 < 1$, we obtain

$$0 < \frac{1}{1 + h[d_2 + \gamma - \beta f(S^0)g'(0)]} = \frac{1}{1 + h(d_2 + \gamma)(1 - \mathcal{R}_0)} < 1.$$

Therefore, from (11) we have $\lim_{n\to\infty} v_n = 0$. From the first and third equations of system (10) we have

$$\begin{pmatrix} u_{n+1} \\ w_{n+1} \end{pmatrix} = A^{-1} \begin{pmatrix} u_n \\ w_n \end{pmatrix} + A^{-1} \begin{pmatrix} -h\beta f(S^0)g'(0) \\ h\gamma \end{pmatrix} v_{n+1}, \tag{12}$$

where

$$A = \begin{pmatrix} 1 + h[d_1 + \eta] & -h\delta \\ -h\eta & 1 + h[d_3 + \delta] \end{pmatrix}.$$

It is clear that two eigenvalues λ_i $(i = 1, 2)$ of matrix A satisfy $|\lambda_i| > 1$. Hence, norms of two eigenvalues of A^{-1} are less than one. Since $\lim_{n\to\infty} v_n = 0$, from (12) we can obtain $\lim_{n\to\infty} u_n = 0$ and $\lim_{n\to\infty} w_n = 0$. This shows that equilibrium E^0 of model (3) is locally stable.

When $\mathscr{R}_0 > 1$, since $|\frac{1}{1+h[d_2+\gamma-\beta f(S^0)g'(0)]}| > 1$, from (11) we obtain $\lim_{n\to\infty} v_n = \infty$. Therefore, E^0 is unstable.

Calculating the linearization system of model (3) at equilibrium E^*, we have

$$\begin{aligned} u_{n+1} = \;& u_n - h[\beta f'(S^*)g(I^*)u_{n+1} - \beta f(S^*)g'(I^*)v_{n+1} \\ & -(d_1 + \eta)u_{n+1} + \delta w_{n+1}], \\ v_{n+1} = \;& v_n + h[\beta f'(S^*)g(I^*)u_{n+1} + \beta f(S^*)g'(I^*)v_{n+1} - (d_2 + \gamma)v_{n+1}], \\ w_{n+1} = \;& w_n + h[\eta u_{n+1} + \gamma v_{n+1} - (d_3 + \delta)w_{n+1}]. \end{aligned} \tag{13}$$

Let

$$A = \begin{pmatrix} 1 + h[\beta f'(S^*)g(I^*) + d_1 + \eta] & h\beta f(S^*)g'(I^*) & -h\delta \\ -h\beta f'(S^*)g(I^*) & 1 - h[\beta f(S^*)g'(I^*) - d_2 - \gamma] & 0 \\ -h\eta & -h\gamma & 1 + h[d_3 + \delta] \end{pmatrix}$$

and $X_n = (u_n, v_n, w_n)^T$, then equation (13) can be rewrote into

$$X_{n+1} = A^{-1}X_n. \tag{14}$$

It is clear that if all eigenvalues λ of matrix $-A$ satisfy $|\lambda| > 1$, then all eigenvalues σ of matrix A^{-1} will satisfy $|\sigma| < 1$. The characteristic equation of $-A$ is $|\lambda E + A| = 0$, where E is the unit matrix. Let $r = \frac{\lambda+1}{h}$, then we easily obtain

$$|\lambda E + A| = r^3 + ar^2 + br + c,$$

where

$$a = d_1 + \eta + \beta f'(S^*)g(I^*) + d_3 + \delta + d_2 + \gamma - \beta f(S^*)g'(I^*),$$

$$\begin{aligned} b = \ & d_1(d_3 + \delta) + d_3\eta + (d_3 + \delta)\beta f'(S^*)g(I^*) + (d_2 + \gamma)\beta f'(S^*)g(I^*) \\ & (d_3 + \delta + d_1 + \eta)(d_2 + \gamma - \beta f(S^*)g'(I^*)) \end{aligned}$$

and

$$c = (d_1(d_3 + \delta) + d_3\eta)(d_2 + \gamma - \beta f(S^*)g'(I^*)) + (d_2(d_3 + \delta) + d_3\gamma)\beta f'(S^*)g(I^*).$$

From assumption (H_1), we easily obtain $\frac{g(I)}{I} - g'(I) \geq 0$ for all $I > 0$. Since $\beta f(S^*)g(I^*) - (d_2 + \gamma)I^* = 0$, we obtain $d_2 + \gamma - \beta f(S^*)g'(I^*) \geq 0$. Hence, we have $a > 0$, $b > 0$ and $c > 0$. By calculating, we further obtain

$$\begin{aligned} ab - c = \ & (d_1 + \eta + d_3 + \delta)[(d_2 + \gamma - \beta f(S^*)g'(I^*))^2 \\ & +(d_3 + \delta)(d_1 + \beta f'(S^*)g(I^*)) + d_3\eta] \\ & +(d_2 + \gamma - \beta f(S^*)g'(I^*))[(d_3 + \delta + d_1 + \eta)^2 \\ & +\beta f'(S^*)g(I^*)(2(d_3 + \delta) + d_2 + \gamma + d_1 + \eta)] \\ & +\beta f'(S^*)g(I^*)[(d_1 + \eta)(d_2 + \gamma) + d_1(d_3 + \delta) + \gamma\delta] \\ & +(\beta f'(S^*)g(I^*))^2(d_2 + \gamma + d_3 + \delta) > 0. \end{aligned}$$

Therefore, by the Routh-Hurwitz criterion all roots of equation

$$r^3 + ar^2 + br + c = 0$$

have the negative real parts. Since $\lambda = hr - 1$, we further obtain that all eigenvalues λ of matrix $-A$ satisfy $|\lambda| > 1$. Therefore, the zero solution $X = 0$ of equation (14) is asymptotically stable. This shows that equilibrium E^* is locally asymptotically stable. This completes the proof.

Remark 5. From Theorems 3 we directly see that assumptions (H_2) and (H_3) only are used to obtain the global asymptotic stability of equilibria of model (3).

Remark 6. From the results obtained in this section, we easily see that the backward difference scheme, that is discrete dynamical model (3), for a class of SIRS epidemic models (1) with nonlinear incidence is provided for us with excellent properties in the local stability of equilibria and the permanence of disease. These properties nearly are same to corresponding continuous-time model (1).

4. The global stability

Now, we study the stability of equilibria of model (3). Firstly, on the global stability of disease-free equilibrium E^0, we have the following result:

Theorem 4. *Suppose that $(H_1)-(H_3)$ hold. If $\mathscr{R}_0 \leq 1$, then for any time step size $h > 0$ disease-free equilibrium E^0 of model (3) is globally asymptotically stable.*

Proof: Model (3) can be rewritten as the following form

$$\begin{aligned}
S_{n+1} - S_n = \;& h[-(d_1+\eta)(S_{n+1} - S^0) - \beta g(I_{n+1})(f(S_{n+1}) - f(S^0)) \\
& +\delta(R_{n+1} - R^0) - \beta f(S^0) g(I_{n+1})], \\
I_{n+1} - I_n = \;& h[\beta g(I_{n+1})(f(S_{n+1}) - f(S^0)) \\
& -(d_2+\gamma)I_{n+1} + \beta f(S^0) g(I_{n+1})], \\
R_{n+1} - R_n = \;& h[\eta(S_{n+1} - S^0) + \gamma I_{n+1} - (d_3+\delta)(R_{n+1} - R^0)].
\end{aligned} \tag{15}$$

We consider the following Lyapunov function

$$\begin{aligned}
W_n = \;& \frac{1}{2}(S_n - S^0 + I_n + R_n - R^0)^2 + k_1 \int_{S^0}^{S_n} (f(\tau) - f(S^0))d\tau \\
& +(k_2+k_3)I_n + \frac{k_4}{2}(R_n - R^0)^2,
\end{aligned}$$

where $k_i\,(i = 1, 2, 3, 4)$ are positive constants which will be determined in the following. Calculating difference of W_n along solutions of equation (15), by assumption (H_1) we have

$$\begin{aligned}
W_{n+1} - W_n = \;& k_1 \int_{S_n}^{S_{n+1}} (f(\tau) - f(S^0))d\tau + (k_2+k_3)(I_{n+1} - I_n) \\
& +\frac{k_4}{2}[(R_{n+1} - R^0)^2 - (R_n - R^0)^2] \\
& +\frac{1}{2}[(S_{n+1} - S^0 + I_{n+1} + R_{n+1} - R^0)^2 \\
& -(S_n - S^0 + I_n + R_n - R^0)^2] \\
= \;& k_1(S_{n+1} - S_n)(f(S_{n+1}) - f(S^0)) + (k_2+k_3)(I_{n+1} - I_n) \\
& +\frac{k_4}{2}[(R_{n+1} - R_n)(R_n - R_{n+1} + 2(R_{n+1} - R^0)] \\
& +\frac{1}{2}[(S_{n+1} - S_n + I_{n+1} - I_n + R_{n+1} - R_n) \\
& \times(S_n - S_{n+1} + 2(S_{n+1} - S^0) + I_n - I_{n+1} \\
& +2I_{n+1} + R_n - R_{n+1} + 2(R_{n+1} - R^0)] \\
\leq \;& k_1(S_{n+1} - S_n)(f(S_{n+1}) - f(S^0)) + (k_2+k_3)(I_{n+1} - I_n) \\
& +k_4(R_{n+1} - R_n)(R_{n+1} - R^0) + (S_{n+1} - S^0 + I_{n+1} + R_{n+1} - R^0) \\
& \times(S_{n+1} - S_n + I_{n+1} - I_n + R_{n+1} - R_n) \\
= \;& k_1 h[-(d_1+\eta)(S_{n+1} - S^0) - \beta g(I_{n+1})(f(S_{n+1}) - f(S^0))
\end{aligned}$$

$$
\begin{aligned}
&+\delta(R_{n+1}-R^0)-\beta f(S^0)g(I_{n+1})](f(S_{n+1})-f(S^0))\\
&+k_2h[\beta g(I_{n+1})(f(S_{n+1})-f(S^0))-(d_2+\gamma)I_{n+1}\\
&+\beta f(S^0)g(I_{n+1})]+k_3h(\beta f(S_{n+1})g(I_{n+1})-(d_2+\gamma)I_{n+1})\\
&+k_4h[\eta(S_{n+1}-S^0)+\gamma I_{n+1}-(d_3+\delta)(R_{n+1}-R^0)]\\
&\times(R_{n+1}-R^0)+h[-d_1(S_{n+1}-S^0)-d_2I_{n+1}\\
&-d_3(R_{n+1}-R^0)]\times(S_{n+1}-S^0+I_{n+1}+R_{n+1}-R^0)
\end{aligned}
$$

Since $\mathscr{R}_0=\frac{\beta f(S^0)g'(0)}{d_2+\gamma}\leq 1$, we have $\beta f(S^0)g'(0)\leq d_2+\gamma$. Under assumption (H_1), we have

$$\frac{g(I_{n+1})}{I_{n+1}}\leq\lim_{I\to 0^+}\frac{g(I)}{I}=g'(0).$$

Choosing constants $k_2=k_1f(S^0)$ and $k_4=\frac{d_2+d_3}{\gamma}$, we further have

$$
\begin{aligned}
W_{n+1}-W_n\leq\;&-k_1h\beta g(I_{n+1})(f(S_{n+1})-f(S^0))^2-d_1h(S_{n+1}-S^0)^2\\
&-d_2hI_{n+1}^2-k_4h(d_3+\delta)(R_{n+1}-R^0)^2\\
&-d_3h(R_{n+1}-R^0)^2-h(d_1+d_2)(S_{n+1}-S^0)I_{n+1}\\
&-h(d_1+d_3-k_4\eta)(S_{n+1}-S^0)(R_{n+1}-R^0)\\
&-(d_1+\eta)k_1h[f(S_{n+1})-f(S^0)](S_{n+1}-S^0)\\
&+k_1h\delta(f(S_{n+1})-f(S^0))(R_{n+1}-R^0)\\
&+k_2h\beta f(S^0)I_{n+1}[\frac{g(I_{n+1})}{I_{n+1}}-g'(0)]\\
&+k_3h\beta I_{n+1}[f(S_{n+1})(\frac{g(I_{n+1})}{I_{n+1}}-g'(0))+g'(0)(f(S_{n+1})-f(S^0))]\\
\leq\;&-d_1h(S_{n+1}-S^0)^2-d_2hI_{n+1}^2-(k_4(d_3+\delta)+d_3)h(R_{n+1}-R^0)^2\\
&+k_1h\delta(f(S_{n+1})-f(S^0))(R_{n+1}-R^0)\\
&-(d_1+\eta)k_1h[f(S_{n+1})-f(S^0)](S_{n+1}-S^0)\\
&-(d_1+d_2)h(S_{n+1}-S^0)I_{n+1}+k_3h\beta g'(0)I_{n+1}(f(S_{n+1})-f(S^0))\\
&-(d_1+d_3-k_4\eta)h(S_{n+1}-S^0)(R_{n+1}-R^0)\\
=\;&-d_1h(S_{n+1}-S^0)^2-d_2hI_{n+1}^2-(k_4(d_3+\delta)+d_3)h(R_{n+1}-R^0)^2\\
&-(d_1+d_3-k_4\eta)h(S_{n+1}-S^0)(R_{n+1}-R^0)\\
&-(d_1+d_2)h(S_{n+1}-S^0)I_{n+1}\\
&-(d_1+\eta)k_1h[\frac{f(S_{n+1})-f(S^0)}{S_{n+1}-S^0}](S_{n+1}-S^0)^2\\
&+k_1h\delta(\frac{f(S_{n+1})-f(S^0)}{S_{n+1}-S^0})(S_{n+1}-S^0)(R_{n+1}-R^0)
\end{aligned}
$$

$$
\begin{aligned}
&+k_3h\beta g'(0)I_{n+1}(S_{n+1}-S^0)(\frac{f(S_{n+1})-f(S^0)}{S_{n+1}-S^0}) \\
= &-h[(S_{n+1}-S^0,I_{n+1})P(S_{n+1}-S^0,I_{n+1})^T \\
&+(S_{n+1}-S^0,R_{n+1}-R^0)Q(S_{n+1}-S^0,R_{n+1}-R^0)^T],
\end{aligned}
$$

where

$$
P=\begin{pmatrix} k_1(d_1+\eta)F(S_{n+1},S^0) & p_{12} \\ p_{12} & d_2, \end{pmatrix}, \quad Q=\begin{pmatrix} d_1 & q_{12} \\ q_{12} & [d_3+k_4(d_3+\delta)] \end{pmatrix}
$$

with

$$
\begin{aligned}
p_{12}&=\frac{1}{2}(d_1+d_2-k_3\beta g'(0)F(S_{n+1},S^0)), \\
q_{12}&=\frac{1}{2}(d_1+d_3-k_4\eta-k_1\delta F(S_{n+1},S^0)).
\end{aligned}
$$

Further, we choose $k_1=K_1$ and $k_3=\frac{K_3}{g'(0)}$, then assumption (H_3) implies that matrices P and Q are positive definite. This implies that

$$
W_{n+1}-W_n<0 \quad \text{for all} \quad (S_n,I_n,R_n)\neq(S^0,0,R^0).
$$

By the Lyapunov's theorems on the global asymptotical stability for difference e-quations [28], we obtain that disease-free equilibrium E^0 is globally asymptotically stable. This completes the proof.

Remark 7. In articles [1-3,5], the authors studied the global properties of solutions for the various discrete difference scheme, such as the nonstandard finite difference scheme, backward difference scheme and forward difference scheme, for continuous-time SIRS epidemic models. The condition that the death rate (d_1) of susceptible is less than or equal to the death rate (d_2) of infected and the death rate (d_3) of recovered, that is, $d_1\leq\min\{d_2,d_3\}$ is required. Therefore, the global asymptotic stability of the disease-free equilibrium can be established only when the basic reproduction number $\mathscr{R}_0\leq 1$, except for some basic assumptions, for example, such as assumption (H_1) for model (3).

However, in this paper we do not require the condition $d_1\leq\min\{d_2,d_3\}$ for model (3). Therefore, in order to obtain the global stability of the disease-free equilibrium of model (3), a new Lyapunov function is constructed and the assumption (H_3) is introduced.

On the global stability of the endemic equilibrium E^*, we have the following result.

Theorem 5. *Suppose that $(H_1)-(H_3)$ hold. If $\mathscr{R}_0 > 1$, then for any time step size $h > 0$ endemic equilibrium E^* of model (3) is globally asymptotically stable.*

Proof: The model (3) can be rewritten as the following form:

$$\begin{aligned} S_{n+1}-S_n = &\ h[-\beta g(I_{n+1})[f(S_{n+1})-f(S^*)]-(d_1+\eta)(S_{n+1}-S^*)\\ &+\delta(R_{n+1}-R^*)-\beta f(S^*)(g(I_{n+1})-g(I^*))],\\ I_{n+1}-I_n = &\ h[\beta g(I_{n+1})(f(S_{n+1})-f(S^*))-(d_2+\gamma)(I_{n+1}-I^*)\\ &+\beta f(S^*)(g(I_{n+1})-g(I^*))],\\ R_{n+1}-R_n = &\ h[\eta(S_{n+1}-S^*)+\gamma(I_{n+1}-I^*)-(d_3+\delta)(R_{n+1}-R^*)]. \end{aligned} \tag{16}$$

Since $(d_2+\gamma)I^* = \beta f(S^*)g(I^*)$, we also have

$$\begin{aligned} I_{n+1}-I_n = &\ h[\beta f(S_{n+1})g(I_{n+1})-(d_2+\gamma)I_{n+1}]\\ = &\ \beta h I_{n+1}[f(S_{n+1})\frac{g(I_{n+1})}{I_{n+1}}-f(S^*)\frac{g(I^*)}{I^*}]\\ = &\ \beta h I_{n+1}[f(S_{n+1})(\frac{g(I_{n+1})}{I_{n+1}}-\frac{g(I^*)}{I^*})+\frac{g(I^*)}{I^*}(f(S_{n+1})-f(S^*))] \end{aligned} \tag{17}$$

and

$$\begin{aligned} &S_{n+1}-S_n+I_{n+1}-I_n+R_{n+1}-R_n\\ = &\ -d_1(S_{n+1}-S^*)-d_2(I_{n+1}-I^*)-d_3(R_{n+1}-R^*). \end{aligned} \tag{18}$$

We consider the following Lyapunov function

$$\begin{aligned} V_n = &\ \frac{1}{2}(S_n-S^*+I_n-I^*+R_n-R^*)^2+k_1\int_{S^*}^{S_n}(f(\tau)-f(S^*))d\tau\\ &+k_2\int_{I^*}^{I_n}\frac{g(\tau)-g(I^*)}{g(\tau)}d\tau+k_3(I_n-I^*-I^*\ln\frac{I_n}{I^*})+\frac{k_4}{2}(R_n-R^*)^2, \end{aligned}$$

where $k_i\,(i=1,2,3,4)$ are positive constants which will be determined in the following. Calculating difference of V_n along equation (16), then by (17) and (18) we have

$$\begin{aligned} V_{n+1}-V_n = &\ k_1[\int_{S_n}^{S_{n+1}}(f(\tau)-f(S^*))d\tau]+k_2\int_{I_n}^{I_{n+1}}\frac{g(\tau)-g(I^*)}{g(\tau)}d\tau\\ &+k_3(I_{n+1}-I_n-I^*\ln\frac{I_{n+1}}{I_n})+\frac{k_4}{2}[(R_{n+1}-R^*)^2-(R_n-R^*)^2]\\ &+\frac{1}{2}[(S_{n+1}-S^*+I_{n+1}-I^*+R_{n+1}-R^*)^2\\ &-(S_n-S^*+I_n-I^*+R_n-R^*)^2]\\ \le &\ k_1(S_{n+1}-S_n)(f(S_{n+1})-f(S^*))\\ &+k_2(I_{n+1}-I_n)(\frac{g(I_{n+1})-g(I^*)}{g(I_{n+1})})+k_3(I_{n+1}-I_n)\frac{I_{n+1}-I^*}{I_{n+1}}\\ &+\frac{k_4}{2}(R_{n+1}-R_n)(R_n-R_{n+1}+2(R_{n+1}-R^*))\\ &+\frac{1}{2}(S_{n+1}-S_n+I_{n+1}-I_n+R_{n+1}-R_n) \end{aligned}$$

$$
\begin{aligned}
& \times(S_n - S_{n+1} + 2(S_{n+1} - S^*) + I_n - I_{n+1} \\
& +2(I_{n+1} - I^*) + R_n - R_{n+1} + 2(R_{n+1} - R^*)) \\
\leq\ & k_1(S_{n+1} - S_n)(f(S_{n+1}) - f(S^*)) + k_4(R_{n+1} - R_n)(R_{n+1} - R^*) \\
& +k_2(I_{n+1} - I_n)(\frac{g(I_{n+1}) - g(I^*)}{g(I_{n+1})}) + k_3(I_{n+1} - I_n)\frac{I_{n+1} - I^*}{I_{n+1}} \\
& +(S_{n+1} - S_n + I_{n+1} - I_n + R_{n+1} - R_n) \\
& \times(S_{n+1} - S^* + I_{n+1} - I^* + R_{n+1} - R^*) \\
=\ & k_1 h[-\beta g(I_{n+1})[f(S_{n+1}) - f(S^*)] - (d_1 + \eta)(S_{n+1} - S^*) + \delta(R_{n+1} - R^*) \\
& -\beta f(S^*)(g(I_{n+1}) - g(I^*))](f(S_{n+1}) - f(S^*)) \\
& +k_2 h[\beta g(I_{n+1})(f(S_{n+1}) - f(S^*)) - (d_2 + \gamma)(I_{n+1} - I^*) \\
& +\beta f(S^*)(g(I_{n+1}) - g(I^*))](\frac{g(I_{n+1}) - g(I^*)}{g(I_{n+1})}) \\
& +k_3 h[\beta f(S_{n+1})(\frac{g(I_{n+1})}{I_{n+1}} - \frac{g(I^*)}{I^*}) + \beta\frac{g(I^*)}{I^*}(f(S_{n+1}) - f(S^*))](I_{n+1} - I^*) \\
& +k_4 h[\eta(S_{n+1} - S^*) + \gamma(I_{n+1} - I^*) - (d_3 + \delta)(R_{n+1} - R^*)](R_{n+1} - R^*) \\
& -h(d_1(S_{n+1} - S^*) + d_2(I_{n+1} - I^*) + d_3(R_{n+1} - R^*)) \\
& \times(S_{n+1} - S^* + I_{n+1} - I^* + R_{n+1} - R^*)
\end{aligned}
$$

Choosing constants $k_2 = k_1 f(S^*)$ and $k_4 = \frac{d_2+d_3}{\gamma}$, we further have

$$
\begin{aligned}
V_{n+1} - V_n \leq\ & -d_1 h(S_{n+1} - S^*)^2 - d_2 h(I_{n+1} - I^*)^2 \\
& -h[k_4(d_3 + \delta) + d_3](R_{n+1} - R^*)^2 - (d_1 + d_2)h(S_{n+1} - S^*)(I_{n+1} - I^*) \\
& -(d_1 + d_3 - k_4\eta)h(R_{n+1} - R^*)(S_{n+1} - S^*) \\
& -k_1(d_1 + \eta)h(S_{n+1} - S^*)(f(S_{n+1}) - f(S^*)) \\
& +k_1 h\delta(R_{n+1} - R^*)(f(S_{n+1}) - f(S^*)) \\
& +k_2 h\frac{g(I_{n+1}) - g(I^*)}{g(I_{n+1})}[\beta f(S^*)(g(I_{n+1})) - g(I^*)) - (d_2 + \gamma)(I_{n+1} - I^*)] \\
& +k_3 h\beta f(S_{n+1})(I_{n+1} - I^*)(\frac{g(I_{n+1})}{I_{n+1}} - \frac{g(I^*)}{I^*}) \\
& +k_3 h\beta\frac{g(I^*)}{I^*}(I_{n+1} - I^*)(f(S_{n+1}) - f(S^*)).
\end{aligned}
$$

From assumption (H_1) and $d_2 + \gamma = \beta f(S^*)\frac{g(I^*)}{I^*}$, we have

$$
\begin{aligned}
& k_2\frac{g(I_{n+1}) - g(I^*)}{g(I_{n+1})}[\beta f(S^*)(g(I_{n+1}) - g(I^*)) - (d_2 + \gamma)(I_{n+1} - I^*)] \\
=\ & k_2\frac{g(I_{n+1}) - g(I^*)}{g(I_{n+1})}[\beta f(S^*)g(I_{n+1}) - (d_2 + \gamma)I_{n+1}] \\
=\ & \frac{k_2\beta f(S^*)I_{n+1}}{g(I_{n+1})}(g(I_{n+1}) - g(I^*))[\frac{g(I_{n+1})}{I_{n+1}} - \frac{g(I^*)}{I^*}] \leq 0
\end{aligned}
$$

and

$$k_3\beta f(S_{n+1})(I_{n+1}-I^*)(\frac{g(I_{n+1})}{I_{n+1}}-\frac{g(I^*)}{I^*})\leq 0.$$

Hence,

$$\begin{aligned}
V_{n+1}-V_n \leq\ & -d_1h(S_{n+1}-S^*)^2-d_2h(I_{n+1}-I^*)^2\\
& -h[k_4(d_3+\delta)+d_3](R_{n+1}-R^*)^2-(d_1+d_2)h(S_{n+1}-S^*)(I_{n+1}-I^*)\\
& -(d_1+d_3-k_4\eta)h(R_{n+1}-R^*)(S_{n+1}-S^*)\\
& -k_1(d_1+\eta)h(S_{n+1}-S^*)(f(S_{n+1})-f(S^*))\\
& +k_1h\delta(R_{n+1}-R^*)(f(S_{n+1})-f(S^*))\\
& +k_3h\beta\frac{g(I^*)}{I^*}(I_{n+1}-I^*)(f(S_{n+1})-f(S^*))\\
=\ & -d_1h(S_{n+1}-S^*)^2-d_2h(I_{n+1}-I^*)^2\\
& -h[k_4(d_3+\delta)+d_3](R_{n+1}-R^*)^2-(d_1+d_2)h(S_{n+1}-S^*)(I_{n+1}-I^*)\\
& -(d_1+d_3-k_4\eta)h(R_{n+1}-R^*)(S_{n+1}-S^*)\\
& -k_1(d_1+\eta)h(S_{n+1}-S^*)^2\frac{f(S_{n+1})-f(S^*)}{S_{n+1}-S^*}\\
& +k_1h\delta(R_{n+1}-R^*)(S_{n+1}-S^*)\frac{f(S_{n+1})-f(S^*)}{S_{n+1}-S^*}\\
& +k_3h\beta\frac{g(I^*)}{I^*}(I_{n+1}-I^*)(S_{n+1}-S^*)\frac{f(S_{n+1})-f(S^*)}{S_{n+1}-S^*}\\
=\ & -h[(S_{n+1}-S^*,I_{n+1}-I^*)P(S_{n+1}-S^*,I_{n+1}-I^*)^T\\
& +(S_{n+1}-S^*,R_{n+1}-R^*)Q(S_{n+1}-S^*,R_{n+1}-R^*)^T],
\end{aligned}$$

where

$$P=\begin{pmatrix} k_1(d_1+\eta)F(S_{n+1},S^*) & p_{12}\\ p_{12} & d_2\end{pmatrix},\quad Q=\begin{pmatrix} d_1 & q_{12}\\ q_{12} & [d_3+k_4(d_3+\delta)]\end{pmatrix}$$

with

$$\begin{aligned}
p_{12}&=\frac{1}{2}(d_1+d_2-k_3\beta\frac{g(I^*)}{I^*}F(S_{n+1},S^*)),\\
q_{12}&=\frac{1}{2}(d_1+d_3-k_4\eta-k_1\delta F(S_{n+1},S^*)).
\end{aligned}$$

Further, we choose $k_1=K_1$ and $k_3=K_3\frac{I^*}{g(I^*)}$, then assumption (H_3) implies that matrices P and Q are positive definite. This implies that

$$V_{n+1}-V_n<0 \quad \text{for all} \quad (S_n,I_n,R_n)\neq(S^*,I^*,R^*).$$

By the Lyapunov's theorems on the globally asymptotical stability for difference equations [28], we directly obtained that the endemic equilibrium E^* is globally asymptotically stable. This completes the proof.

Remark 8. From the above discussion we immediately see that constant $\mathscr{R}_0$ is the basic reproduction number of model (3) and it can completely determine the global asymptotic stability of model (3).

Remark 9. From the above discussions we easily see that assumption (H_2) only is used to ensure the positivity of constant K_1. When $\eta = 0$, that is, there is not vaccination in susceptible, then assumption (H_2) naturally holds.

As consequences of Theorems 4 and 5, combining Remarks 3 and 4 we have the following corollaries.

Corollary 1. *Assume that in model (3) $f(S) \equiv S$ and (H_1) and (H_2) hold.*

(1). If $\mathscr{R}_0 \le 1$, then disease-free equilibrium E^0 of model (3) is globally asymptotically stable.

(2). If $\mathscr{R}_0 > 1$, then endemic equilibrium E^* of model (3) is globally asymptotically stable.

Corollary 2. *Assume that in model (3) $f(S) \equiv \frac{S}{1+\omega S}$, (H_1) and (H_2), and conditions (5) and (6) hold.*

(1). If $\mathscr{R}_0 \le 1$, then disease-free equilibrium E^0 of model (3) is globally asymptotically stable.

(2). If $\mathscr{R}_0 > 1$, then endemic equilibrium E^* of model (3) is globally asymptotically stable.

Remark 10. In [3], the following backward difference scheme for SIRS epidemic model with the bilinear incidence is studied

$$\begin{aligned} S_{n+1} = &\ S_n + B - \mu_1 S_{n+1} - \beta S_{n+1} I_{n+1} + \delta R_{n+1} \\ I_{n+1} = &\ I_n + \beta S_{n+1} I_{n+1} - (\mu_2 + \gamma) I_{n+1} \\ R_{n+1} = &\ R_n + \gamma I_{n+1} - (\mu_3 + \delta) R_{n+1}. \end{aligned} \tag{19}$$

The condition $\mu_1 \le \min\{\mu_2, \mu_3\}$ is required. By constructing the discrete Lyapunov functions $U_\delta^{E^0}$ and $U_\delta^{E^*}$ (see the proof of Theorem 2.1 in [3]), the authors established that if the basic reproduction number $\mathscr{R}_0 \le 1$, then disease-free equilibrium E^0 of model (19) is globally asymptotically stable, and if $\mathscr{R}_0 > 1$, then endemic equilibrium E^* of model (19) is globally asymptotically stable.

By computing, we easily see that the Lyapunov functions $U_\delta^{E^0}$ and $U_\delta^{E^*}$ are not applicable for model (3). Therefore, in this paper we construct a class of new Lyapunov functions to study the global asymptotic stability of model (3).

Furthermore, we also see that above Corollary 1 is an extension of the main results given in [3] in the nonlinear incidence case.

Remark 11. Analyzing the conditions and results given in Corollary 1 and Theorems 3 given in Section 3, we can propose an important and interesting open problem for general model (3): whether only when assumption (H_1) holds, we can obtain that the disease-free equilibrium is globally asymptotically stable if and only if $\mathscr{R}_0 \leq 1$, and the endemic equilibrium is globally asymptotically stable if and only if $\mathscr{R}_0 > 1$.

5. Conclusion

In [4], the dynamical properties of the forward difference scheme for a class of SIRS epidemic models with general nonlinear incidence are investigated. It is shown that when step size h is small enough the disease-free equilibrium and endemic equiloibrium are local asymptotically stable, and along step size h increase, the scheme will occur the bifurcation phenomena.

In this paper, the dynamical properties of the backward difference scheme for a class of SIRS epidemic models with nonlinear incidence $\beta f(S)g(I)$ are investigated. From the main results obtained in this paper, we see that the backward difference scheme, that is discrete dynamical model (3), is provided for us with excellent dynamical properties for any step size h in the local and global stability of equilibria. These properties nearly are same to corresponding continuous-time model (1).

Furthermore, we also see that the results on the global asymptotic stability of the endemic equilibrium for the backward difference scheme for SIRS epidemic model with bilinear incidence obtained in [3] are directly extended. By constructing new discrete Lyapunov functions we established the sufficient and necessary conditions on the global asymptotic stability of the disease-free equilibrium and endemic equilibrium for a class of discrete SIRS epidemic models with general nonlinear incidence $\beta f(S)g(I)$, vaccination in susceptible and different death rates d_1, d_2 and d_3. That is, under assumptions $(H_1) - (H_3)$, the disease-free equilibrium is globally asymptotically stable if and only if basic reproduction number $\mathscr{R}_0 \leq 1$, and the endemic equilibrium is globally asymptotically stable if and only if $\mathscr{R}_0 > 1$.

However, we also see that assumption (H_3) is very strong. For the local stability of the disease-free equilibrium and endemic equilibrium for model (3), the assump-

tion (H_3) is not required. Therefore, an interesting and important open problem is whether the assumption (H_3) can be weakened in the studies of the global stability of equilibria of model (3).

On the other hand, we know that there is the nonstandard difference scheme to discretize continuous-time model (1) with nonlinear incidence. For the the nonstandard difference scheme of model (1) whether we also can establish the same results, like in this paper, still is an interesting open problem.

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Bounds for the largest eigenvalue of nonnegative tensors

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Abstract

In this paper, we establish some eigenvalue properties of nonnegative tensors. We derive new bounds for the largest eigenvalue (Z-eigenvalue, H-eigenvalue, and B-eigenvalue) of nonnegative tensors. Numerical examples show the efficiency of these bounds.

Key words: Nonnegative tensor; Spectral radius; Eigenvalue; Bound

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1 Introduction

Eigenvalue problems of higher order tensors have become an important topic of study in a new applied mathematics branch, numerical multilinear algebra, and they have a wide range of practical applications [2, 5, 6, 7, 8, 9, 10, 17].

First, we recall some definitions on tensors. Let $\mathbb{R}$ be the real field. An m-th order n dimensional square tensor $\mathcal{A}$ consists of n^m entries in $\mathbb{R}$, which is defined as follows:

$$\mathcal{A} = (\mathcal{A}_{i_1 i_2 \cdots i_m}), \quad \mathcal{A}_{i_1 i_2 \cdots i_m} \in \mathbb{R}, \quad 1 \leq i_1, i_2, \cdots i_m \leq n.$$

$\mathcal{A}$ is called nonnegative if $\mathcal{A}_{i_1 i_2 \cdots i_m} \geq 0$. To an n-vector x, real or complex, we define the n-vector:

$$\mathcal{A}x^{m-1} = \left(\sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_n} \right)_{1 \leq i \leq n},$$

$$x^{[m-1]} = (x_i^{m-1})_{1 \leq i \leq n}.$$

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In this paper, we continue this research on the eigenvalue problems for tensors. In section 2, bounds for the largest Z-eigenvalue are obtained, and proved to be tighter than that in Corollary 4.5 in [16]. In section 3, bounds for the largest H-eigenvalue are given. Moreover, the upper bound for the largest B-eigenvalue is presented in section 4.

2 Notation and preliminaries.

The following two definitions were first introduced and studied by Qi and Lim [4, 11].

Definition 2.1. *Let $\mathcal{A}$ be an m-order and n-dimensional tensor. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an eigenvalue-eigenvector (or simply eigenpair) of $\mathcal{A}$ if they satisfy the equation*

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}.$$

We call (λ, x) an H-eigenpair if they are both real.

Definition 2.2. *Let $\mathcal{A}$ be an m-order and n-dimensional tensor. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an E-eigenvalue and E-eigenvector (or simply E-eigenpair) of $\mathcal{A}$ if they satisfy the equation*

$$\begin{cases} \mathcal{A}x^{m-1} = \lambda x, \\ x^T x = 1. \end{cases} \tag{1}$$

We call (λ, x) an Z-eigenpair if they are both real.

Recently, Chang et al. [1, 2] generalized the notion of eigenvalues of higher order tensors to tensor pairs (or tensor pencils).

Definition 2.3. *Let $\mathcal{A}$, $\mathcal{B}$ be two m-order and n-dimensional tensors. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an B-eigenvalue and B-eigenvector of $\mathcal{A}$ relative to $\mathcal{B}$ if they satisfy the equation*

$$\mathcal{A}x^{m-1} = \lambda \mathcal{B}x^{[m-1]}.$$

The following definition for irreducibility has been introduced in [1, 11].

Definition 2.4. *The tensor $\mathcal{A}$ is called reducible if there exists a nonempty proper index subset $\mathbb{J} \subset \{1, 2, \cdots, n\}$ such that $a_{i_1,i_2,\cdots,i_m} = 0, \ \forall i_1 \in \mathbb{J}, \ \forall i_2, \cdots, i_m \notin \mathbb{J}$. If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ to be irreducible.*

In this paper, let $N = \{1, 2, \ldots, n\}$, we define the ith row sum of $\mathcal{A}$ as $R_i(\mathcal{A}) = \sum\limits_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m}$, and denote the largest and the smallest row sums of $\mathcal{A}$ by

$$R_{\max}(\mathcal{A}) = \max_{i=1,\cdots,n} R_i(\mathcal{A}), \ R_{\min}(\mathcal{A}) = \min_{i=1,\cdots,n} R_i(\mathcal{A}).$$

Furthermore, a real tensor of order m dimension n is called the unit tensor, if its entries are $\delta_{i_1\ldots i_m}$ for $i_1, \ldots, i_m \in N$, where

$$\delta_{i_1\ldots i_m} = \begin{cases} 1, & \text{if } i_1 = \ldots = i_m \\ 0, & \text{otherwise.} \end{cases}$$

And we define

$$r_i(\mathcal{A}) = \sum_{\delta_{ii_2\ldots i_m}=0} a_{ii_2\ldots i_m},\quad r_i^j(\mathcal{A}) = \sum_{\substack{\delta_{ii_2\ldots i_m}=0,\\ \delta_{ji_2\ldots i_m}=0}} a_{ii_2\ldots i_m} = r_i(\mathcal{A}) - a_{ij\ldots j}.$$

3 Bounds for the largest Z-eigenvalue.

First, we list some results about the largest Z-eigenvalue of tensors.

Definition 3.1. *Let $\mathcal{A}$ be an m-order and n-dimensional tensor. We define $\sigma(\mathcal{A})$ the Z-spectrum of $\mathcal{A}$ by the set of all Z-eigenvalues of $\mathcal{A}$. Assume $\sigma(\mathcal{A}) \neq \emptyset$, then the Z-spectral radius of $\mathcal{A}$ is denoted by*

$$\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}.$$

In [3], Chang, Pearson, and Zhang gave the following bounds for the Z-eigenvalues of an m-order n-dimensional tensor $\mathcal{A}$.

Lemma 3.2. *(Proposition 3.3 in [3]) Let $\mathcal{A}$ be an m-order and n-dimensional tensor. Then*

$$\rho(\mathcal{A}) \leq \sqrt{n} \max_{i\in N} \sum_{i_2,\ldots,i_m=1}^{n} \left|a_{ii_2\ldots i_m}\right|. \tag{2}$$

For the positively homogeneous operators, Song and Qi [16] studied the relationship between the Gelfand formula and the spectral radius as well as the upper bound of the spectral radius. From Corollary 4.5 in [16], we can get the following Lemma:

Lemma 3.3. *(Corollary 4.5 in [16]) Let $\mathcal{A}$ be an m-order and n-dimensional tensor. Then*

$$\rho(\mathcal{A}) \leq \max_{i\in N} \sum_{i_2,\ldots,i_m=1}^{n} \left|a_{ii_2\ldots i_m}\right|. \tag{3}$$

Obviously, the bound in (3) is better than the bound in (2). Here, we give another proof of Lemma 3.3, which is very simple.

Proof. Suppose that λ is an Z-eigenvalue of $\mathcal{A}$ with eigenvector x. Assume that

$$|x_i| = \max_{j\in N} |x_j|.$$

Consider the i-th equation of (1). We have

$$\lambda x_i = \sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2} \ldots x_{i_m}.$$

By $|x_i| \leq \sqrt[m-1]{|x_i|} \leq 1$, we can get

$$|\lambda| \leq \sum_{i_2,\cdots,i_m=1}^{n} |a_{ii_2\cdots i_m}| \left|\frac{x_{i_2}}{\sqrt[m-1]{|x_i|}}\right| \ldots \left|\frac{x_{i_m}}{\sqrt[m-1]{|x_i|}}\right| \leq \sum_{i_2,\ldots,i_m=1}^{n} \left|a_{ii_2\ldots i_m}\right|.$$

3

Thus, we complete the proof. □

Note that λ and x may be non-real here.

A tensor $\mathcal{A}$ is called *weakly symmetric* if the associated homogeneous polynomial $\mathcal{A}x^m$ satisfies

$$\nabla\mathcal{A}x^m = m\mathcal{A}x^{m-1}.$$

This concept was first introduced and used by Chang, Pearson and Zhang [3] for studying the properties of Z-eigenvalue of nonnegative tensors and presented the following Perron-Frobenius Theorem for the Z-eigenvalue of nonnegative tensors, which was later reproved as Lemma 4.7 by Song and Qi in [16], using a different technique.

Lemma 3.4. *Suppose that m-order n-dimensional tensor $\mathcal{A}$ is weakly symmetric, nonnegative and irreducible. Then $\rho(\mathcal{A})$ is a positive Z-eigenvalue with a positive Z-eigenvector.*

Based on the above Lemma, we give the main result of this section.

Theorem 3.5. *Suppose that m-order n-dimensional tensor $\mathcal{A}$ is weakly symmetric, nonnegative and irreducible. Then*

$$\rho(\mathcal{A}) \le \max_{i,j\in N, j\neq i} \frac{1}{2}\{a_{i\ldots i} + a_{j\ldots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\},$$

where

$$\Delta_{i,j}(\mathcal{A}) = (a_{i\ldots i} - a_{j\ldots j} + r_i^j(\mathcal{A}))^2 + 4a_{ij\ldots j}r_j(\mathcal{A}).$$

Proof. Let $x = (x_1, \ldots, x_n)^T$ be an Z-eigenvector of $\mathcal{A}$ corresponding to $\rho(\mathcal{A})$, that is,

$$\mathcal{A}x^{m-1} = \rho(\mathcal{A})x, \tag{4}$$

Let

$$x_t \ge x_s \ge \max\{x_k,\ \ k = 1, \ldots, n,\ \ k \neq t, s\}.$$

Obviously, by Lemma 3.4, we have $x_t > 0$, $x_s > 0$. From Corollary 4.10 in [3], we have

$$\rho(\mathcal{A}) - a_{i\ldots i} \ge 0,\ \ i = 1, \ldots, n.$$

Consider the equation of (1), by $x_t^{m-1} \le x_t$, $x_s^{m-1} \le x_s$, we can get

$$\begin{aligned}
(\rho(\mathcal{A}) - a_{t\ldots t})x_t &= \sum_{\substack{\delta_{ti_2\ldots i_m}=0,\\ \delta_{si_2\ldots i_m}=0}} a_{ti_2\ldots i_m}x_{i_2}\ldots x_{i_m} + a_{ts\ldots s}x_s^{m-1} + a_{t\ldots t}(x_t^{m-1} - x_t)\\
&\le \sum_{\substack{\delta_{ti_2\ldots i_m}=0,\\ \delta_{si_2\ldots i_m}=0}} a_{ti_2\ldots i_m}x_t^{m-1} + a_{ts\ldots s}x_s^{m-1}\\
&\le r_t^s(\mathcal{A})x_t + a_{ts\ldots s}x_s, \qquad (5)
\end{aligned}$$

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equivalently,

$$(\rho(\mathcal{A}) - a_{t\ldots t} - r_t^s(\mathcal{A}))x_t \leq a_{ts\ldots s}x_s.$$

Moreover, from equality (1), we similarly get

$$\begin{aligned}(\rho(\mathcal{A}) - a_{s\ldots s})x_s &= \sum_{\delta_{si_2\ldots i_m}=0} a_{si_2\ldots i_m}x_{i_2}\ldots x_{i_m} + a_{s\ldots s}(x_s^{m-1} - x_s)\\ &\leq r_s(\mathcal{A})x_t^{m-1} + a_{s\ldots s}(x_s^{m-1} - x_s)\\ &\leq r_s(\mathcal{A})x_t. \end{aligned} \tag{6}$$

Multiplying equation (5) and (6), we get

$$(\rho(\mathcal{A}) - a_{t\ldots t} - r_t^s(\mathcal{A}))(\rho(\mathcal{A}) - a_{s\ldots s}) \leq a_{ts\ldots s}r_s(\mathcal{A}).$$

Then, solving for $\rho(\mathcal{A})$,

$$\rho(\mathcal{A}) \leq \frac{1}{2}\{a_{i\ldots i} + a_{j\ldots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\} \leq \max_{i,j\in N, j\neq i} \frac{1}{2}\{a_{i\ldots i} + a_{j\ldots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\}.$$

Thus, we complete the proof. □

From Theorem 3.5 in [12], we know that

$$\max_{i,j\in N, j\neq i} \frac{1}{2}\{a_{i\ldots i} + a_{j\ldots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\} \leq \max_{i\in N} \sum\nolimits_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m},$$

that is to say, our new bound in Theorem 3.5 is always better than the bound in Lemma 3.3. We now show the efficiency of the upper bound in Theorem 3.5 by the following example which was introduced in [3].

Example 3.1. Consider the tensor $\mathcal{A} = (a_{ijkl})$ of order 4 dimension 2 with entries defined as follows:

$$a_{1111} = \frac{1}{2},\ \ a_{2222} = 3,\ \ \text{and}\ \ a_{ijkl} = \frac{1}{3}\ \ \text{elsewhere}.$$

By Lemma 3.2, we have

$$\rho(\mathcal{A}) \leq 10.6666.$$

By Lemma 3.3, we have

$$\rho(\mathcal{A}) \leq 5.3333.$$

By Theorem 3.5, we have

$$\rho(\mathcal{A}) \leq 5.1667.$$

In fact, $\rho(\mathcal{A}) = 3.1092$. Hence, the bound in Theorem 3.5 is tight and sharper.

4 Bounds for the largest H-eigenvalue

In this section, we give the lower bound and the upper bound for the largest H-eigenvalue of an m-order n-dimensional nonnegative tensor $\mathcal{A}$.

Definition 4.1. *Let $\mathcal{A}$ be an m-order and n-dimensional tensor. We define the H-spectrum of $\mathcal{A}$, denoted $H(\mathcal{A})$ to be the set of all H-eigenvalues of $\mathcal{A}$. Assume $H(\mathcal{A}) \neq 0$, then the H-spectral radius of $\mathcal{A}$, denoted $\mu(\mathcal{A})$, is defined as* $\mu(\mathcal{A}) = \max\{|\lambda| : \lambda \in H(\mathcal{A})\}$.

First, we introduce some results for H-eigenvalue of nonnegative tensors [1, 13, 14], which are generalized from nonnegative matrices.

Theorem 4.2. *If $\mathcal{A}$ is irreducible and nonnegative, then there exists a number $\mu(\mathcal{A}) > 0$ and a vector $x_0 > 0$, such that $\mathcal{A}x_0^{m-1} = \mu(\mathcal{A})x_0^{[m-1]}$. Moreover, if λ is an eigenvalue of $\mathcal{A}$, then $|\lambda| \leq \mu(\mathcal{A})$.*

Lemma 4.3. *(Lemma 5.2 in [13]) Let $\mathcal{A}$ be an m-order and n-dimensional nonnegative tensor. Then*

$$R_{\min}(\mathcal{A}) \leq \mu(\mathcal{A}) \leq R_{\max}(\mathcal{A}). \tag{7}$$

According to some eigenvalue inclusion theorems, Li, Li and Kong [12] obtained the following upper bound for the spectral radius of a nonnegative tensor, which is sharper than the upper bound in Lemma 4.3.

Lemma 4.4. *(Theorem 3.3 in [12]) Suppose that m-order n-dimensional tensor $\mathcal{A}$ is nonnegative. Then*

$$\mu(\mathcal{A}) \leq \max_{i,j\in N, j\neq i} \frac{1}{2}\left\{a_{i\ldots i} + a_{j\ldots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\right\},$$

where

$$\Delta_{i,j}(\mathcal{A}) = \left(a_{i\ldots i} - a_{j\ldots j} + r_i^j(\mathcal{A})\right)^2 + 4a_{ij\ldots j}r_j(\mathcal{A}).$$

In the following Theorem, we give new bounds for the spectral radius of a nonnegative tensor.

Theorem 4.5. *Suppose that m-order n-dimensional tensor $\mathcal{A}$ is nonnegative. Then*

$$\min_{i\in N} \max_{j\in N, j\neq i} \frac{1}{2}\left\{a_{i\ldots i} + a_{j\ldots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\right\} \leq \mu(\mathcal{A}) \leq \max_{i\in N} \min_{j\in N, j\neq i} \frac{1}{2}\left\{a_{i\ldots i} + a_{j\ldots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\right\},$$

where

$$\Delta_{i,j}(\mathcal{A}) = \left(a_{i\ldots i} - a_{j\ldots j} + r_i^j(\mathcal{A})\right)^2 + 4a_{ij\ldots j}r_j(\mathcal{A}).$$

Proof. First, we assume that tensor $\mathcal{A}$ is strictly positive and let x be the unique positive eigenvector corresponding to $\mu(\mathcal{A})$, i.e.

$$\mathcal{A}x^{m-1} = \mu(\mathcal{A})x^{[m-1]}.$$

Assume $0 < x_t = \max_{i \in N} x_i$, then, for any $s \neq t$, we can get

$$(\mu(\mathcal{A}) - a_{t\ldots t})x_t^{m-1} - a_{ts\ldots s}x_s^{m-1} = \sum_{\substack{\delta_{ti_2\ldots i_m}=0,\\ \delta_{si_2\ldots i_m}=0}} a_{ti_2\ldots i_m}x_{i_2}\ldots x_{i_m},$$

$$(\mu(\mathcal{A}) - a_{s\ldots s})x_s^{m-1} - a_{st\ldots t}x_t^{m-1} = \sum_{\substack{\delta_{ti_2\ldots i_m}=0,\\ \delta_{si_2\ldots i_m}=0}} a_{si_2\ldots i_m}x_{i_2}\ldots x_{i_m}.$$

Solving for x_t we obtain

$$((\mu(\mathcal{A}) - a_{s\ldots s})(\mu(\mathcal{A}) - a_{t\ldots t}) - a_{st\ldots t}a_{ts\ldots s})\, x_t^{m-1} = (\mu(\mathcal{A}) - a_{s\ldots s}) \sum_{\substack{\delta_{ti_2\ldots i_m}=0,\\ \delta_{si_2\ldots i_m}=0}} a_{ti_2\ldots i_m}x_{i_2}\ldots x_{i_m} + a_{ts\ldots s} \sum_{\substack{\delta_{ti_2\ldots i_m}=0,\\ \delta_{si_2\ldots i_m}=0}} a_{si_2\ldots i_m}x_{i_2}\ldots x_{i_m}. \quad (8)$$

Recalling that $0 < x_t = \max_{i \in N} x_i$, we have

$$\begin{aligned}(\mu(\mathcal{A}) - a_{s\ldots s})(\mu(\mathcal{A}) - a_{t\ldots t}) - a_{st\ldots t}a_{ts\ldots s} &= (\mu(\mathcal{A}) - a_{s\ldots s}) \sum_{\substack{\delta_{ti_2\ldots i_m}=0,\\ \delta_{si_2\ldots i_m}=0}} a_{ti_2\ldots i_m}\frac{x_{i_2}}{x_s}\ldots\frac{x_{i_m}}{x_s} \\ &\quad + a_{ts\ldots s} \sum_{\substack{\delta_{ti_2\ldots i_m}=0,\\ \delta_{si_2\ldots i_m}=0}} a_{si_2\ldots i_m}\frac{x_{i_2}}{x_s}\ldots\frac{x_{i_m}}{x_s} \\ &\leq (\mu(\mathcal{A}) - a_{s\ldots s})r_t^s(\mathcal{A}) + a_{ts\ldots s}r_s^t(\mathcal{A}). \end{aligned} \quad (9)$$

Therefore

$$\mu(\mathcal{A}) \leq \frac{1}{2}\left\{a_{t\ldots t} + a_{s\ldots s} + r_t^s(\mathcal{A}) + \Delta_{t,s}^{\frac{1}{2}}(\mathcal{A})\right\}.$$

This must be true for every $s \neq t$, then, we get

$$\mu(\mathcal{A}) \leq \min_{j \in N, j \neq t} \frac{1}{2}\left\{a_{t\ldots t} + a_{j\ldots j} + r_t^j(\mathcal{A}) + \Delta_{t,j}^{\frac{1}{2}}(\mathcal{A})\right\}.$$

And this could be true for any $t \in N$, that is

$$\mu(\mathcal{A}) \leq \max_{i \in N} \min_{j \in N, j \neq i} \frac{1}{2}\left\{a_{i\ldots i} + a_{j\ldots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\right\}.$$

Similarly, assume $0 < x_T = \min_{i \in N} x_i$, we can get

$$\mu(\mathcal{A}) \geq \min_{i \in N} \max_{j \in N, j \neq i} \frac{1}{2}\left\{a_{i\ldots i} + a_{j\ldots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\right\}.$$

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If the tensor $\mathcal{A}$ is not strictly positive, we denote by $\mathcal{D} = (d_{i_1\ldots i_m})$ the m-order n-dimensional tensor with $d_{i_1\ldots i_m} = 1$, for all $i_1 \in N, \ldots, i_m \in N$. Hence, $\mathcal{A} + t\mathcal{D}$ is strictly positive for any chosen positive real number t, and then letting $t \to 0$, the result follows by continuity. □

From the proof of the Theorem 3.5 in [12], we can get the following result:

$$\begin{aligned}\max_{i\in N}\min_{j\in N, j\neq i}\tfrac{1}{2}\left\{a_{i\ldots i} + a_{j\ldots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\right\} &\leq \max_{i,j\in N, j\neq i}\tfrac{1}{2}\left\{a_{i\ldots i} + a_{j\ldots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\right\} \\ &\leq R,\end{aligned}$$

where

$$\Delta_{i,j}(\mathcal{A}) = \left(a_{i\ldots i} - a_{j\ldots j} + r_i^j(\mathcal{A})\right)^2 + 4a_{ij\ldots j}r_j(\mathcal{A}).$$

We now compare the lower bound in Theorems 4.5 with that in Lemma 4.3.

Theorem 4.6. *Suppose that m-order n-dimensional tensor $\mathcal{A}$ is nonnegative. Then*

$$R_{\min}(\mathcal{A}) \leq \min_{i\in N}\max_{j\in N, j\neq i}\frac{1}{2}\left\{a_{i\ldots i} + a_{j\ldots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\right\},$$

where

$$\Delta_{i,j}(\mathcal{A}) = \left(a_{i\ldots i} - a_{j\ldots j} + r_i^j(\mathcal{A})\right)^2 + 4a_{ij\ldots j}r_j(\mathcal{A}).$$

Proof. First, we assume that tensor $\mathcal{A}$ is strictly positive. Equivalently, we will prove that, if

$$\mu(\mathcal{A}) \geq \min_{i\in N}\max_{j\in N, j\neq i}\frac{1}{2}\left\{a_{i\ldots i} + a_{j\ldots j} + r_i^j(\mathcal{A}) + \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\right\}.$$

Then, we can get

$$\mu(\mathcal{A}) \geq R_{\min}(\mathcal{A}).$$

If $\mu(\mathcal{A})$ satisfies the lower bound in the Theorem 4.5 and the matrix is positive, similar to the proof of Theorem 4.5, if assume $0 < x_T = \min_{i\in N} x_i$, for any $s \neq T$, we can get

$$(\mu(\mathcal{A}) - a_{s\ldots s})(\mu(\mathcal{A}) - a_{T\ldots T}) \geq (\mu(\mathcal{A}) - a_{s\ldots s})r_T^s(\mathcal{A}) + a_{Ts\ldots s}r_s(\mathcal{A}).$$

If we assumed that $\mu(\mathcal{A}) \leq R_s(\mathcal{A})$, then we have that $\mu(\mathcal{A}) - a_{s\ldots s} \leq r_s(\mathcal{A})$. So,

$$(\mu(\mathcal{A}) - a_{s\ldots s})(\mu(\mathcal{A}) - a_{T\ldots T} - r_T^s(\mathcal{A}) \geq a_{Ts\ldots s}r_s(\mathcal{A}) \geq a_{Ts\ldots s}(\mu(\mathcal{A}) - a_{s\ldots s}),$$

that is

$$(\mu(\mathcal{A}) - a_{s\ldots s})(\mu(\mathcal{A}) - a_{T\ldots T} - r_T^s(\mathcal{A}) - a_{Ts\ldots s}) \geq 0.$$

From Lemma 3.2 in [12], we know $\mu(\mathcal{A}) - a_{s\ldots s} \geq 0$, then, we obtain

$$\mu(\mathcal{A}) - a_{T\ldots T} - r_T^s(\mathcal{A}) - a_{Ts\ldots s} \geq 0,$$

that is

$$\mu(\mathcal{A}) \geq R_T(\mathcal{A}) \geq R_{\min}(\mathcal{A}).$$

If the tensor $\mathcal{A}$ is not strictly positive, we denote by $\mathcal{D} = (d_{i_1\ldots i_m})$ the m-order n-dimensional tensor with $d_{i_1\ldots i_m} = 1$, for all $i_1 \in N, \ldots, i_m \in N$. Hence, $\mathcal{A} + t\mathcal{D}$ is strictly positive for any chosen positive real number t, and then letting $t \to 0$, the result follows by continuity. □

We now show the efficiency of the bounds in Theorem 4.5 by the following example.

Example 4.1. Consider the tensor $\mathcal{A} = (a_{ijkl})$ of order 4 dimension 3 with entries defined as follows:

$$a_{1111} = 1, \;\; a_{1222} = 1, \;\; a_{1333} = 1,$$

$$a_{2111} = 2, \;\; a_{2222} = 2, \;\; a_{2333} = 2,$$

$$a_{3111} = 3, \;\; a_{3222} = 3, \;\; a_{3333} = 3,$$

and $a_{ijkl} = 0$ elsewhere. By Lemma 4.3, we have

$$3 \leq \mu(\mathcal{A}) \leq 9.$$

By Lemma 4.4, we have

$$\mu(\mathcal{A}) \leq 8.$$

By Theorem 4.5, we have

$$5 \leq \mu(\mathcal{A}) \leq 7.$$

In fact, $\mu(\mathcal{A}) = 6$. Hence, the bound in Theorem 4.5 is tight and sharper.

5 Bounds for the largest B-eigenvalue

In this section, we focus our attention on the largest B-eigenvalue of a m-order n-dimensional nonnegative tensor $\mathcal{A}$ relative to $\mathcal{B}$.

Definition 5.1. *Let $\mathcal{A}$, $\mathcal{B}$ be two m-order and n-dimensional tensors. We define the B-spectrum of $\mathcal{A}$ relative to $\mathcal{B}$, denoted $T(\mathcal{A})$ to be the set of all B-eigenvalues of $\mathcal{A}$ relative to $\mathcal{B}$. Assume $T(\mathcal{A}) \neq 0$, then the B-spectral radius of $\mathcal{A}$, denoted $\nu(\mathcal{A})$, is defined as $\nu(\mathcal{A}) = \max\{|\lambda| : \lambda \in T(\mathcal{A})\}$.*

For an m-order n-dimensional tensor $\mathcal{A}$, let

$$F_{\mathcal{A}} = \left(\mathcal{A}x^{m-1}\right)^{[\frac{1}{m-1}]},$$

Song and Qi [15] proved the Perron-Frobenius property for nonnegative tensor pairs $(\mathcal{A}, \mathcal{B})$ without the requirement of the tensor inversion.

Lemma 5.2. *(Corollary 4.2 in [15]) Let $\mathcal{A}$, $\mathcal{B}$ be two weakly irreducible and nonnegative tensors with order m and dimension n and $F_{\mathcal{A}}F_{\mathcal{B}} = F_{\mathcal{B}}F_{\mathcal{A}}$. If $\exists x > 0$ such*

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that $\mathcal{B}x^{m-1} \geq x^{[m-1]}$, then $\mathcal{A}$ has a unique positive B-eigenvalue with a corresponding positive B-eigenvector.

Based on the above Lemma, we give the main results of this section.

Theorem 5.3. *Under the conditions of Lemma 5.2 and $b_{i\dots i} > 0$ for all $i \in N$. Then*

$$\nu(\mathcal{A}) \leq \max_{i\in N} \frac{R_i(\mathcal{A})}{b_{i\dots i}}.$$

Proof. Let x be the unique positive eigenvector corresponding to $\nu(\mathcal{A})$, i.e.

$$\mathcal{A}x^{m-1} = \nu(\mathcal{A})\mathcal{B}x^{m-1}.$$

Assume $0 < x_t = \max_{i\in N} x_i$, then, from the i-th equation of the above equation, we can get

$$\sum_{\delta_{ti_2\dots i_m}=0} a_{ti_2\dots i_m} x_{i_2}\dots x_{i_m} + a_{t\dots t}x_t^{m-1} = \nu(\mathcal{A})\left(\sum_{\delta_{ti_2\dots i_m}=0} b_{ti_2\dots i_m} x_{i_2}\dots x_{i_m} + b_{t\dots t}x_t^{m-1}\right),$$

Dividing both sides by x_t^{m-1} and rearranging yields

$$\nu(\mathcal{A})\left(\sum_{\delta_{ti_2\dots i_m}=0} b_{ti_2\dots i_m} \frac{x_{i_2}}{x_t}\dots\frac{x_{i_m}}{x_t} + b_{t\dots t}\right) - a_{t\dots t} = \sum_{\delta_{ti_2\dots i_m}=0} a_{ti_2\dots i_m} \frac{x_{i_2}}{x_t}\dots\frac{x_{i_m}}{x_t} \leq r_t(\mathcal{A}).$$

Hence,

$$\nu(\mathcal{A}) \leq \frac{R_t(\mathcal{A})}{\sum_{\delta_{ti_2\dots i_m}=0} b_{ti_2\dots i_m} \frac{x_{i_2}}{x_t}\dots\frac{x_{i_m}}{x_t} + b_{t\dots t}} \leq \frac{R_t(\mathcal{A})}{b_{t\dots t}}.$$

Thus, we complete the proof. □

If $\exists x > 0$ such that $\mathcal{B}x^{m-1} \geq x^{[m-1]}$ and suppose that m-order n-dimensional tensor $\mathcal{B}$ is nonnegative and diagonal, we can get

$$b_{i\dots i} \geq 1,$$

for all $i \in N$. Similar to the proof of Theorem 4.5, we can get some new bounds for $\nu(\mathcal{A})$, including the upper bound and the lower bound.

Theorem 5.4. *Under the conditions of Lemma 5.2 and suppose that m-order n-dimensional tensor $\mathcal{B}$ is nonnegative and diagonal. Then*

$$\begin{aligned}\min_{i\in N}\max_{j\in N, j\neq i} \frac{a_{i\dots i}b_{j\dots j}+a_{j\dots j}b_{i\dots i}+b_{j\dots j}r_i^j(\mathcal{A})+\Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})}{2b_{i\dots i}b_{j\dots j}} &\leq \nu(\mathcal{A}) \\ &\leq \max_{i\in N}\min_{j\in N, j\neq i} \frac{a_{i\dots i}b_{j\dots j}+a_{j\dots j}b_{i\dots i}+b_{j\dots j}r_i^j(\mathcal{A})+\Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})}{2b_{i\dots i}b_{j\dots j}},\end{aligned}$$

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where

$$\Delta_{i,j}(\mathcal{A}) = \left(a_{i\ldots i}b_{j\ldots j} - a_{j\ldots j}b_{i\ldots i} + b_{j\ldots j}r_i^j(\mathcal{A})\right)^2 + 4a_{ij\ldots j}b_{i\ldots i}b_{j\ldots j}r_j(\mathcal{A}).$$

If $b_{i\ldots i} = 1$ for all $i \in N$, then, the results in Theorem 5.4 reduce to the result in Theorem 4.5.

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A note on fractional neutral integro-differential inclusions with state-dependent delay in Banach spaces

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Abstract

We have applied different fixed point theorems to examine the existence results for fractional neutral integro-differential inclusions (FNIDI) with state-dependent delay (SDD) in Banach spaces. We tend to conjointly discuss the cases once the multivalued nonlinear term takes convex values further as nonconvex values. An example is offered to demonstrate the obtained results.

Keywords: Fractional order differential equations, state-dependent delay, multivalued map, fixed point theorem, Banach spaces, semigroup theory.

2010 Mathematics Subject Classification: 26A33, 34A08, 35R12, 34A60, 34G20, 34K05, 45J05.

1 Introduction

The aim of the manuscript is to investigate the existence of mild solutions for neutral integro-differential inclusions of fractional order as given below

$$\frac{d}{dt}\big[x(t)-\mathscr{G}(t,x_{\varrho(t,x_t)})\big] \in \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\mathscr{A}\big[x(s)-\mathscr{G}(s,x_{\varrho(s,x_s)})\big]ds + \mathscr{F}(t,x_{\varrho(t,x_t)}), \quad \text{a.e.} \quad t\in\mathscr{I}=[0,b], \tag{1.1}$$

$$x_0=\varsigma\in\mathscr{B}, \tag{1.2}$$

such that $1<\alpha<2$ and $\mathscr{A}: D(\mathscr{A})\subset\mathbb{X}\to\mathbb{X}$ denotes a linear densely defined operator of sectorial type on a complex Banach space $(\mathbb{X},|\cdot|)$, the convolution integral within the equation is understood because the Riemann-Liouville fractional integral (see [4]) and $\mathscr{F}:\mathscr{I}\times\mathscr{B}\to\mathscr{P}(\mathbb{X})$ represents a multivalued map ($\mathscr{P}(\mathbb{X})$ is the family of nonempty subsets of $\mathbb{X}$), $\mathscr{G}:\mathscr{I}\times\mathscr{B}\to\mathbb{X}$, and $\varrho:\mathscr{I}\times\mathscr{B}\to(-\infty,b]$ are apposite functions, and $\mathscr{B}$ is a theoretical phase space axioms outlined in Preliminaries.

We recall that for any continuous function x defined on $(-\infty,b]$ and for any $t\geq 0$, we designate by x_t the part of $\mathscr{B}$ defined by $x_t(\theta)=x(t+\theta)$ for $\theta\leq 0$. Here $x_t(\cdot)$ speaks to the historical backdrop of the state from every $\theta\in(-\infty,0]$ likely the current time t.

Fractional differential equations have picked up hefty grandness as a final result of their exertion in numerous field of science and engineering. In the latest years, there has been a major growth in differential systems

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comprising fractional derivatives, e.g. the monographs of Abbas et al. [5], Baleanu et al. [6], Podlubny [7], Diethelm [8], Kilbas et al. [9], Tarasov [10] and Anastassiou [11], and the papers [12, 13, 14, 15, 16, 17, 18, 19], and the references cited therein.

As it is known, a delay differential equation (DDE) may be a special sort of functional differential equation (FDE). FDEs with SDD seem often in applications as models of equations and for this intention the report of this kind of equations received nice care in latest years. For points of interest, we recommend the reader to check the papers by by Abada et al. [20], Ait Dads et al. [21], Anguraj et al. [22], Benchohra et al. [23], Cuevas et al. [24], Hernandez et al. [25, 26], Mallika Arjunan et al. [27] and Yan et al. [28].

In the situation where $\mathscr{F}$ is either a single or a multivalued map, the problem (1.1)-(1.2) with $\mathscr{G} = 0$ was investigated on a compact interval in Agarwal et al. [29], Benchohra et al. [30, 31]. On unbounded interval when $\mathscr{F}$ is a single map, the problem (1.1)-(1.2) with $\mathscr{G} = 0$ was discussed by Benchohra et al. [32]. According to the knowledge of the authors , there is no work on the existence results for FNIDI with SDD in Banach spaces, which is communicated in the structure (1.1)-(1.2). Roused by this thought, in this paper, we concentrate on this problem, which is common generalizations of the idea of mild solution for fractional neutral equations well known in the theory of integer order systems.

This manuscript has the following structure. In section 2, we show some preliminaries and lemmas to be utilized to demonstrate our primary results. In section 3, we show two results for the problem (1.1)-(1.2) when the right-hand side is convex valued. The principal result is focused on a fixed point theorem of Bohnenblust-Karlin [1], and the second one on the nonlinear alternative of Leray-Schauder type [2]. The final existence result is given for a nonconvex valued right-hand side by utilizing a fixed point theorem for contraction multivalued maps thanks to Covitz and Nadler [3]. An application is presented in Section 4.

2 Preliminaries

Let $C(\mathscr{I}, \mathbb{X})$ be the Banach space of all continuous functions from $\mathscr{I}$ into $\mathbb{X}$ with the norm

$$\|x\|_\infty = \sup\{|x(t)| : t \in \mathscr{I}\}.$$

Let $B(\mathbb{X})$ signifies the Banach space of all bounded linear operators from $\mathbb{X}$ into $\mathbb{X}$.

A measurable function $x : \mathscr{I} \to \mathbb{X}$ is Bochner integrable if and only if $|x|$ is Lebesgue integrable. (For extra insights about Bochner integral, see Yosida [33]).

Let $L^1(\mathscr{I}, \mathbb{X})$ signify the Banach space of all continuous functions $x : \mathscr{I} \to \mathbb{X}$ which are Bochner integrable and have norm

$$\|x\|_{L^1} = \int_0^b |x(t)|dt \quad \text{for all} \quad x \in L^1(\mathscr{I}, \mathbb{X}).$$

We expect that the phase space $(\mathscr{B}, \|\cdot\|_{\mathscr{B}})$ is a semi-normed linear space of functions mapping $(-\infty, 0]$ into $\mathbb{X}$, and fulfilling the subsequent elementary adages as a result of Hale and Kato (see more details in [34, 35]).

(P_1) If $x : (-\infty, b] \to \mathbb{X}, b > 0$, is continuous on $\mathscr{I}$ and $x_0 \in \mathscr{B}$, then for every $t \in \mathscr{I}$ the accompanying conditions hold:

(i) x_t is in $\mathscr{B}$;

(ii) $|x(t)| \leq H\|x_t\|_{\mathscr{B}}$;

(iii) $\|x_t\|_{\mathscr{B}} \leq \mathscr{D}_1(t)\sup\{|x(s)| : 0 \leq s \leq t\} + \mathscr{D}_2(t)\|x_0\|_{\mathscr{B}}$, where $H > 0$ is a constant and $\mathscr{D}_1(\cdot) : [0,+\infty) \to [0,+\infty)$ is continuous, $\mathscr{D}_2(\cdot) : [0,+\infty) \to [0,+\infty)$ is locally bounded, and $\mathscr{D}_1, \mathscr{D}_2$ are independent of $x(\cdot)$.

(P_2) For the function $x(\cdot)$ in (P_1), x_t is a $\mathscr{B}$-valued continuous function on $\mathscr{I}$.

(P_3) The space $\mathscr{B}$ is complete.

Designate $\mathscr{D}_1^* = \sup\{\mathscr{D}_1(t) : t \in \mathscr{I}\}$ and $\mathscr{D}_2^* = \sup\{\mathscr{D}_2(t) : t \in \mathscr{I}\}$.
Now, we briefly review some known results from the solution operator. The Laplace transformation of a function $f \in L^1_{loc}(\mathbb{R}^+, \mathbb{X})$ is defined by

$$\mathcal{L}(f)(\lambda) = \widehat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt, \quad \mathrm{Re}(\lambda) > \omega,$$

if the integral is definitely convergent for $Re(\lambda) > \omega$. We mention the subsequent definition [4].

Definition 2.1. *Let $\mathscr{A} : D(\mathscr{A}) \subset \mathbb{X} \to \mathbb{X}$ be a closed and linear operator on a Banach space $\mathbb{X}$. We call $\mathscr{A}$ is the generator of a solution if there exist $\omega \in \mathbb{R}$ and a strongly continuous function $S_\alpha : \mathbb{R}^+ \to B(\mathbb{X})$ such that*

$$\{\lambda^\alpha : \quad Re\,(\lambda) > \omega\} \subset \rho(\mathscr{A}),$$

and

$$\lambda^{\alpha-1}(\lambda^\alpha - \mathscr{A})^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x dt, \quad Re\,\lambda > \omega, \quad x \in \mathbb{X}.$$

In this case, $S_\alpha(t)$ is called operator function created by $\mathscr{A}$. The idea of a solution operator, as characterized above, is nearly identified with the ones of a resolvent family [37]. Having in mind the uniqueness of the Laplace transform, in the fringe case $\alpha = 1$, the family $S_\alpha(t)$ relates to a strongly continuous semigroup (see Pazy [38]), while in the case $\alpha = 2$ a solution operator relates to the idea of a cosine family; see [39]. The subsequent result is an immediate outcome of [40, Proposition 3.1 and Lemma 2.2].

Proposition 2.1. *Let $S_\alpha(t)$ be a solution operator on $\mathbb{X}$ with generator $\mathscr{A}$. Then, we have*

(a) $S_\alpha(t)$ is strongly continuous for $t \geq 0$ and $S(0) = I$;

(b) $S_\alpha(t)D(\mathscr{A}) \subset D(\mathscr{A})$ and $\mathscr{A}S_\alpha(t)x = S_\alpha(t)\mathscr{A}x$ for all $x \in D(\mathscr{A})$, $t \geq 0$;

(c) For every $x \in D(\mathscr{A})$ and $t \geq 0$,

$$S_\alpha(t)x = x + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \mathscr{A} S_\alpha(s)x ds.$$

(d) Let $x \in D(\mathscr{A})$. Then $\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(s)x ds \in D(\mathscr{A})$ and

$$S_\alpha(t)x = x + \mathscr{A}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(s)x ds.$$

Definition 2.2. *A solution operator $\{S_\alpha(t)\}_{t>0}$ is called uniformly continuous if*

$$\lim_{t\to s} \|S_\alpha(t) - S_\alpha(s)\|_{B(\mathbb{X})} = 0.$$

Before we finish this section, we review some known results from multivalued analysis that we will apply in the spin-off. We recall that

$$\mathscr{P}(\mathbb{X}) = \{Y \subset \mathbb{X} : Y \neq \emptyset\}, \mathscr{P}_{cl}(\mathbb{X}) = \{Y \in \mathscr{P}(\mathbb{X}) : Y \text{ closed}\}, \quad \mathscr{P}_b(\mathbb{X}) = \{Y \in \mathscr{P}(\mathbb{X}) : Y \text{ bounded }\},$$
$$\mathscr{P}_{cp}(\mathbb{X}) = \{Y \in \mathscr{P}(\mathbb{X}) : Y \text{ compact}\},$$
$$\mathscr{P}_{cp,c}(\mathbb{X}) = \{Y \in \mathscr{P}(\mathbb{X}) : Y \text{ compact and convex}\}.$$

Remark 2.1. *For extra points of interest on this, we suggest the reader to [13].*

Definition 2.3. *The multivalued map $\mathscr{F} : \mathscr{I} \times \mathscr{B} \to \mathscr{P}(\mathbb{X})$ is said to be Carathéodory if*

(i) $t \mapsto \mathscr{F}(t, u)$ *is measurable for each* $u \in \mathscr{B}$*;*

(ii) $u \mapsto \mathscr{F}(t, u)$ *is upper semicontinuous for almost all* $t \in \mathscr{I}$*.*

Let $S_{\mathscr{F},x}$ be a set characterized by

$$S_{\mathscr{F},x} = \{v \in L^1(\mathscr{I}, \mathbb{X}) : v(t) \in \mathscr{F}(t, x_{\varrho(t,x_t)}) \text{ a.e. } t \in \mathscr{I}\}.$$

Definition 2.4. *A multivalued operator* $\Upsilon : \mathbb{X} \to \mathscr{P}_{cl}(\mathbb{X})$ *is called:*

(a) Λ*-Lipschitz if there exists* $\Lambda > 0$ *such that*

$$H_d(\Upsilon(x), \Upsilon(\overline{x})) \leq \Lambda\, d(x, \overline{x}) \quad \textit{for all} \quad x, \overline{x} \in \mathbb{X};$$

(b) a contraction if it is Λ*-Lipschitz with* $\Lambda < 1$*.*

Presently, we express the accompanying lemmas which are important to make our primary result.

Lemma 2.1 ([41]). *Let $\mathbb{X}$ be a Banach space. Let $\mathscr{F} : \mathscr{I} \times \mathscr{B} \to \mathscr{P}_{cp,c}(\mathbb{X})$ be an L^1-Carathéodory multivalued map and let Ψ be a linear continuous mapping from $L^1(\mathscr{I}, \mathbb{X})$ to $C(\mathscr{I}, \mathbb{X})$, then the operator*

$$\Psi \circ S_F : C(\mathscr{I}, \mathbb{X}) \to \mathscr{P}_{cp,c}(C(\mathscr{I}, \mathbb{X})),$$
$$x \quad \mapsto (\Psi \circ S_{\mathscr{F}})(x) := \Psi(S_{\mathscr{F},x})$$

has a closed graph operator in $C(\mathscr{I}, \mathbb{X}) \times C(\mathscr{I}, \mathbb{X})$.

Lemma 2.2 (Bohnenblust-Karlin's [1]). *Let $\mathbb{X}$ be a Banach space and $D \in \mathscr{P}_{cl,c}(\mathbb{X})$. Suppose that the operator $G : D \to \mathscr{P}_{cl,c}(D)$ is upper semicontinuous and the set $G(D)$ is relatively compact in $\mathbb{X}$. Then G has a fixed point in D.*

Lemma 2.3 (Covitz and Nadler [3]). *Let $(\mathbb{X}, d)$ be a complete metric space. If $\Upsilon : \mathbb{X} \to \mathscr{P}_{cl}(\mathbb{X})$ is a contraction, then Fix $\Upsilon \neq \emptyset$.*

For more details on multivalued maps see the books of Graef et al. [42] and Castaing et al. [43].

3 Existence results

We demonstrate below the existence of solutions for the problem (1.1)-(1.2). To start with, we delineate the mild solution for the problem (1.1)-(1.2).

Definition 3.1. *We affirm that the function $x : (-\infty, b] \to \mathbb{X}$ is a mild solution of (1.1)-(1.2) if $x(t) = \varsigma(t)$ for all $t \leq 0$, the constraint of $x(\cdot)$ to the interval $[0, b]$ is continuous and there exists $v(\cdot) \in L^1(\mathscr{I}, \mathbb{X})$, such that $v(t) \in \mathscr{F}(t, x_{\varrho(t,x_t)})$ a.e. $t \in [0, b]$, and x fulfills the consecutive integral equation:*

$$x(t) = S_\alpha(t)\big[\varsigma(0) - \mathscr{G}(0, \varsigma(0))\big] + \mathscr{G}(t, x_{\varrho(t,x_t)}) + \int_0^t S_\alpha(t-s)v(s)ds \quad \textit{for each} \quad t \in \mathscr{I}. \tag{3.1}$$

Let us set

$$\mathcal{R}(\varrho^-) = \{\varrho(s,\varsigma) : (s,\varsigma) \in \mathscr{I} \times \mathscr{B},\ \varrho(s,\varsigma) \leq 0\}.$$

We generally expect that $\varrho : \mathscr{I} \times \mathscr{B} \to (-\infty, b]$ is continuous. Moreover, we suppose:

(H_ς) The function $t \to \varsigma_t$ is continuous from $\mathcal{R}(\varrho^-)$ into $\mathscr{B}$ and there exists a continuous and bounded function $L^\varsigma : \mathcal{R}(\varrho^-) \to (0, \infty)$ such that

$$\|\varsigma_t\|_{\mathscr{B}} \leq L^\varsigma(t)\|\varsigma\|_{\mathscr{B}} \quad \text{for every} \quad t \in \mathcal{R}(\varrho^-).$$

Lemma 3.4. *[26, Lemma 3.1] If $x : (-\infty, b] \to \mathbb{X}$ is a function such that $x_0 = \varsigma$, then*

$$\|x_s\|_{\mathscr{B}} \leq (\mathscr{D}_2^* + L^\varsigma)\|\varsigma\|_{\mathscr{B}} + \mathscr{D}_1^* \sup\{|x(\theta)| : \theta \in [0, \max\{0, s\}]\}, \quad s \in \mathcal{R}(\varrho^-) \cup \mathscr{I},$$

where $L^\varsigma = \sup\limits_{t \in \mathcal{R}(\varrho^-)} L^\varsigma(t)$.

3.1 Existence results: The convex case

In this section, we are dealing with the existence results for the structure (1.1)-(1.2). We expect that $\mathscr{F}$ is a compact and convex valued multivalued map and we apply Lemma 2.2 to build our first result. Thus, we have:

(H1) The solution operator $S_\alpha(t)_{t \in \mathscr{I}}$ is compact for $t > 0$, and there is $M > 0$ such that

$$\|S_\alpha(t)\|_{B(\mathbb{X})} \leq M, \quad \text{for each} \quad t \in \mathscr{I}.$$

(H2) The multivalued map $\mathscr{F} : \mathscr{I} \times \mathscr{B} \to \mathscr{P}_{cp,c}(\mathbb{X})$ is Carathéodory.

(H3) There exists a continuous function $k : \mathscr{I} \to \mathbb{R}^+$ such that

$$|\mathscr{F}(t,u) - \mathscr{F}(t,v)| \leq k(t)\|u - v\|_{\mathscr{B}}, \quad t \in \mathscr{I},\ u, v, \in \mathscr{B},$$

and

$$k^* = \sup_{t \in \mathscr{I}} \int_0^t k(s)ds < \infty.$$

(H4) The function $t \to \mathscr{F}(t, 0) = \mathscr{F}_0 \in L^1(\mathscr{I}, \mathbb{R}^+)$ with $\mathscr{F}^* = \|\mathscr{F}_0\|_{L^1}$.

(H5) The function $\mathscr{G}(t, \cdot)$ is continuous on $\mathscr{I}$ and there exists a constant $K_{\mathscr{G}} > 0$ such that

$$|\mathscr{G}(t,u) - \mathscr{G}(t,v)| \leq K_{\mathscr{G}}\|u - v\|_{\mathscr{B}}, \quad \text{for each} \quad u, v \in \mathscr{B},$$

and

$$\mathscr{G}^* = \sup_{t \in \mathscr{I}} |\mathscr{G}(t, 0)| < \infty.$$

(H6) For each $t \in \mathscr{I}$ and any bounded set $\mathcal{V} \subset \mathscr{B}$, the set $\{\mathscr{F}(t,u), \mathscr{G}(t,u) : u \in B\}$ is relatively compact in $\mathbb{X}$.

(H7) For any bounded set $\mathcal{V} \subset \mathscr{B}$, the function $\{t \to \mathscr{G}(t, x_{\varrho(t,x_t)}) : x \in \mathcal{V}\}$ is equicontinuous on $\mathscr{I}$.

Theorem 3.1. *Assume that (H1)-(H7) and (H_ς) hold. Then, the problem (1.1)-(1.2) has a mild solution on $(-\infty, b]$ provided that*

$$\left[\mathscr{D}_1^*(K_{\mathscr{G}} + Mk^*)\right] < 1. \tag{3.2}$$

Proof. We will transmute the structure (1.1)-(1.2) into a fixed point problem. We conceive the set

$$\mathcal{V} = \{x : (-\infty, b] \to \mathbb{X} : x|_{\mathscr{I}} \quad \text{is continuous and} \quad x_0 \in \mathscr{B}\},$$

where $x|_{\mathscr{I}}$ is the constraint of x to the real compact interval on $\mathscr{I}$. Recognize the multivalued operator $\Upsilon : \mathcal{V} \to \mathscr{P}(\mathcal{V})$ defined by $\Upsilon(h) = \{h \in \mathcal{V}\}$ with

$$h(t) = \begin{cases} \varsigma(t), & t \leq 0; \\ S_\alpha(t)\big[\varsigma(0) - \mathscr{G}(0, \varsigma(0))\big] + \mathscr{G}(t, x_{\varrho(t,x_t)}) + \displaystyle\int_0^t S_\alpha(t-s)v(s)ds, & t \in \mathscr{I}, \end{cases}$$

where $v \in S_{\mathscr{F}, x_{\varrho(s,x_s)}}$. For $\varsigma \in \mathscr{B}$, we express the function $y(\cdot) : (-\infty, b] \to \mathbb{X}$ by

$$y(t) = \begin{cases} \varsigma(t), & t \leq 0; \\ S_\alpha(t)\varsigma(0), & t \in \mathscr{I}, \end{cases}$$

then $y_0 = \varsigma$. For every function $z \in \mathcal{V}$ with $z_0 = 0$, we designate by $\overline{z}$ the function clear by

$$\overline{z}(t) = \begin{cases} 0, & t \leq 0; \\ z(t), & t \in \mathscr{I}. \end{cases}$$

If $x(\cdot)$ fulfills (3.1), we are able to decompose it as $x(t) = z(t) + y(t)$, $t \in \mathscr{I}$, which suggests $x_t = z_t + y_t$, for each $t \in \mathscr{I}$ and also the function $z(\cdot)$ fulfills

$$z(t) = \mathscr{G}(t, z_{\rho(t,z_t+y_t)} + y_{\rho(t,z_t+y_t)}) - S_\alpha(t)\mathscr{G}(0, \varsigma(0)) + \int_0^t S_\alpha(t-s)v(s)ds, \quad t \in \mathscr{I},$$

where $v(s) \in S_{\mathscr{F}, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}}$.

Let $\mathcal{V}_b^0 = \{z \in \mathcal{V} : z(0) = 0 \in \mathscr{B}\}$. For some $z \in \mathcal{V}_b^0$, we have

$$\|z\|_b = \sup_{t \in \mathscr{I}} \|z(t)\| + \|z_0\|_{\mathscr{B}} = \sup_{t \in \mathscr{I}} \|z(t)\|.$$

Thus $\mathcal{V}_b^0$ is a Banach space with the norm $\|\cdot\|_b$. We delimit the operator $\overline{\Upsilon} : \mathcal{V}_b^0 \to \mathscr{P}(\mathcal{V}_b^0)$ by $\overline{\Upsilon}(z) = \{h \in \mathcal{V}_b^0\}$ with

$$h(t) = \mathscr{G}(t, z_{\rho(t,z_t+y_t)} + y_{\rho(t,z_t+y_t)}) - S_\alpha(t)\mathscr{G}(0, \varsigma(0)) + \int_0^t S_\alpha(t-s)v(s)ds, \quad t \in J,$$

where $v(s) \in S_{\mathscr{F}, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}}$.

We recall that the operator Υ has a fixed point if and only if $\overline{\Upsilon}$ has a fixed point. Thus, let us demonstrate that $\overline{\Upsilon}$ has a fixed point. Let

$$B_r = \{z \in \mathcal{V}_b^0 : z(0) = 0, \|z\|_b \leq r\},$$

where r is any fixed real number. It is perfect that B_r is a closed, convex, bounded set in $\mathcal{V}_b^0$.

Remark 3.1. *By hypotheses (H3)-(H5) we obtain:*

(i)

$$\begin{aligned}
&M\int_0^t |\mathscr{F}(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}) - \mathscr{F}(s,0) + \mathscr{F}(s,0)|ds \\
&\leq M\int_0^t k(s)\|z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}\|_{\mathscr{B}}ds + M\int_0^t |\mathscr{F}(s,0)|ds. \qquad (3.3)
\end{aligned}$$

Since

$$\begin{aligned}
\|z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}\|_{\mathscr{B}} &\leq \|z_{\varrho(s,z_s+y_s)}\|_{\mathscr{B}} + \|y_{\varrho(s,z_s+y_s)}\|_{\mathscr{B}} \\
&\leq \mathscr{D}_1^*|z(s)| + (\mathscr{D}_2^* + L^{\varsigma})\|z_0\|_{\mathscr{B}} + \mathscr{D}_1^*|y(s)| + (\mathscr{D}_2^* + L^{\varsigma})\|\varsigma\|_{\mathscr{B}} \\
&\leq \mathscr{D}_1^*|z(s)| + (\mathscr{D}_2^* + L^{\varsigma} + \mathscr{D}_1^* MH)\|\varsigma\|_{\mathscr{B}} \\
&\leq \mathscr{D}_1^*|z(s)| + C_1,
\end{aligned}$$

where $C_1 = (\mathscr{D}_2^* + L^{\varsigma} + \mathscr{D}_1^* MH)\|\varsigma\|_{\mathscr{B}}$. *Then (3.3) becomes*

$$\begin{aligned}
&M\int_0^t |\mathscr{F}(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}) - \mathscr{F}(s,0) + \mathscr{F}(s,0)|ds \\
&\leq M\int_0^t k(s)\|z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}\|_{\mathscr{B}}ds + M\int_0^t |\mathscr{F}(s,0)|ds \\
&\leq M\int_0^t k(s)\big[\mathscr{D}_1^*|z(s)| + C_1\big]ds + M\mathscr{F}^* \\
&\leq M\mathscr{D}_1^* r k^* + MC_1k^* + M\mathscr{F}^*.
\end{aligned}$$

(ii)

$$\begin{aligned}
&|\mathscr{G}(t, z_{\rho(t,z_t+y_t)} + y_{\rho(t,z_t+y_t)}) - \mathscr{G}(t,0) + \mathscr{G}(t,0)| \\
&\leq K_{\mathscr{G}}\|z_{\rho(t,z_t+y_t)} + y_{\rho(t,z_t+y_t)}\|_{\mathscr{B}} + \mathscr{G}^* \\
&\leq K_{\mathscr{G}}\mathscr{D}_1^* r + K_{\mathscr{G}} C_1 + \mathscr{G}^*.
\end{aligned}$$

(iii)

$$\begin{aligned}
&\int_0^{\eta_1} \|S_\alpha(\eta_2 - s) - S_\alpha(\eta_1 - s)\|_{B(\mathbb{X})}|\mathscr{F}(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}) - \mathscr{F}(s,0)|ds \\
&\quad + \int_0^{\eta_1} \|S_\alpha(\eta_2 - s) - S_\alpha(\eta_1 - s)\|_{B(\mathbb{X})}|\mathscr{F}(s,0)|ds \\
&\quad + \int_{\eta_1}^{\eta_2} \|S_\alpha(\eta_2 - s)\|_{B(\mathbb{X})}|\mathscr{F}(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}) - \mathscr{F}(s,0)|ds \\
&\quad + \int_{\eta_1}^{\eta_2} \|S_\alpha(\eta_2 - s)\|_{B(\mathbb{X})}|\mathscr{F}(s,0)|ds \\
&\leq \mathscr{D}_1^* r\int_0^{\eta_1} \|S_\alpha(\eta_2 - s) - S_\alpha(\eta_1 - s)\|_{B(\mathbb{X})}k(s)ds \\
&\quad + C_1\int_0^{\eta_1} \|S_\alpha(\eta_2 - s) - S_\alpha(\eta_1 - s)\|_{B(\mathbb{X})}k(s)ds \\
&\quad + \int_0^{\eta_1} \|S_\alpha(\eta_2 - s) - S_\alpha(\eta_1 - s)\|_{B(\mathbb{X})}|\mathscr{F}(s,0)|ds
\end{aligned}$$

$$
\begin{aligned}
&+ \mathscr{D}_1^* r \int_{\eta_1}^{\eta_2} \|S_\alpha(\eta_2 - s)\|_{B(\mathbb{X})} k(s) ds \\
&+ C_1 \int_{\eta_1}^{\eta_2} \|S_\alpha(\eta_2 - s)\|_{B(\mathbb{X})} k(s) ds \\
&+ \int_{\eta_1}^{\eta_2} \|S_\alpha(\eta_2 - s)\|_{B(\mathbb{X})} |\mathscr{F}(s, 0)| ds.
\end{aligned}
$$

Presently, we might demonstrate that $\overline{\Upsilon}$ fulfills all the assumptions of Bohnenblust-Karlin's theorem. For better comprehensibility, we break the verification into succession of steps.

Step 1: $\overline{\Upsilon}(z)$ is convex for each $z \in \mathcal{V}_b^0$.

In fact, if h_1 and h_2 have a place with $\overline{\Upsilon}(z)$, then there exists $v_1, v_2 \in S_{\mathscr{F}, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}}$ such that, for $t \in \mathscr{I}$, we have

$$
h_i(t) = \mathscr{G}(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)}) - S_\alpha(t)\mathscr{G}(0, \varsigma(0)) + \int_0^t S_\alpha(t-s) v_i(s) ds, \quad i = 1, 2.
$$

Let $0 \leq d \leq 1$. Then, for every $t \in \mathscr{I}$, we have

$$
(\lambda h_1 + (1-\lambda)h_2)(t) = \mathscr{G}(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)}) - S_\alpha(t)\mathscr{G}(0, \varsigma(0)) + \int_0^t S_\alpha(t-s)\big[\lambda v_1(s) + (1-\lambda) v_2(s)\big] ds.
$$

Since $\mathscr{F}$ has convex values, $S_{\mathscr{F}, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}}$ is convex, we see that $(\lambda h_1 + (1-\lambda)h_2) \in \overline{\Upsilon}(z)$.

Step 2: $\overline{\Upsilon}(B_r) \subseteq B_r$ for some $r > 0$.

We assert that there exists a positive number r such that $\overline{\Upsilon}(b_r) \subseteq B_r$. On the off chance that it is not true, then for every positive number r, there exists a function $z_r \in B_r$ and $h \in \overline{\Upsilon}(z_r)$ such that $|h(t)| > r$ for some $t \in \mathscr{I}$. Then from Remark 3.1, we have

$$
\begin{aligned}
r < |h(t)| &\leq |\mathscr{G}(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)}) - \mathscr{G}(t,0) + \mathscr{G}(t,0)| + \|S_\alpha(t)\|_{B(\mathbb{X})} |\mathscr{G}(0, \varsigma(0))| \\
&\quad + \int_0^t \|S_\alpha(t-s)\|_{B(\mathbb{X})} |\mathscr{F}(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}) - \mathscr{F}(s,0) + \mathscr{F}(s,0)| ds \\
&\leq K_{\mathscr{G}} \mathscr{D}_1^* r + K_{\mathscr{G}} C_1 + \mathscr{G}^* + M|\mathscr{G}(0, \varsigma(0))| + M k^* \mathscr{D}_1^* r + M C_1 k^* + M \mathscr{F}^* \\
&\leq K_{\mathscr{G}} \mathscr{D}_1^* r + M k^* \mathscr{D}_1^* r + C_2,
\end{aligned}
$$

where $C_2 = K_{\mathscr{G}} C_1 + \mathscr{G}^* + M|\mathscr{G}(0, \varsigma(0))| + M C_1 k^* + M \mathscr{F}^*$ is independent of r. Dividing both sides by r and taking the lower limit, we have

$$
\Big[\mathscr{D}_1^* (K_{\mathscr{G}} + M k^*)\Big] \geq 1.
$$

This contradicts to (3.2). Hence for some positive number r, $\overline{\Upsilon}(B_r) \subseteq B_r$.

Step 3: $\overline{\Upsilon}(B_r)$ is relatively compact.

We know that B_r is bounded and $\overline{\Upsilon}(B_r) \subseteq B_r$, it is clear that $\overline{\Upsilon}(B_r)$ is bounded. It remains to show that $\overline{\Upsilon}(B_r)$ is equicontinuous.

Let $\eta_1, \eta_2 \in \mathscr{I}$ with $\eta_1 < \eta_2$ and $z \in \overline{\Upsilon}(B_r)$. Then from the remark 3.1 (iii), we have

$$
\begin{aligned}
|h(\eta_2) - h(\eta_1)| &\leq |\mathscr{G}(\eta_2, z_{\varrho(\eta_2, z_{\eta_2} + y_{\eta_2})} + y_{\varrho(\eta_2, z_{\eta_2} + y_{\eta_2})}) - \mathscr{G}(\eta_1, z_{\varrho(\eta_1, z_{\eta_1} + y_{\eta_1})} + y_{\varrho(\eta_1, z_{\eta_1} + y_{\eta_1})})| \\
&\quad + \|S_\alpha(\eta_2) - S_\alpha(\eta_1)\|_{B(\mathbb{X})} |\mathscr{G}(0, \varsigma(0))| \\
&\quad + \int_0^{\eta_1} \|S_\alpha(\eta_2 - s) - S_\alpha(\eta_1 - s)\|_{B(\mathbb{X})} |\mathscr{F}(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)})| ds \\
&\quad + \int_{\eta_1}^{\eta_2} \|S_\alpha(\eta_2 - s)\|_{B(\mathbb{X})} |\mathscr{F}(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)})| ds
\end{aligned}
$$

$$\begin{aligned}
&\leq |\mathscr{G}(\eta_2, z_{\varrho(\eta_2, z_{\eta_2}+y_{\eta_2})} + y_{\varrho(\eta_2, z_{\eta_2}+y_{\eta_2})}) - \mathscr{G}(\eta_1, z_{\varrho(\eta_1, z_{\eta_1}+y_{\eta_1})} + y_{\varrho(\eta_1, z_{\eta_1}+y_{\eta_1})})| \\
&\quad + \|S_\alpha(\eta_2) - S_\alpha(\eta_1)\|_{B(\mathbb{X})} |\mathscr{G}(0, \varsigma(0))| \\
&\quad + \mathscr{D}_1^* r \int_0^{\eta_1} \|S_\alpha(\eta_2 - s) - S_\alpha(\eta_1 - s)\|_{B(\mathbb{X})} k(s) ds \\
&\quad + C_1 \int_0^{\eta_1} \|S_\alpha(\eta_2 - s) - S_\alpha(\eta_1 - s)\|_{B(\mathbb{X})} k(s) ds \\
&\quad + \int_0^{\eta_1} \|S_\alpha(\eta_2 - s) - S_\alpha(\eta_1 - s)\|_{B(\mathbb{X})} |\mathscr{F}(s, 0)| ds \\
&\quad + \mathscr{D}_1^* r \int_{\eta_1}^{\eta_2} \|S_\alpha(\eta_2 - s)\|_{B(\mathbb{X})} k(s) ds \\
&\quad + C_1 \int_{\eta_1}^{\eta_2} \|S_\alpha(\eta_2 - s)\|_{B(\mathbb{X})} k(s) ds \\
&\quad + \int_{\eta_1}^{\eta_2} \|S_\alpha(\eta_2 - s)\|_{B(\mathbb{X})} |\mathscr{F}(s, 0)| ds.
\end{aligned}$$

At the point when $\eta_2 \to \eta_1$, the right-hand side of the overhead inequality has a tendency to zero, subsequent to by (H7) and $S_\alpha(t)$ is uniformly continuous, this demonstrates the equicontinuity. As a result of Steps 1-3, together with the Arzela-Ascoli's theorem, we conclude that the operator $\overline{\Upsilon}$ is completely continuous.

Step 4: $\overline{\Upsilon}$ has a closed graph.

Suppose that $z^n \to z^*,\ h_n \in \overline{\Upsilon}(z^n)$ with $h_n \to h_*$. We claim that $h_* \in \overline{\Upsilon}(z^*)$. In fact, assumption $h_n \in \overline{\Upsilon}(z^n)$ suggests that there exists $v_n \in S_{\mathscr{F}, z^n_{\varrho(s,z^n_s+y_s)}+y_{\varrho(s,z^n_s+y_s)}}$ such that, for every $t \in \mathscr{I}$,

$$h_n(t) = \mathscr{G}(t, z^n_{\varrho(t,z^n_t+y_t)} + y_{\varrho(t,z^n_t+y_t)}) - S_\alpha(t)\mathscr{G}(0, \varsigma(0)) + \int_0^t S_\alpha(t-s) v_n(s) ds.$$

We must demonstrate that there exists $v_* \in S_{\mathscr{F}, z^*_{\varrho(s,z^*_s+y_s)}+y_{\varrho(s,z^*_s+y_s)}}$ such that, for each $t \in \mathscr{I}$,

$$h_*(t) = \mathscr{G}(t, z^*_{\varrho(t,z^*_t+y_t)} + y_{\varrho(t,z^*_t+y_t)}) - S_\alpha(t)\mathscr{G}(0, \varsigma(0)) + \int_0^t S_\alpha(t-s) v_*(s) ds.$$

Set

$$\Theta_n(t) = h_n(t) - \mathscr{G}(t, z^n_{\varrho(t,z^n_t+y_t)} + y_{\varrho(t,z^n_t+y_t)}) - S_\alpha(t)\mathscr{G}(0, \varsigma(0)) + \int_0^t S_\alpha(t-s) v_n(s) ds,$$
$$\Theta_*(t) = h_*(t) - G(t, z^*_{\varrho(t,z^*_t+y_t)} + y_{\varrho(t,z^*_t+y_t)}) - S_\alpha(t)\mathscr{G}(0, \varsigma(0)) + \int_0^t S_\alpha(t-s) v_*(s) ds.$$

We have, for every $t \in \mathscr{I}$,

$$\|\Theta_n(t) - \Theta_*(t)\| \to 0 \quad \text{as} \quad n \to \infty.$$

Recognize the linear continuous operator $\Psi : L^1(\mathscr{I}, \mathbb{X}) \to \mathcal{V}_b^0$ defined by

$$\Psi(v)(t) = \int_0^t S_\alpha(t-s) v(s) ds.$$

From Lemma 2.1 and the definition of Ψ, it follows that $\Psi \circ S_{\mathscr{F}}$ is a closed graph operator, and for every $t \in \mathscr{I}$,

$$\Theta_n(t) \in \Psi(S_{\mathscr{F}, z^n_{\varrho(s,z^n_s+y_s)}+y_{\varrho(s,z^n_s+y_s)}}).$$

Since $z^n \to z^*$ and $\Psi \circ S_{\mathscr{F}}$ is a closed graph operator, then there exists $v_* \in S_{\mathscr{F}, z^*_{\varrho(s,z^*_s+y_s)}+y_{\varrho(s,z^*_s+y_s)}}$ such that, for each $t \in \mathscr{I}$,

$$h_*(t) - \mathscr{G}(t, z^*_{\varrho(t,z^*_t+y_t)} + y_{\varrho(t,z^*_t+y_t)}) + S_\alpha(t)\mathscr{G}(0, \varsigma(0)) = \int_0^t S_\alpha(t-s) v_*(s) ds.$$

Hence $h_* \in \overline{\Upsilon}(z^*)$.

As a result of Lemma 2.2, we find that $\overline{\Upsilon}$ has a fixed point z on the interval $(-\infty, b]$. Along these lines, $x = \overline{z} + y$ is a fixed point of the operator Υ which is the mild solution of the structure (1.1)-(1.2). □

Our next result is focused on the Leray-Schauder's alternative fixed point theorem [2]. In order to utilize this theorem, we require the subsequent further hypothesis:

$(H3^*)$ There exists a function $\vartheta \in L^1(\mathscr{I}, \mathbb{R}^+)$ and a continuous non-decreasing function $\Omega : \mathbb{R}^+ \to (0, \infty)$ such that

$$|\mathscr{F}(t, u)| \leq \vartheta(t)\Omega(\|u\|_{\mathscr{B}}) \quad \text{for a.e.} \quad t \in \mathscr{I} \quad \text{and each} \quad u \in \mathscr{B}.$$

If $\mu = 1 - \mathscr{D}_1^* K_{\mathscr{G}} > 0$ and

$$\frac{\mathscr{D}_1^* M}{\mu} \int_0^b \vartheta(s) ds < \int_C^\infty \frac{ds}{\Omega(s)},$$

where $C = C_1 + \frac{\mathscr{D}_1^*}{\mu}\Big[M|\mathscr{G}(0, \varsigma(0))| + K_{\mathscr{G}} C_1 + \mathscr{G}^*\Big]$.

Theorem 3.2. *Assume that (H1), (H2), $(H3^*)$ and (H5)-(H8) are fulfilled. Then, the problem (1.1)-(1.2) has a mild solution on $(-\infty, b]$.*

Proof. Let z be solutions of the inclusion $z \in \lambda\Upsilon(z)$, for any $\lambda \in (0,1)$, then there exists $v \in S_{F, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}}$ such that

$$\begin{aligned}
|z(t)| &\leq \|S_\alpha(t)\|_{B(\mathbb{X})} |\mathscr{G}(0, \varsigma(0))| + |\mathscr{G}(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)}) - \mathscr{G}(t, 0) + \mathscr{G}(t, 0)| \\
&\quad + \int_0^t \|S_\alpha(t-s)\|_{B(\mathbb{X})} |\mathscr{F}(s, z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)})| ds \\
&\leq M|\mathscr{G}(0, \varsigma(0))| + K_{\mathscr{G}} \|z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)}\|_{\mathscr{B}} + \mathscr{G}^* \\
&\quad + M \int_0^t \vartheta(s) \Omega(\|z_{\varrho(s,z_s+y_s)} + y_{\varrho(s,z_s+y_s)}\|_{\mathscr{B}}) ds.
\end{aligned}$$

From the remark 3.1, we have

$$\begin{aligned}
|z(t)| &\leq M|\mathscr{G}(0, \varsigma(0))| + K_{\mathscr{G}} C_1 + K_{\mathscr{G}} \mathscr{D}_1^* |z(t)| + \mathscr{G}^* + M \int_0^t \vartheta(s) \Omega(\mathscr{D}_1^* |z(s)| + C_1) ds \\
&\leq \frac{1}{\mu}\Big[M|\mathscr{G}(0, \varsigma(0))| + K_{\mathscr{G}} C_1 + \mathscr{G}^*\Big] + \frac{M}{\mu} \int_0^t \vartheta(s) \Omega(\mathscr{D}_1^* |z(s)| + C_1) ds.
\end{aligned}$$

Then

$$\mathscr{D}_1^* |z(t)| + C_1 \leq C_1 + \frac{\mathscr{D}_1^*}{\mu}\Big[M|\mathscr{G}(0, \varsigma(0))| + K_{\mathscr{G}} C_1 + \mathscr{G}^*\Big] + \frac{\mathscr{D}_1^* M}{\mu} \int_0^t \vartheta(s) \Omega(\mathscr{D}_1^* |z(s)| + C_1) ds.$$

We conceive the function β characterized by

$$\beta(t) = \sup\{\mathscr{D}_1^* |z(s)| + C_1 : 0 \leq s \leq b\}, \quad t \in \mathscr{I}.$$

Let $t^* \in [0, t]$ be such that $\beta(t) = \mathscr{D}_1^* |z(t^*)| + C_1 \|\varsigma\|_{\mathscr{B}}$. By the aforementioned inequality, we sustain

$$\beta(t) \leq C_1 + \frac{\mathscr{D}_1^*}{\mu}\Big[M|\mathscr{G}(0, \varsigma(0))| + K_{\mathscr{G}} C_1 + \mathscr{G}^*\Big] + \frac{\mathscr{D}_1^* M}{\mu} \int_0^t \vartheta(s) \Omega(\beta(s)) ds.$$

Let us occupy the right-hand side of the overhead inequality as $v(t)$, for all $t \in \mathscr{I}$. Then, we sustain

$$\beta(t) \leq v(t), \quad \text{for all} \quad t \in \mathscr{I}.$$

From the meaning of v, we obtain

$$v(0) = C_1 + \frac{\mathscr{D}_1^*}{\mu}\Big[M|\mathscr{G}(0,\varsigma(0))| + K_{\mathscr{G}}C_1 + \mathscr{G}^*\Big] = C$$

and

$$v'(t) = \frac{\mathscr{D}_1^* M}{\mu}\vartheta(t)\Omega(\beta(t)), \quad \text{a.e.} \quad t \in \mathscr{I}.$$

Applying the non-decreasing character of Ω, we conclude

$$v'(t) \le \frac{\mathscr{D}_1^* M}{\mu}\vartheta(t)\Omega(v(t)), \quad \text{a.e.} \quad t \in \mathscr{I},$$

and hence

$$\int_{v(0)=C}^{v(t)} \frac{ds}{\Omega(s)} \le \frac{\mathscr{D}_1^* M}{\mu}\int_0^t \vartheta(s)ds \le \frac{\mathscr{D}_1^* M}{\mu}\int_0^b \vartheta(s)ds < \int_C^\infty \frac{ds}{\Omega(s)}.$$

In this manner, for each $t \in \mathscr{I}$, there exists a constant Λ_* such that $v(t) \le \Lambda_*$ and henceforth $\beta(t) \le \Lambda_*$. Since $\|z\|_{\mathscr{B}} \le \beta(t)$, we have $\|z\|_{\mathscr{B}} \le \Lambda_*$. Set

$$\mathcal{U} = \{z \in \mathcal{V}_b^0 : \|z\|_\infty < \Lambda_* + 1\}.$$

From Theorem 3.1, we realize that the operator $\overline{\Upsilon} : \overline{\mathcal{U}} \to \overline{\Upsilon}(z)$ is completely continuous. Besides, from the decision of $\mathcal{U}$, there is no $z \in \partial\mathcal{U}$ such that $z = \lambda\overline{\Upsilon}(z)$, for $\lambda \in (0,1)$. As an outcome of the nonlinear alternative of Leray-Schauder type [2], we conclude that Υ has a fixed point z in $\mathcal{U}$, then the structure (1.1)-(1.2) has at least one mild solution on $(-\infty, b]$. □

3.2 Existence results: Nonconvex case

The next step is to demonstrate the existence results for the structure (1.1)-(1.2). Our result is focused around the Lemma 2.3.

Theorem 3.3. *Assume that the subsequent hypotheses hold:*

(H8) $\mathscr{F} : \mathscr{I} \times \mathscr{B} \to \mathscr{P}_{cp}(\mathbb{X})$ *has the assets that* $\mathscr{F}(\cdot, u) : \mathscr{I} \to \mathscr{P}_{cp}(\mathbb{X})$ *is measurable, for each* $u \in \mathscr{B}$.

(H9) There exists $\wp \in L^1(\mathscr{I}, \mathbb{R}^+)$ *such that*

$$H_d(\mathscr{F}(t,u), \mathscr{F}(t,\overline{u})) \le \wp(t)\|u - \overline{u}\|_{\mathscr{B}}, \quad \textit{for every} \quad u, \overline{u} \in \mathscr{B},$$

and

$$d(0, \mathscr{F}(t,0)) \le \wp(t) \quad \textit{a.e.} \quad t \in \mathscr{I}.$$

(H10) There exists a positive constant $L_* > 0$ *such that*

$$|\mathscr{G}(t,u) - \mathscr{G}(t,\overline{u})| \le L_*\|u - \overline{u}\|_{\mathscr{B}}, \quad \textit{a.e} \quad t \in \mathscr{I} \quad \textit{and for all} \quad u.\overline{u} \in \mathscr{B}.$$

Then the problem (1.1)-(1.2) has at least one mild solution on $(-\infty, b]$.

Remark 3.2. *For every* $z \in \mathcal{V}_b^0$*, the set* $S_{\mathscr{F},z}$ *is nonempty, since, by (H8),* $\mathscr{F}$ *has a measurable choice [43, Theorem III.6].*

Proof. Let $\overline{\Upsilon} : \mathcal{V}_b^0 \to \mathscr{P}(\mathcal{V}_b^0)$, where $\overline{\Upsilon}$ is defined in Theorem 3.1 be solutions of the problem (1.1)-(1.2). Presently, we might demonstrate that the operator $\overline{\Upsilon}$ fulfills all the states of Lemma 2.3. For our comfort, we split up the proof into two steps:

Step 1: $\overline{\Upsilon}(z) \in \mathscr{P}_{cl}(\mathcal{V}_b^0)$ for all $z \in \mathcal{V}_b^0$.

In fact, let $(z^n)_{n\geq 0} \in \overline{\Upsilon}(z)$ be such that $z^n \to \tilde{z} \in \mathcal{V}_b^0$. Then $\tilde{z} \in \mathcal{V}_b^0$ and there exists $v_n \in S_{\mathscr{F}, z_{\varrho(s,z_s+y_s)}+y_{\varrho(s,z_s+y_s)}}$ such that, for every $t \in \mathscr{I}$,

$$z^n(t) = \mathscr{G}(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)}) - S_\alpha(t)\mathscr{G}(0, \varsigma(0)) + \int_0^t S_\alpha(t-s)v_n(s)ds.$$

Utilizing the way that $\mathscr{F}$ has compact values and from (H9), we may go to a subsequence if important to get that v_n converges to v in $L^1(\mathscr{I}, \mathbb{X})$ and consequently $v \in S_{\mathscr{F}, z_{\varrho(s,z_s+y_s)}+y_{\varrho(s,z_s+y_s)}}$. Then, for every $t \in \mathscr{I}$,

$$z^n(t) \to \tilde{z} = \mathscr{G}(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)}) - S_\alpha(t)\mathscr{G}(0, \varsigma(0)) + \int_0^t S_\alpha(t-s)v(s)ds.$$

So $\tilde{z} \in \overline{\Upsilon}(z)$.

Step 2: There exists $\Lambda < 1$ such that

$$H_d(\mathscr{F}(z), \mathscr{F}(\overline{z})) \leq \Lambda\|z - \overline{z}\|_\infty \quad \text{for all} \quad z, \overline{z} \in \mathcal{V}_b^0.$$

Let $z, \overline{z} \in \mathcal{V}_b^0$ and $h \in \overline{\Upsilon}(z)$. Then there exists $v(t) \in \mathscr{F}(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)})$ such that, for every $t \in \mathscr{I}$,

$$h(t) = \mathscr{G}(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)}) - S_\alpha(t)\mathscr{G}(0, \varsigma(0)) + \int_0^t S(t-s)v(s)ds.$$

From (H9), it takes after that

$$H_d\Big(\mathscr{F}(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)}), \mathscr{F}(t, \overline{z}_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)})\Big) \leq \wp(t)\|z_{\varrho(t,z_t+y_t)} - \overline{z}_{\varrho(t,z_t+y_t)}\|_{\mathscr{B}}.$$

Therefore, there is $w \in \mathscr{F}(t, \overline{z}_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)})$ such that

$$|v(t) - w| \leq \wp(t)\|z_{\varrho(t,z_t+y_t)} - \overline{z}_{\varrho(t,z_t+y_t)}\|_{\mathscr{B}}, \quad t \in \mathscr{I}.$$

Recognize $U : \mathscr{I} \to \mathscr{P}(\mathbb{X})$ specified by

$$U(t) = \{w \in \mathbb{X} : |v(t) - w| \leq \wp(t)\|z_{\varrho(t,z_t+y_t)} - \overline{z}_{\varrho(t,z_t+y_t)}\|_{\mathscr{B}}\}.$$

Since the multivalued operator $V(t) = U(t) \bigcap \mathscr{F}(t, \overline{z}_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)})$ is measurable [43, Proposition III.4], there exists a function $\overline{v}(t)$, which is measurable choice v. Along these lines, $\overline{v}(t) \in \mathscr{F}(t, \overline{z}_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)})$, and utilizing phase space axioms, for every $t \in J$, we obtain

$$\begin{aligned}|v(t) - \overline{v}(t)| &\leq \wp(t)\|z_{\varrho(t,z_t+y_t)} - \overline{z}_{\varrho(t,z_t+y_t)}\|_{\mathscr{B}} \\ &\leq \wp(t)\mathscr{D}_1^*|z(t) - \overline{z}(t)|.\end{aligned}$$

For every $t \in \mathscr{I}$, give us a chance to characterize

$$\overline{h}(t) = \mathscr{G}(t, \overline{z}_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)}) - S_\alpha(t)\mathscr{G}(0, \varsigma(0)) + \int_0^t S(t-s)\overline{v}(s)ds.$$

Then, for every $t \in \mathscr{I}$,

$$\begin{aligned}|h(t) - \overline{h}(t)| &\leq |\mathscr{G}(t, z_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)}) - \mathscr{G}(t, \overline{z}_{\varrho(t,z_t+y_t)} + y_{\varrho(t,z_t+y_t)})| \\ &\quad + \int_0^t \|S_\alpha(t-s)\|_{B(\mathbb{X})}|v(s) - \overline{v}(s)|ds \\ &\leq L_*\|z_{\varrho(t,z_t+y_t)} - \overline{z}_{\varrho(t,z_t+y_t)}\|_{\mathscr{B}} + M\mathscr{D}_1^*\int_0^t \wp(s)|z(s) - \overline{z}(s)|ds \\ &\leq L_*\mathscr{D}_1^*|z(t) - \overline{z}(t)| + \int_0^t \overline{\wp}(s)|z(s) - \overline{z}(s)|ds\end{aligned}$$

$$\begin{aligned}
&\leq \Big[L_*\mathscr{D}_1^* e^{\tau L(t)}\Big]\Big[e^{-\tau L(t)}|z(t)-\overline{z}(t)|\Big] + \int_0^t \Big[\overline{\wp}(s)e^{\tau L(s)}\Big]\Big[e^{-\tau L(s)}|z(s)-\overline{z}(s)|\Big]ds \\
&\leq L_*\mathscr{D}_1^* e^{\tau L(t)}\|z-\overline{z}\|_{\overline{\mathcal{V}}} + \|z-\overline{z}\|_{\overline{\mathcal{V}}}\int_0^t \Big[\frac{e^{\tau L(s)}}{\tau}\Big]' ds \\
&\leq \Big[L_*\mathscr{D}_1^* + \frac{1}{\tau}\Big]e^{\tau L(t)}\|z-\overline{z}\|_{\overline{\mathcal{V}}},
\end{aligned}$$

where $\tau > 0$, $L(t) = \displaystyle\int_0^t \overline{\wp}(s)ds$, $\overline{\wp}(t) = M\mathscr{D}_1^*\wp(t)$, and $\|\cdot\|_{\overline{\mathcal{V}}}$ is the Bielecki-type norm on $\mathcal{V}_b^0$ defined by $\|z\|_{\overline{\mathcal{V}}} = \sup\{e^{-\tau L(t)}\|z(t)\| : t \in \mathscr{I}\}$.

Thus, we obtain

$$\|h-\overline{h}\|_{\overline{\mathcal{V}}} \leq \Big[L_*\mathscr{D}_1^* + \frac{1}{\tau}\Big]\|z-\overline{z}\|_{\overline{\mathcal{V}}}.$$

By exchanging the parts of z and $\overline{z}$, we have

$$H_d(\overline{\Upsilon}(z), \overline{\Upsilon}(\overline{z})) \leq \Big[L_*\mathscr{D}_1^* + \frac{1}{\tau}\Big]\|z-\overline{z}\|_{\overline{\mathcal{V}}}.$$

Settling $\tau > 0$ and for $\big[L_*\mathscr{D}_1^* + \frac{1}{\tau}\big] < 1$, implies $\overline{\Upsilon}$ is a contraction, and by Lemma 2.3, it has a fixed point z, which represents a mild solution (1.1)-(1.2). □

4 Application

We consider the FNIDI with SDD, namely:

$$\begin{aligned}
\frac{\partial}{\partial t}\Big[u(t,\xi) - g(t,u(t-\sigma(u(t,0)),\xi))\Big] &\in \int_t^0 \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\Big(\frac{\partial^2}{\partial \xi^2} - r\Big)\Big[u(s,\xi) - g(s,u(s-\sigma(u(s,0)),\xi))\Big]ds \\
&+ \Big[f_1(t,u(t-\sigma(u(t,0)),\xi)), f_2(t,u(t-\sigma(u(t,0)),\xi))\Big], \quad 0 \leq t \leq b, \quad 0 \leq \xi \leq \pi, \qquad (4.1)
\end{aligned}$$

$$u(t,0) = u(t,\pi) = 0, \quad t \in \mathscr{I}, \qquad (4.2)$$

$$u(\theta,\xi) = u_0(\theta,\xi), \quad \theta \in (-\infty,0], \quad \xi \in \mathscr{J} = [0,\pi], \qquad (4.3)$$

where $1 < \alpha < 2$, $(u_0,\sigma) \in C(\mathbb{R},[0,\infty))$, $L_\xi = \Big(\dfrac{\partial^2}{\partial\xi^2} - r\Big)$, $r > 0$ stands for the operator with respect to the special variable ξ, $f_1, f_2 : \mathscr{I} \times \mathscr{B} \to \mathbb{R}$ are measurable in t and continuous in x, and $g : \mathscr{I} \times \mathscr{B} \to \mathbb{R}$ are appropriate functions. We expect that for every $t \in \mathscr{I}$, $f_1(t,\cdot)$ is lower semicontinuous , and assume that for each $t \in \mathscr{I}$, $f_2(t,\cdot)$ is upper semicontinuous.

Consider $\mathbb{X} = L^2([0,\pi],\mathbb{R})$ and the operator $\mathscr{A} : L_\xi : D(\mathscr{A}) \subset \mathbb{X} \to \mathbb{X}$ with domain

$$D(\mathscr{A}) = \{u \in \mathbb{X} : u'' \in \mathbb{X}, \quad u(0) = u(\pi) = 0\}.$$

$\mathscr{A}$ is densely defined in $\mathbb{X}$ and is sectorial. As a result $\mathscr{A}$ represents a generator of a solution operator on $\mathbb{X}$. For the phase space, we pick $\mathscr{B} = C_\gamma = \{\varsigma \in C((-\infty,0] : \mathbb{X}) : \lim_{\theta\to-\infty} e^{\gamma\theta}\varsigma(\theta) \quad \text{exists in } \mathbb{X}\}$ invested with the norm

$$|\varsigma| = \sup_{-\infty<\theta\leq 0} e^{\gamma\theta}|\varsigma(\theta)|.$$

We note that the phase space C_γ satisfies the conditions $(P_1), (P_2)$ and (P_3). Set

$$\begin{aligned}
x(t)(\xi) &= u(t,\xi), \quad t \in \mathscr{I}, \quad \xi \in \mathscr{J}, \\
\varsigma(0)(\xi) &= u_0(\theta,\xi), \quad t \in \mathscr{I}, \quad \theta \leq 0, \\
\mathscr{G}(t,\varsigma)(\xi) &= g(t,\varsigma(0,\xi)), \quad t \in \mathscr{I}, \quad \xi \in \mathscr{J},
\end{aligned}$$

$$\mathscr{F}(t,\varsigma)(\xi) = \Big[f_1(t,\varsigma(0,\xi)), f_2(t,\varsigma(0,\xi))\Big], \quad t \in \mathscr{I}, \quad \xi \in \mathscr{J},$$
$$\varrho(t,\varsigma) = t - \sigma(\varsigma(0,0)).$$

The multivalued map $\mathscr{F}$ is u.s.c. with compact convex values [43]. Hence (H1) and (H2) are satisfied.

At this stage, the existence of mild solutions can be reasoned from an immediate application of Theorem 3.2.

Theorem 4.1. *Let $\varsigma \in C_\gamma$ be such that (H_ς) holds, and let $t \to \varsigma_t$ be continuous on $\mathcal{R}(\varrho^-)$. Moreover, we assume that $(H3^*)$ is fulfilled. Thus, there exists at least one mild solution of (4.1)-(4.3).*

Corollary 4.1. *Let $\varsigma \in C_\gamma$ be continuous and bounded and assume that $(H3^*)$ holds. Thus, there exists at least one mild solution of (4.1)-(4.3) on $(-\infty, b]$.*

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NEW HERMITE–HADAMARD'S INEQUALITIES FOR PREINVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

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ABSTRACT. In this paper,we have established some Hermite− Hadamard inequalities for preinvex functions via fractional integrals and these results have some relationship with the obtained results. Application of the obtained results are given as well.

1. Introduction

The usefulness of inequalities involving convex functions is realized from the very beginning and is now widely acknowledged as one of the prime driving forces behind the development of several modern branches of mathematics and has been given considerable attention. One of the most famous inequalities for convex functions is Hermite−Hadamard's inequality, stated as [12]:

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction for f to be concave.

In [19] Pearce and J. Pecaric established the following result connected with the right part of (1).

Theorem 1. *Let $f : I^0 \subseteq R \to R$ is a differentiable function on $I^0 \quad a, b \in I^0$ with $a < b$,and If $|f'|^q$ is convex function on $[a, b]$,for some fixed $q \geq 1$.then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{\frac{1}{q}}.$$

If $|f'|^q$ is concave function on $[a, b]$, for some fixed $q \geq 1$.then

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right|.$$

Some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper.

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Definition 1. *Let $f \in L^1[a,b]$. The left−sided and right−sided Riemann−Liouville fractional integrals of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \qquad a < x$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \qquad x < b$$

respectively, where $\Gamma(.)$ is Gamma function and its definition is $\Gamma(\alpha) = \int_0^{\infty} e^{-u} u^{\alpha-1} du$. It is to be noted that $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. Properties concerning this operator can be found in [13] and for some recent results connected with fractional integral inequalities, see [8], [9], [10], [17], [22],[24]. Hermite−Hadamard Inequality has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found in [11], [15], [16], [19] and the references cited therein.

In [24] Sarikaya et. al. proved a variant of Hermite−Hadamard's inequalities in fractional integral forms as follows:

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L^1[a,b]$. If f is convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{2}$$

with $\alpha > 0$

Remark 1. *For $\alpha = 1$, inequality* (2) *reduces to inequality* (1)*.*

Using the following identity Sarikaya et. al. established the following result which hold for convex functions.

Lemma 1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b), with $a < b$ and $f' \in L[a,b]$, then the following identity holds:*

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] = \frac{b-a}{2} \int_0^{1/2} \left[(1-t)^{\alpha} - t^{\alpha}\right] f'(ta+(1-t)b) dt, \tag{3}$$

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b), with $a < b$. If $|f'|$ is convex function on $[a,b]$, then the following inequalities for fractional integrals holds:*

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \leq \frac{b-a}{2(\alpha+1)} \left(1 - \frac{1}{2^{\alpha}}\right) \left(|f'(a)| + |f'(b)| \right). \tag{4}$$

In recent years several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of invex functions introduced by Hanson in [14]. Weir and Mond [18] introduced the concept of preinvex functions and applied it to the establishment of the sufficient

optimality conditions and duality in nonlinear programming. Pini [20] introduced the concept of prequasiinvex as a generalization of invex functions. Later, Mohan and Neogy[18] obtained some properties of generalized preinvex functions. Noor [2]-[4] has established some Hermite-Hadamard type inequalities for preinvex and log-preinvex functions. In recent papers Barani et al. in [5] presented some estimates of the right hand side of a Hermite-Hadamard type inequality in which some preinvex functions are involved. In this paper we generalized the results in [15] and [16] for preinvex functions via fractional integrals. Let K be a closed set R^n and let $f : K \to \mathrm{R}$ and $\eta : K \times K \to \mathrm{R}$ be continuous functions. Let $x \in K$, then the set K is said to be invex at x with respect to $\eta(.,.)$,

If

$$x + t\eta(y, x) \in K, \forall x, y \in K, t \in [0, 1] .$$

K is said to be an invex set with respect to η if K is invex at each $x \in K$. The invex set K is also called a η-connected set.

Definition 2. *The function f on the invex set $K$is said to be preinvex with respect to η, if*

$$f(u + t\eta(v, u)) \leq +(1 - t) f(u) + tf(v), \forall u, v \in K, t \in [0, 1] .$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

In the recent paper, Noor [4] has obtained the following Hermite-Hadamard inequalities for the preinvex functions:

Theorem 4. *Let $f : [a, a + \eta(b, a)] \to (0, \infty)$ be a open preinvex function on the interval of real numbers K^0 (the interior of K^0) and $a, b \in K^0$ with $a < a+\eta(b, a)$. the following inequality holds:*

$$f\left(\frac{2a + \eta(b, a)}{2}\right) \leq \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)dx \leq \frac{f(a) + f(b)}{2}.$$

Barani, Ghazanfari and Dragomir in [5], presented the following estimates of the right-side of a Hermite-Hadamard type inequality in which some preinvex functions are involved.

Theorem 5. *Let $K \subseteq R$ be an open invex subset with respect to $\eta : K \times K \to R$. Suppose that $f : K \to R$ is a differentiable function. If $|f'|$ is preinvex on K, then for every $a, b \in K$ with $\eta(b, a) \neq 0$ the following inequality holds:*

$$\left|\frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{\eta(b, a)}{8} \{|f'(a)| + |f'(b)|\} .$$

Theorem 6. *Let $K \subseteq R$ be an open invex subset with respect to $\eta : K \times K \to R$. Suppose that $f : K \to R$ is a differentiable function. Assume $p \in R$ with $p > 1$. If $|f'|^{\frac{p}{(p-1)}}$ is preinvex on K, then for every $a, b \in K$ with $\eta(b, a) \neq 0$ the following inequality holds:*

$$\left|\frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{\eta(b, a)}{2(1 + p)^{1/p}} \left\{\frac{|f'(a)|^{\frac{p}{(p-1)}} + |f'(b)|^{\frac{p}{(p-1)}}}{2}\right\}^{\frac{p-1}{p}} .$$

The aim of this paper is to establish left Hermite−Hadamard type inequalities for Riemann−Liouville fractional integral using the identity obtained for fractional integrals.

2. Main Results

In order to obtain our results, we modified [15, Lemma 2.1] as following:

Lemma 2. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to R$ and $a, b \in A$ with $a < a + \eta(b, a)$. Suppose that $f : A \to \mathbb{R}$ be a differentiable function. If f' is preinvex function on A and $f' \in L[a, a + \eta(b, a)]$, then the following identity for Riemann−Liouville fractional integrals holds:*

$$f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a+} f(a+\eta(b,a)) + J^{\alpha}_{a+\eta(b,a)-} f(a)\right] = \frac{\eta(b,a)}{2}\sum_{k=1}^{4} I_k, \quad (5)$$

where

$$I_1 = \int_0^{1/2} t^{\alpha} f'(a+t\eta(b,a))dt, \qquad I_2 = \int_0^{1/2} (-t^{\alpha}) f'(b+t\eta(a,b))dt,$$
$$I_3 = \int_{1/2}^{1} (t^{\alpha}-1) f'(a+t\eta(b,a))dt, \quad I_4 = \int_{1/2}^{1} (1-t^{\alpha}) f'(b+t\eta(a,b))dt.$$

Proof. Integrating by parts

$$\begin{aligned}
I_1 &= \int_0^{1/2} t^{\alpha} f'(a+t\eta(b,a))dt \\
&= \frac{t^{\alpha} f(a+t\eta(b,a))dt}{\eta(b,a)}\Bigg|_0^{1/2} - \frac{\alpha}{\eta(b,a)}\int_0^{1/2} t^{\alpha-1} f(a+t\eta(b,a))dt \\
&= \frac{2^{-\alpha}}{\eta(b,a)} f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{\alpha}{\eta(b,a)}\int_0^{1/2} t^{\alpha-1} f(a+t\eta(b,a))dt.
\end{aligned}$$

Analogously:

$$I_2 = \frac{2^{-\alpha}}{\eta(b,a)} f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{\alpha}{\eta(b,a)}\int_0^{1/2} t^{\alpha-1} f(b+t\eta(a,b))dt$$

and

$$\begin{aligned}
I_3 &= \int_{1/2}^{1} (t^{\alpha}-1) f'(a+t\eta(b,a))dt \\
&= \frac{(t^{\alpha}-1) f((a+t\eta(b,a))}{\eta(b,a)}\Bigg|_{1/2}^{1} - \frac{\alpha}{\eta(b,a)}\int_{1/2}^{1} t^{\alpha-1} f(a+t\eta(b,a))dt \\
&= \frac{1-2^{-\alpha}}{\eta(b,a)} f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{\alpha}{\eta(b,a)}\int_{1/2}^{1} t^{\alpha-1} f(a+t\eta(b,a))dt.
\end{aligned}$$

Analogously:

$$I_4 = \frac{1-2^{-\alpha}}{\eta(b,a)} f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{\alpha}{\eta(b,a)}\int_{1/2}^{1} t^{\alpha-1} f(b+t\eta(a,b))dt$$

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Adding above equalities, we get

$$\frac{2}{\eta(b,a)}f\left(\frac{2a+\eta(b,a)}{2}\right)-\frac{\alpha}{\eta(b,a)}\left[\int_0^1 t^{\alpha-1}f(a+t\eta(b,a))dt+\int_0^1 t^{\alpha-1}f((b+t\eta(a,b))dt\right]$$
$$=I_1+I_2+I_3+I_4.$$

Now making substitution $u=(a+t\eta(b,a))$, we have

$$\int_0^1 t^{\alpha-1}f(a+t\eta(b,a))dt=\frac{1}{\eta^\alpha(b,a)}\int_a^{a+\eta(b,a)}(u-a)^{\alpha-1}f(u)du$$
$$=\frac{\Gamma(\alpha)}{\eta^\alpha(b,a)}J^\alpha_{b^-}f(a),$$

likewise

$$\int_0^{1/2} t^{\alpha-1}f(b+t\eta(a,b))=\frac{\Gamma(\alpha)}{\eta^\alpha(b,a)}J^\alpha_{a^+}f(a+\eta(b,a)),$$

which completes our proof. ∎

New upper bound for the left-hand side of (2) for convex functions is proposed in the following theorem.

Theorem 7. *Let $A\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:A\times A\to R$ and $a,b\in A$ with $a<a+\eta(b,a)$. Such that $f'\in L[a,a+\eta(b,a)]$. Suppose that $f:A\to\mathbb{R}$ be a differentiable function. If $|f'|$ is preinvex function on A then the following inequality for fractional integrals holds for $0<\alpha\leq 1$:*

$$\left|f\left(\frac{2a+\eta(b,a)}{2}\right)-\frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)}\left[J^\alpha_{a^+}f(a+\eta(b,a))+J^\alpha_{a+\eta(b,a)^-}f(a)\right]\right|$$
$$\leq\frac{\eta(b,a)}{2^{\alpha+1}(\alpha+1)}(\ |f'(a)|+|f'(b)|\ ) \quad (6)$$

Proof. By using the properties of modulus on Lemma 2, we have

$$\left|f\left(\frac{2a+\eta(b,a)}{2}\right)-\frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)}\left[J^\alpha_{a^+}f(a+\eta(b,a))+J^\alpha_{a+\eta(b,a)^-}f(a)\right]\right|\leq\frac{\eta(b,a)}{2}\sum_{k=1}^4|I_k|.$$

Now, using preinvexity of $|f'|$, we have

$$|I_1|\leq\int_0^{1/2}t^\alpha|f'(a+t\eta(b,a))|dt\leq\int_0^{1/2}t^\alpha|f'(1-t)a+tb|dt$$
$$\leq|f'(a)|\int_0^{1/2}t^\alpha(1-t)dt+|f'(b)|\int_0^{1/2}t^{\alpha+1}dt$$
$$=\frac{\alpha+3}{2^{\alpha+2}(\alpha+1)(\alpha+2)}|f'(a)|+\frac{1}{2^{\alpha+2}(\alpha+2)}|f'(b)|.$$

Analogously:

$$|I_2|\leq\frac{\alpha+3}{2^{\alpha+2}(\alpha+1)(\alpha+2)}|f'(b)|+\frac{1}{2^{\alpha+2}(\alpha+2)}|f'(a)|.$$

By using preinvexity on $|f'|$ and fact that for $\alpha\in(0,1]$ and $\forall\ t_1,t_2\in[0,1]$,

$$|{t_1}^\alpha-{t_2}^\alpha|\leq|t_1-t_2|^\alpha,$$

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$$
\begin{aligned}
|I_3| &\leq |f'(a)| \int_{1/2}^{1} (1-t^{\alpha})(1-t)\ dt + |f'(b)| \int_{1/2}^{1} (1-t^{\alpha})t\ dt \\
&\leq |f'(a)| \int_{1/2}^{1} (1-t)^{\alpha+1}\ dt + |f'(b)| \int_{1/2}^{1} (t-t^{\alpha+1})\ dt \\
&= \frac{1}{2^{\alpha+2}(\alpha+2)} |f'(a)| + \frac{\alpha+3}{2^{\alpha+2}(\alpha+1)(\alpha+2)} |f'(b)|,
\end{aligned}
$$

similarly

$$
|I_4| \leq \frac{1}{2^{\alpha+2}(\alpha+2)} |f'(b)| + \frac{\alpha+3}{2^{\alpha+2}(\alpha+1)(\alpha+2)} |f'(a)|,
$$

which completes the proof.

Corollary 1. *If we take $\eta(b,a) = b-a$ in Theorem 7, then inequality* (6) *becomes inequality as*

$$
\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a) \right] \right| \leq \frac{(b-a)}{2^{\alpha+1}(\alpha+1)} \left(|f'(a)| + |f'(b)| \right). \quad (7)
$$

∎

Remark 2. *If we take $\alpha = 1$, in Corollary 1 then inequality* (7) *becomes inequality as obtained in* [15, *Theorem* 2.2] .

The corresponding version for powers of the absolute value of the derivative is incorporated in the following theorem.

Theorem 8. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to R$ and $a, b \in A$ with $a < a + \eta(b,a)$ .Such that $f' \in L[a, a+\eta(b,a)]$.Suppose that $f : A \to \mathbb{R}$ be a differentiable function. If $|f'|^{\frac{p}{p-1}}$ is preinvex function on A for some fixed $p \geq 1$ with $q = \frac{p}{p-1}$, then the following inequality for fractional integrals holds for $0 < \alpha \leq 1$:*

$$
\begin{aligned}
&\left| f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)} \left[J_{a^+}^{\alpha} f(a+\eta(b,a)) + J_{a+\eta(b,a)^-}^{\alpha} f(a) \right] \right| \\
&\qquad \leq \frac{\eta(b,a)}{2^{\alpha+1}(\alpha p+1)^{1/p}} \left[\left(\frac{|f'(b)|^q + 3|f'(a)|^q}{4} \right)^{1/q} + \left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4} \right)^{1/q} \right]. \quad (8)
\end{aligned}
$$

Proof. From Lemma 2 and using Hölder inequality with properties of modulus, we have

$$
\left| f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f(a+\eta(b,a)) + J_{a+\eta(b,a)^-}^{\alpha} f(a) \right] \right| \leq \frac{\eta(b,a)}{2} \sum_{k=1}^{4} |I_k|.
$$

By using the convexity of $|f'|^q$, we have

$$
\begin{aligned}
|I_1| &\leq \left(\int_0^{1/2} t^{\alpha p} dt \right)^{1/p} \left(\int_0^{1/2} |f'(a+t\eta(b,a))|^q\ dt \right)^{1/q} \\
&\leq \left(\frac{1}{2^{\alpha p+1}(\alpha p+1)} \right)^{1/p} \left(|f'(a)|^q \int_0^{1/2} (1-t) dt + |f'(b)|^q \int_0^{1/2} t\ dt \right)^{1/q} \\
&= \left(\frac{1}{2^{\alpha p+1}(\alpha p+1)} \right)^{1/p} \left(\frac{|f'(b)|^q + 3|f'(a)|^q}{8} \right)^{1/q},
\end{aligned}
$$

similarly

$$|I_2| \leq \left(\frac{1}{2^{\alpha p+1}(\alpha p+1)}\right)^{1/p}\left(\frac{3|f'(b)|^q+|f'(a)|^q}{8}\right)^{1/q},$$

now

$$|I_3| \leq \left(\int_{1/2}^{1}(1-t^\alpha)^p dt\right)^{1/p}\left(\int_{1/2}^{1}|f'(tb+(1-t)a)|^q\,dt\right)^{1/q}.$$

Let $\alpha \in (0,1]$ and $\forall\ t_1, t_2 \in [0,1]$,

$$|{t_1}^\alpha - {t_2}^\alpha| \leq |t_1 - t_2|^\alpha,$$

therefore

$$\int_{1/2}^{1}(1-t^\alpha)^p dt \leq \int_{1/2}^{1}(1-t)^{\alpha p} dt = \frac{1}{2^{\alpha p+1}(\alpha p+1)}$$

Hence

$$|I_3| \leq \left(\frac{1}{2^{\alpha p+1}(\alpha p+1)}\right)^{1/p}\left(\frac{|f'(b)|^q+3|f'(a)|^q}{8}\right)^{1/q},$$

and

$$|I_4| \leq \left(\frac{1}{2^{\alpha p+1}(\alpha p+1)}\right)^{1/p}\left(\frac{3|f'(b)|^q+|f'(a)|^q}{8}\right)^{1/q},$$

which completes the proof. ◻

Corollary 2. *.If we take $\eta(b,a) = b-a$ in Theorem 8, then inequality* (8) *becomes inequality* (2.1) *of* [16, Theorem 2.3]

$$\left|f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+}f(b) + J^\alpha_{b^-}f(a)\right]\right|$$
$$\leq \frac{(b-a)}{2^{\alpha+1}(\alpha p+1)^{1/p}}\left[\left(\frac{|f'(b)|^q+3|f'(a)|^q}{4}\right)^{1/q} + \left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{1/q}\right]. \quad (9)$$

Remark 3. *If we take $\alpha = 1$, in Corollary 2 then inequality* (9) *becomes inequality* (2.1) *of* [15, *Theorem* 2.3].

Another similar result may be extended in the following theorem.

Theorem 9. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to R$ and $a, b \in A$ with $a < a + \eta(b,a)$.Such that $f' \in L[a, a+\eta(b,a)]$.Suppose that $f : A \to \mathbb{R}$ be a differentiable function. If $|f'|^{\frac{p}{p-1}}$ is preinvex function on A for some fixed $p > 1$ with $q = \frac{p}{p-1}$, then the following inequality for fractional integrals holds for $\alpha > 0$:*

$$\left|f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)}\left[J^\alpha_{a^+}f(b) + J^\alpha_{b^-}f(a)\right]\right| \leq \frac{\eta(b,a)}{2^{\alpha+1}(\alpha+1)}\times$$
$$\left[\left(\frac{(\alpha+3)\,|f'(b)|^q+(\alpha+1)\,|f'(a)|^q}{2(\alpha+2)}\right)^{1/q} + \left(\frac{(\alpha+3)\,|f'(a)|^q+(\alpha+1)\,|f'(b)|^q}{2(\alpha+2)}\right)^{1/q}\right]. \quad (10)$$

Proof. Using the well-known power-mean integral inequality for $q > 1$ we have

$$|I_1| \leq \left(\int_0^{1/2} t^\alpha dt\right)^{1-1/q}\left(\int_0^{1/2} t^\alpha\,|f'(a+t\eta(b,a))|^q\,dt\right)^{1/q}$$

By preinvexity of $|f'|^q$

$$|I_1| \leq \left(\frac{1}{2^{\alpha+1}(\alpha+1)}\right)^{1-1/q}\left(\frac{\alpha+3}{2^{\alpha+2}(\alpha+1)(\alpha+2)}|f'(a)|^q+\frac{1}{2^{\alpha+2}(\alpha+2)}|f'(b)|^q\right)^{1/q}$$
$$=\frac{1}{2^{\alpha+1}(\alpha+1)}\left(\frac{(\alpha+3)|f'(a)|^q+(\alpha+1)|f'(b)|^q}{2(\alpha+2)}\right)^{1/q}$$

Analogously:

$$|I_2| \leq \frac{1}{2^{\alpha+1}(\alpha+1)}\left(\frac{(\alpha+3)|f'(b)|^q+(\alpha+1)|f'(a)|^q}{2(\alpha+2)}\right)^{1/q}$$
$$|I_3| \leq \frac{1}{2^{\alpha+1}(\alpha+1)}\left(\frac{(\alpha+3)|f'(b)|^q+(\alpha+1)|f'(a)|^q}{2(\alpha+2)}\right)^{1/q}$$

and

$$|I_4| \leq \frac{1}{2^{\alpha+1}(\alpha+1)}\left(\frac{(\alpha+3)|f'(a)|^q+(\alpha+1)|f'(b)|^q}{2(\alpha+2)}\right)^{1/q}$$

Combining all the obtained inequalities, we get desired inequality. Which completes the proof. ◻

Corollary 3. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to R$ and $a, b \in A$ with $a < a + \eta(b,a)$ .Such that $f' \in L[a, a+\eta(b,a)]$.Suppose that $f : A \to \mathbb{R}$ be a differentiable function. If $|f'|^q$ is preinvex function on A for some fixed $q > 1$ then the following inequality for fractional integrals holds for*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx - f\left(\frac{2a+\eta(b,a)}{2}\right)\right| \leq \frac{\eta(b,a)}{8}\left(\frac{1+2^{1/q}}{3^{1/p}}\right)[|f'(a)|+|f'(b)|], \quad (11)$$

Proof. If we take $\alpha = 1$ in Theorem 9, then inequality (10) becomes as:

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx - f\left(\frac{2a+\eta(b,a)}{2}\right)\right|$$
$$\leq \frac{\eta(b,a)}{8}\left[\left(\frac{2|f'(a)|^q+|f'(b)|^q}{3}\right)^{1/q}+\left(\frac{|f'(a)|^q+2|f'(b)|^q}{3}\right)^{1/q}\right],$$

which can be made equivalent to (11) by using the fact:

$$\sum_{i=1}^{n}(a_i+b_i)^r \leq \sum_{i=1}^{n} a_i{}^r + \sum_{i=1}^{n} b_i{}^r,$$

for $0 \leq r < 1$, $a_1, a_2, ..., a_n \geq 0$ and $b_1, b_2, ..., b_n \geq 0$. ◻

Remark 4. *Inequality* (11) *is an improvement of obtained inequality as in* [16, *Theorem* 2.1].

3. *Applications to special means*

In what follows we give certain generalizations of some notions for a positive valued function of a positive variable.

Definition 3. *A function $M : R_+ \to R_+$,is called a Mean function if it has the following properties:*

(1) Homogeneity : $M(ax, ay) = aM(x,y)$ *, for all* $a > 0$,
(2) Symmetry : $M(x,y) = M(y,x)$,
(3) Reflexivity : $M(x,x) = x$,

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(4) Monotonicity : If $x \leqslant x$ *and* $y \leqslant y'$ *then* $M(x,y) = M(x',y')$,
(5) Internality : min $\{x,y\} \leqslant M(x,y) \leqslant max\{x,y\}$.

We consider some mens for arbitrary positive real numbers a and b (see for instance [7]).

The arithmetic mean

$$A(a,b) = \frac{a+b}{2} \,, \qquad a,b \in \mathbf{R}$$

The geometric mean

$$G(a,b) = \sqrt{ab} \,, \qquad a,b \in \mathbf{R}$$

The harmonic mean

$$H(a,b) = \frac{2ab}{a+b} \,, \qquad a,b \in \mathbf{R}\backslash\{0\}$$

The power mean

$$P(a,b) = \left(\frac{a^r + b^r}{2}\right)^{\frac{1}{r}} \,, \qquad r \geqslant 1$$

.The identric mean

$$I(a,b) = \begin{cases} a & \text{if } a = b \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } a \neq b \end{cases} \,, \quad a,b > 0$$

The logarithmic mean

$$L(a,b) = \begin{cases} a & \text{if } a = b \\ \frac{b-a}{lnb - lna} & \text{if } a \neq b \end{cases} \,,$$

Generalized logarithmic mean

$$L_n(a,b) = \begin{cases} a & \text{if } a = b \\ \left[\frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)}\right]^{\frac{1}{n}} & \text{if } a \neq b \end{cases} \,, n \in \mathbf{Z}\backslash\{-1,0\}; \quad a,b > 0$$

Now, using the results of Section Main Results, some new inequalities are derived for the above means.It is well known that Lp is monotonic nondecreasing over $p \in R$,with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leqslant G \leqslant L \leqslant I \leqslant A$.

Now letting a and b be positive real numbers such that $a < b$.Consider the function $M := M(a,b) : [a, a+\eta(b,a)] \times [a, a+\eta(b,a)] \to R^+$,which is one of the above mentioned means, therefore one can obtain variant inequalities for these means as follows:

Setting $\eta(b,a) = M(b-a)$ in $(6), (8), (10)$, one can obtain the following interesting inequalities involving means:

$$\left| f\left(\frac{2a + M(b,a)}{2}\right) - \frac{\Gamma(\alpha+1)}{2M^\alpha(b,a)} \left[J^\alpha_{a^+} f(a + M(b,a)) + J^\alpha_{a+M(b,a)-} f(a) \right] \right| \leq \frac{M(b,a)}{2^{\alpha+1}(\alpha+1)} \left(\ |f'(a)| + |f'(b)| \ \right). \quad (12)$$

$$\left|f\left(\frac{2a+M(b,a)}{2}\right)-\frac{\Gamma(\alpha+1)}{2M^{\alpha}(b,a)}\left[J_{a+}^{\alpha}f(a+M(b,a))+J_{a+M(b,a)-}^{\alpha}f(a)\right]\right| \leq \frac{M(b,a)}{2^{\alpha+1}(\alpha p+1)^{1/p}}\left[\left(\frac{|f'(b)|^{q}+3|f'(a)|^{q}}{4}\right)^{1/q}+\left(\frac{|f'(a)|^{q}+3|f'(b)|^{q}}{4}\right)^{1/q}\right]. \quad (13)$$

$$\left|f\left(\frac{2a+M(b,a)}{2}\right)-\frac{\Gamma(\alpha+1)}{2M^{\alpha}(b,a)}\left[J_{a+}^{\alpha}f(a+M(b,a))+J_{a+M(b,a)-}^{\alpha}f(a)\right]\right| \leq \frac{M(b,a)}{2^{\alpha+1}(\alpha+1)}\left[\left(\frac{(\alpha+3)|f'(a)|^{q}+(\alpha+1)|f'(b)|^{q}}{2(\alpha+2)}\right)^{\frac{1}{q}}+\left(\frac{(\alpha+3)|f'(b)|^{q}+(\alpha+1)|f'(a)|^{q}}{2(\alpha+2)}\right)^{\frac{1}{q}}\right]. \quad (14)$$

Proof. Letting $M = A, G, H, P, I, L, L_P$ in (12) , (13) , and (14) , we can get the required inequalities, and the details are left to interested reader. □

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The Borel direction and uniqueness of meromorphic function *

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Abstract

The purpose of this paper is to investigate the relationship between Borel directions and shared-set of meromorphic functions and obtain some results of meromorphic functions sharing one finite set in an angular domain containing a Borel line.

Key words: Meromorphic function; Borel direction; Uniqueness.
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1 Introduction and main results

We use $\mathbb{C}$ to denote the open complex plane, $\widehat{\mathbb{C}}(=\mathbb{C}\bigcup\{\infty\})$ to denote the extended complex plane, and $\Omega(\subset\mathbb{C})$ to denote an angular domain. We assume that the readers are familiar with the standard notations and fundamental results of Nevanlinna value distribution theory of meromorphic functions (see [7, 16]).

Let S be a set of distinct elements in $\widehat{\mathbb{C}}$ and $\Omega := \{z:\ \alpha \le \arg z \le \beta\} \subseteq \mathbb{C}$. Define

$$E(S,\Omega,f)=\bigcup_{a\in S}\{z\in\Omega|f_a(z)=0,\ \textit{counting multiplicities}\},$$

$$\overline{E}(S,\Omega,f)=\bigcup_{a\in S}\{z\in\Omega|f_a(z)=0,\ \textit{ignoring multiplicities}\},$$

where $f_a(z)=f(z)-a$ if $a\in\mathbb{C}$ and $f_\infty(z)=1/f(z)$.

Let f and g be two non-constant meromorphic functions in $\mathbb{C}$. We say f and g share the set S CM (counting multiplicities) in Ω if $E(S,\Omega,f)=E(S,\Omega,g)$; we say f and g share the set S IM (ignoring multiplicities) in Ω if $\overline{E}(S,\Omega,f)=\overline{E}(S,\Omega,g)$. If $S=\{a\}$, where $a\in\widehat{\mathbb{C}}$, we say f and g share the value a CM in Ω if $E(S,\Omega,f)=E(S,\Omega,g)$, and we say f and g share the value a IM in Ω if $\overline{E}(S,\Omega,f)=\overline{E}(S,\Omega,g)$. If $\Omega=\mathbb{C}$, we give the simple notation as before, $E(S,f),\overline{E}(S,f)$ and so on(see [17]).

In 1926, R. Nevanlinna (see [11]) proved his famous five-value and four-value theorems. After this very work, many investigations studied the uniqueness of meromorphic functions with shared values in the whole complex plane (see [17]). Around 2003, Zheng [18, 19] was the first to study the uniqueness of meromorphic functions sharing five values and four values in some angular domain under some condition.

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Theorem 1.1 *([18, Theorem 1.1]). Let $f(z)$ and $g(z)$ be both transcendental meromorphic functions and let $f(z)$ be of the finite lower order μ and such that for some $a \in \widehat{\mathbb{C}}$ and an integer $p \geq 0$, $\delta = \delta(a, f^{(p)}) > 0$. For q pair of real numbers α_j, β_j satisfying*

$$-\pi \leq \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \cdots \leq \alpha_q < \beta_q \leq \pi,$$

and

$$\sum_{j=1}^{q} (\alpha_{j+1} - \beta_j) < \frac{4}{\sigma} \arcsin \sqrt{\frac{\delta}{2}}, \tag{1}$$

where $\sigma = \max\{\omega, \mu\}$ and $\omega = \max\{\frac{\pi}{\beta_1 - \alpha_1}, \cdots, \frac{\pi}{\beta_q - \alpha_q}\}$, assume that $f(z)$ and $g(z)$ have five distinct IM shared values in $\Omega = \bigcup_{j=1}^{q} \{z : \alpha_j \leq \arg z \leq \beta_j\}$. If $\omega < \lambda(f)$, then $f(z) \equiv g(z)$.

After Zheng's work, there were many interesting results about the uniqueness with shared values in the angular domain, see [1, 9, 14, 15, 20].

In 2006, Lin, Mori and Tohge [9] dealt with the uniqueness problem on meromorphic functions sharing three finite sets in an angular domain and obtained the following theorems.

Theorem 1.2 *(see [9, Thereom 1]). Let $S_1 = \{\infty\}$, $S_2 = \{\omega | \omega^{n-1}(\omega + a) - b = 0\}$, $S_3 = \{0\}$, where $n(\geq 4)$ is an integer, and a, b are two nonzero constants, such that the algebraic equation $\omega^{n-1}(\omega + a) - b = 0$ has no multiple roots. Assume that f is a meromorphic function of lower order $\mu(f) \in (1/2, \infty)$ in $\widehat{\mathbb{C}}$ and $\delta := \delta(\iota, f) > 0$ for some $\iota \in \widehat{\mathbb{C}} \backslash \{0, -a\}$. Then, for each $\sigma < \infty$ with $\mu(f) \leq \sigma \leq \lambda(f)$, there exists an angular domain $\Omega = \Omega(\alpha, \beta) := \{z : \alpha < \arg z < \beta\}$ with $0 \leq \alpha < \beta \leq 2\pi$ and*

$$\beta - \alpha > \max\left\{\frac{\pi}{\sigma}, 2\pi - \frac{4}{\sigma} \arcsin \sqrt{\frac{\delta}{2}}\right\} \tag{2}$$

such that if the conditions $E(S_3, f) = E(S_3, g)$ and $E(S_j, \Omega, f) = E(S_j, \Omega, g)$ $(j = 1, 2)$ hold for a meromorphic function g of finite order or, more generally, with the growth satisfying either $\log T(r, g) = O(\log T(r, f))$ or

$$\lim_{r \to \infty} \frac{\log \log T(r, g)}{\min\{\log r, \log T(r, f)\}} = 0, \quad r \notin E_1, \tag{3}$$

where E_1 is a set of finite linear measure, then $f \equiv g$.

It is well known that Borel direction is an important singular direction for meromorphic function in the fields of complex analysis, and Borel directions played an important role in the topic of angular distribution(see [8, 12, 13]). Valiron [16] proved that every meromorphic function of finite order $\rho > 0$ has at least one Borel direction of order ρ, where the order of meromorphic function f is defined by $\rho = \rho(f) = \limsup\limits_{r \to \infty} \frac{\log T(r,f)}{\log r}$.

In 2012, Long and Wu[10] was the first to investigate the problem concerning Borel direction and shared value of meromorphic functions and obtained the following theorems.

Theorem 1.3 (see [10, Theorem 1.1]) *Let f be meromorphic function of infinite order $\rho(r)$, $g \in M(\rho(r))$, $\arg z = \theta (0 \leq \theta < 2\pi)$ be one Borel direction of $\rho(r)$ order of meromorphic function f, $a_i \in \widehat{\mathbb{C}} (i = 1, 2, 3, 4, 5)$ be five distinct complex numbers. If f and g share $a_i (i = 1, 2, 3, 4, 5)$ IM in the angular domain $\Omega(\theta - \varepsilon, \theta + \varepsilon)$ for any $\varepsilon (0 < \varepsilon < \pi)$, then $f \equiv g$.*

Definition 1.1 *[2]. Let f be a meromorphic function of infinite order, $\rho(r)$ is a real function satisfying the following conditions:*

(i) $\rho(r)$ is continuous, non-decreasing for $r \geq r_0$ and $\rho(r) \to \infty$ as $r \to \infty$;

(ii)

$$\lim_{r \to \infty} \frac{\log U(R)}{\log U(r)} = 1, \quad R = r + \frac{r}{\log U(r)},$$

where $U(r) = r^{\rho(r)}$ $(r \geq r_0)$*;*

(iii)

$$\limsup_{r\to\infty} \frac{\log T(r,f)}{\log U(r)} = 1.$$

Then $\rho(r)$ *is called infinite order of meromorphic function* f*. This definition is given by Xiong Qinglai[2].*

Let $\rho(r)$ be infinite order of meromorphic function f, we will denote by $M(\rho(r))$ the set of meromorphic function g satisfying $0 < \limsup_{r\to\infty} \frac{\log T(r,g)}{\rho(r)\log r} \leq 1$, that is,

$$M(\rho(r)) := \left\{ g : 0 < \limsup_{r\to\infty} \frac{\log T(r,g)}{\rho(r)\log r} \leq 1 \right\}.$$

Let $\alpha < \beta, \beta - \alpha < 2\pi, r > 0$, and $\Omega(\alpha, \beta, r) := \{z : \alpha \leq \arg z \leq \beta, 0 < |z| \leq r\}$. The definition of Borel direction of meromorphic functions f of infinite order $\rho(r)$ is defined as follows.

Definition 1.2 *[2]. Let* f *be meromorphic functions of infinite order* $\rho(r)$*, if for any* $\varepsilon (0 < \varepsilon < \pi)$*, the equality*

$$\limsup_{r\to\infty} \frac{\log n(\Omega(\theta-\varepsilon, \theta+\varepsilon, r), f = a)}{\rho(r)\log r} = 1,$$

holds for any complex number $a \in \widehat{\mathbb{C}}$*, at most except two exception, where* $n(\Omega(\theta-\varepsilon, \theta+\varepsilon, r), f = a)$ *is the counting function of zero of the function* $f - a$ *in the angular domain* $\Omega(\theta-\varepsilon, \theta+\varepsilon)$*, counting multiplicities. Then the ray* $\arg z = \theta$ *is called a Borel direction of* $\rho(r)$ *order of meromorphic function* f*.*

Remark 1.1 *Chuang [2] proved that every meromorphic function* f *with infinite order* $\rho(r)$ *has as least one Borel direction of infinite order* $\rho(r)$*.*

In this paper, we will investigate the uniqueness problem of meromorphic functions sharing one finite set in an angular domain containing a Borel line. We will mainly consider the following finite set $S = \{w \in \mathbb{A} : P_1(w) = 0\}$, where

$$P_1(w) = \frac{(n-1)(n-2)}{2} w^n - n(n-2)w^{n-1} + \frac{n(n-1)}{2} w^{n-2} - c,$$

c is a complex number satisfying $c \neq 0, 1$.

Theorem 1.4 *Let* f *be meromorphic function of infinite order* $\rho(r)$*,* $g \in M(\rho(r))$*,* $\arg z = \theta (0 \leq \theta < 2\pi)$ *be one Borel direction of* $\rho(r)$ *order of meromorphic function* f*, if* $E(S, \Omega(\theta-\varepsilon, \theta+\varepsilon), f) = E(S, \Omega(\theta-\varepsilon, \theta+\varepsilon), g)$ *and* n *is an integer* ≥ 11*, then* $f \equiv g$*.*

A set S is called a unique range set for meromorphic functions on the Borel direction $\arg z = \theta$, if for any two nonconstant meromorphic functions f and g the condition $E(S, \Omega(\theta-\varepsilon, \theta+\varepsilon), f) = E(S, \Omega(\theta-\varepsilon, \theta+\varepsilon), g)$ implies $f \equiv g$. We denote by $\sharp S$ the cardinality of a set S. Thus, from Theorem 1.4, we can get the following corollary

Corollary 1.1 *There exists one finite set* S *with* $\sharp S = 11$*, such that any two meromorphic functions* f *and* g *on the Borel direction, which* $f(z)$ *is meromorphic function of infinite order* $\rho(r)$*,* $g \in M(\rho(r))$*,* $\arg z = \theta (0 \leq \theta < 2\pi)$ *be one Borel direction of* $\rho(r)$ *order of meromorphic function* f*, and* $E(S, \Omega(\theta-\varepsilon, \theta+\varepsilon), f) = E(S, \Omega(\theta-\varepsilon, \theta+\varepsilon), g)$*.*

2 Some Lemmas

We first introduce the basic notations and definitions of meromorphic functions in an angular domain as follows(see [7, 18, 19]).

Let f be a meromorphic function on the angular domain $\Omega(\alpha,\beta)=\{z:\alpha\leq\arg z\leq\beta\}$ and $0<\beta-\alpha\leq 2\pi$. Define

$$A_{\alpha,\beta}(r,f)=\frac{\omega}{\pi}\int_1^r(\frac{1}{t^\omega}-\frac{t^\omega}{r^{2\omega}})\{\log^+|f(te^{i\alpha})|+\log^+|f(te^{i\beta})|\}\frac{dt}{t},$$

$$B_{\alpha,\beta}(r,f)=\frac{2\omega}{\pi r^\omega}\int_\alpha^\beta\log^+|f(re^{i\theta})|\sin\omega(\theta-\alpha)d\theta,$$

$$C_{\alpha,\beta}(r,f)=2\sum_{1<|b_\mu|<r}(\frac{1}{|b_\mu|^\omega}-\frac{|b_\mu|^\omega}{r^{2\omega}})\sin\omega(\theta_\mu-\alpha),$$

$$S_{\alpha,\beta}(r,f)=A_{\alpha,\beta}(r,f)+B_{\alpha,\beta}(r,f)+C_{\alpha,\beta}(r,f),$$

where $\omega=\frac{\pi}{\beta-\alpha}$ and $b_\mu=|b_\mu|e^{i\theta_\mu}(\mu=1,2,\cdots)$ are the poles of f on $\Omega(\alpha,\beta)$ counted according to their multiplicities. $S_{\alpha,\beta}(r,f)$ is called the Nevanlinna's angular characteristic, and $C_{\alpha,\beta}(r,f)$ is called the angular counting function of the poles of f on $\Omega(\alpha,\beta)$, and $\overline{C}_{\alpha,\beta}(r,f)$ is the reduced function of $C_{\alpha,\beta}(r,f)$. Similarly, when $a\neq\infty$, we will use the notations $A_{\alpha,\beta}(r,\frac{1}{f-a})$, $B_{\alpha,\beta}(r,\frac{1}{f-a})$, $C_{\alpha,\beta}(r,\frac{1}{f-a})$, $S_{\alpha,\beta}(r,\frac{1}{f-a})$ and so on.

To prove our result, we require the following Lemmas.

Lemma 2.1 *(see [6]). Let f be a nonconstant meromorphic function on $\Omega(\alpha,\beta)$. Then for arbitrary complex number a, we have*

$$S_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)=S_{\alpha,\beta}(r,f)+\varepsilon(r,a),$$

where $\varepsilon(r,a)=O(1)$ as $r\to\infty$.

Lemma 2.2 *(see [5, 19]). Suppose that f is a non-constant meromorphic function in one angular domain $\Omega(\alpha,\beta)$ with $0<\beta-\alpha\leq 2\pi$, then for arbitrary q distinct $a_j\in\widehat{\mathbb{C}}(1\leq j\leq q)$, we have*

$$(q-2)S_{\alpha,\beta}(r,f)\leq\sum_{j=1}^q\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right)-C^0_{\alpha,\beta}(r,\frac{1}{f'})+R_{\alpha,\beta}(r,f),$$

where $C^0_{\alpha,\beta}(r,\frac{1}{f'})$ is the counting function of the zeros of f' in Ω where f does not take anyone of the values $a_j(j=1,2,\ldots,q)$, and the term $\overline{C}_{\alpha,\beta}(r,\frac{1}{f-a_j})$ will be replaced by $\overline{C}_{\alpha,\beta}(r,f)$ when some $a_j=\infty$ and

$$\begin{aligned}R_{\alpha,\beta}(r,f)=&A_{\alpha,\beta}\left(r,\frac{f'}{f}\right)+B_{\alpha,\beta}\left(r,\frac{f'}{f}\right)\\&+\sum_{j=1}^q\left\{A_{\alpha,\beta}\left(r,\frac{f'}{f-a_j}\right)+B_{\alpha,\beta}\left(r,\frac{f'}{f-a_j}\right)\right\}+O(1).\end{aligned}\tag{4}$$

Lemma 2.3 *(see [6, P138].) Let f be a nonconstant meromorphic function in the whole complex plane $\mathbb{C}$. Given one angular domain on $\Omega(\alpha,\beta)$. Then for any $1\leq r<R$, we have*

$$A_{\alpha,\beta}\left(r,\frac{f'}{f}\right)\leq K\left\{\left(\frac{R}{r}\right)^\omega\int_1^R\frac{\log^+T(r,f)}{t^{1+\omega}}dt+\log^+\frac{r}{R-r}+\log\frac{R}{r}+1\right\},$$

and

$$B_{\alpha,\beta}\left(r,\frac{f'}{f}\right)\leq\frac{4\omega}{r^{\omega}}m\left(r,\frac{f'}{f}\right),$$

where $\omega=\frac{\pi}{\beta-\alpha}$ *and* K *is a positive constant not depending on* r *and* R.

Remark 2.1 *Nevanlinna conjectured that*

$$A_{\alpha,\beta}\left(r,\frac{f'}{f}\right)+B_{\alpha,\beta}\left(r,\frac{f'}{f}\right)=o\left(S_{\alpha,\beta}\left(r,f\right)\right) \tag{5}$$

when r *tends to* $+\infty$ *outside an exceptional set of finite linear measure, and he proved that* $A_{\alpha,\beta}\left(r,\frac{f'}{f}\right)+B_{\alpha,\beta}\left(r,\frac{f'}{f}\right)=O(1)$ *when the function* f *is meromorphic in* $\mathbb{C}$ *and has finite order. In 1974, Gol'dberg [5] constructed a counter-example to show that (5) is not valid.*

Remark 2.2 *From Lemma 2.2 and Lemma 2.3, we can get the following conclusion:*

$$R_{\alpha,\beta}\left(r,f\right)=\begin{cases}O(1), & f\ is\ of\ finite\ order,\\ O(\log U(r)),\ r\notin E, & f\ is\ of\ infinite\ order,\end{cases}$$

where $R_{\alpha,\beta}\left(r,f\right)$ *is stated as in (1),* $U(r)=r^{\rho(r)}$, $\rho(r)$ *is infinite order of meromorphic function* f, E *is a set of finite linear measure.*

Lemma 2.4 *(see [3]). Let* f *be meromorphic function of infinite order* $\rho(r)$*. Then the ray* $\arg z=\theta$ *is one Borel direction of* $\rho(r)$ *order of meromorphic function* f *if and only if* f *satisfies the equality*

$$\limsup_{r\to\infty}\frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)}{\rho(r)\log r}=1, \tag{6}$$

for any $\varepsilon(0<\varepsilon<\frac{\pi}{2})$.

Similar to discuss as in [1, Lemma 1] and [17], we can get the lemma below easily.

Lemma 2.5 *Suppose that* f *is a non-constant meromorphic function with infinite order* $\rho(r)$*, the ray* $\arg z=\theta$ *is one Borel direction of* $\rho(r)$ *order of meromorphic function* f*. Let* $P(f)=a_0f^p+a_1f^{p-1}+\cdots+a_p(a_0\neq 0)$ *be a polynomial of* f *with degree* p*, where the coefficients* $a_j(j=0,1,\ldots,p)$ *are constants, and let* $b_j(j=1,2,\ldots,q)$ *be* $q(q\geq p+1)$ *distinct finite complex numbers. Then for any* $\varepsilon(0<\varepsilon<\pi/2)$,

$$D_{\theta-\varepsilon,\theta+\varepsilon}\left(r,\frac{P(f)\cdot f'}{(f-b_1)(f-b_2)\cdots(f-b_q)}\right)=R_{\theta-\varepsilon,\theta+\varepsilon}(r,f),$$

where $D_{\theta-\varepsilon,\theta+\varepsilon}(r,\bullet)=A_{\theta-\varepsilon,\theta+\varepsilon}(r,\bullet)+B_{\theta-\varepsilon,\theta+\varepsilon}(r,\bullet)$.

Lemma 2.6 *Suppose that* f *is a non-constant meromorphic function with infinite order* $\rho(r)$*, the ray* $\arg z=\theta$ *is one Borel direction of* $\rho(r)$ *order of meromorphic function* f*. Then for any* $\varepsilon(0<\varepsilon<\pi/2)$*, we have*

$$C_{\theta-\varepsilon,\theta+\varepsilon}\left(r,\frac{1}{f'}\right)\leq C_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{f})+\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,f)+R_{\theta-\varepsilon,\theta+\varepsilon}(r,f)+O(1).$$

Proof: By Lemma 2.1, Lemma 2.3 and Lemma 2.4, we have

$$D_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{f})\leq D_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{f'})+D_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{f'}{f})=D_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{f'})+R_{\theta-\varepsilon,\theta+\varepsilon}(r,f),$$

then from the definition of $S(r,f)$, we have

$$S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)-C_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{f})\leq S_{\theta-\varepsilon,\theta+\varepsilon}(r,f')-C_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{f'})+R_{\theta-\varepsilon,\theta+\varepsilon}(r,f)+O(1),$$

i.e.,

$$\begin{aligned}C_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{f'})\leq & S_{\theta-\varepsilon,\theta+\varepsilon}(r,f')-S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\\ &+C_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{f})+R_{\theta-\varepsilon,\theta+\varepsilon}(r,f)+O(1).\end{aligned} \tag{7}$$

Since

$$\begin{aligned}S_{\theta-\varepsilon,\theta+\varepsilon}(r,f') &= D_{\theta-\varepsilon,\theta+\varepsilon}(r,f')+D_{\theta-\varepsilon,\theta+\varepsilon}(r,f') \\ &\leq D_{\theta-\varepsilon,\theta+\varepsilon}(r,f)+D_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{f'}{f})+C_{\theta-\varepsilon,\theta+\varepsilon}(r,f)+\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\\ &\leq S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)+\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,f)+R_{\theta-\varepsilon,\theta+\varepsilon}(r,f)+O(1),\end{aligned} \tag{8}$$

then from (7) and (8), we can get the conclusion of this lemma. □

Lemma 2.7 *Let F be transcendental entire function of infinite order $\rho(r)$, $G\in M(\rho(r))$, $\arg z=\theta(0\leq\theta<2\pi)$ be one Borel direction of $\rho(r)$ order of function f and $\Omega:=\Omega(\theta-\varepsilon,\theta+\varepsilon)$ for any $\varepsilon(0<\varepsilon<\pi)$. If F and G satisfy $E(0,\Omega,F)=E(0,\Omega,G)$ and $c_1,c_2,\ldots,c_q$ are $q(\geq 2)$ distinct non-zero complex numbers. If*

$$\limsup_{r\to\infty,r\in I}\frac{3\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,F)+\sum_{j=1}^{q}\overline{C}^{(2)}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F-c_j})+\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F'})}{S_{\theta-\varepsilon,\theta+\varepsilon}(r,F)}<q, \tag{9}$$

and

$$\limsup_{r\to\infty,r\in I}\frac{3\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,G)+\sum_{j=1}^{q}\overline{C}^{(2)}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{G-c_j})+\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{G'})}{S_{\theta-\varepsilon,\theta+\varepsilon}(r,G)}<q, \tag{10}$$

where $\overline{C}^{(2)}_{\theta-\varepsilon,\theta+\varepsilon}(r,\cdot)=\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\cdot)+\overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r,\cdot)$, $\overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r,\cdot)=\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\cdot)-C^{1)}(r,\cdot)$, $C^{1)}_{\theta-\varepsilon,\theta+\varepsilon}(r,\cdot)$ is the counting function which only counts simple zeros of the function $\cdot$ in Ω and I is some set of r of infinite linear measure, then

$$F=\frac{aG+b}{cG+d},$$

where $a,b,c,d\in\mathbb{C}$ are constants with $ad-bc\neq 0$.

Proof: Since meromorphic function F is of infinite order $\rho(r)$ and $\arg z=\theta(0\leq\theta<2\pi)$ is one Borel direction of $\rho(r)$ order of F, then we can get by Lemma 2.4 for any $\varepsilon(0<\varepsilon<\pi)$

$$\limsup_{r\to\infty}\frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,F)}{\rho(r)\log r}=1. \tag{11}$$

And since $G\in M(\rho(r))$, we have

$$\limsup_{r\to\infty}\frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,G)}{\rho(r)\log r}\leq 1. \tag{12}$$

Set $R(r)=O(\rho(r)\log r))$ as $r\to\infty,(r\notin E)$, where E is a set of finite linear measure, then from (11) and (12), we have $R(r)=o(S_{\theta-\varepsilon,\theta+\varepsilon}(r,F))=o(S_{\theta-\varepsilon,\theta+\varepsilon}(r,G))$ as $r\to\infty,r\notin E$.

Set

$$H \equiv \frac{F''}{F'} - 2\frac{F'}{F} - \left(\frac{G''}{G'} - 2\frac{G'}{G}\right). \tag{13}$$

Suppose that $H \not\equiv 0$, from Lemma 2.1 and Lemma 2.2, we have

$$D_{\theta-\varepsilon,\theta+\varepsilon}(r, H) = R(r). \tag{14}$$

Since $E(0,\Omega,F) = E(0,\Omega,G)$, and by an elementary calculation, we can conclude that if z_0 is a common simple zero of F and G in Ω, then $H(z_0) = 0$. Thus, from (13) we have

$$C^{1)}_{\theta-\varepsilon,\theta+\varepsilon}(r) \le C_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{H}) \le S_{\theta-\varepsilon,\theta+\varepsilon}(r, H) + O(1) \le C_{\theta-\varepsilon,\theta+\varepsilon}(r, H) + R(r), \tag{15}$$

where $C^{1)}_{\theta-\varepsilon,\theta+\varepsilon}(r) = C^{1)}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F}) = C^{1)}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{G})$. The poles of H in $\Omega(\theta-\varepsilon, \theta+\varepsilon)$ can only occur at zeros of F' and G' in $\Omega(\theta-\varepsilon, \theta+\varepsilon)$ or poles of F and G in $\Omega(\theta-\varepsilon, \theta+\varepsilon)$. Moreover, H only has simple zeros in $\Omega(\theta-\varepsilon, \theta+\varepsilon)$. Hence, from (13), we have

$$\begin{aligned} C^{1)}_{\theta-\varepsilon,\theta+\varepsilon}(r) \le& \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, F) + \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, G) + \overline{C}^{0}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F'}) + \overline{C}^{0}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{G'}) \\ &+ \sum_{j=1}^{q} \overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F-c_j}) + \sum_{j=1}^{q} \overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{G-c_j}) + R(r), \end{aligned} \tag{16}$$

where $\overline{C}^{0}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F'})$ is the reduced counting function for the zeros of F' in Ω where F does not take one of the values $0, c_1, c_2, \ldots, c_q$.

Since

$$\begin{aligned} &\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F}) + \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{G}) \\ &\qquad = 2C^{1)}_{\theta-\varepsilon,\theta+\varepsilon}(r) + \overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F}) + \overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{G}), \end{aligned} \tag{17}$$

then from (15)-(17), we have

$$\begin{aligned} &\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F}) + \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{G}) \\ \le& 2\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, F) + 2\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, G) + 2\overline{C}^{0}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F'}) + \\ &+ 2\overline{C}^{0}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{G'}) + \overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F}) + \overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{G}) + \\ &+ 2\sum_{j=1}^{q} \overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F-c_j}) + 2\sum_{j=1}^{q} \overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{G-c_j}) + R(r). \end{aligned} \tag{18}$$

By Lemma 2.2, we have

$$\begin{aligned} qS_{\theta-\varepsilon,\theta+\varepsilon}(r, F) \le& \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, F) + \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F}) + \sum_{j=1}^{q} \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F-c_j}) \\ &- C^{0}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F'}) + R(r),\ r \notin E, \end{aligned} \tag{19}$$

$$\begin{aligned} qS_{\theta-\varepsilon,\theta+\varepsilon}(r, G) \le& \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, G) + \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{G}) + \sum_{j=1}^{q} \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{G-c_j}) \\ &- C^{0}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{G'}) + R(r),\ r \notin E, \end{aligned} \tag{20}$$

where E is a set of r of finite linear measure, and it needs not be the same at each occurrence. From (18)-(20), it follows for $r \notin E$,

$$\begin{aligned}&q\{S_{\theta-\varepsilon,\theta+\varepsilon}(r,F)+S_{\theta-\varepsilon,\theta+\varepsilon}(r,G)\}\\ \leq&3\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,F)+3\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,G)+\sum_{j=1}^{q}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F-c_j})+\sum_{j=1}^{q}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{G-c_j})\\ &+2\sum_{j=1}^{q}\overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F-c_j})+2\sum_{j=1}^{q}\overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{G-c_j})+\overline{C}^{0}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F'})\\ &+\overline{C}^{0}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{G'})+\overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F})+\overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{G})+R(r). \qquad (21)\end{aligned}$$

Since

$$\sum_{j=1}^{q}\overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F-c_j})+\overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F})+\overline{C}^{0}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F'})=\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F'}), \qquad (22)$$

and

$$\sum_{j=1}^{q}\overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{G-c_j})+\overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{G})+\overline{C}^{0}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{G'})=\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{G'}), \qquad (23)$$

and from (21)-(23), we have for $r \notin E$,

$$\begin{aligned}&q\{S_{\theta-\varepsilon,\theta+\varepsilon}(r,F)+S_{\theta-\varepsilon,\theta+\varepsilon}(r,G)\} \qquad (24)\\ \leq&3\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,F)+3\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,G)+\sum_{j=1}^{q}\overline{C}^{(2)}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F-c_j})+\\ &\sum_{j=1}^{q}\overline{C}^{(2)}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{G-c_j})+\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{F'})+\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{G'})+R(r).\end{aligned}$$

From (9), (10) and (24), and since Let F be transcendental entire function of infinite order $\rho(r)$, $G \in M(\rho(r))$, $\arg z = \theta (0 \leq \theta < 2\pi)$ be one Borel direction of $\rho(r)$ order of function f, we have

$$S_{\theta-\varepsilon,\theta+\varepsilon}(r,F)+S_{\theta-\varepsilon,\theta+\varepsilon}(r,G)\leq o\{S_{\theta-\varepsilon,\theta+\varepsilon}(r,F)+S_{\theta-\varepsilon,\theta+\varepsilon}(r,G)\},\quad r\notin E,\ r\in I.$$

Thus, we can get a contradiction. Therefore, $H(z) \equiv 0$, that is,

$$\frac{F''}{F'}-2\frac{F'}{F}\equiv\frac{G''}{G'}-2\frac{G'}{G}.$$

For above equality, by integration, it follows that

$$F\equiv\frac{aG+b}{cG+d},$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. □

The following result can be derived from the proof of Frank-Reinders' theorem in [4].

Lemma 2.8 *Let* $n \geq 6$ *and*

$$P(w)=\frac{(n-1)(n-2)}{2}w^{n}-n(n-2)w^{n-1}+\frac{n(n-1)}{2}w^{n-2}.$$

Then $P(w)$ *is a unique polynomial for transcendental meromorphic functions, i.e., for any two transcendental meromorphic functions* f *and* g, $P(f) \equiv P(g)$ *implies* $f \equiv g$.

3 The Proof of Theorem 1.4

Proof: Since f is a meromorphic function of infinite order $\rho(r)$ and $\arg z = \theta(0 \le \theta < 2\pi)$ be one Borel direction of $\rho(r)$ order of meromorphic function f, similar to (11) and (12), for any $\varepsilon(0 < \varepsilon < \pi)$, we have

$$\limsup_{r\to\infty} \frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)}{\rho(r)\log r} = 1, \tag{25}$$

and

$$\limsup_{r\to\infty} \frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,g)}{\rho(r)\log r} \le 1. \tag{26}$$

Set $R(r) = O(\rho(r)\log r))$ as $r \to \infty, r \notin E$, where E is a set of finite linear measure, then we have $R(r) = o(S_{\theta-\varepsilon,\theta+\varepsilon}(r,f))$ as $r \to \infty, r \notin E$ from (25) and (26).

From the definition of $P_1(w)$, we have $P_1(1) = 1 - c := c_1 \ne 0$, $P_1(0) = -c := c_2 \ne 0$ and

$$P_1'(w) = \frac{n(n-1)(n-2)}{2}(w-1)^2 w^{n-3}, \tag{27}$$

$$P_1(w) - c_1 = (w-1)^3 Q_1(w), \quad Q_1(1) \ne 0, \tag{28}$$

$$P_1(w) - c_2 = w^{n-2} Q_2(w), \quad Q_2(0) \ne 0, \tag{29}$$

where Q_1, Q_2 are polynomials of degree $n-3$ and 2, respectively. We also see that $Q_i(i = 1, 2)$ and P_1 have only simple zeros.

Let F and G be defined as $F = P_1(f)$ and $G = P_1(g)$. Since $E(S, \Omega, f) = E(S, \Omega, g)$, we have $E(0, \Omega, F) = E(0, \Omega, G)$. From (28) and (29), we have

$$\begin{aligned}
\overline{C}^{(2)}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F-c_1}) &= \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F-c_1}) + \overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F-c_1}) \\
&\le 2\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{f-1}) + \sum_{i=1}^{n-3} C_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{f-a_i}) \\
&\le (n-1)S_{\theta-\varepsilon,\theta+\varepsilon}(r,f) + R(r),
\end{aligned} \tag{30}$$

and

$$\begin{aligned}
\overline{C}^{(2)}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F-c_2}) &= \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F-c_2}) + \overline{C}^{(2}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F-c_2}) \\
&\le 2\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{f}) + \sum_{j=1}^{2} C_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{f-b_j}) \\
&\le 4S_{\theta-\varepsilon,\theta+\varepsilon}(r,f) + R(r),
\end{aligned} \tag{31}$$

where $a_i(i = 1, \ldots, n-3)$ and $b_j(j = 1, 2)$ are the zeros of $Q_1(w)$ and $Q_2(w)$ in Ω, respectively.

From (27), we have

$$\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F'}) \le \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{f}) + \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{f-1}) + \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{f'}). \tag{32}$$

From [6, Theorem 6.3], we have $S_{\theta-\varepsilon,\theta+\varepsilon}(r,F) = nS_{\theta-\varepsilon,\theta+\varepsilon}(r,f) + R(r)$. Thus, combining (30), (31) and (32), by Lemma 2.6, Lemma 2.7 and $n \ge 11$, we have

$$\begin{aligned}
&\limsup_{r\to\infty, r\notin E} \frac{3\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,F) + \sum_{j=1}^{2} \overline{C}^{(2)}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F-c_j}) + \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{F'})}{S_{\theta-\varepsilon,\theta+\varepsilon}(r,F)} \\
\le &\limsup_{r\to\infty, r\notin E} \frac{4\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,f) + (n+6)S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)}{nS_{\theta-\varepsilon,\theta+\varepsilon}(r,f)} < 2.
\end{aligned} \tag{33}$$

Similarly, we have

$$\begin{aligned}&\limsup_{r\to\infty, r\notin E}\frac{3\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,G)+\sum_{j=1}^{2}\overline{C}^{(2)}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{G-c_j})+\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{G'})}{S_{\theta-\varepsilon,\theta+\varepsilon}(r,G)}\\ \leq&\limsup_{r\to\infty, r\notin E}\frac{4\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,g)+(n+6)S_{\theta-\varepsilon,\theta+\varepsilon}(r,g)}{nS_{\theta-\varepsilon,\theta+\varepsilon}(r,g)}<2.\end{aligned} \tag{34}$$

Thus, we have by Lemma 2.7

$$\frac{F''}{F'}-2\frac{F'}{F}\equiv\frac{G''}{G'}-2\frac{G'}{G}.$$

For above equality, by integration, it follows that

$$F\equiv\frac{aG+b}{cG+d},$$

where $a,b,c,d\in\mathbb{C}$ and $ad-bc\neq 0$. Since $E(S,\Omega,f)$ is nonempty, and $E(S,\Omega,f)=E(S,\Omega,g)$, we have $b=0, a\neq 0$. Hence

$$F\equiv\frac{aG}{cG+d}\equiv\frac{G}{AG+B}, \tag{35}$$

where $A=\frac{c}{a}$ and $B=\frac{d}{a}\neq 0$.

Two cases will be considered as follows:

Case 1: $A\neq 0$. From the definition of $P_1(w)$ and (35), we know that every zero of $P_1(g)+\frac{B}{A}$ in Ω has a multiplicity of at least n. Here, we wiil consider the three following subcases.

Subcase 1.1: $\frac{B}{A}=-c_1$. From (28), we have

$$P_1(g)+\frac{B}{A}=(g-1)^3(g-a_1)(g-a_2)\cdots(g-a_{n-3}),$$

where $a_i\neq 0,1$ are distinct values. It follows that

$$\Theta_{\theta-\varepsilon,\theta+\varepsilon}(a_i,f)=1-\limsup_{r\to\infty}\frac{\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a)}{S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)}\geq 1-\limsup_{r\to\infty}\frac{\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a)}{C_{\theta-\varepsilon,\theta+\varepsilon}(r,f)}\geq\frac{1}{2}.$$

We can see that it has $n-2$ values satisfying the above inequality. Thus, from Lemma 2.2 and $n\geq 11$, we can get a contradiction.

Subcase 1.2: $\frac{B}{A}=-c_2$. From (28), we have

$$P_1(g)+\frac{B}{A}=g^{n-2}(g-b_1)(g-b_2),$$

where $b_1\neq b_2, b_i\neq 0,1 (i=1,2)$. It follows that every zero of g in Ω has a multiplicity at least 2 and every zero of $g-b_i (i=1,2)$ in Ω has a multiplicity at least n. Then, by Lemma 2.2, we have

$$\begin{aligned}S_{\theta-\varepsilon,\theta+\varepsilon}(r,g)&\leq\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{g})+\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{g-b_1})+\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{g-b_2})+R(r)\\&\leq\frac{1}{2}C_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{g})+\frac{1}{n}C_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{g-b_1})+\frac{1}{n}C_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{1}{g-b_1})+R(r)\\&\leq(\frac{1}{2}+\frac{2}{n})S_{\theta-\varepsilon,\theta+\varepsilon}(r,g)+R(r).\end{aligned}$$

Since $g\in M(\rho(r))$ and $n\geq 11$, we can get a contradiction.

Subcase 1.3: $\frac{B}{A}\neq -c_1,-c_2$. Similar to discuss as in Subcase 1.1 or Subcase 1.2, we can get a contradiction.

Case 2: $A = 0$. If $B \neq 1$, from (35) we have $F = \frac{G}{B}$ that is,

$$P_1(f) = \frac{1}{B}P_1(g). \tag{36}$$

From (29) and (36), we have

$$P_1(f) - \frac{c_2}{B} = \frac{1}{B}(P_1(g) - c_2) = \frac{1}{B}g^{n-2}(g - b_1)(g - b_2). \tag{37}$$

Since $\frac{c_2}{B} \neq c_2$, from (27), it follows that $P_1(f) - \frac{c_2}{B}$ at least $n - 2$ distinct zeros $e_1, e_2, \dots, e_{n-2}$. Then, we have from Lemma 2.2

$$\begin{aligned}(n-4)S_{\theta-\varepsilon,\theta+\varepsilon}(r, f) &\leq \sum_{i=1}^{n-2} \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{f - e_i}) + R(r) \qquad (38)\\ &\leq \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{g}) + \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{g - b_1}) + \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, \frac{1}{g - b_2}) + R(r)\\ &\leq 3S_{\theta-\varepsilon,\theta+\varepsilon}(r, g) + R(r).\end{aligned}$$

Since f is a meromorphic function of infinite order $\rho(r)$, $\arg z = \theta(0 \leq \theta < 2\pi)$ is one Borel direction of $\rho(r)$ order of meromorphic function f, we have from Lemma 2.4 $S(r, f) = O(\rho(r) \log r)$. Then, applying Lemma 2.7 to (36), and from (38) and $n \geq 11$, we can get a contradiction.

Thus, we have $A = 0$ and $B = 1$, that is, $P_1(f) = P_1(g)$. Note the form of $P_1(w)$, we can get that $P(f) = P(g)$. Then, by Lemma 2.8, we get $f \equiv g$.

Therefore, the proof of Theorem 1.4 is completed.

□

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Pseudo-valuations on BCH-algebras with respect to subalgebras and (closed) ideals

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Abstract. The notion of pseudo-valuations (valuations) on a BCH-algebra is introduced by using the Buşneag's model ([1, 2, 3]), and a pseudo-metric is induced by a pseudo-valuation on BCH-algebras. Conditions for a real-valued function to be an I-pseudo-valuation are provided. The fact that the binary operation in BCH-algebras is uniformly continuous is provided based on the notion of (pseudo) valuation.

1. Introduction

Buşneag [2] defined pseudo-valuation on a Hilbert algebra, and proved that every pseudo-valuation induces a pseudo metric on a Hilbert algebra. Also, Buşneag [3] provided several theorems on extensions of pseudo-valuations. Buşneag [1] introduced the notions of pseudo-valuations (valuations) on residuated lattices, and proved some theorems of extension for these (using the model of Hilbert algebras ([3])). As a generalization of BCK/BCI-algebras, Hu and Li [5, 6] introduced the notion of BCH-algebras, and Chaudhry [4] investigated several properties of BCH-algebras.

In this paper, using the Buşneag's model, we introduce the notion of pseudo-valuations (valuations) on BCH-algebras, and we induce a pseudo-metric by using a pseudo-valuation on BCH-algebras. We provide conditions for a real-valued function on a BCH-algebra X to be an I-pseudo-valuation on X. Based on the notion of (pseudo) valuation, we show that the binary operation $*$ in BCH-algebras is uniformly continuous.

2. Preliminaries

An algebra $(X;*,0)$ of type $(2,0)$ is called a *BCH-algebra* if it satisfies the following conditions:

(I) $(\forall x \in X)\ (x * x = 0)$,

(II) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0\ \Rightarrow\ x = y)$,

(III) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$.

Any BCH-algebra X satisfies the following conditions:

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TABLE 1. $*$-operation

$*$	0	a	b	c	d
0	0	0	0	0	d
a	a	0	0	a	d
b	b	b	0	0	d
c	c	c	c	0	d
d	d	d	d	d	0

(a1) $(\forall x, y \in X)$ $((x * (x * y)) * y = 0)$,
(a2) $(\forall x \in X)$ $(x * 0 = 0 \ \Rightarrow x = 0)$,
(a3) $(\forall x, y \in X)$ $(0 * (x * y) = (0 * x) * (0 * y))$,
(a4) $(\forall x \in X)$ $(x * 0 = x)$.

A *BCH_0-algebra* is a BCH-algebra X which satisfies $0 * x = 0$ for all $x \in X$.

We can define a relation $\leq$ on a BCH-algebra X by

$$(\forall x, y \in X) \ (x \leq y \ \Leftrightarrow \ x * y = 0)$$

This relation is reflexive and anti-symmetric but not transitive in general. A non-empty subset S of a BCH-algebra X is called *a subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

A subset I of a BCH-algebra X is called an *ideal* if it satisfies:

(I1) $0 \in I$,
(I2) $(\forall x \in X)$ $(\forall y \in I)$ $(x * y \in I \ \Rightarrow \ x \in I)$.

Note that an ideal of a BCH-algebra may not be a subalgebra. An ideal I of a BCH-algebra X is said to be *closed* if $0 * x \in I$ for all $x \in I$.

3. Pseudo-valuations on BCH-algebras

In what follows let X denote a BCH-algebra unless otherwise specified.

Definition 3.1. A real-valued function ϑ on X is called a *pseudo-valuation* on X with respect to a subalgebra (briefly, *S-pseudo-valuation* on X) if it satisfies the following condition:

$$(\forall x, y \in X) \ (\vartheta(x * y) \leq \vartheta(x) + \vartheta(y)). \tag{3.1}$$

Example 3.2. Let $X = \{0, a, b, c, d\}$ be a BCH-algebra with the $*$-operation given by Table 1 (see [4]). Let ϑ be a real-valued function on X defined by

$$\vartheta = \begin{pmatrix} 0 & a & b & c & d \\ 0 & 3 & 1 & 2 & 0 \end{pmatrix}.$$

Then ϑ is an S-pseudo-valuation on X.

Proposition 3.3. *Every S-pseudo-valuation ϑ on X satisfies the following inequality:*

$$(\forall x \in X) \ (\vartheta(0 * x) \leq 3\vartheta(x)). \tag{3.2}$$

Proof. For any $x \in X$, we have

$$\begin{aligned}\vartheta(0 * x) &\leq \vartheta(0) + \vartheta(x) = \vartheta(x * x) + \vartheta(x) \\ &\leq \vartheta(x) + \vartheta(x) + \vartheta(x) = 3\vartheta(x)\end{aligned}$$

by using (I). □

Corollary 3.4. *For any S-pseudo-valuation ϑ on a BCH_0-algebra X, we have*

$$(\forall x \in X)\ (\vartheta(x) \geq 0). \tag{3.3}$$

Proof. Straightforward. □

The following example shows that Corollary 3.4 may not be true in BCH-algebras.

Example 3.5. Consider the BCH-algebra X which is given in Example 3.2. Let φ be a real-valued function on X defined by

$$\varphi = \begin{pmatrix} 0 & a & b & c & d \\ -3 & 3 & 1 & 2 & -3 \end{pmatrix}.$$

Then φ is an S-pseudo-valuation on X with negative values.

Theorem 3.6. *Let S be a subalgebra of X. For any real numbers t_1 and t_2 with $0 \leq t_1 < t_2$, let ϑ_S be a real-valued function on X defined by*

$$\vartheta_S(x) = \begin{cases} t_1 & \text{if } x \in S, \\ t_2 & \text{if } x \notin S \end{cases}$$

for all $x \in X$. Then ϑ_S is an S-pseudo-valuation on X.

Proof. Straightforward. □

Theorem 3.7. *If a real-valued function ϑ on X is an S-pseudo-valuation on X, then the set*

$$A := \{x \in X \mid \vartheta(x) \leq 0\}$$

is a subalgebra of X.

Proof. Let $x, y \in A$. Then $\vartheta(x) \leq 0$ and $\vartheta(y) \leq 0$. It follows from (3.1) that

$$\vartheta(x * y) \leq \vartheta(x) + \vartheta(y) \leq 0$$

so that $x * y \in A$. Hence A is a subalgebra of X. □

The following example shows that the converse of Theorem 3.7 may not be true, that is, there exist a BCH-algebra X and a mapping $\vartheta : X \to \mathbb{R}$ such that

(1) ϑ is not an S-pseudo-valuation on X,
(2) $A := \{x \in X \mid \vartheta(x) \leq 0\}$ is a subalgebra of X.

Table 2. $*$-operation

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	b
c	c	c	0	0

Example 3.8. Consider the BCH-algebra X which is given in Example 3.2. Let φ be a real-valued function on X defined by

$$\varphi = \begin{pmatrix} 0 & a & b & c & d \\ -3 & 1 & -5 & 4 & -3 \end{pmatrix}.$$

Then φ is not an S-pseudo-valuation on X since

$$\varphi(c * b) = \varphi(c) = 4 \nleq -1 = 4 - 5 = \varphi(c) + \varphi(b).$$

But $A := \{x \in X \mid \varphi(x) \leq 0\} = \{0, b, d\}$ is a subalgebra of X.

Corollary 3.9. *If a real-valued function ϑ on a BCH_0-algebra X is an S-pseudo-valuation on X, then the set*

$$A := \{x \in X \mid \vartheta(x) = 0\}$$

is a subalgebra of X.

Definition 3.10. A real-valued function ϑ on X is called a *pseudo-valuation* on X with respect to an ideal (briefly, *I-pseudo-valuation* on X) if it satisfies the following two conditions:

(i) $\vartheta(0) = 0$,
(ii) $(\forall x, y \in X)$ $(\vartheta(x) \leq \vartheta(x * y) + \vartheta(y))$.

Definition 3.11. A real-valued function ϑ on X is called a *pseudo-valuation* on X with respect to a closed ideal (briefly, *CI-pseudo-valuation* on X) if it satisfies the following condition:

$$(\forall x, y \in X)\ (\vartheta(0 * x) \leq \vartheta(x) \leq \vartheta(x * y) + \vartheta(y)).$$

If ϑ is an S-pseudo-valuation (resp. I-pseudo-valuation and CI-pseudo-valuation) on X satisfying the following condition:

$$(\forall x \in X)\ (x \neq 0 \ \Rightarrow\ \vartheta(x) \neq 0)$$

then we say that ϑ is an *S-valuation* (resp. *I-valuation* and *CI-valuation*) on X.

Example 3.12. Let $X = \{0, a, b, c\}$ be a BCH-algebra with the $*$-operation given by Table 2. Let ϑ be a real-valued function on X defined by

$$\vartheta = \begin{pmatrix} 0 & a & b & c \\ 0 & 1 & 3 & 2 \end{pmatrix}.$$

Then ϑ is both a CI-pseudo-valuation and an I-pseudo-valuation on X.

Theorem 3.13. *Every CI-pseudo-valuation is an S-pseudo-valuation.*

Proof. Let ϑ be a CI-pseudo-valuation on X and let $x, y \in X$. Then

$$\begin{aligned}\vartheta(x * y) &\leq \vartheta((x * y) * x) + \vartheta(x)\\ &= \vartheta((x * x) * y) + \vartheta(x)\\ &= \vartheta(0 * y) + \vartheta(x) \leq \vartheta(x) + \vartheta(y)\end{aligned}$$

Hence ϑ is an S-pseudo-valuation on X. □

The following example shows that the converse of Theorem 3.13 may not be true.

Example 3.14. Consider an S-pseudo-valuation ϑ which is given in Example 3.2. Then ϑ is not a CI-pseudo-valuation since $\vartheta(a) \nleq \vartheta(a * b) + \vartheta(b)$.

The following example shows that a CI-pseudo-valuation may not be an I-pseudo-valuation.

Example 3.15. Consider the BCH-algebra X which is given Example 3.12. Let ϑ be a real-valued function on X defined by

$$\vartheta = \begin{pmatrix} 0 & a & b & c \\ 1 & 1 & 3 & 2 \end{pmatrix}.$$

Then ϑ is a CI-pseudo-valuation on X, but not an I-pseudo-valuation on X since $\vartheta(0) = 1 \neq 0$.

Proposition 3.16. *In a BCH_0-algebra, every CI-pseudo-valuation ϑ satisfies the following inequality:*

$$(\forall x, y)\ (\vartheta(x * y) \leq \vartheta(0) + \vartheta(x)).$$

Proof. For any $x, y \in X$, we have

$$\begin{aligned}\vartheta(x * y) &\leq \vartheta((x * y) * x) + \vartheta(x)\\ &= \vartheta((x * x) * y) + \vartheta(x)\\ &= \vartheta(0 * y) + \vartheta(x) = \vartheta(0) + \vartheta(x)\end{aligned}$$

This completes the proof. □

Theorem 3.17. *In a BCH_0-algebra, every I-pseudo-valuation is an S-pseudo-valuation.*

Proof. Let ϑ be an I-pseudo-valuation on a BCH_0-algebra X. Since

$$((x * y) * x) * y = ((x * x) * y) * y = (0 * y) * y = 0$$

for all $x, y \in X$, we have

$$\begin{aligned}0 = \vartheta(0) &= \vartheta(((x * y) * x) * y)\\ &\geq \vartheta((x * y) * x) - \vartheta(y)\\ &\geq \vartheta(x * y) - \vartheta(x) - \vartheta(y).\end{aligned}$$

Hence $\vartheta(x * y) \leq \vartheta(x) + \vartheta(y)$, and therefore ϑ is an S-pseudo-valuation on X. □

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Table 3. $*$-operation

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	c	0

The converse of Theorem 3.17 is not true as shown by the following example.

Example 3.18. Let $X = \{0, a, b, c\}$ be a BCH_0-algebra with the $*$-operation given by Table 3. Let ϑ be a real-valued function on X defined by

$$\vartheta = \begin{pmatrix} 0 & a & b & c \\ 0 & 4 & 1 & 3 \end{pmatrix}.$$

Then ϑ is an S-pseudo-valuation , but not an I-pseudo-valuation on X since $\vartheta(a) = 4 \not\leq 1 = \vartheta(0) + \vartheta(b) = \vartheta(a * b) + \vartheta(b)$.

In general, Theorem 3.17 may not be true in a BCH-algebra as shown by the following example.

Example 3.19. Consider a commutative group $(\mathbb{R}, +, 0)$. Then $(\mathbb{R}, *, 0)$, where $x * y = x - y$, is a BCH-algebra which is not a BCH_0-algebra. Define a real-valued function ϑ on $\mathbb{R}$ by

$$\vartheta(x) = \begin{cases} 0 & \text{if } x = 0, \\ -3x + 1 & \text{otherwise} \end{cases}$$

for all $x \in \mathbb{R}$. Then ϑ is an I-pseudo-valuation on $\mathbb{R}$, but it is not an S-pseudo-valuation on $\mathbb{R}$ since $\vartheta(1 * 2) = \vartheta(-1) = 4 \not\leq -7 = \vartheta(1) + \vartheta(2)$. Moreover ϑ is not a CI-pseudo-valuation on $\mathbb{R}$.

Proposition 3.20. *For any I-pseudo-valuation ϑ on X, we have the following inequalities:*

(1) *ϑ is order preserving.*
(2) $(\forall x, y \in X)\ (\vartheta(x * y) + \vartheta(y * x) \geq 0)$.

Proof. (1) Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 0$, and so

$$\vartheta(x) \leq \vartheta(x * y) + \vartheta(y) = \vartheta(0) + \vartheta(y) = \vartheta(y).$$

(2) Let $x, y \in X$. Using Definition 3.10(ii), we have $\vartheta(x*y) \geq \vartheta(x) - \vartheta(y)$ and $\vartheta(y*x) \geq \vartheta(y) - \vartheta(x)$. It follows that $\vartheta(x * y) + \vartheta(y * x) \geq 0$. □

We provide conditions for a real-valued function on X to be an I-pseudo-valuation on X.

Theorem 3.21. *If a real-valued function ϑ on X satisfies Definition* 3.10(i) *and*

$$(\forall x, y, z \in X)\ (\vartheta(((x * y) * y) * z) \geq \vartheta(x * y) - \vartheta(z)) \tag{3.4}$$

then ϑ is an I-pseudo-valuation on X.

Proof. Taking $y = 0$ in (3.4) and using (a4), we have

$$\vartheta(x * z) = \vartheta(((x * 0) * 0) * z) \geq \vartheta(x * 0) - \vartheta(z) = \vartheta(x) - \vartheta(z),$$

that is, $\vartheta(x) \leq \vartheta(x * z) + \vartheta(z)$ for all $x, z \in X$. Hence ϑ is an I-pseudo-valuation on X. □

In a BCH_0-algebra X, every I-pseudo-valuation ϑ on X satisfies the inequality (3.3). We know from Example 3.19 that an I-pseudo-valuation ϑ on a BCH-algebra X does not satisfy the inequality (3.3).

Definition 3.22. An I-pseudo-valuation on a BCH-algebra X is said to be *positive* if it satisfies the inequality (3.3).

Example 3.23. Consider a CI-pseudo-valuation ϑ which is given in Example 3.12. Then ϑ is a positive I-pseudo-valuation.

Theorem 3.24. *Let ϑ be a real-valued function on X such that*

(1) $(\forall x, y \in X)\ (x \leq y \ \Rightarrow\ \vartheta(x) \leq \vartheta(y))$,
(2) $(\forall x, y \in X)\ (\vartheta(x) \leq \vartheta(x * y) + \vartheta(y))$.

Then the set

$$A := \{x \in X \mid \vartheta(x) \leq 0\} \cup \{0\}$$

is an ideal of X.

Proof. Let $x, y \in X$ be such that $x * y \in A$ and $y \in A$. If $y = 0$, then $x = x * 0 \in A$. Assume that $y \neq 0$. Then $\vartheta(y) \leq 0$. If $x * y = 0$, then $x \leq y$ which implies from (1) that $\vartheta(x) \leq \vartheta(y) \leq 0$. Hence $x \in A$. If $x * y \neq 0$, then $\vartheta(x * y) \leq 0$. Using (2), we have $\vartheta(x) \leq \vartheta(x * y) + \vartheta(y) \leq 0$ and so $x \in A$. Therefore A is an ideal of X. □

Corollary 3.25. *For any I-pseudo-valuation ϑ on X, the set*

$$A := \{x \in X \mid \vartheta(x) \leq 0\}$$

is an ideal of X.

Proof. Straightforward. □

The following example illustrates Corollary 3.25.

Example 3.26. Consider the I-pseudo-valuation ϑ on $\mathbb{R}$ which is described in Example 3.19. Then $A = \{x \in \mathbb{R} \mid x \geq \frac{1}{3}\} \cup \{0\}$ which is an ideal of $\mathbb{R}$.

Theorem 3.27. *If an I-pseudo-valuation ϑ on X is positive, then the set*

$$A := \{x \in X \mid \vartheta(x) = 0\}$$

is an ideal of X.

Proof. Obviously, $0 \in A$. Let $x, y \in X$ be such that $x * y \in A$ and $y \in A$. Then $\vartheta(x * y) = 0$ and $\vartheta(y) = 0$. It follows from Definition 3.10(ii) that $\vartheta(x) \leq \vartheta(x * y) + \vartheta(y) = 0$ so that $\vartheta(x) = 0$ since ϑ is positive. Hence $x \in A$, which shows that A is an ideal of X. □

Table 4. $*$-operation

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	b	0

Combining Theorems 3.17 and 3.27, and Corollary 3.4, we have the following corollary.

Corollary 3.28. *Let X be a BCH_0-algebra. If a real-valued function ϑ on X is an I-pseudo-valuation on X, then the set*

$$A := \{x \in X \mid \vartheta(x) = 0\}$$

is an ideal of X.

Given a real-valued function ϑ on X, consider the following condition:

$$(\forall x, y, z \in X)\ (\vartheta(x * y) \leq \vartheta(x * z) + \vartheta(z * y)). \tag{3.5}$$

In general, a real-valued function ϑ on X does not satisfy the condition (3.5) as shown by the following example.

Example 3.29. Let ϑ be a real-valued function on X which is given in Example 3.12. Then ϑ does not satisfy the condition (3.5) since $\vartheta(0*b) = \vartheta(b) = 3 \not\leq 2 = 2+0 = \vartheta(c)+\vartheta(0) = \vartheta(0*c)+\vartheta(c*b)$.

Example 3.30. Let $X = \{0, a, b, c\}$ be a BCH-algebra with the $*$-operation given by Table 4. Let ϑ be a real-valued function on X defined by

$$\vartheta = \begin{pmatrix} 0 & a & b & c \\ 1 & 1 & 2 & 3 \end{pmatrix}.$$

Then ϑ satisfies the condition (3.5).

By a *pseudo-metric space* we mean an ordered pair (M, d), where M is a non-empty set and $d : M \times M \to \mathbb{R}$ is a positive function such that the following properties are satisfied: $d(x, x) = 0, d(x, y) = d(y, x)$ and $d(x, z) \leq d(x, y) + d(y, z)$ for every $x, y, z \in M$. If in the pseudo-metric space (M, d) the implication $d(x, y) = 0 \Rightarrow x = y$ holds, then (M, d) is called a *metric space*. For a real-valued function ϑ on X, define a mapping $d_\vartheta : X \times X \to \mathbb{R}$ by $d_\vartheta(x, y) = \vartheta(x * y) + \vartheta(y * x)$ for all $(x, y) \in X \times X$.

Theorem 3.31. *If a real-valued function ϑ on X is an I-pseudo-valuation on X and satisfies the condition* (3.5)*, then d_ϑ is a pseudo-metric on X, and so (X, d_ϑ) is a pseudo-metric space.*

We say d_ϑ is the I-pseudo-metric induced by an I-pseudo-valuation ϑ.

Proof. Obviously, $d_\vartheta(x, y) \geq 0$, $d_\vartheta(x, x) = 0$ and $d_\vartheta(x, y) = d_\vartheta(y, x)$ for all $x, y \in X$. Let $x, y, z \in X$. Using the condition (3.5), we have

$$\begin{aligned} d_\vartheta(x, y) + d_\vartheta(y, z) &= [\vartheta(x * y) + \vartheta(y * x)] + [\vartheta(y * z) + \vartheta(z * y)] \\ &= [\vartheta(x * y) + \vartheta(y * z)] + [\vartheta(z * y) + \vartheta(y * x)] \\ &\geq \vartheta(x * z) + \vartheta(z * x) = d_\vartheta(x, z). \end{aligned}$$

Therefore (X, d_ϑ) is a pseudo-metric space. □

Theorem 3.32. *For a real-valued function ϑ on X satisfying the condition* (3.5)*, if d_ϑ is a pseudo-metric on X, then $(X \times X, d^*_\vartheta)$ is a pseudo-metric space, where*

$$d^*_\vartheta((x, y), (a, b)) = \max\{d_\vartheta(x, a), d_\vartheta(y, b)\}$$

for all $(x, y), (a, b) \in X \times X$.

Proof. Suppose d_ϑ is a pseudo-metric on X. For any $(x, y), (a, b) \in X \times X$, we have

$$d^*_\vartheta((x, y), (x, y)) = \max\{d_\vartheta(x, x), d_\vartheta(y, y)\} = 0$$

and

$$\begin{aligned} d^*_\vartheta((x, y), (a, b)) &= \max\{d_\vartheta(x, a), d_\vartheta(y, b)\} \\ &= \max\{d_\vartheta(a, x), d_\vartheta(b, y)\} \\ &= d^*_\vartheta((a, b), (x, y)). \end{aligned}$$

Now let $(x, y), (a, b), (u, v) \in X \times X$. Then

$$\begin{aligned} & d^*_\vartheta((x, y), (u, v)) + d^*_\vartheta((u, v), (a, b)) \\ &= \max\{d_\vartheta(x, u), d_\vartheta(y, v)\} + \max\{d_\vartheta(u, a), d_\vartheta(v, b)\} \\ &\geq \max\{d_\vartheta(x, u) + d_\vartheta(u, a), d_\vartheta(y, v) + d_\vartheta(v, b)\} \\ &\geq \max\{d_\vartheta(x, a), d_\vartheta(y, b)\} \\ &= d^*_\vartheta((x, y), (a, b)). \end{aligned}$$

Therefore $(X \times X, d^*_\vartheta)$ is a pseudo-metric space. □

Corollary 3.33. *If $\vartheta : X \to \mathbb{R}$ is an I-pseudo-valuation on X and satisfies the condition* (3.5)*, then $(X \times X, d^*_\vartheta)$ is a pseudo-metric space.*

Theorem 3.34. *If $\vartheta : X \to \mathbb{R}$ is a positive I-valuation on Xsatisfying the condition* (3.5)*, then (X, d_ϑ) is a metric space.*

Proof. Suppose ϑ is a positive I-valuation on X. Then (X, d_ϑ) is a pseudo-metric space by Theorem 3.31. Let $x, y \in X$ be such that $d_\vartheta(x, y) = 0$. Then $0 = d_\vartheta(x, y) = \vartheta(x * y) + \vartheta(y * x)$, and so $\vartheta(x * y) = 0$ and $\vartheta(y * x) = 0$ since ϑ is positive. Also, since ϑ is an I-valuation on X, it follows that $x * y = 0$ and $y * x = 0$ so from (II) that $x = y$. Therefore (X, d_ϑ) is a metric space. □

Corollary 3.35. *If a real-valued function ϑ on a BCH_0-algebra X is an I-valuation and satisfies the condition* (3.5)*, then (X, d_ϑ) is a metric space.*

Theorem 3.36. *If $\vartheta : X \to \mathbb{R}$ is a positive I-valuation on X which satisfies the condition* (3.5)*, then $(X \times X, d^*_\vartheta)$ is a metric space.*

Proof. Note from Corollary 3.33 that $(X \times X, d^*_\vartheta)$ is a pseudo-metric space. Let $(x, y), (a, b) \in X \times X$ be such that $d^*_\vartheta((x, y), (a, b)) = 0$. Then

$$0 = d^*_\vartheta((x, y), (a, b)) = \max\{d_\vartheta(x, a), d_\vartheta(y, b)\},$$

and so $d_\vartheta(x, a) = 0 = d_\vartheta(y, b)$ since $d_\vartheta(x, y) \geq 0$ for all $(x, y) \in X \times X$. Hence

$$0 = d_\vartheta(x, a) = \vartheta(x * a) + \vartheta(a * x)$$

and

$$0 = d_\vartheta(y, b) = \vartheta(y * b) + \vartheta(b * y).$$

Since ϑ is positive, it follows that $\vartheta(x * a) = 0 = \vartheta(a * x)$ and $\vartheta(y * b) = 0 = \vartheta(b * y)$ so that $x * a = 0 = a * x$ and $y * b = 0 = b * y$. Using (II), we have $a = x$ and $b = y$, and so $(x, y) = (a, b)$. Therefore $(X \times X, d^*_\vartheta)$ is a metric space. □

Corollary 3.37. *If $\vartheta : X \to \mathbb{R}$ is an I-valuation on a BCH_0-algebra X satisfying the condition* (3.5)*, then $(X \times X, d^*_\vartheta)$ is a metric space.*

Proposition 3.38. *Let ϑ be an I-pseudo-valuation on X which satisfies the condition* (3.5)*. Then the I-pseudo-metric d_ϑ induced by ϑ satisfies the following inequality:*

$$d_\vartheta(x * y, a * b) \leq d_\vartheta(x * y, a * y) + d_\vartheta(a * y, a * b) \tag{3.6}$$

for all $a, b, x, y \in X$.

Proof. Using the condition (3.5), we have

$$\vartheta((x * y) * (a * b)) \leq \vartheta((x * y) * (a * y)) + \vartheta((a * y) * (a * b)),$$

$$\vartheta((a * b) * (x * y)) \leq \vartheta((a * b) * (a * y)) + \vartheta((a * y) * (x * y))$$

for all $x, y, a, b \in X$. Hence

$$\begin{aligned} d_\vartheta(x * y, a * b) &= \vartheta((x * y) * (a * b)) + \vartheta((a * b) * (x * y)) \\ &\leq [\vartheta((x * y) * (a * y)) + \vartheta((a * y) * (a * b))] \\ &\qquad + [\vartheta((a * b) * (a * y)) + \vartheta((a * y) * (x * y))] \\ &= [\vartheta((x * y) * (a * y)) + \vartheta((a * y) * (x * y))] \\ &\qquad + [\vartheta((a * b) * (a * y)) + \vartheta((a * y) * (a * b))] \\ &= d_\vartheta(x * y, a * y) + d_\vartheta(a * y, a * b) \end{aligned}$$

for all $x, y, a, b \in X$. □

Theorem 3.39. *Let ϑ be a positive I-valuation on X which satisfies the condition* (3.5)*. If the I-pseudo-metric d_ϑ induced by ϑ satisfies the following inequality:*

$$d_\vartheta(x, y) \geq \max\{d_\vartheta(x * a, y * a), d_\vartheta(a * x, a * y)\} \tag{3.7}$$

for all $x, y, a, b \in X$, then the operation $*$ in X is uniformly continuous.

Proof. Note that (X, d_ϑ) and $(X \times X, d^*_\vartheta)$ are metric spaces (see Theorems 3.34 and 3.36). For any $\varepsilon > 0$, if $d^*_\vartheta((x, y),(a, b)) < \frac{\varepsilon}{2}$, then $d_\vartheta(x, a) < \frac{\varepsilon}{2}$ and $d_\vartheta(y, b) < \frac{\varepsilon}{2}$. Using (3.7) and Proposition 3.38, we have

$$\begin{aligned} d_\vartheta(x * y, a * b) &\leq d_\vartheta(x * y, a * y) + d_\vartheta(a * y, a * b) \\ &\leq d_\vartheta(x, a) + d_\vartheta(y, b) < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore the operation $* : X \times X \to X$ is uniformly continuous □

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DERIVATIVES OF DECREASING FUNCTIONS WITH RESPECT TO FUZZY MEASURES

H.M. KIM, Y.H. KIM, AND J. CHOI

Abstract In this paper, we consider Choquet integrals and derivatives of non-negative, continuous and decreasing functions on the non-positive real line with respect to fuzzy measures. We prove some properties of derivatives of those functions and some examples.

1. Introduction

The concept of fuzzy measure was introduced by Sugeno([7]). We note that Choquet first studied Choquet integral and T. Murofushi et al. studied Choquet integrals with respect to a fuzzy measure([2], [6]). In [8], Sugeno introduced the concept of derivatives of non-negative, continuous and incresing functions on the non-negative real line $\mathbb{R}^+ = [0, \infty)$ with respect to fuzzy measures. Choi showed some basic properties of derivatives of those functions([1]). In this paper, we consider Choquet integrals and derivatives of non-negative, continuous and decreasing functions on the non-positive real line $\mathbb{R}^- = (\infty, 0]$ with respect to fuzzy measures.

We assume that X is a nonempty set. Let $\mathcal{A}$ be the any σ-algebra of subsets of X. Then $(X, \mathcal{A})$ is called a measurable space. A fuzzy measure is a set function $\nu : \mathcal{A} \to \mathbb{R}^+$ which satisfies
(1) $\nu(\phi)$=0,
(2) $\nu(A) \leq \nu(B)$, whenever $A, B \in \mathcal{A}$ and $A \subset B$,
(3) if for every increasing sequence $\{A_n\}$ of measurable sets, then $\nu(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \nu(A_n)$ and if for every decreasing sequence $\{A_n\}$ of measurable sets and $\nu(A_1) < \infty$, then $\nu(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \nu(A_n)$. Recall that a function $h : X \to \mathbb{R}^+$ is said to be measurable if $\{t|h(t) > \alpha\} \in \mathcal{A}$ for all $\alpha \in (-\infty, \infty)$.

In section 2, we give Choquet integrals of non-negative, continuous and decreasing functions on $\mathbb{R}^-$ with respect to fuzzy measure according to the ideas of [8]. In section 3, we investigate some properties and examples of derivatives of those functions with respect to distorted

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Lebesgue measures. Finally, we show that existence and non-existence of derivatives depend on fuzzy measures.

2. Choquet integral of $g \in \mathcal{M}^-$

In this paper, we assume that $X = \mathbb{R}^-$. Let $\mathcal{A}$ be the smallest σ-algebra of subsets of X. Then $(X, \mathcal{A}, \nu)$ is called a fuzzy measure space.

As in [8], Choquet integral of g with respect to a fuzzy measure ν on a set A is defined by

$$(C) \int_A g(t) d\nu = \int_0^\infty \nu\left(\{t | g(t) \geq \alpha\} \cap A\right) d\alpha. \tag{2.1}$$

Let $\mathcal{M}^-$ be the set of measurable, non-negative, continuous and decreasing functions such that $g : \mathbb{R}^- \to \mathbb{R}^+$.

Definition 2.1. *([8]) Let $m : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous and increasing function $m(0) = 0$. A fuzzy measure ν_m, a distorted Lebesgue measure, is defined by*

$$\nu_m(\cdot) = m(\lambda(\cdot)), \tag{2.2}$$

where $\lambda([a, b]) = b - a$ for all $[a, b] \subset \mathbb{R}^-$.

From Definition 2.1 and (2.1), we have the following theorem([8]).

Theorem 1. *We assume that $\nu([t, t]) = 0$ for all $t \in \mathbb{R}^-$. Let $g \in \mathcal{M}^-$, then Choquet integral of g with respect to ν on $[t, 0]$ is represented as*

$$\int_0^\infty \nu(\{t | g(t) \geq \alpha\} \cap [t, 0]) d\alpha = \int_t^0 \nu'([t, \tau]) g(\tau) d\tau. \tag{2.3}$$

In particular, for $\nu = \nu_m$,

$$\int_0^\infty \nu(\{t | g(t) \geq \alpha\} \cap [t, 0]) d\alpha = \int_t^0 m'(\tau - t) g(\tau) d\tau. \tag{2.4}$$

Proof. Let $\alpha = g(\tau), d\alpha = g'(\tau)d\tau$ and $g^{-1}(\alpha) = \tau$. By the definition of Choquet integral, we have that

$$\begin{aligned}
&(C)\int_{[t,0]} g(\tau)d\nu(\tau) = \int_0^\infty \nu(\{t|g(t) \geq \alpha\} \cap [t,0])d\alpha \\
&= \int_0^{g(0)} \nu(\{t|g(t) \geq \alpha\} \cap [t,0])d\alpha + \int_{g(0)}^{g(t)} \nu(\{t|g(t) \geq \alpha\} \cap [t,0])d\alpha \\
&= \nu([t,0])g(0) + \int_{g(0)}^{g(t)} \nu([t, g^{-1}(\alpha)])d\alpha \\
&= \nu([t,0])g(0) + \int_0^t \nu([t,\tau])g'(\tau)d\tau \\
&= \nu([t,0])g(0) + \Big[\nu([t,\tau])g(\tau)\Big]_0^t - \int_0^t \nu'([t,\tau])g(\tau)d\tau \\
&= \nu([t,0])g(0) + \nu([t,t])g(t) - \nu([t,0])g(0) + \int_t^0 \nu'([t,\tau])g(\tau)d\tau \\
&= \int_t^0 \nu'([t,\tau])g(\tau)d\tau.
\end{aligned}$$

For $\nu = \nu_m$, we obtain

$$(C)\int_{[t,0]} g(\tau)d\nu(\tau) = \int_t^0 \nu'([t,\tau])g(\tau)d\tau = \int_t^0 m'(\tau - t)g(\tau)d\tau.$$

□

From Theorem 1, we have the following corollary.

Corollary 2. *Let $g(t) = k$ be a constant function for all $t \in \mathbb{R}^-$ and $k \in \mathbb{R}^+$. Then*

$$\int_0^\infty \nu(\{t|g(t) \geq \alpha\} \cap [t,0])d\alpha = k\nu([t,0]).$$

In particular, for $\nu = \nu_m$,

$$\int_0^\infty \nu(\{t|g(t) \geq \alpha\} \cap [t,0])d\alpha = km(-t).$$

Proof. From (2.1) we have that

$$\begin{aligned}
(C)\int_{[t,0]} g(\tau)d\nu(\tau) &= \int_0^\infty \nu(\{t|g(t) = k \geq \alpha\} \cap [t,0])d\alpha \\
&= \int_0^k \nu([t,0])d\alpha = k\nu([t,0]).
\end{aligned}$$

For $\nu = \nu_m$, we obtain that

$$(C)\int_{[t,0]} g(\tau)d\nu(\tau) = k\nu([t,0]) = km(-t).$$

□

Definition 2.2. *Let f be a continuous and decreasing function with $f(0) = 0$. The derivative of f with respect to a fuzzy measure ν_m is defined as the inverse operation of Choquet integral based on (2.4) by*

$$\frac{df(t)}{d\nu_m(t)} = D_m(f) = g(t), \tag{2.5}$$

if $g(t)$ is found to be an element of $\mathcal{M}^-$.

From (2.5), let us consider a class of f's for a given $m(t)$ such that

$$\mathcal{T}_m(\mathcal{M}^-) = \Big\{f|f(t) = \int_t^0 m'(\tau - t)g(\tau)d\tau, \quad g \in \mathcal{M}^-\Big\}. \tag{2.6}$$

3. Derivatives with respect to distorted Lebesgue measures

In this section, we discuss some properties of derivatives of continuous and decreasing functions with respect to distorted Lebesgue measures. From the conditions of $g(t)$ in (2.5), we obtain the following theorem.

Theorem 3. *$D_m(f)$ is linear for $f \in \mathcal{T}_m(\mathcal{M}^-)$ and non-negative constants.*

Proof. Let $f_1(t), f_2(t) \in \mathcal{T}_m(\mathcal{M}^-)$ and $k \in \mathbb{R}^+$. From the condition of $f_1(t)$ and $f_2(t)$, we have that

$$f_1(t) = \int_t^0 m'(\tau - t)D_m(f_1(\tau))d\tau \tag{3.1}$$

and

$$f_2(t) = \int_t^0 m'(\tau - t)D_m(f_2(\tau))d\tau. \tag{3.2}$$

Adding (3.1) and (3.2), we obtain that

$$f_1(t) + f_2(t) = \int_t^0 m'(\tau - t)\Big\{D_m(f_1(\tau)) + D_m(f_2(\tau))\Big\}d\tau.$$

By the definitions of $\mathcal{M}^-$ and $T_m(\mathcal{M}^-)$, we know that

$$D_m(f_1(t)) + D_m(f_2(t)) \in \mathcal{M}^- \quad \text{and} \quad f_1(t) + f_2(t) \in \mathcal{T}_m(\mathcal{M}^-).$$

From (3.1), we see that

$$kf_1(t) = \int_t^0 m'(\tau - t)kD_m(f_1(\tau))d\tau.$$

Since $kD_m(f_1(t)) \in \mathcal{M}^-$, we have $kf_1(t) \in \mathcal{T}_m(\mathcal{M}^-)$. □

From Definition 2.2 and Theorem 3, we have the following theorem.

Theorem 4. *For $t \in \mathbb{R}^-$, we have the followings:*

(1) $\quad \dfrac{d}{d\nu_m} m(-t) = 1,$

(2) $\quad \dfrac{d}{d\nu_m}\left(n\displaystyle\int_t^0 m(\tau - t)(-\tau)^{n-1}d\tau\right) = (-t)^n, \quad n = 1, 2, \ldots$

(3) $\quad \dfrac{d}{d\nu_m}\left(m(-t) - a\displaystyle\int_t^0 m(\tau - t)e^{a\tau}d\tau\right) = e^{at}, \quad a \le 0,$

(4) $\quad \dfrac{d}{d\nu_m}\left(\displaystyle\int_t^0 \frac{m(\tau - t)}{1 - \tau}d\tau\right) = \ln(1 - t).$

Proof. (1) From (2.4), we have that

$$\int_t^0 m'(\tau - t)d\tau = m(-t).$$

(2) By $(-t)^n \in \mathcal{M}^-$, $n = 1, 2, \ldots$, we obtain that

$$\begin{aligned}
&\int_t^0 m'(\tau - t)(-\tau)^n d\tau \\
&= \Big[m(\tau - t)(-\tau)^n\Big]_t^0 + n\int_t^0 m(\tau - t)(-\tau)^{n-1}d\tau \\
&= n\int_t^0 m(\tau - t)(-\tau)^{n-1}d\tau.
\end{aligned}$$

(3) Similarly, by $e^{at} \in \mathcal{M}^-$ for all $a \le 0$, we have that

$$\begin{aligned}
\int_t^0 m'(\tau - t)e^{a\tau}d\tau &= \Big[m(\tau - t)e^{a\tau}\Big]_t^0 - a\int_t^0 m(\tau - t)e^{a\tau}d\tau \\
&= m(-t) - a\int_t^0 m(\tau - t)e^{a\tau}d\tau.
\end{aligned}$$

(4) Since $\ln(1 - t) \in \mathcal{M}^-$, we have

$$\begin{aligned}
\int_t^0 m'(\tau - t)\ln(1 - \tau)d\tau &= [m(\tau - t)\ln(1 - \tau)]_t^0 + \int_t^0 \frac{m(\tau - t)}{1 - \tau}d\tau \\
&= \int_t^0 \frac{m(\tau - t)}{1 - \tau}d\tau.
\end{aligned}$$

Thus (4) is proved. □

By Definition 2.1 and (2.3), we have the following remark.

Remark 1. *We assume that $g(t)$ is even function. Let $\tau - t = p$, $d\tau = dp$, and $0 \leq p \leq -t$. Then, we have*

$$\begin{aligned}(C)\int_{[t,0]} g(t)d\nu &= \int_t^0 \nu'([t,\tau])g(\tau)d\tau = \int_t^0 \nu'([t,\tau])g(-\tau)d\tau \\ &= \int_0^{-t} \nu'(p)g(-p-t)dp = \int_0^{\alpha} \nu'(\tau)g(\alpha-\tau)d\tau \\ &= \int_0^{\alpha} \nu'(\alpha-\tau)g(\tau)d\tau.\end{aligned}$$

In particular, for $\nu = \nu_m$,

$$(C)\int_{[t,0]} g(t)d\nu = -\int_0^{\alpha} m'(\alpha-\tau)g(\tau)d\tau.$$

From Remark 1, the Choquet integral (2.3) on $\mathbb{R}^-$ is considered as a convolution. That is, under the assumption of even function, we can apply the Laplace transformation for calculations of the Choquet integrals.

As you see (2.5) and (2.6) of Definition 2.2., to find $g(t)$ is same with the solvability of a Volterra integral equation of the first kind for given $f(t)$ and a fuzzy measure ν_m ([4]). In fact, fuzzy measures play important roles in the existence of derivatives. Now we give a theorem to explain the relation with the existence of derivatives and fuzzy measures.

Theorem 5. *(The dependence on fuzzy measures)*

Let $t \in \mathbb{R}^-$.

(1) If $\nu_m([t,0]) = m(-t) = e^{-t} - 1$, then $-t \notin \mathcal{T}_m(\mathcal{M}^-)$, that is, $\nexists D_m(-t)$.

(2) If $\nu_m([t,0]) = m(-t) = -t$, then $-t \in \mathcal{T}_m(\mathcal{M}^-)$, that is, $\exists D_m(-t)$.

Proof. Suppose that $-t \in \mathcal{T}_m(\mathcal{M}^-)$. From (2.6) of Definition 2.2., we know that

$$-t = \int_t^0 m'(\tau-t)x_1(\tau)d\tau = \int_t^0 e^{(\tau-t)}x_1(\tau)d\tau, \tag{3.3}$$

where $D_m(-t) = x_1(t) \in \mathcal{M}^-$.

But differentiating (3.3), we obtain

$$-1 = -e^{-t}\int_t^0 e^{\tau}x_1(\tau)d\tau - x_1(t) = t - x_1(t).$$

From the definition of $\mathcal{M}^-$, we obtain

$$x_1(t) = t + 1 \notin \mathcal{M}^-.$$

(1) is proved by this contradiction.

To prove (2), it is sufficient to find $x_2(t) \in \mathcal{M}^-$ such that

$$-t = \int_t^0 m'(\tau - t)x_2(\tau)d\tau = \int_t^0 x_2(\tau)d\tau. \tag{3.4}$$

Differentiating (3.4), we have

$$x_2(t) = 1 \in \mathcal{M}^-.$$

□

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Some new inequalities for the gamma function

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Abstract

In this paper, we present some new inequalities for the gamma function. The main tools are the multiple-correction method developed in [6, 7] and a generalized Mortici's lemma.

1 Introduction

Duo to its importance in mathematics, the problem of finding new and sharp inequalities for the gamma function and, in particular for large values of x

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt, \quad x>0, \tag{1.1}$$

has attracted the attention of many researchers (see [2, 3, 8, 9, 12, 14, 15, 16, 17, 18] and the references therein). Let's recall some of the classical results. Maybe one of the most well-known formula for approximation the gamma function is the Stirling's formula

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x, \quad x\to+\infty. \tag{1.2}$$

See, e.g. [1, p. 253]. The following two formulas give slightly better estimates than Stirling's formula,

$$\Gamma(x+1) \sim \sqrt{2\pi}\left(\frac{x+\frac{1}{2}}{e}\right)^{x+\frac{1}{2}}, \quad \text{(Burnside [5], 1917)}, \tag{1.3}$$

$$\Gamma(x+1) \sim \sqrt{2\pi}\left(\frac{x}{e}\right)^x\sqrt{x+\frac{1}{6}}, \quad \text{(Gosper [10], 1978)}. \tag{1.4}$$

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Ramanujian [22] proposed the claim (without proof) for the gamma function

$$\Gamma(x+1) = \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{\theta_x}{30}\right)^{\frac{1}{6}}, \tag{1.5}$$

where $\theta_x \to 1$ as $x \to +\infty$ and $\frac{3}{10} < \theta_x < 1$. This open problem was solved by Karatsuba[13]. Thus (1.5) provides a more accurate estimate for the gamma function (see Sec. 2 below).

In this paper, we will continue the previous works [6, 7], and introduce a class of new approximations to improve these inequalities.

Throughout the paper, the notation $\Psi(k;x)$ denotes a polynomial of degree k in x with all coefficients non-negative, which may be different at each occurrence. Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 0}$ be two sequences of real numbers with $a_n \neq 0$ for all $n \in \mathbb{N}$. The generalized continued fraction

$$\tau = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots}} = b_0 + \frac{a_1}{b_1+}\frac{a_2}{b_2+}\cdots = b_0 + \mathop{\mathbf{K}}_{n=0}^{\infty}\frac{a_n}{b_n}$$

is defined as the limit of the nth approximant

$$\frac{A_n}{B_n} = b_0 + \mathop{\mathbf{K}}_{k=1}^{n}\frac{a_k}{b_k}$$

as n tends to infinity. See [2, p.105].

2 A generalized Mortici's lemma

Mortici [14] established a very useful tool for measuring the rate of convergence, which says that a sequence $(x_n)_{n\geq 1}$ converging to zero is the fastest possible when the difference $(x_n - x_{n+1})_{n\geq 1}$ is the fastest possible. Since then, Mortici's lemma has been effectively applied in many paper such as [6, 7, 17, 18]. The following lemma is a generalization of Mortici's lemma.

Lemma 1. *If* $\lim_{x\to+\infty} f(x) = 0$, *and there exists the limit*

$$\lim_{x\to+\infty} x^{\lambda}\left(f(x) - f(x+1)\right) = l \in \mathbb{R}, \tag{2.1}$$

with $\lambda > 1$, *then there exists the limit*

$$\lim_{x\to+\infty} x^{\lambda-1} f(x) = \frac{l}{\lambda - 1}. \tag{2.2}$$

Proof. It is not very difficult to prove that for $x > 2$

$$\frac{1}{(\lambda-1)x^{\lambda-1}} = \int_x^{+\infty}\frac{dt}{t^{\lambda}} \leq \sum_{j=0}^{\infty}\frac{1}{(x+j)^{\lambda}} \leq \int_{x-1}^{+\infty}\frac{dt}{t^{\lambda}} = \frac{1}{(\lambda-1)(x-1)^{\lambda-1}}. \tag{2.3}$$

2

For $\varepsilon > 0$, we assume that $l - \varepsilon \le x^{\lambda}\left(f(x) - f(x+1)\right) \le l + \varepsilon$ for every real number x greater than or equal to the rank $X_0 > 0$. By adding the inequalities of the form

$$(l-\varepsilon)\frac{1}{x^{\lambda}} \le f(x) - f(x+1) \le (l+\varepsilon)\frac{1}{x^{\lambda}}, \tag{2.4}$$

we get

$$(l-\varepsilon)\sum_{j=0}^{m-1}\frac{1}{(x+j)^{\lambda}} \le f(x) - f(x+m) \le (l+\varepsilon)\sum_{j=0}^{m-1}\frac{1}{(x+j)^{\lambda}} \tag{2.5}$$

for every $x \ge X_0$ and $m \ge 1$. By taking the limit as $m \to \infty$, then multiplying by $x^{\lambda-1}$, we obtain

$$(l-\varepsilon)x^{\lambda-1}\sum_{j=0}^{\infty}\frac{1}{(x+j)^{\lambda}} \le x^{\lambda-1}f(x) \le (l+\varepsilon)x^{\lambda-1}\sum_{j=0}^{\infty}\frac{1}{(x+j)^{\lambda}}. \tag{2.6}$$

It follows from (2.3) that

$$\frac{l-\varepsilon}{\lambda-1} \le x^{\lambda-1}f(x) \le \frac{l+\varepsilon}{\lambda-1}\frac{x^{\lambda-1}}{(x-1)^{\lambda-1}}. \tag{2.7}$$

Now by taking the limit as $x \to +\infty$, this completes the proof of the lemma at once. □

An example Let's consider the Ramanujan's asymptotic formula (1.5). Let the error term $E(x)$ be defined by the following relation

$$\Gamma(x+1) = \sqrt{\pi}\left(\frac{x}{e}\right)^{x}\left(8x^3 + 4x^2 + x + \frac{1}{30}\right)^{\frac{1}{6}}\left(1 + E(x)\right). \tag{2.8}$$

It follows readily from the recurrence formula $\Gamma(x+1) = x\Gamma(x)$ that

$$\begin{aligned}\ln\left(1 + E(x)\right) - \ln\left(1 + E(x+1)\right) = &-1 + x\ln\left(1 + \frac{1}{x}\right) \\ &+ \frac{1}{6}\ln\frac{8(x+1)^3 + 4(x+1)^2 + (x+1) + \frac{1}{30}}{8x^3 + 4x^2 + x + \frac{1}{30}}.\end{aligned} \tag{2.9}$$

By using the *Mathematica* software, we expand the right-hand function in the above formula as a power series in terms of $1/x$:

$$\ln\left(1 + E(x)\right) - \ln\left(1 + E(x+1)\right) = \frac{11}{2880x^5} + O(\frac{1}{x^6}). \tag{2.10}$$

Thus, by Lemma 1 we have

$$\lim_{x\to+\infty} x^4\ln\left(1 + E(x)\right) = \frac{11}{11520}. \tag{2.11}$$

Noting that $\lim_{u\to 0}\frac{\ln(1+u)}{u} = 1$, one get finally

$$\lim_{x\to+\infty} x^4 E(x) = \frac{11}{11520}. \tag{2.12}$$

Remark 1. Just as Motici's lemma, Lemma 1 also provides a method for finding the limit of a function as x tends to infinity.

3 Gosper-type inequalities

In this section, we use an example to illustrate the idea of this paper. To this end, we introduce some class of correction function $(\mathrm{MC}_k(x))_{k\ge 0}$ such that the relative error function $E_k(x)$ has the fastest possible rate of convergence, which are defined by the relations

$$\Gamma(x+1)=\sqrt{2\pi}\left(\frac{x}{e}\right)^x\sqrt{x+\frac{1}{6}+\mathrm{MC}_k(x)}\cdot\exp(E_k(x)). \tag{3.1}$$

If $\lim_{x\to+\infty}x^\mu f(x)=l\neq 0$ with constant $\mu>0$, we say that the function $f(x)$ is order $x^{-\mu}$, and write the exponent of convergence $\mu=\mu(f(x))$. Clearly if $\mu(E_k(x))=\mu_k$, we have the following asymptotic formula

$$\Gamma(x+1)=\sqrt{2\pi}\left(\frac{x}{e}\right)^x\sqrt{x+\frac{1}{6}+\mathrm{MC}_k(x)}\cdot\left(1+O(x^{-\mu_k})\right),\quad x\to+\infty. \tag{3.2}$$

Let us briefly review a so-called *multiple-correction method* presented in our previous paper [6, 7]. Actually, the *multiple-correction method* is a recursive algorithm, and one of its advantages is that by repeating correction process we always can accelerate the convergence, i.e. the sequence $(\mu(E_k(x)))_{k\ge 0}$ is a strictly increasing. The key step is to find a suitable structure of $\mathrm{MC}_k(x)$. In general, the correction function $\mathrm{MC}_k(x)$ is a finite *generalized* continued fraction (see [7] or (3.8) below) or a *hyper-power* series (see [6] or (4.7) below) in x.

It is not difficult to see that (3.1) is equivalent to

$$\ln\Gamma(x+1)=\frac{1}{2}\ln(2\pi)+x\left(\ln x-1\right)+\frac{1}{2}\ln\left(x+\mathrm{MC}_k(x)\right)+E_k(x). \tag{3.3}$$

By the recurrence formula $\Gamma(x+1)=x\Gamma(x)$, we have for $x>0$

$$E_k(x)-E_k(x+1)=-1+x\ln\left(1+\frac{1}{x}\right)+\frac{1}{2}\ln\frac{(x+1)+\frac{1}{6}+\mathrm{MC}_k(x+1)}{x+\frac{1}{6}+\mathrm{MC}_k(x)}. \tag{3.4}$$

Now by taking the initial-correction function $\mathrm{MC}_0(x)=\frac{\kappa_0}{x+\lambda_0}$ and using *Mathematica* software, we expand $E_k(x)-E_k(x+1)$ into a power series in terms of $1/x$:

$$\begin{aligned}E_0(x)-E_0(x+1)=&\frac{-\frac{1}{72}+\kappa_0}{x^3}+\frac{17-945\kappa_0-810\kappa_0\lambda_0}{540x^4}+\\&\frac{-641+33120\kappa_0-12960\kappa_0^2+43200\kappa_0\lambda_0+25920\kappa_0\lambda_0^2}{12960x^5}+O\left(\frac{1}{x^6}\right).\end{aligned} \tag{3.5}$$

The fastest possible function $E_0(x)-E_0(x+1)$ is obtained when the first two coefficients in the above formula vanish. In this case, we find $\kappa_0=\frac{1}{72},\lambda_0=\frac{31}{90}$, and

$$E_0(x)-E_0(x+1)=\frac{5929}{1166400x^5}+O\left(\frac{1}{x^6}\right). \tag{3.6}$$

By Lemma 1, we can check that

$$\lim_{x\to+\infty} x^4 E_0(x) = \frac{5929}{4665600}. \tag{3.7}$$

We continue the above correction process to successively determine the correction function $\mathrm{MC}_k(x)$ until some k^* you want. On one hand, to find the related coefficients, we often use an appropriate symbolic computations software because it's huge of computations. On the other hand, the exact expressions at each occurrence also need lot of space. Hence in this paper we omit many related details. For interesting readers, see our previous paper [6, 7]. In fact, we can prove that for $0 \le k \le 3$

$$\mathrm{MC}_k(x) = \mathop{\mathbf{K}}_{j=0}^{k} \frac{\kappa_j}{x+\lambda_j}, \tag{3.8}$$

where

$$\kappa_0 = \frac{1}{72}, \qquad \lambda_0 = \frac{31}{90},$$
$$\kappa_1 = \frac{5929}{32400}, \qquad \lambda_1 = \frac{481937}{3735270},$$
$$\kappa_2 = \frac{76899172249}{248039857296}, \qquad \lambda_2 = \frac{7745462509019287}{19149278075101482},$$
$$\kappa_3 = \frac{7868734172706312117499 21}{851541507731717527392144}, \qquad \lambda_3 = \frac{209833574581775168536420106727907 1}{303110888724869214663347815892549 70}.$$

By Lemma 1 again, we get for some constant $C_k \neq 0$

$$\lim_{x\to+\infty} x^{2k+4} E_k(x) = C_k, \quad (k = 0, 1, 2, 3), \tag{3.9}$$

i.e. $\mu(E_k(x)) = 2k + 4$ for $k = 0, 1, 2, 3$. Thus we obtain more accurate approximation formulas:

$$\Gamma(x+1) = \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt{x + \frac{1}{6} + \mathrm{MC}_k(x)} \cdot \left(1 + O(x^{-(2k+4)})\right), \quad x \to +\infty. \tag{3.10}$$

It should be noted that if we rewrite $\mathrm{MC}_k(x)$ in the form of $\frac{P_r(m)}{Q_s(m)}$, where P, Q are polynomials with $r = k$ and $s = k + 1$, theoretically at least, for a large x the above formula may reduce or eliminate numerically computations compared with the previous results, see e.g. [9, 12]. This is the main advantage of the *multiple-correction method.*

The following theorem tells us how to obtain sharp inequalities.

Theorem 1. *Let* $\mathrm{MC}_k(x)$ *be defined as* (3.8). *Let* $x \ge 1$, *then we have for* $k = 0, 2$,

$$\Gamma(x+1) > \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt{x + \frac{1}{6} + \mathrm{MC}_k(x)}, \tag{3.11}$$

and for $k = 1, 3$,

$$\Gamma(x+1) < \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt{x + \frac{1}{6} + \mathrm{MC}_k(x)}. \tag{3.12}$$

Proof. We let $f_k(x) = E_k(x) - E_k(x+1)$. Clearly if $\lim_{x\to+\infty} E_k(x) = 0$, then $E_k(x) = \sum_{j=0}^{\infty} f_k(x+j)$. This transformation plays an important role in this paper (essentially, it is a difference method). Hence, in order to prove inequality $E_k(x) > 0$ (or $E_k(x) < 0$), it suffices to show that the equality $f_k(x) > 0$ (or $f_k(x) < 0$) holds under the condition $\lim_{x\to+\infty} E_k(x) = 0$. By the Stirling's formula (1.2), we can show that the condition $\lim_{x\to+\infty} E_k(x) = 0$ always holds. In what follows, we will apply this condition many times.

By using *Mathematica* software, we may prove that for $x \ge 1$

$$
\begin{aligned}
f_0''(x) &= \frac{\Psi_1(8;x)}{x(1+x)^2(31+90x)^2(121+90x)^2(77+552x+1080x^2)^2(1709+2712x+1080x^2)^2} > 0,\\
f_1''(x) &= -\frac{\Psi_2(13;x)(x-1)+1463\cdots9447}{x(1+x)^2)(1359251+2829648x+5976432x^2)^2\Psi_3(16;x)} < 0,\\
f_2''(x) &= \frac{\Psi_4(20;x)}{x(1+x)^2\Psi_5(28;x)} > 0,\\
f_3''(x) &= \frac{\Psi_6(25;x)(x-1)+17135\cdots66999}{x(1+x)^2\Psi_7(36;x)} < 0.
\end{aligned}
$$

We only give the proof of inequalities in case $k = 3$, other may be proved similarly. In this case, we see that for $x \ge 1$ the inequality (3.12) is equivalent to $E_3(x) < 0$. As $\lim_{x\to+\infty} E_3(x) = 0$, it suffices to prove that $f_3(x) < 0$ for $x \ge 1$. Since $f_3'(x)$ is strictly decreasing, but $\lim_{x\to+\infty} f_3'(x) = 0$, so $f_3'(x) > 0$. Thus $f_3(x)$ is strictly increasing with $\lim_{x\to+\infty} f_3(x) = 0$, so $f_3(x) < 0$. This completes the proof of Theorem 1. $\square$

By the *multiple-correction* method, we also find another kind of inequalities.

Theorem 2. *Let the k-th correction function* $\mathrm{MC}_k(x)$ *be defined by*

$$
\begin{aligned}
\mathrm{MC}_0(x) &= \frac{\kappa_0}{(x+\frac{23}{90})^2+\lambda_0},\\
\mathrm{MC}_k(x) &= \frac{\kappa_0}{(x+\frac{23}{90})^2+\lambda_0+} \mathop{\mathbf{K}}_{j=1}^{k} \frac{\kappa_j}{x+\lambda_j}, \quad (k \ge 1),
\end{aligned}
$$

where

$$
\begin{aligned}
\kappa_0 &= -\frac{1}{144}, & \lambda_0 &= \frac{4007}{21600},\\
\kappa_1 &= \frac{4394}{637875}, & \lambda_1 &= \frac{130311599}{15575040},\\
\kappa_2 &= \frac{7894414898425}{119793516544}, & \lambda_2 &= -\frac{265702682899837009577}{344276317894782873 60},\\
\kappa_3 &= \frac{18975608492521061778584657 92}{77174813342532578267347147395}, & \lambda_3 &= \frac{303203804556162930048989281631315632448119 79}{613436431567206532574665270824029803422720 0}.
\end{aligned}
$$

Then we have

$$
\Gamma(x+1) < \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt{x+\frac{1}{6}}\left(1+\mathrm{MC}_0(x)\right), \quad x \ge 13, \tag{3.13}
$$

$$\Gamma(x+1) < \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt{x+\frac{1}{6}}\left(1+\mathrm{MC}_2(x)\right), \quad x \ge 6, \tag{3.14}$$

and for $k = 1, 3$,

$$\Gamma(x+1) > \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt{x+\frac{1}{6}}\left(1+\mathrm{MC}_k(x)\right), \quad x \ge 1. \tag{3.15}$$

Proof. Since the proof of Theorem 2 is very similar to that of Theorem 1, here we only give the outline of the proof. First, let the relative error function $E_k(x)$ be defined by

$$\Gamma(x+1) = \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt{x+\frac{1}{6}}\left(1+\mathrm{MC}_k(x)\right)\exp(E_k(x)). \tag{3.16}$$

Hence

$$E_k(x) - E_k(x+1) = -1 + x\ln\left(1+\frac{1}{x}\right) + \ln\frac{1+\mathrm{MC}_k(x+1)}{1+\mathrm{MC}_k(x)}. \tag{3.17}$$

By making use of *Mathematica* software and Lemma 1, we can prove

$$\mu(E_k(x)) = 2k+5, \quad (k = 0, 1, 2, 3). \tag{3.18}$$

Next we let $g_k(x) = E_k(x) - E_k(x+1)$. By using *Mathematica* software, it isn't difficult to check that

$$\begin{aligned}
g_0''(x) &= \frac{\Psi_1(14;x)(x-13) + 29707\cdots 81369}{x(1+x)^2(1+6x)^2(7+6x)^2\Psi_2(16;x)} > 0, \quad x \ge 13,\\
g_1''(x) &= -\frac{\Psi_3(20;x)(x-1) + 13798\cdots 89479}{x(1+x)^2(1+6x)^2(7+6x)^2\Psi_4(24;x)} < 0, \quad x \ge 1,\\
g_2''(x) &= \frac{\Psi_5(26;x)(x-6) + 97250\cdots 34321}{x(1+x)^2(1+6x)^2(7+6x)^2\Psi_6(32;x)} > 0, \quad x \ge 6,\\
g_3''(x) &= -\frac{\Psi_7(32;x)(x-1) + 836559\cdots 37479}{x(1+x)^2(1+6x)^2(7+6x)^2\Psi_8(40;x)} < 0, \quad x \ge 1.
\end{aligned}$$

Lastly, just as the proof of Theorem1, Theorem 2 follows from the above inequalities readily. □

4 Ramanujan-type inequalities

Theorem 3. *Let the* k*-th correction function* $\mathrm{MC}_k(x)$ *be defined as*

$$\mathrm{MC}_k(x) = \mathop{\mathbf{K}}_{j=0}^{k} \frac{a_j}{x+b_j}, \tag{4.1}$$

7

where

$$a_0 = -\frac{11}{240}, \qquad b_0 = \frac{79}{154},$$
$$a_1 = \frac{459733}{711480}, \qquad b_1 = -\frac{1455925}{70798882},$$
$$a_2 = \frac{49600874140433}{101450127018720}, \qquad b_2 = \frac{10259108965771635091}{19545564575317443762},$$
$$a_3 = \frac{1690853053361525271315110039 63}{101221579151797375403194730976}, \quad b_3 = -\frac{614144853590800271121992001648883 4171}{203275987838924050801436670299517447102}.$$

Let $x \geq 1$, then for $k = 0, 2$,

$$\Gamma(x+1) < \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1}{30} + \mathrm{MC}_k(x)\right)^{\frac{1}{6}}, \tag{4.2}$$

and for $k = 1, 3$,

$$\Gamma(x+1) > \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1}{30} + \mathrm{MC}_k(x)\right)^{\frac{1}{6}}. \tag{4.3}$$

Proof. We define the relative error function $E_k(x)$ by the relation

$$\Gamma(x+1) = \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1}{30} + \mathrm{MC}_k(x)\right)^{\frac{1}{6}} \exp(E_k(x)). \tag{4.4}$$

Thus

$$\begin{aligned} E_k(x) - E_k(x+1) = & -1 + x\ln\left(1 + \frac{1}{x}\right) \\ & + \frac{1}{6}\ln\frac{8(x+1)^3 + 4(x+1)^2 + (x+1) + \frac{1}{30} + \mathrm{MC}_k(x+1)}{8x^3 + 4x^2 + x + \frac{1}{30} + \mathrm{MC}_k(x)}. \end{aligned} \tag{4.5}$$

By using *Mathematica* software and Lemma 1, we can check

$$\mu(E_k(x)) = 2k + 6, \quad (k = 0, 1, 2, 3). \tag{4.6}$$

We let $U_k(x) = E_k(x) - E_k(x+1)$. By making use of *Mathematica* software again, we can prove

$$U_0''(x) = \frac{\Psi_1(13;x)(x-1) + 41683855850929775426161473171 5717}{3x(1+x)^2(79+154x)^2(233+154x)^2\left(\Psi_{21}(3;x)(x-1) + 363565\right)^2 \Psi_{22}^2(4;x)} < 0,$$
$$U_1''(x) = \frac{\Psi_3(19;x)(x-1) + 85653\cdots 25001}{x(1+x)^2\Psi_4(28;x)} > 0,$$
$$U_2''(x) = -\frac{\Psi_5(25;x)(x-1) + 32968\cdots 13479}{x(1+x)^2\Psi_6(36;x)} < 0,$$
$$U_3''(x) = \frac{\Psi_7(31;x)(x-1) + 17145\cdots 57723}{3x(1+x)^2\Psi_8(44;x)} > 0.$$

Similar to the proof of Theorem 1, we can get the desired assertions from the above inequalities. $\square$

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Theorem 4. *Let the first-correction function* $\mathrm{MC}_1^*(x)$ *be defined by*

$$\mathrm{MC}_1^*(x) = \frac{\kappa_0}{x+\lambda_0} + \frac{\kappa_1}{x^3+\lambda_{10}x^2+\lambda_{11}x+\lambda_{12}}, \tag{4.7}$$

where

$$\kappa_0 = -\frac{11}{240}, \qquad \lambda_0 = \frac{79}{154},$$
$$\kappa_1 = \frac{459733}{15523200}, \qquad \lambda_{10} = \frac{71181889}{70798882},$$
$$\lambda_{11} = \frac{717183502490887}{520777318696096}, \qquad \lambda_{12} = \frac{1118629052995381153799}{1958878792277282473920}.$$

Then for $x \geq 1$, *the following inequality holds true*

$$\Gamma(x+1) < \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3+4x^2+x+\frac{1}{30}+\mathrm{MC}_1^*(x)\right)^{\frac{1}{6}}. \tag{4.8}$$

Proof. Let the first-correction error function $E_1^*(x)$ be defined by

$$\Gamma(x+1) = \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3+4x^2+x+\frac{1}{30}+\mathrm{MC}_1^*(x)\right)^{\frac{1}{6}} \exp(E_1^*(x)). \tag{4.9}$$

Hence

$$\begin{aligned} E_1^*(x) - E_1^*(x+1) = &-1 + x\ln\left(1+\frac{1}{x}\right) \\ &+ \frac{1}{6}\ln\frac{8(x+1)^3+4(x+1)^2+(x+1)+\frac{1}{30}+\mathrm{MC}_1^*(x+1)}{8x^3+4x^2+x+\frac{1}{30}+\mathrm{MC}_1^*(x)}. \end{aligned} \tag{4.10}$$

By using *Mathematica* software and Lemma 1, we have

$$\mu(E_1^*(x)) = 10. \tag{4.11}$$

Now we let $V(x) = E_1^*(x) - E_1^*(x+1)$. By using *Mathematica* again, we have

$$V_1''(x) = -\frac{\Psi_1(33;x)(x-1)+96057\cdots27429}{3x\left(\Psi_2(3;x)\right)^2\Psi_3(12;x)\left(\Psi_4(6;x)(x-1)+2169\cdots3461\right)^2\Psi_5(14;x)} < 0. \tag{4.12}$$

By the same approach as the proof of Theorem 1, the inequality (4.8) follows from the (4.12). □

Remark 2. It is an interesting question whether our method may be used to obtain some sharp bounds for the ratio of the gamma functions, see e.g. [11, 19, 20, 21].

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Mathematical analysis of humoral immunity viral infection model with Hill type infection rate

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Abstract

In this paper, we propose and analyze a viral infection model with humoral immunity. The incidence rate is given by Hill type infection rate. We have derived two threshold parameters, R_0 and R_1 which completely determined the global properties of the model. By constructing suitable Lyapunov functions and applying LaSalle's invariance principle we have established the global asymptotic stability of all steady states of the model. We have proven that, if $R_0 \leq 1$, then the infection-free steady state is globally asymptotically stable (GAS), if $R_1 \leq 1 < R_0$, then the chronic-infection steady state without humoral immune response is GAS, and if $R_1 > 1$, then the chronic-infection steady state with humoral immune response is GAS.

Keywords: Virus infection; Global stability; Immune response; Lyapunov function; Hill type infection rate.

1 Introduction

In recent years, considerable attention has been paid to study the dynamical behaviors of viruses such as human immunodeficiency virus (HIV) (see e.g. [1]-[11]), hepatitis B virus (HBV) [12]-[14], hepatitis C virus (HCV) [15]-[17], human T cell leukemia (HTLV) [18] and dengue virus [19], etc. There are many benifits from mathematical models of viral infection include: (i) it provide important quantitative insights into viral dynamics in vivo, (ii) it can improve diagnosis and treatment strategies which yield to raise hopes of patients with viruses, (iii) it can be used to estimate key parameter values that control the infection process.

Nowak and Bangham [2] proposed the basic viral infection model which contains three variables x, y and v representing the populations of the uninfected target cells, infected cells and free virus particles, respectively. In [20]-[26], the basic model has been modified to take into consideration the humoral immune response. The basic model of viral infection with humoral immune response has

been introduced by Murase et. al. [20] and Shifi Wang [26] as:

$$\dot{x} = \lambda - dx - \beta xv, \tag{1}$$

$$\dot{y} = \beta xv - ay, \tag{2}$$

$$\dot{v} = ky - cv - rzv, \tag{3}$$

$$\dot{z} = gzv - \mu z, \tag{4}$$

where z denotes the population of the B cells. Parameters λ, k and g represent, respectively, the rate at which new healthy cells are generated from the source within the body, the generation rate constant of free viruses produced from the infected cells and the proliferation rate constant of B cells. Parameters d, a, c and μ are the natural death rate constants of the uninfected cells, infected cells, free virus particles and B cells, respectively. Parameter β is the infection rate constant and r is the removal rate constant of the virus due to humoral immune response. All the parameters given in model (1)-(4) are positive.

In model (1)-(4), the incidence rate is supposed to be bilinear, βxv, which is based on the law of mass action. In reality, bilinear incidence rate is not accurate to describe the viral dynamics during the full course of infection. In [27], the incidence rate is given by Hill type infection rate. However, the humoral immune response has been neglected.

Our aim in this paper is to propose a viral infection model with humoral immune response and investigate its global stability analysis. The incidence rate is given by Hill type infection rate. Using Lyapunov functions, we prove that the global stability of the model is determined by two threshold parameters, the basic infection reproduction number R_0 and the humoral immune response activation number R_1. We have proven that, if $R_0 \leq 1$, then the uninfected steady state is globally asymptotically stable (GAS), if $R_1 \leq 1 < R_0$, then the infected steady state without humoral immune response is GAS, and if $R_1 > 1$, then the infected steady state with humoral immune response is GAS.

2 The model

In this section, we propose a viral infection model with humoral immune response. The incidence rate is given by a Hill type infection rate.

$$\dot{x} = \lambda - dx - \frac{\beta x^n v}{\gamma^n + x^n}, \tag{5}$$

$$\dot{y} = \frac{\beta x^n v}{\gamma^n + x^n} - ay, \tag{6}$$

$$\dot{v} = ky - cv - rvz, \tag{7}$$

$$\dot{z} = gvz - \mu z, \tag{8}$$

where γ and n are positive constants. Next, we study the properties of the solutions of the model.

2.1 Positive invariance

We note that model (5)-(8) is biologically acceptable in the sense that no population goes negative. It is straightforward to check the positive invariance of the non-negative orthant $\mathbb{R}^4_{\geq 0}$ by model (5)-(8). In the following, we show the boundednes of the solution of model (5)-(8).

Proposition 1. There exist positive numbers $L_i, i = 1, 2, 3$ such that the compact set $\Omega = \left\{(x, y, v, z) \in \mathbb{R}^4_{\geq 0} : 0 \leq x, y \leq L_1, 0 \leq v \leq L_2, 0 \leq z \leq L_3\right\}$

is positively invariant.

Proof. Let $X_1(t) = x(t) + y(t)$, then

$$\dot{X}_1 = \lambda - dx - ay \leq \lambda - s_1 X_1,$$

where $s_1 = \min\{d, a\}$. Hence $X_1(t) \leq L_1$, if $X_1(0) \leq L_1$, where $L_1 = \dfrac{\lambda}{s_1}$. Since $x(t) > 0$ and $y(t) \geq 0$, then $0 \leq x(t)$, $y(t) \leq L_1$ if $0 \leq x(0) + y(0) \leq L_1$. On the other hand, let $X_2(t) = v(t) + \frac{r}{g} z(t)$, then

$$\dot{X}_2 = ky - cv - \frac{r\mu}{g} z \leq kL_1 - s_2(v + \frac{r}{g} z) = kL_1 - s_2 X_2,$$

where $s_2 = \min\{c, \mu\}$. Hence $X_2(t) \leq L_2$, if $X_2(0) \leq L_2$, where $L_2 = \dfrac{kL_1}{s_2}$. Since $v(t) \geq 0$ and $z(t) \geq 0$, then $0 \leq v(t) \leq L_2$ and $0 \leq z(t) \leq L_3$ if $0 \leq v(0) + \frac{r}{g} z(0) \leq L_2$, where $L_3 = \frac{gL_2}{r}$.

2.2 Steady states

In this subsection, we calculate the steady states of model (5)-(8) and derive two thresholds parameters. The steady states of model (5)-(8) satisfy the following equations:

$$\lambda - dx - \frac{\beta x^n v}{\gamma^n + x^n} = 0, \tag{9}$$

$$\frac{\beta x^n v}{\gamma^n + x^n} - ay = 0, \tag{10}$$

$$ky - cv - rvz = 0, \tag{11}$$

$$(gv - \mu)z = 0. \tag{12}$$

Equation (12) has two possible solutions, $z = 0$ or $v = \mu/g$. If $z = 0$, then from Eqs. (10)-(11) we obtain

$$\frac{k\beta x^n v}{a(\gamma^n + x^n)} - cv = 0. \tag{13}$$

Equation (13) has two possibilities, $v = 0$ or $v \neq 0$. If $v = 0$, then $y = 0$ and $x = \frac{\lambda}{d}$ which leads to the uninfected steady state $E_0 = (x_0, 0, 0, 0)$, where $x_0 = \frac{\lambda}{d}$. If $v \neq 0$, then from Eqs. (9) and (13) we obtain

$$v = \frac{k}{ac} \frac{\beta x^n v}{\gamma^n + x^n} = \frac{k(\lambda - dx)}{ac} \tag{14}$$

$$\Rightarrow x = x_0 - \frac{ac}{dk} v. \tag{15}$$

Then, Eq. (13) becomes

$$\frac{k\beta\left(x_0-\frac{ac}{dk}v\right)^n v}{a\gamma^n+a\left(x_0-\frac{ac}{dk}v\right)^n}-cv=0.$$

Let us define a function Ψ_1 as

$$\Psi_1(v)=\frac{k\beta\left(x_0-\frac{ac}{dk}v\right)^n v}{a\gamma^n+a\left(x_0-\frac{ac}{dk}v\right)^n}-cv=0.$$

It is clear that $\Psi_1(0)=0$, and when $v=\overline{v}=\frac{x_0dk}{ac}>0$, then $\Psi_1(\overline{v})=-c\overline{v}<0$. Since $\Psi_1(v)$ is continuous for all $v\geq 0$, then we have

$$\Psi_1'(0)=c\left(\frac{k}{ac}\frac{\beta x_0^n}{\gamma^n+x_0^n}-1\right).$$

Therefore, if $\Psi_1'(0)>0$ i.e. $\frac{k}{ac}\frac{\beta x_0^n}{\gamma^n+x_0^n}>1$, then there exist a $v_1\in(0,\overline{v})$ such that $\Psi_1(v_1)=0$. From Eq. (11) we obtain $y_1=\frac{c}{k}v_1>0$ and from Eq. (9) we define a function Ψ_2 as:

$$\Psi_2(x)=\lambda-dx-\frac{\beta x^n v_1}{\gamma^n+x^n}=0.$$

We have $\Psi_2(0)=\lambda>0$ and $\Psi_2(x_0)=-\frac{\beta x_0^n}{\gamma^n+x_0^n}v_1<0$. Since $f(x)=\frac{x^n}{\gamma^n+x^n}$ is a strictly increasing function of x, then Ψ_2 is a strictly decreasing function of x, then there exist a unique $x_1\in(0,x_0)$ such that $\Psi_2(x_1)=0$. It means that, an infected steady state without humoral immune response $E_1=(x_1,y_1,v_1,0)$ exists when $\frac{k}{ac}\frac{\beta x_0^n}{\gamma^n+x_0^n}>1$. Now we are ready to define the basic infection reproduction number as:

$$R_0=\frac{k}{ac}\frac{\beta x_0^n}{\gamma^n+x_0^n}.$$

The other possibility of Eq. (12) $z\neq 0$ leads to $v_2=\dfrac{\mu}{g}$. Inserting v_2 in Eq. (9) we define a function Ψ_3 as:

$$\Psi_3(x)=\lambda-dx-\frac{\beta x^n v_2}{\gamma^n+x^n}=0.$$

Note that Ψ_3 is a strictly decreasing function of x. Clearly, $\Psi_3(0)=\lambda>0$ and $\Psi_3(x_0)=-\frac{\beta x_0^n v_2}{\gamma^n+x_0^n}<0$. Thus, there exists a unique $x_2\in(0,x_0)$ such that $\Psi_3(x_2)=0$. It follows from Eq. (11) that,

$$y_2=\frac{\beta x_2^n v_2}{a(\gamma^n+x_2^n)},\quad z_2=\frac{c}{r}\left[\frac{k}{ac}\frac{\beta x_2^n}{\gamma^n+x_2^n}-1\right].$$

Thus $y_2>0$, and if $\frac{k}{acv_2}\frac{\beta x_2^n v_2}{\gamma^n+x_2^n}>1$, then $z_2>0$ when . Now we define the humoral immune response activation number as:

$$R_1=\frac{k}{ac}\frac{\beta x_2^n}{\gamma^n+x_2^n},$$

Hence, z_2 can be rewritten as $z_2=\frac{c}{r}(R_1-1)$. It follows that, there exists an infected steady state with humoral immune response $E_2(x_2,y_2,v_2,z_2)$ when $R_1>1$. Since $x_1<x_0$, then

$$R_1=\frac{k}{ac}\frac{\beta x_2^n}{\gamma^n+x_2^n}<\frac{k}{ac}\frac{\beta x_0^n}{\gamma^n+x_0^n}=R_0.$$

From above we have the following result.

Lemma 1 (i) if $R_0 \leq 1$, then there exists only one positive equilibrium E_0,

(ii) if $R_1 \leq 1 < R_0$, then there exist two positive steady states E_0 and E_1, and

(iii) if $R_1 > 1$, then there exist three positive steady states E_0, E_1 and E_2.

3 Global stability analysis

In this section, we establish the global stability of the three steady states of system (5)-(8) employing the direct Lyapunov method and LaSalle's invariance principle.

Theorem 1. If $R_0 \leq 1$, then E_0 is GAS.

Proof. Define a Lyapunov functional W_0 as follows:

$$W_0 = x - x_0 - \int_{x_0}^{x} \frac{x_0^n(\gamma^n + s^n)}{s^n(\gamma^n + x_0^n)} ds + y + \frac{a}{k} v + \frac{ar}{kg} z.$$

Calculating $\frac{dW_0}{dt}$ along the trajectories of (5)-(8) as:

$$\begin{aligned}
\frac{dW_0}{dt} &= \left(1 - \frac{x_0^n(\gamma^n + x^n)}{x^n(\gamma^n + x_0^n)}\right)\left(\lambda - dx - \frac{\beta x^n v}{\gamma^n + x^n}\right) + \frac{\beta x^n v}{\gamma^n + x^n} - ay \\
&\quad + \frac{a}{k}(ky - cv - rvz) + \frac{ar}{kg}(gvz - \mu z) && (16) \\
&= \lambda\left(1 - \frac{x_0^n(\gamma^n + x^n)}{x^n(\gamma^n + x_0^n)}\right)\left(1 - \frac{x}{x_0}\right) + \frac{\beta x_0^n v}{\gamma^n + x_0^n} - \frac{ac}{k} v - \frac{ar\mu}{kg} z \\
&= \lambda\left(1 - \frac{x_0^n(\gamma^n + x^n)}{x^n(\gamma^n + x_0^n)}\right)\left(1 - \frac{x}{x_0}\right) + \frac{ac}{k}\left(\frac{k}{ac}\frac{\beta x_0^n}{(\gamma^n + x_0^n)} - 1\right) v - \frac{ar\mu}{kg} z \\
&= \frac{\lambda \gamma^n (x^n - x_0^n)(x_0 - x)}{x^n x_0 (\gamma^n + x_0^n)} + \frac{ac}{k}(R_0 - 1)v - \frac{ar\mu}{kg} z. && (17)
\end{aligned}$$

We have $(x^n - x_0^n)(x_0 - x) \leq 0$ for all $x, n > 0$. Then if $R_0 \leq 1$ then $\frac{dW_0}{dt} \leq 0$ for all $x, v, z > 0$. Thus, the solutions of system (5)-(8) converge to Ω, the largest invariant subset of $\left\{\frac{dW_0}{dt} = 0\right\}$ [28]. Clearly, it follows from Eq. (17) that $\frac{dW_0}{dt} = 0$ if and only if $x = x_0$, $v = 0$ and $z = 0$. The set Ω is invariant and for any element belongs to Ω satisfies $v = 0$ and $z = 0$, then $\dot{v} = 0$. We can see from Eq. (7) that $0 = \dot{v} = ky$, and thus $y = 0$. Hence $\frac{dW_0}{dt} = 0$ if and only if $x = x_0$, $y = 0$, $v = 0$ and $z = 0$. From LaSalle's invariance principle, E_0 is GAS.

Theorem 2. If $R_1 \leq 1 < R_0$, then E_1 is GAS.

Proof. We construct the following Lyapunov functional

$$W_1 = x - x_1 - \int_{x_1}^{x} \frac{x_1^n(\gamma^n + s^n)}{s^n(\gamma^n + x_1^n)} ds + y_1 H\left(\frac{y}{y_1}\right) + \frac{a}{k} v_1 H\left(\frac{v}{v_1}\right) + \frac{ar}{kg} z.$$

The time derivative of W_1 along the trajectories of (5)-(8) is given by

$$\frac{dW_1}{dt} = \left(1 - \frac{x_1^n(\gamma^n + x^n)}{x^n(\gamma^n + x_1^n)}\right)\left(\lambda - dx - \frac{\beta x^n v}{\gamma^n + x^n}\right) + \left(1 - \frac{y_1}{y}\right)\left(\frac{\beta x^n v}{\gamma^n + x^n} - ay\right)$$
$$+ \frac{a}{k}\left(1 - \frac{v_1}{v}\right)(ky - cv - rvz) + \frac{ar}{kg}(gvz - \mu z). \tag{18}$$

Applying $\lambda = dx_1 + \frac{\beta x_1^n v_1}{\gamma^n + x_1^n}$ and collecting terms of Eq. (18) we get

$$\frac{dW_1}{dt} = \left(1 - \frac{x_1^n(\gamma^n + x^n)}{x^n(\gamma^n + x_1^n)}\right)(dx_1 - dx) + \frac{\beta x_1^n v_1}{\gamma^n + x_1^n}\left(1 - \frac{x_1^n(\gamma^n + x^n)}{x^n(\gamma^n + x_1^n)}\right)$$
$$- \frac{\beta x^n v}{\gamma^n + x^n}\frac{y_1}{y} + ay_1 - a\frac{yv_1}{v} + \frac{ac}{k}v_1 + \frac{ar}{k}v_1 z - \frac{ar\mu}{kg}z.$$

Using the equilibrium conditions for E_1, $\frac{\beta x_1^n v_1}{\gamma^n + x_1^n} = ay_1$, $cv_1 = ky_1$, we obtain

$$\frac{dW_1}{dt} = dx_1\left(1 - \frac{x_1^n(\gamma^n + x^n)}{x^n(\gamma^n + x_1^n)}\right)\left(1 - \frac{x}{x_1}\right) + \frac{\beta x_1^n v_1}{\gamma^n + x_1^n}\left(1 - \frac{x_1^n(\gamma^n + x^n)}{x^n(\gamma^n + x_1^n)}\right)$$
$$- \frac{\beta x_1^n v_1}{\gamma^n + x_1^n}\frac{x^n(\gamma^n + x_1^n)vy_1}{x_1^n(\gamma^n + x^n)v_1 y} + \frac{\beta x_1^n v_1}{\gamma^n + x_1^n} - \frac{\beta x_1^n v_1}{\gamma^n + x_1^n}\frac{yv_1}{y_1 v} + \frac{\beta x_1^n v_1}{\gamma^n + x_1^n} + \frac{ar}{k}\left(v_1 - \frac{\mu}{g}\right)z$$
$$= \frac{d\gamma^n(x^n - x_1^n)(x_1 - x)}{x^n(\gamma^n + x_1^n)} + \frac{ar}{k}\left(v_1 - \frac{\mu}{g}\right)z$$
$$+ \frac{\beta x_1^n v_1}{\gamma^n + x_1^n}\left[3 - \frac{x_1^n(\gamma^n + x^n)}{x^n(\gamma^n + x_1^n)} - \frac{x^n(\gamma^n + x_1^n)vy_1}{x_1^n(\gamma^n + x^n)v_1 y} - \frac{yv_1}{y_1 v}\right]. \tag{19}$$

Clearly, the first term of Eq. (19) is less than or equal zero. Because the geometrical mean is less than or equal to the arithmetical mean, then the third and fourth terms of Eq. (19) are less than or equal zero. Now we show that if $R_1 \leq 1$ then $v_1 \leq \frac{\mu}{r} = v_2$. This can be achieved if we show that

$$sgn\,(x_2 - x_1) = sgn\,(v_1 - v_2) = sgn\,(R_1 - 1).$$

We have

$$\left(\frac{x_2^n}{\gamma^n + x_2^n} - \frac{x_1^n}{\gamma^n + x_1^n}\right)(x_2 - x_1) > 0, \tag{20}$$

Suppose that, $sgn\,(x_2 - x_1) = sgn\,(v_2 - v_1)$. Using the conditions of the steady states E_1 and E_2 we have

$$(\lambda - dx_2) - (\lambda - dx_1) = \frac{\beta x_2^n v_2}{\gamma^n + x_2^n} - \frac{\beta x_1^n v_1}{\gamma^n + x_1^n}$$
$$= \frac{\beta x_2^n v_2}{\gamma^n + x_2^n} - \frac{\beta x_2^n v_1}{\gamma^n + x_2^n} + \frac{\beta x_2^n v_1}{\gamma^n + x_2^n} - \frac{\beta x_1^n v_1}{\gamma^n + x_1^n}$$
$$= \frac{\beta x_2^n}{\gamma^n + x_2^n}(v_2 - v_1) + \beta v_1\left(\frac{x_2^n}{\gamma^n + x_2^n} - \frac{x_1^n}{\gamma^n + x_1^n}\right),$$

and from inequality (20) we get $sgn\,(x_1 - x_2) = sgn\,(x_2 - x_1)$, which leads to contradiction. Thus, $sgn\,(x_2 - x_1) = sgn\,(v_1 - v_2)$. Using the steady state conditions for E_1 we have $\frac{k}{ac}\frac{\beta x_1^n}{\gamma^n + x_1^n} = 1$, then

$$R_1 - 1 = \frac{k}{ac}\left(\frac{\beta x_2^n}{\gamma^n + x_2^n} - \frac{\beta x_1^n}{\gamma^n + x_1^n}\right).$$

From inequality (20) we get:

$$sgn\left(R_1-1\right)=sgn\left(v_1-v_2\right).$$

It follows that, if $R_1 \leq 1$ then $v_1 \leq \frac{\mu}{r}=v_2$. Therefore, if $R_1 \leq 1$ then $\frac{dW_1}{dt} \leq 0$ for all $x,y,v,z>0$, where the equality occurs at the equilibrium E_1. LaSalle's invariance principle implies the global stability of E_1.

Theorem 3. If $R_1>1$,then E_2 is GAS.

Proof. We construct the following Lyapunov functional

$$W_2=x-x_2-\int\limits_{x_2}^{x}\frac{x_2^n(\gamma^n+s^n)}{s^n(\gamma^n+x_2^n)}ds+y_2H\left(\frac{y}{y_2}\right)+\frac{a}{k}v_2H\left(\frac{v}{v_2}\right)+\frac{ar}{kg}z_2H\left(\frac{z}{z_2}\right).$$

We calculate the time derivative of W_2 along the trajectories of (5)-(8) as:

$$\begin{aligned}\frac{dW_2}{dt}&=\left(1-\frac{x_2^n(\gamma^n+x^n)}{x^n(\gamma^n+x_2^n)}\right)\left(\lambda-dx-\frac{\beta x^n v}{\gamma^n+x^n}\right)+\left(1-\frac{y_2}{y}\right)\left(\frac{\beta x^n v}{\gamma^n+x^n}-ay\right)\\&\quad+\frac{a}{k}\left(1-\frac{v_2}{v}\right)(ky-cv-rvz)+\frac{ar}{kg}\left(1-\frac{z_2}{z}\right)(gvz-\mu z).\end{aligned} \tag{21}$$

Applying $\lambda=dx_2+\frac{\beta x_2^n v_2}{\gamma^n+x_2^n}$ and collecting terms of Eq. (21) we get

$$\begin{aligned}\frac{dW_2}{dt}&=\left(1-\frac{x_2^n(\gamma^n+x^n)}{x^n(\gamma^n+x_2^n)}\right)(dx_2-dx)+\frac{\beta x_2^n v_2}{\gamma^n+x_2^n}\left(1-\frac{x_2^n(\gamma^n+x^n)}{x^n(\gamma^n+x_2^n)}\right)\\&\quad+\frac{\beta x_2^n v}{\gamma^n+x_2^n}-\frac{\beta x^n v}{\gamma^n+x^n}\frac{y_2}{y}+ay_2-\frac{ac}{k}v-a\frac{yv_2}{v}\\&\quad+\frac{ac}{k}v_2+\frac{ar}{k}v_2z-\frac{ar\mu}{kg}z-\frac{ar}{k}z_2v+\frac{ar\mu}{kg}z_2.\end{aligned}$$

Using the equilibrium conditions for E_2

$$\frac{\beta x_2^n v_2}{\gamma^n+x_2^n}=ay_2,ky_2=cv_2+rv_2z_2,\ \mu=gv_2,$$

we get

$$\begin{aligned}\frac{dW_2}{dt}&=dx_2\left(1-\frac{x_2^n(\gamma^n+x^n)}{x^n(\gamma^n+x_2^n)}\right)\left(1-\frac{x}{x_2}\right)+\frac{\beta x_2^n v_2}{\gamma^n+x_2^n}\left(1-\frac{x_2^n(\gamma^n+x^n)}{x^n(\gamma^n+x_2^n)}\right)\\&\quad-\frac{\beta x_2^n v_2}{\gamma^n+x_2^n}\frac{x^n(\gamma^n+x_2^n)vy_2}{x_2^n(\gamma^n+x^n)v_2y}+\frac{\beta x_2^n v_2}{\gamma^n+x_2^n}-\frac{\beta x_2^n v_2}{\gamma^n+x_2^n}\frac{yv_2}{y_2v}+\frac{\beta x_2^n v_2}{\gamma^n+x_2^n}\\&=\frac{d\gamma^n(x^n-x_2^n)(x_2-x)}{x^n(\gamma^n+x_2^n)}+\frac{\beta x_2^n v_2}{\gamma^n+x_2^n}\left[3-\frac{x_2^n(\gamma^n+x^n)}{x^n(\gamma^n+x_2^n)}-\frac{x^n(\gamma^n+x_2^n)vy_2}{x_2^n(\gamma^n+x^n)v_2y}-\frac{yv_2}{y_2v}\right].\end{aligned} \tag{22}$$

Thus, if $R_1>1$ then x_2,y_2,v_2 and $z_2>0$. Clearly, we get that the first and second terms of Eq. (22) are less than or equal zero. Since the arithmetical mean is greater than or equal to the geometrical mean, then $\frac{dW_2}{dt} \leq 0$. It can be seen that, $\frac{dW_2}{dt}=0$ if and only if $x=x_2$, $y=y_2$ and $v=v_2$. From Eq. (7), if $v=v_2$ and $y=y_2$, then $\dot{v}=0$ and $0=ky_2-cv_2-rv_2z$, which yields $z=z_2$ and hence $\frac{dW_2}{dt}$ equal to zero at E_2. LaSalle's invariance principle implies global stability of E_2.

4 Conclusion

In this paper, we have proposed and analyzed a viral infection model with humoral immune response. The model is a four dimensional that describe the interaction between the uninfected target cells, infected cells, free virus particles and B cells. The incidence rate has been represented by Hill type infection rate. We have derived two threshold parameters, the basic reproduction number R_0 and the humoral immune response number R_1 which completely determined the basic and global properties of the viral infection model. Using Lyapunov method and applying LaSalle's invariance principle we have proven that if $R_0 \leq 1$, then the uninfected steady state is GAS, if $R_1 \leq 1 < R_0$, then the infected steady state without humoral immune response is GAS, and if $R_1 > 1$, then the infected steady state with humoral immune response is GAS.

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A Parameterized Series Representation for Apéry's Constant $\zeta(3)$

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Abstract. We prove that if $\lambda \leq 1/2$, then

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=1}^{n} \binom{n}{k} (-\lambda)^{n-k} \delta_k$$

with

$$\delta_k = \frac{H_k}{k^2} - \frac{1}{k}\left(\frac{\pi^2}{6} - H_k^{(2)}\right),$$

where H_k and $H_k^{(2)}$ denote the harmonic numbers and the generalized harmonic numbers of order 2, respectively.

Keywords. Apéry's constant, series representation, harmonic numbers.

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1. Introduction

The famous Riemann zeta function is defined for all complex numbers s with $\Re s > 1$ by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots . \tag{1}$$

In this note we are concerned with the special case $s = 3$, that is, with

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1.20205\ldots .$$

This number is known in the literature as Apéry's constant. It is named after the Greek-French mathematician Roger Apéry (1916–1994), who proved in 1979 that $\zeta(3)$ is irrational; see [4]. A central role in his proof is played by the elegant series representation

$$\zeta(3) = \frac{5}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}}.$$

Apéry's constant has been the subject of much attention. It appears in the solution of finding sharp bounds for the Mathieu series $\sum_{n=1}^{\infty} 2n(n^2+r^2)^{-2}$, it has applications in physics and it also occurs in the solution of probability problems; see [1], [8] and [10, A002117]. Euler, Ramanujan and numerous other researchers provided various integral and series representations for $\zeta(3)$ and related constants. We refer to Srivastava's survey paper [12] and the references therein; see also [2].

As is well-known, Euler proved that the numbers $\zeta(2n)$ $(n = 1, 2, 3, \ldots)$ are irrational. Thus, it is natural to ask whether the values $\zeta(2n+1)$ $(n = 2, 3, 4, \ldots)$ are also irrational. This is a classical open problem. Recent progress on this subject was made by Rivoal [9], who established that infinitely many of the numbers $\zeta(2n+1)$ are irrational, and Zudilin [14], who proved that at least one of the numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational.

It is the aim of this note to present a new singly-parameterized series representation for $\zeta(3)$ in terms of the classical harmonic numbers

$$H_k = \sum_{j=1}^{k} \frac{1}{j} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} \quad (k = 1, 2, \ldots).$$

and the generalized harmonic numbers of order 2

$$H_k^{(2)} = \sum_{j=1}^{k} \frac{1}{j^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} \quad (k = 1, 2, \ldots).$$

Our method of proof, which can be used to obtain series representations for other mathematical constants as well, is explained in detail in [3]. A key

role is played by the remarkable integral representation

$$\zeta(3) = \frac{1}{2}\int_0^1 \frac{\log t \, \log(1-t)}{t\,(1-t)}dt, \tag{2}$$

which was published by Janous [7] in 2006.

2. Main Result

The following series representation for Apéry's constant is valid.

Theorem. *Let λ be a real number with $\lambda \le 1/2$. Then,*

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=1}^{n} \binom{n}{k} (-\lambda)^{n-k} \delta_k \tag{3}$$

with

$$\delta_k = \frac{H_k}{k^2} - \frac{1}{k}\left(\frac{\pi^2}{6} - H_k^{(2)}\right). \tag{4}$$

Proof. Let $\lambda \le 1/2$ and $0 < t < 1$. Then,

$$-1 < \frac{t-\lambda}{1-\lambda} < 1.$$

Expanding in a geometric series, we obtain

$$\frac{1}{1-t} = \frac{1}{1-\lambda}\cdot\frac{1}{1-\frac{t-\lambda}{1-\lambda}} = \frac{1}{1-\lambda}\sum_{n=0}^{\infty}\left(\frac{t-\lambda}{1-\lambda}\right)^n.$$

Since

$$(t-\lambda)^n = \sum_{k=0}^{n}\binom{n}{k} t^k (-\lambda)^{n-k}$$

(where $(-\lambda)^{n-k} = 1$ if $\lambda = n-k = 0$), we find that

$$\frac{1}{1-t} = \sum_{n=0}^{\infty}\frac{1}{(1-\lambda)^{n+1}}\sum_{k=0}^{n}\binom{n}{k} t^k(-\lambda)^{n-k}.$$

Substituting this into (2) gives

$$\begin{aligned} \zeta(3) &= \frac{1}{2}\int_0^1 \frac{\log t \, \log(1-t)}{t} \sum_{n=0}^{\infty}\frac{1}{(1-\lambda)^{n+1}}\sum_{k=0}^{n}\binom{n}{k} t^k(-\lambda)^{n-k} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(1-\lambda)^{n+1}}\sum_{k=0}^{n}\binom{n}{k}(-\lambda)^{n-k}\delta_k, \end{aligned} \tag{5}$$

where

$$\delta_k = \int_0^1 t^{k-1}\log t \log(1-t)dt.$$

Here if we substitute the series

$$\log(1-t) = -\sum_{\nu=1}^{\infty} \frac{t^\nu}{\nu}$$

we obtain

$$\delta_k = -\sum_{\nu=1}^{\infty} \frac{1}{\nu} \int_0^1 t^{k+\nu-1} \log t\, dt = \sum_{\nu=1}^{\infty} \frac{1}{\nu(k+\nu)^2} \tag{6}$$

using integration by parts.

For $k = 0$ we have

$$\delta_0 = \sum_{\nu=1}^{\infty} \frac{1}{\nu^3} = \zeta(3)$$

and for $k \geq 1$ we find that

$$\delta_k = \sum_{\nu=1}^{\infty} \left(\frac{1}{k^2} \left(\frac{1}{\nu} - \frac{1}{k+\nu} \right) - \frac{1}{k} \frac{1}{(k+\nu)^2} \right) = \frac{H_k}{k^2} - \frac{\zeta(2) - H_k^{(2)}}{k}.$$

Applying (5) gives

$$\begin{aligned} \zeta(3) &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \left((-\lambda)^n \delta_0 + \sum_{k=1}^{n} \binom{n}{k} (-\lambda)^{n-k} \delta_k \right) \\ &= \frac{\delta_0}{2(1-\lambda)} \sum_{n=0}^{\infty} \left(\frac{-\lambda}{1-\lambda} \right)^n + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=1}^{n} \binom{n}{k} (-\lambda)^{n-k} \delta_k. \end{aligned}$$

Since

$$\frac{\delta_0}{2(1-\lambda)} \sum_{n=0}^{\infty} \left(\frac{-\lambda}{1-\lambda} \right)^n = \frac{1}{2}\delta_0 = \frac{1}{2}\zeta(3),$$

and $\zeta(2) = \pi^2/6$, we conclude that (3) is valid with δ_k as given in (4). □

3. Examples

We consider the cases $\lambda = 0, -1, \pm 1/2$, and $1/4$.

Example 1. The special case $\lambda = 0$ leads to the representation

$$\zeta(3) = \sum_{n=1}^{\infty} \delta_n = \sum_{n=1}^{\infty} \left(\frac{H_n}{n^2} - \frac{\zeta(2) - H_n^{(2)}}{n} \right).$$

This formula can also be proved without using the Theorem. In fact, the last expression can be written as the difference of two series whose terms all

cancel, except for those in series (1) for $\zeta(3)$. Indeed,

$$\begin{aligned}\sum_{n=1}^{\infty}\delta_n &= \sum_{i=1}^{\infty}\frac{H_i}{i^2} - \sum_{j=1}^{\infty}\frac{\zeta(2)-H_j^{(2)}}{j} \\ &= \sum_{i=1}^{\infty}\frac{1}{i^2}\sum_{j=1}^{i}\frac{1}{j} - \sum_{j=1}^{\infty}\frac{1}{j}\sum_{i=j+1}^{\infty}\frac{1}{i^2} \\ &= \sum_{i\geq j\geq 1}\frac{1}{i^2 j} - \sum_{i>j\geq 1}\frac{1}{i^2 j} \\ &= \sum_{i=j\geq 1}\frac{1}{i^2 j} = \sum_{i\geq 1}\frac{1}{i^3} = \zeta(3),\end{aligned} \tag{7}$$

as claimed.

If we combine this with (6) and reverse the order of summation, we get

$$\zeta(3) = \sum_{n=1}^{\infty}\delta_n = \sum_{\nu=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\nu(n+\nu)^2} = \sum_{\nu=1}^{\infty}\frac{\zeta(2)-H_\nu^{(2)}}{\nu}.$$

Together with (7), this proves Euler's famous relation [5]

$$\sum_{i=1}^{\infty}\frac{H_n}{n^2} = 2\zeta(3).$$

Example 2. The case $\lambda = -1$ yields

$$2\zeta(3) = \sum_{n=1}^{\infty}\frac{1}{2^n}\sum_{k=1}^{n}\binom{n}{k}\delta_k.$$

This may be compared to the series

$$\frac{3}{2}\,\zeta(3) = \sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{(k+1)^3}$$

which in turn is the case $s = 3$ of a global series for $\zeta(s)$ due to Hasse [6] and rediscovered in [11].

Example 3. The cases $\lambda = 1/2, -1/2, 1/4$ give

$$\begin{aligned}\zeta(3) &= \sum_{n=1}^{\infty}(-1)^n\sum_{k=1}^{n}(-1)^k\binom{n}{k}2^{k+1}\delta_k, \\ \frac{3}{2}\zeta(3) &= \sum_{n=1}^{\infty}\frac{1}{3^n}\sum_{k=1}^{n}\binom{n}{k}2^k\,\delta_k, \\ \frac{3}{4}\zeta(3) &= \sum_{n=1}^{\infty}\frac{1}{(-3)^n}\sum_{k=1}^{n}\binom{n}{k}(-4)^k\,\delta_k,\end{aligned}$$

respectively.

4. Concluding Remarks

We conclude the paper with three remarks.

Remark 1. If we multiply both sides of (3) by $(1-\lambda)^{a+1}$ $(a \in \mathbb{R})$ and differentiate with respect to λ, then we find that

$$(a+1)\,\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{(1-\lambda)^{n+1}} \sum_{k=1}^{n} \binom{n}{k} (-\lambda)^{n-k-1} \big(n - a\lambda + (\lambda-1)k\big)\,\delta_k.$$

Applying this with $a=1$, $\lambda=-1/4$ and (3) with $\lambda=-1/4$ yields

$$\frac{5}{4}\,\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{5^n} \sum_{k=1}^{n} \binom{n}{k} 4^k (4n-5k)\,\delta_k.$$

Remark 2. Using the asymptotic formulas

$$H_k \sim \log k \quad \text{and} \quad \zeta(2) - H_k^{(2)} \sim \frac{1}{k} \quad (k \to \infty)$$

we obtain

$$\delta_k \sim \frac{\log k}{k^2} \quad (k \to \infty).$$

For $k = 1, 2, \dots, 10$, we have the values

$$\delta_k = 0.35506\ldots,\ 0.17753\ldots,\ 0.10909\ldots, 0.07487\ldots,\ 0.05506\ldots,$$
$$0.04246\ldots,\ 0.03389\ldots,\ 0.02777\ldots,\ 0.02324\ldots,\ 0.01977\ldots\ .$$

Remark 3. Applying the series representation (6) and [13, Theorem 11d] we conclude that the sequence $\{\delta_k\}_{k=0}^{\infty}$ is not only decreasing and convex but even *completely monotonic*, that is,

$$(-1)^n\,\Delta^n \delta_k \geq 0 \quad \text{for} \quad k, n = 0, 1, 2, ...,$$

where Δ denotes the forward difference operator defined by

$$\Delta^0 \delta_k = \delta_k, \quad \Delta^n \delta_k = \Delta^{n-1}\delta_{k+1} - \Delta^{n-1}\delta_k \quad (k = 0, 1, 2, ...;\ n = 1, 2, ...).$$

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